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Geometric algebra approach to analyzing the singularity of six-DOF parallel mechanism

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Abstract

Singular configurations must be avoided in the practical manipulation of parallel mechanism. This paper presents an approach of singularities analysis of six degree of freedom (DOF) parallel mechanism applying geometric algebra. Twists of all passive joints in each limb are described in geometric algebra while all the active joints are locked. And wrenches produced from each limb and acting on the moving platform are derived from the calculations of the outer product and its dual of the corresponding twists. The singular condition of parallel mechanism depends on whether the wrenches acting on the moving platform failed to constrain all the motions, which can be accomplished by the outer product with its duality followed. Two six DOF parallel mechanisms 6-UPS and 6-PUS are introduced to verify the approach proposed in this paper. The results indicate that geometric algebra can also be used for singularity analysis of 6-DOF parallel mechanism.

Keywords : Singularity analysis, Geometric algebra, Parallel mechanism, Screw, 6-UPS, 6-PUS, Duality

1. Introduction

A parallel mechanism is composed of a moving platform with *n* degrees of freedom (DOF), and of a fixed base, connected by no less than two independent kinematic chains (Merlet, 2006). Because of very good performances, such as higher accuracy, higher stiffness, larger payload capacity, parallel mechanisms have been used in a great deal of applications ranging from parallel manipulators to airplane simulators, and have attracted more and more attention from the academic and industrial communities.

Singularity is the inherent characteristic of parallel mechanism. In singular configurations, the motion behavior of a parallel mechanism degrades to a large extent. And the motions of moving platform in some directions will be uncontrollable. The study of singularity is aimed to avoid these configurations, or to manipulate close to them when required to accomplish certain tasks. There are some methods can be used to analyze the singularity of parallel mechanism, such as Jacobian matrix analysis, Grassmann line geometry and screw theory.

Singular configurations can be identified by calculating the determinant of the Jacobian matrix. The singularity of parallel mechanism had classified into three categories by using Jacobian matrix (Gosselin and Angeles, 1990). And those three kinds of singularities had been renamed to be inverse kinematic singularity, forward kinematic singularity and combined singularity (Tsai, 1999). This classification had been refined and six types of singularities were introduced (Zlatanov et al., 1994). The concept of architecture singularities was introduced by Ma and Angeles (1991). Hayawi et al. (2015) studied the detection and avoiding method of the architecture singularities of 6-DOF parallel mechanism. Choi et al. (2010) proposed an expanded Jacobian matrix to study the singularity of a four-DOF H4 parallel manipulator. Jiang and Gosselin (2009) studied the singularity representation and the maximal singularity-free zones in the six-dimensional workspace of the general Gough–Stewart platform. Cheng et al. (2011) studied the singularity of Stewart-Gough platform based on a new eight order Jacobian matrix. Cao et al. (2013) proposed the Jacobian matrix of Stewart-Gough platform to analysis the singularity and orientation ability. Karimi et al. (2014)

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studied singularity-free workspace analysis of general 6-UPS parallel mechanisms via Jacobian matrix and convex optimization. Corinaldi et al. (2018) identified all kinds of singular configurations of a class of spherical parallel mechanisms by a single 3 ×3 Jacobian matrix.

Screw theory was introduced into studying the singularity of parallel mechanism (Hunt, 1978). Gallardo-Alvarado et al. (2006) applied screw theory in studying the singularity of 4-DOF parallel mechanism. Fang et al. (2012) analyzed the singularity of a 3-DOF parallel mechanism using reciprocal screw theory. Enrique and Rodriguez-Leal (2013) proposed an approach for identification of the singular configurations of the 3-CUP parallel mechanism by applying screw theory. Chen et al. (2015) applied screw theory into analyzing the singularity of lower-mobility parallel mechanisms taking the motion/force transmissibility and constrainability into account. Sánchez-Alonsob (2016) applied the screw theory as a tool for the determination of the singular configurations of a reconfigurable parallel mechanism. Based on screw theory, Liu et al. (2014) studied two types of singularity of 6-UCU parallel manipulator which are caused by both the active joints and passive universal joints. Isaksson et al. (2017) analyzed the singularity of a class of kinematically redundant parallel Schönflies motion generators utilizing screw theory. Khalifa et al. (2017) proposed a geometrical/analytical approach for reciprocal screw-based singularity analysis of an endoscopic surgical parallel manipulator (2-PUU_2-PUS).

Grassmann line geometry is also an important approach of singularity analysis. Merlet (1989) proposed the method of singularity analysis based on Grassmann line geometry. Masouleh and Gosselin (2011) applied Grassmann line geometry to predict the singular configurations of 5-RPUR parallel mechanisms (3T2R). Mbarek et al. (2007) studied the singularity of a five-DOF parallel mechanism based on Grassmann line geometry. Cheng et al. (2012)] applied Grassmann line geometry to analyze the singularity of a parallel hip joint simulator. The defect of Grassmann line geometry method is that it is difficult to express singular loci analytically. Grassmann-Cayley algebra (Aminea et al., 2012; Wen et al., 2017) is also utilized to identifying the singular configurations of parallel mechanisms.

Although the concept of geometric algebra which is called Clifford algebra early, can be traced back to nineteenth century, little attention has been paid to it for a long time, especially after Gibbs created vector analysis (Hestenes, 1999). Hestenes (Li, 2008) launched a new approach to Clifford algebra and formulated a geometric version of Clifford algebra which is called geometric algebra. Many scholars (Dorst et al., 2009; Doran and Lasenby, 2003; Perwass, 2009; Bayro-Corrochano, 2019) have promoted the development of geometric algebra together.

To the best of our knowledge, only a few scholars have applied geometric algebra to the study of singularities of parallel mechanisms. Tanev (2006, 2008) proposed a methodology of geometric algebra for identifying the singularity configurations for lower-mobility parallel mechanisms. Zhang (2008) and Li et al. (2015) studied the singularity of 3-RPS parallel mechanism using geometric algebra. Yao et al. (2017) applied geometric algebra to discussing the singularity of over-constrained 3-RPR parallel mechanisms with general and special structures. Additionally, Kim et al. (2015) studied the singularities of a redundant motion parallel mechanism using conformal geometric algebra.

There are no literatures on the singularity of 6-DOF parallel mechanisms studied by geometric algebra. Reference to Tanev (2006, 2008), Li et al. (2015) and Yao et al. (2017), this paper presents the geometric algebra approach to obtaining the singularity loci of the 6-DOF parallel mechanisms of 6-UPS and 6-PUS. The remainder of this paper is organized as follows: Section 2, necessary knowledge of geometric algebra used in this paper are introduced. Section 3, presented geometric algebraic approach for singularity analysis of 6-DOF parallel mechanisms of 6-UPS and 6-PUS. Section 4, 6-UPS was introduced as an example to verify the approach presented in this paper. Section 5, the approach was verified by another parallel mechanism of 6-PUS. Finally, Section 6 draws the conclusions.

2. Necessary knowledge of geometric algebra

Vector analysis, which originated from geometric algebra, is a mathematical tool tailored for three-dimensional space. But geometric algebra is a more powerful and more applicable tool, and it can be applied in any dimensional spaces.

2.1 Geometric product

Basically, geometric algebra can manipulate elements, such as scalars, vectors, areas and volumes, by the help of a simple and consistent notation. Multi-vectors, which can be operated by addition, subtraction and multiplication, are the combinations of such elements. In this paper the geometric algebra of *n*-dimensional vector space $\boldsymbol{R}^n$ denotes as

$\boldsymbol{G}^n$.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in n-dimensional space $\boldsymbol{R}^n$ whose orthogonal basis are denoted as $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$. Geometric algebra introduces geometric product to unite the inner and outer products into a single product. The geometric product of $\boldsymbol{a}$ and $\boldsymbol{b}$ yields

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{1}$$

where, the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ is the symmetric part which yields a scalar, the outer product $\boldsymbol{a} \wedge \boldsymbol{b}$ is the antisymmetric part which means $\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}$ spans a space orthogonal to vector $\boldsymbol{a}$ and vector $\boldsymbol{b}$, and the sign "+" works like that in complex numbers, where real and imaginary parts are added (Perwass, 2009). So, the inner product and the outer product are defined as follows

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \\ \boldsymbol{a} \wedge \boldsymbol{b} &= \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) \end{aligned} \tag{2}$$

As for $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ are the orthogonal basis, the following conditions should be satisfied

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\ \boldsymbol{e}_i \wedge \boldsymbol{e}_j &= \begin{cases} 0 & i = j \\ \boldsymbol{e}_i \boldsymbol{e}_j & i \neq j \end{cases} \end{aligned} \tag{3}$$

In geometric algebra, a k-blade is the outer product of k vectors, and k is used to distinguishes the grade of a blade. Scalar is a 0-blade of grade 0, vector is a 1-blade of grade 1, the outer product of two vectors is called a bivector or a 2-blade of grade 2,…, and the blade of highest grade, which equals the dimension of the corresponding space, is called the pseudo-scalar which is expressed as $\boldsymbol{I}_n = \boldsymbol{e}_1 \boldsymbol{e}_2 \ldots \boldsymbol{e}_n$.

2.2 Duality

Let $\boldsymbol{M}_k$ denotes a k-blade which is the outer product of k vectors in n-dimensional space $\boldsymbol{R}^n$ and represents a subspace directly. The blade $\boldsymbol{D}_{n\text{-}k}$ which is the dual of $\boldsymbol{M}_k$ represents of the orthogonal complement space of the subspace $\boldsymbol{M}_k$.

$$\boldsymbol{D}_{n\text{-}k} = \boldsymbol{M}_k \boldsymbol{I}_n^{-1} = -\boldsymbol{M}_k \boldsymbol{I}_n \tag{4}$$

where, $\boldsymbol{I}_n = \boldsymbol{e}_1 \boldsymbol{e}_2 \ldots \boldsymbol{e}_n$ is the unit pseudo-scalar of space $\boldsymbol{R}^n$, and $\boldsymbol{I}_n^{-1} = \boldsymbol{e}_n \boldsymbol{e}_{n-1} \ldots \boldsymbol{e}_1$ is the inversion of $\boldsymbol{I}_n = \boldsymbol{e}_1 \boldsymbol{e}_2 \ldots \boldsymbol{e}_n$.

2.3 Expressions of screw in geometric algebra

In screw theory, any twist can be expressed as

$$\mathcal{S} = [\boldsymbol{u}; \boldsymbol{r} \times \boldsymbol{u} + h\boldsymbol{u}]^T \tag{5}$$

where, $\boldsymbol{u} = (u_x \quad u_y \quad u_z)^T$ is unit direction vector along the line, $\boldsymbol{r} = (r_x \quad r_y \quad r_z)^T$ is the position vector of an arbitrary point on the line, $\boldsymbol{r} \times \boldsymbol{u}$ is a moment vector, and h is the pitch which is a non-negative quantity.

When the same twist expressed in the geometric algebra $\boldsymbol{G}^3$, yields

$$\mathcal{S} = \boldsymbol{u} + \boldsymbol{r} \wedge \boldsymbol{u} + h\boldsymbol{I}_3 \boldsymbol{u} \tag{6}$$

where, $\boldsymbol{I}_3 = \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_3$ is the unit pseudo-scalar of geometric algebra $\boldsymbol{G}^3$, and the vectors $\boldsymbol{r}$ and $\boldsymbol{u}$ can be expressed as follows

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$$\boldsymbol{r} = r_x\boldsymbol{e}_1 + r_y\boldsymbol{e}_2 + r_z\boldsymbol{e}_3$$
$$\boldsymbol{u} = u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3 \tag{7}$$

When $h = 0$, Eq.(6) represents a pure revolute joint, turned to be

$$\begin{aligned}\boldsymbol{\$} &= \boldsymbol{u} + \boldsymbol{r} \wedge \boldsymbol{u} \\ &= u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3 + (r_x\boldsymbol{e}_1 + r_y\boldsymbol{e}_2 + r_z\boldsymbol{e}_3) \wedge (u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3) \\ &= u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3 + v_x\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 + v_y\boldsymbol{e}_3 \wedge \boldsymbol{e}_1 + v_z\boldsymbol{e}_1 \wedge \boldsymbol{e}_2\end{aligned} \tag{8}$$

where, $v_x = r_y u_z - r_z u_y, v_y = r_z u_x - r_x u_z, v_z = r_x u_y - r_y u_x$. It is clearly that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_2 \wedge \boldsymbol{e}_3$, $\boldsymbol{e}_3 \wedge \boldsymbol{e}_1$, $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$ are orthogonal to each other. So, the screw $\boldsymbol{\$}$ can be re-expressed in geometric algebra $\boldsymbol{G}^6$ as a vector R^6, by designating $\boldsymbol{e}_4 = \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$, $\boldsymbol{e}_5 = \boldsymbol{e}_3 \wedge \boldsymbol{e}_1$, $\boldsymbol{e}_6 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2$, yields

$$\boldsymbol{\$} = \boldsymbol{u} + \boldsymbol{r} \wedge \boldsymbol{u} = u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3 + v_x\boldsymbol{e}_4 + v_y\boldsymbol{e}_5 + v_z\boldsymbol{e}_6 \tag{9}$$

When $h = \infty$, Eq.(6) represents a pure prismatic pair, and the components of both vector $\boldsymbol{r}$ and vector $\boldsymbol{u}$ are far less than pitch, so the influence of vector $\boldsymbol{r}$ and moment $\boldsymbol{r} \times \boldsymbol{u}$ on pitch can be neglected. In this case, the twist is normalized to be

$$\begin{aligned}\boldsymbol{\$} &= \boldsymbol{I}_3\boldsymbol{u} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3(u_x\boldsymbol{e}_1 + u_y\boldsymbol{e}_2 + u_z\boldsymbol{e}_3) \\ &= u_x\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 + u_y\boldsymbol{e}_3 \wedge \boldsymbol{e}_1 + u_z\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \\ &= u_x\boldsymbol{e}_4 + u_y\boldsymbol{e}_5 + u_z\boldsymbol{e}_6\end{aligned} \tag{10}$$

When $h \neq 0$ and $h \neq \infty$, it represents helix pair. Other kinematic pairs, such as universal joint, spherical joint and cylindrical pair, can be modeled by the revolute joints, or prismatic pairs or both of them.

3. Geometric algebra approach for singularity analysis of 6-DOF parallel mechanism

The moving platform of a parallel mechanism is connected to the fixed base by several independent kinematic chains or limbs. It is well known that there are six-DOF for a free object in three-dimensional space. Twists of the moving platform which is used to describe the motion of the moving platform, and wrenches of the moving platform which is used to describe the forces and moments acting on it are orthogonal to each other and span a six-dimensional space. It is widely known that when all active joints are locked, the parallel mechanism would degenerate into a structure. Under this circumstance, all the motions of the moving platform will be constrained by the wrenches (forces and moments come from the actuated limbs) applied to it. In the other word, if the wrenches failed to constrain all the motions, the mechanism is singular. As far as the six-DOF parallel mechanism is concerned, the wrenches acting on the moving platform should constitute a six-screw system. So, the singularity can be identified by checking whether the wrenches are linearly depended or not. Fortunately, this task can be accomplished only by the addition and multiplication in geometric algebra $\boldsymbol{G}^6$. In this paper, two types of six-DOF parallel mechanisms 6-UPS and 6-PUS are studied by geometric algebra.

3.1 Wrenches originated from actuated limbs

The two six-DOF parallel mechanisms 6-UPS and 6-PUS are actuated by the prismatic pair of each limbs. When the corresponding prismatic pair is locked, both the UPS chain and the PUS chain turn to be US kinematic chain as shown in Fig.1. The motions of both universal joint and spherical joint are expressed as the five twists $\boldsymbol{\$}_i (i = 1 \sim 5)$. The twists $\boldsymbol{\$}_1$ and $\boldsymbol{\$}_2$ are used to describing the universal joint U, which can be can be decomposed into two revolute joints whose axes are perpendicular to each other. While the twists $\boldsymbol{\$}_3$, $\boldsymbol{\$}_4$ and $\boldsymbol{\$}_5$ are used to describing the spherical joint, which can be expressed by three revolute joints whose axes perpendicular to each other. The five twists $\$_i (i=1\sim5)$ should be use the expressions of screw in geometric algebra as Fig.1.

According to the screw theory, the twists and the wrenches of a kinematic pair should be reciprocal to each other, i.e. the reciprocal product of the twists and wrenches equal zero. In the other word, the motion of a kinematic pair and its constraints should lie in two spaces, which are orthogonal to each other, respectively. As for the wrench of a limb, which is used to drive the moving platform, should perpendicular to all the axes of the passive joints, and can be obtained by solving the reciprocal screws of the twists of all the passive joints. Generally, the conventional methods of solving homogeneous linear equations and multi-dimensional null space and partitioning approach (Dai, 2003) can be used. This procedure is relatively complex in screw theory. However, it is very simple when manipulated in geometric algebra in which only addition and multiplication are needed to calculate the outer products.

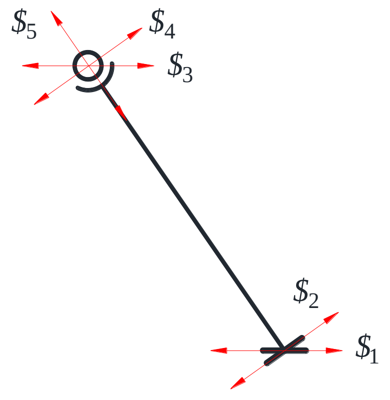

Fig. 1 Sketch of the kinematic chain UPS or PUS with prismatic pair locked. Because the active joint (prismatic pair) is locked, both the kinematic chains UPS and PUS turn to be the kinematic chain US. The universal joint U is described by two revolute joints $\$_1$ and $\$_2$ whose axes are perpendicular to each other. The spherical joint is represented by three revolute joints $\$_3$, $\$_4$ and $\$_5$ whose axes are orthogonal to each other. While the axes of those five revolute joints are described by the twists $\$_i (i=1\sim5)$, respectively.

As mentioned above, the outer product of the twists $\$_i (i=1\sim5)$ span an orthogonal complement space of the twists $\$_i (i=1\sim5)$ in $\boldsymbol{G}^6$. The direction representation of the outer product of the twists $\$_i (i=1\sim5)$ is a five-blades, while the corresponding dual representation is a vector in $\boldsymbol{G}^6$. So, the wrenches of both the kinematic UPS and PUS with the prismatic pairs are actuated, can be obtained by solving the dual representation of the outer product of the twists $\$_i (i=1\sim5)$ in $\boldsymbol{G}^6$.

The coordinates of the centers of the universal joint and the spherical joint are defined as $\boldsymbol{b}=(b_x \quad b_y \quad b_z)^T$ and $\boldsymbol{a}=(a_x \quad a_y \quad a_z)^T$. Without loss of generality, the axes of the three twists $\$_i (i=3,4,5)$ are orthogonal to each other, and $\$_1$ and $\$_3$, $\$_2$ and $\$_4$, are parallel, respectively. The specific expressions of the twists are as follows

$$
\begin{aligned}
\$_1 &= \boldsymbol{e}_1 + b_z\boldsymbol{e}_5 - b_y\boldsymbol{e}_6 \\
\$_2 &= \boldsymbol{e}_2 - b_z\boldsymbol{e}_4 + b_x\boldsymbol{e}_6 \\
\$_3 &= \boldsymbol{e}_1 + a_z\boldsymbol{e}_5 - a_y\boldsymbol{e}_6 \\
\$_4 &= \boldsymbol{e}_2 - a_z\boldsymbol{e}_4 + a_x\boldsymbol{e}_6 \\
\$_5 &= \boldsymbol{e}_3 + a_y\boldsymbol{e}_4 - a_x\boldsymbol{e}_5
\end{aligned}
\tag{11}
$$

The outer product of the twists $\$_i (i=1\sim5)$ is

$$
\boldsymbol{M}_5 = \$_1 \wedge \$_2 \wedge \$_3 \wedge \$_4 \wedge \$_5 \tag{12}
$$

While, its dual representation is

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$$\begin{aligned} &\boldsymbol{D}_1 = \boldsymbol{M}_5 \boldsymbol{I}_6^{-1} = -\boldsymbol{M}_5 \boldsymbol{I}_6 \\ &\boldsymbol{D}_1 = c_0 (m_x \boldsymbol{e}_1 + m_y \boldsymbol{e}_2 + m_z \boldsymbol{e}_3 + v_x \boldsymbol{e}_4 + v_y \boldsymbol{e}_5 + v_z \boldsymbol{e}_6) \end{aligned} \tag{13}$$

Where, $\boldsymbol{D}_1$ is the elliptic polar screw of the wrench of the driven limbs UPS or PUS which is actuated by its prismatic pair, and $m_x = b_y a_z - b_z a_y$, $m_y = b_z a_x - b_x a_z$, $m_z = b_x a_y - a_x b_y$, $v_x = a_x - b_x$, $v_y = a_y - b_y$ and $v_z = a_z - b_z$, while $c_0 = b_z - a_z$ which means that the wrench will vanish when $c_0 = 0$.

The wrench $\tilde{\boldsymbol{D}}_1$ of the driven limbs UPS or PUS can be obtained by the following transformation

$$\tilde{\boldsymbol{D}}_1 = \Delta \boldsymbol{D}_1 = c_0 (v_x \boldsymbol{e}_1 + v_y \boldsymbol{e}_2 + v_z \boldsymbol{e}_3 + m_x \boldsymbol{e}_4 + m_y \boldsymbol{e}_5 + m_z \boldsymbol{e}_6) \tag{14}$$

Where, Δ is an elliptic polar operator which is introduced by Li et al. (2015). Using Eq.(14), the wrench $\tilde{\boldsymbol{D}}_1$ is derived by interchanging the first three components with last three components of its elliptic polar screw $\boldsymbol{D}_1$ in elliptic space.

3.2 Singularity of the mechanism

As stated before, the outer product can be used to judge if the vectors are linearly independent. If there are common constraint, the outer product of the wrenches acting on the moving platform is always zero. So, common constraint should be used in calculating the outer product only once. An algorithm (Li et al., 2016) is proposed to identifying the common constraints. For six-DOF parallel mechanisms 6-UPS or 6-PUS, if the wrenches acting on the moving platform are linearly independent, the motions of the moving platform vanish if all the prismatic pairs are locked. Otherwise, the six wrenches failed to constrain all the motions, i.e. the mechanism is singular.

$$\begin{aligned} \boldsymbol{S}_f &= \left({}^1\tilde{\boldsymbol{D}}_1 \wedge {}^2\tilde{\boldsymbol{D}}_1 \wedge {}^3\tilde{\boldsymbol{D}}_1 \wedge {}^4\tilde{\boldsymbol{D}}_1 \wedge {}^5\tilde{\boldsymbol{D}}_1 \wedge {}^6\tilde{\boldsymbol{D}}_1 \right) \boldsymbol{I}_6^{-1} \\ &= f(z) f_{\sin} \end{aligned} \tag{15}$$

where, ${}^j\tilde{\boldsymbol{D}}_1$ denotes the wrench originates from the j^{th} driven limb, $\boldsymbol{I}_6^{-1} = \boldsymbol{e}_6 \boldsymbol{e}_5 \boldsymbol{e}_4 \boldsymbol{e}_3 \boldsymbol{e}_2 \boldsymbol{e}_1$ is the inversion of the unit pseudo-scalar in geometric algebra $\boldsymbol{G}^6$. There are structural parameters, orientation parameters and position parameters are involved in the function $f_{\sin}$. As for a specific mechanism, the structural parameters are constants, the function $f_{\sin}$ is determined by both orientation and position parameters. The specific expression of function $f(z)$ yields

$$f(z) = \prod_{j=1}^{6} \left({}^j b_z - {}^j a_z \right) \tag{16}$$

which means that when at least one driven limb lies in horizontal plane or the magnitude of $\boldsymbol{b} - \boldsymbol{a}$ equal zero, the parallel mechanism is singular. Especially, the latter case belongs to first kind of singularity in fact.

While the value of the function $f_{\sin}$ determines the parallel mechanism is singular or not. If the function $f_{\sin}$ equals zero, the second kind of singularity occurs. Now, the task is to check the value of $f_{\sin}$ is zero or not. The second kind of singularity of two six-DOF parallel mechanisms 6-UPS and 6-PUS will be analyzed by the approach of calculating the outer product of the wrenches acting on the moving platform in the following. And when singular loci are expressed symbolically completely without any specific parameters substituted in it, the function $f_{\sin}$'s are same.

4. Singularity analysis of 6-UPS parallel mechanism

The general structure of the 6-UPS parallel mechanism is shown in Fig.2. The moving platform is driven by six extendable limbs $\boldsymbol{A}_j\boldsymbol{B}_j$($j$=1~6), and every one of them is connected to the moving platform by a spherical joint and connected to the fixed base by a universal joint. Two coordinate systems ***O-XYZ*** and ***o-xyz*** are fixed to the base and the moving platform respectively. And the origins are placed to the centers of the base and the moving platform respectively as well.

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4.1 Rotation matrix

The rotation matrix $\boldsymbol{R}$ is used to expressed the orientation of the moving coordinates frame ***o-xyz*** with respect to the static coordinates frame ***O-XYZ***. In this paper, $\boldsymbol{R}$ is described by the Roll-Pitch-Yaw angles (R-P-Y angles), yields

$$\boldsymbol{R} = \begin{pmatrix} c_\alpha c_\beta & -c_\gamma s_\alpha + c_\alpha s_\beta s_\gamma & c_\alpha c_\gamma s_\beta + s_\alpha s_\gamma \\ c_\beta s_\alpha & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & c_\gamma s_\alpha s_\beta - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{pmatrix} \tag{17}$$

where *c* is the abbreviation of cosine, *s* is the abbreviation of sine, and α is the yaw which is a anticlockwise rotation about the ***Z*** axis, β is the pitch which is a anticlockwise rotation about the ***Y*** axis, γ is the roll which is a anticlockwise rotation about the ***X*** axis.

And the position vector $\boldsymbol{P}$ is used to expressed the position of the moving coordinates frame ***o-xyz*** with respect to the static coordinates frame ***O-XYZ***, yields

$$\boldsymbol{P} = (X \quad Y \quad Z)^T \tag{18}$$

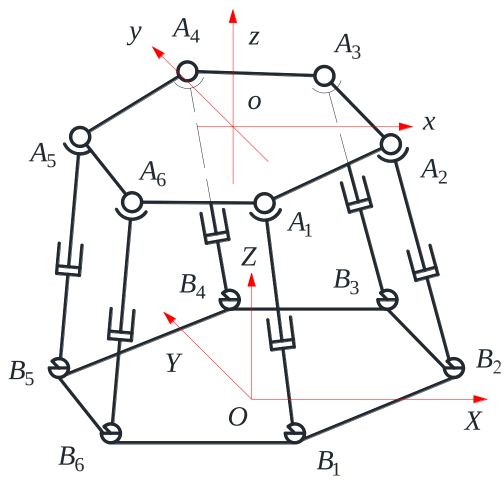

Fig. 2 The structure of the 6-UPS parallel mechanism. The six spherical joints $\boldsymbol{A}_j$(j=1~6), whose centers lie in the same plane, are attached to the moving platform. While the six universal joints $\boldsymbol{B}_j$(j=1~6) are attached to the fixed base with centers lie in another plane. The moving platform is driven by the prismatic pairs of the extendable links $\boldsymbol{A}_j\boldsymbol{B}_j$ (j=1~6). The origins of the static coordinate system ***O-XYZ*** and moving coordinate system ***o-xyz*** are attached to the centers of the fixed base and the moving platform, respectively.

4.2 Position vectors of the universal joints and spherical joints

The postion vectors of centers of those spherical joints $A_j\,(j=1\sim 6)$ with respect to the moving coordinate system ***o-xyz*** can be expressed as follows

$$^{o}\boldsymbol{a}_j = (a_{x,j} \quad a_{y,j} \quad 0)^T \tag{19}$$

where, the pre-super "*o*" indicates that the corressponding vector is expressed in the moving coordinate system ***o-xyz***. The six vectors $^{o}\boldsymbol{a}_j\,(j=1\sim 6)$ should be expressied in the static coordinate system by the following formula

$$\boldsymbol{a}_j = \boldsymbol{R}\cdot {}^{o}\boldsymbol{a}_j + \boldsymbol{P} \qquad (j=1\sim 6) \tag{20}$$

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The postion vectors of centers of those universal joints $B_j\,(j=1\sim 6)$ with respect to the static coordinate system ***O-XYZ*** can be expressed as follows

$$\boldsymbol{b}_j=(b_{x,j}\quad b_{y,j}\quad 0)^T\qquad (j=1\sim 6) \tag{21}$$

4.3 Singularity analysis of 6-UPS parallel mechanism

The wrench ${}^j\tilde{\boldsymbol{D}}_1$ of the j^{th} actuated limb can be obtained by substituting the vectors $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ into Eq.(15). And the analytical expression of singular loci $f_{\sin}$ can be derived from Eq.(15), and its specific expression is very complex and omitted in this paper.

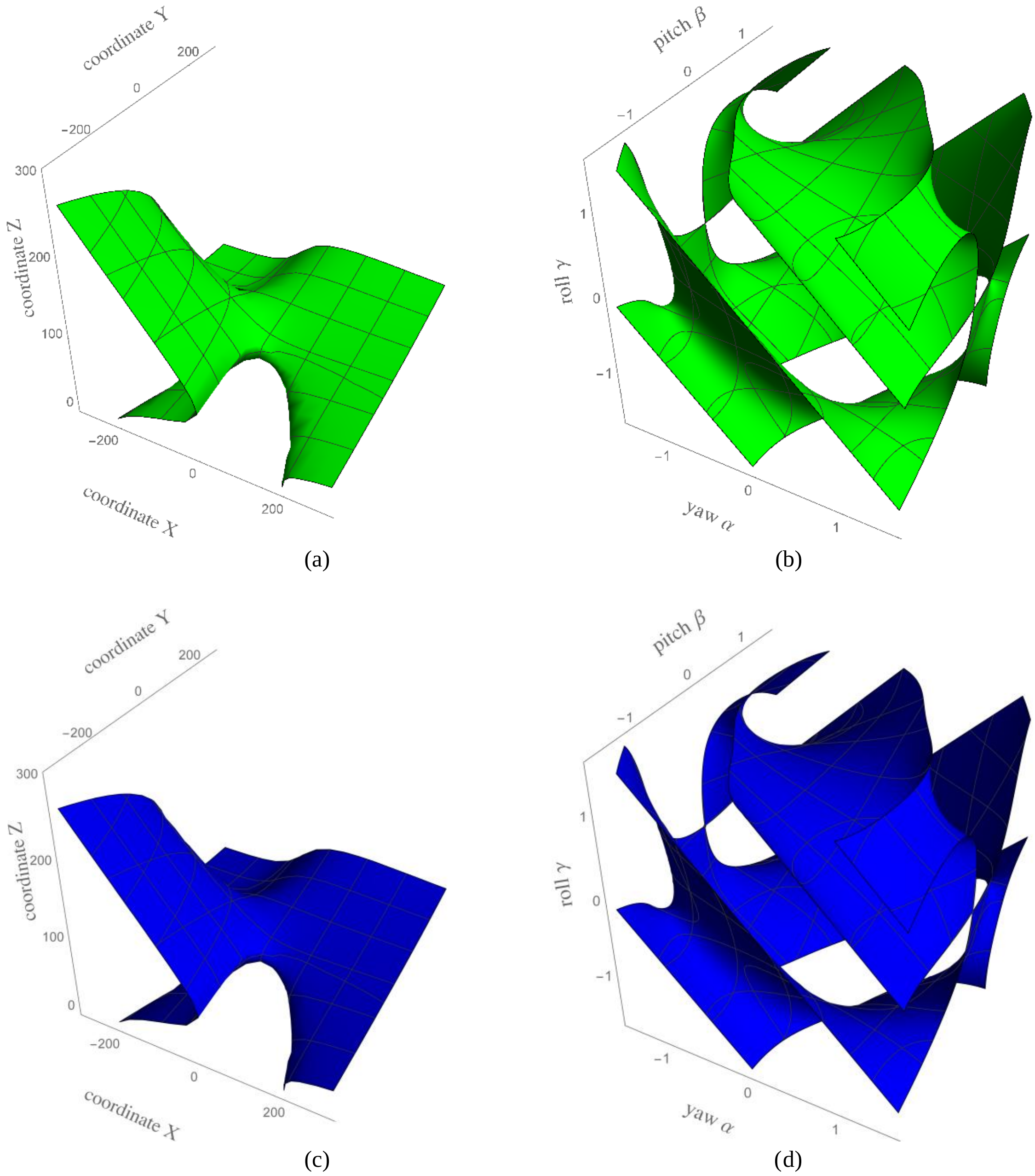

Fig. 3 The singularity loci of 6-UPS parallel mechanism. (a) and (b) are obtained by the approach proposed in this paper, (c) and (d) are obtained by calculating the determinant of the Jacobian matrix. (a) and (c) are the visualizations of the position singularity loci for that the orientation is assumed as $\alpha=0.3\pi$, $\beta=0.15\pi$, $\gamma=0.15\pi$. (b) and (d) are the visualizations of orientation singularity loci for that the position vector is assumed as $\boldsymbol{P}=(0\quad 0\quad 250)^T$.

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Both vectors $^{o}\boldsymbol{a}_j\,(j=1\sim 6)$ and the vectors $\boldsymbol{b}_j\,(j=1\sim 6)$ are determined by the structural parameters of the mechanism. The six vectors $^{o}\boldsymbol{a}_j\,(j=1\sim 6)$ are as follows: $^{o}\boldsymbol{a}_1=(16.00\quad -59.70\quad 0)^T$, $^{o}\boldsymbol{a}_2=(59.70\quad 16.00\quad 0)^T$, $^{o}\boldsymbol{a}_3=(43.70\quad 43.70\quad 0)^T$, $^{o}\boldsymbol{a}_4=(-43.70\quad 43.70\quad 0)^T$, $^{o}\boldsymbol{a}_5=(-59.70\quad 16.00\quad 0)^T$, $^{o}\boldsymbol{a}_6=(-15.00\quad -59.70\quad 0)^T$.

The six vectors $\boldsymbol{b}_j\,(j=1\sim 6)$ are as follows: $\boldsymbol{b}_1=(66.91\quad -74.31\quad 0)^T$, $\boldsymbol{b}_2=(97.81\quad -20.79\quad 0)^T$, $\boldsymbol{b}_3=(30.90\quad 95.11\quad 0)^T$, $\boldsymbol{b}_4=(-30.90\quad 95.11\quad 0)^T$, $\boldsymbol{b}_5=(-97.81\quad -20.79\quad 0)^T$, $\boldsymbol{b}_6=(-66.91\quad -74.31\quad 0)^T$.

The singularity loci, which are obtained by using both the geometric algebra proposed in this paper and the traditional Jacobian matrix method, are visualized in Fig.3. When the orientation are designated as $\alpha=0.3\pi$, $\beta=0.15\pi$ and $\gamma=0.15\pi$, the position singularity loci are shown in Fig.3(a) and Fig.3(c). While Fig.3(a) and Fig.3(c) are the orientation singularity loci corresponding to the position condition that $\boldsymbol{P}=(0\quad 0\quad 250)^T$.

Observing the visualizations of singularity loci in Fig.3, both the position singularity loci and the orientation singularity loci, which are obtained by the geometric algebra approach proposed in this paper, are identical to those which are derived from the traditional Jacobian matrix method. The results verified the validity of geometric algebraic method. So, the geometric algebraic method can be used to analyzing the singularity of 6-UPS parallel mechanism.

5. Singularity analysis of 6-PUS parallel mechanism

A different six-DOF parallel mechanism 6-PUS emerged by changing the sequence of the prismatic pair and the universal joint of every actuated limbs. Because the driving devices are all fixed on the fixed base, 6-PUS has the advantages of lighter mass of the moving components, lower inertia and relatively large workspace. As shown in Fig.4, the moving platform is driven by six identical kinematics chains PUS. The link $\boldsymbol{A}_j\boldsymbol{B}_j$($j$=1~6), of which the length is a constant, is connected to the moving platform by a spherical joint and connected to a slider by a universal joint. While the slider is a part of the prismatic pair which is served as the active joint. And in this paper, all the directions of six prismatic pairs are constructed vertically. The atatic coordinate system ***O-XYZ*** and the moving coordinate system ***o-xyz*** are fixed to the fixed base and the moving platform respectively, with the origins are placed to the centers of the fixed base and the moving platform respectively as well.

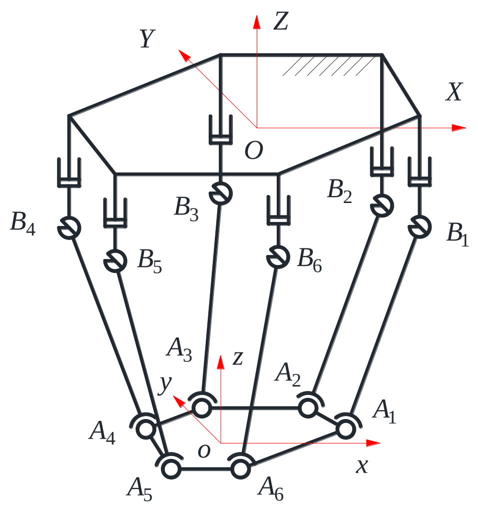

Fig. 4 The structure of the 6-PUS parallel mechanism. The mechanism is driven by the six prismatic pairs which are made up of the guide rails and sliders. The six spherical joints $\boldsymbol{A}_j$(j=1~6), whose centers lie in the same plane, are attached to the moving platform. Every universal joint $\boldsymbol{B}_j$(j=1~6) is attached to the slider, which can be moved along the vertical guide rail. The origins of the static coordinate system ***O-XYZ*** and the moving coordinate system ***o-xyz*** are attached to the centers of the fixed base and the moving platform, respectively.

5.1 Position vectors of universal joints and spherical joints

The centers of all spherical jionts $\boldsymbol{A}_j\,(j=1\sim 6)$ are placed in the same plane. The postion vectors $\boldsymbol{a}_j\,(j=1\sim 6)$, which decibe the centers of spherical jionts $\boldsymbol{A}_j\,(j=1\sim 6)$ in the static coordinate system, can be obtained by Eq.(20). The forms and meaning of the vectors $^{o}\boldsymbol{a}_j\,(j=1\sim 6)$, the rotation matrix $\boldsymbol{R}$ and the position vector $\boldsymbol{P}$ are identical

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to those in Section 4.

While the postion vectors of centers of those universal joints $\boldsymbol{B}_j\,(j=1\sim6)$, decribed in the static coordinate system ***O-XYZ*** yields

$$\boldsymbol{b}_j=(b_{x,j}\quad b_{y,j}\quad -z_j)^T\qquad (j=1\sim6) \tag{22}$$

where, $b_{x,j}$ and $b_{y,j}$ are determined by the structural parameters of the mechanism, while z_j is the input variable which represents the relative translation between the two parts common to this joint and is used to actuated the mechanism.

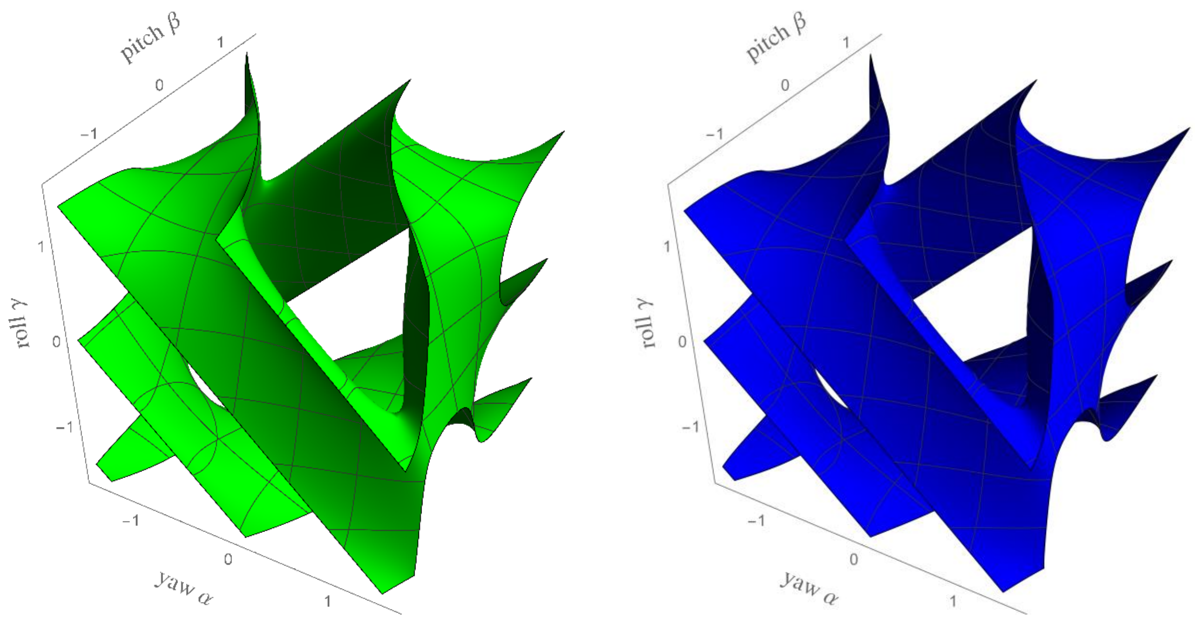
5.2 Singularity analysis of 6-PUS parallel mechanism

As for the magnitudes of links $\boldsymbol{A}_j\boldsymbol{B}_j(j=1\sim6)$ are all constants, it is impossible that the vectors $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ are to be coincided. It is means that the mechanism is singular when at least one of the links $\boldsymbol{A}_j\boldsymbol{B}_j(j=1\sim6)$ lie in the horizontal plane.

The six vectors ${}^o\boldsymbol{a}_j\,(j=1\sim6)$ are defined as follows: ${}^o\boldsymbol{a}_1=(16.95\quad -63.27\quad 0)^T$, ${}^o\boldsymbol{a}_2=(63.27\quad 16.95\quad 0)^T$, ${}^o\boldsymbol{a}_3=(46.32\quad 46.32\quad 0)^T$, ${}^o\boldsymbol{a}_4=(-46.32\quad 46.32\quad 0)^T$, ${}^o\boldsymbol{a}_5=(-63.27\quad 16.95\quad 0)^T$, ${}^o\boldsymbol{a}_6=(-16.95\quad -63.27\quad 0)^T$.

The six vectors $\boldsymbol{b}_j\,(j=1\sim6)$ are defined as follows: $\boldsymbol{b}_1=(91.22\quad -91.22\quad 0)^T$, $\boldsymbol{b}_2=(124.60\quad -33.39\quad 0)^T$, $\boldsymbol{b}_3=(33.39\quad 124.60\quad 0)^T$, $\boldsymbol{b}_4=(-33.39\quad 124.60\quad 0)^T$, $\boldsymbol{b}_5=(-124.60\quad -33.39\quad 0)^T$, $\boldsymbol{b}_6=(-91.22\quad -91.22\quad 0)^T$.

As the previous example, the analytical expression of the singular loci $f_{\sin}=0$ can be derived from Eq.(15), by substituting the wrenches ${}^j\tilde{\boldsymbol{D}}_1\,(j=1\sim6)$ of the six actuated limbs, which can be obtained from Eq.(14) when the vectors $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ are specified, into Eq.(15).



(a) (b)

Fig. 5 The singularity loci with respect to orientation of the 6-PUS parallel mechanism as the position vector is designated as $\boldsymbol{P}=(-10\quad -10\quad -450)^T$. The orientation of the moving platform with respect to the fixed base is described by the Roll-Pitch-Yaw angles. (a) was obtained by the geometric algebra approach proposed in this paper, while (b) was derived from the traditional Jacobian matrix method.

When the orientation is given, the singularity with respect to the position is simple and there is a relatively lager singular free space can be utilized to accomplish the task of manipulation. There is no visualization of position singularity presented in this paper.

When the position is specified, the singularity with respect to the orientation can be visualized in 3-dimensiaonal space easily. Fig.5 shows the singularity loci as the position vector is specified as $\boldsymbol{P}=(-10\quad -10\quad -450)^T$. The

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singularity loci, which is obtained by the geometric algebra approach, is shown in Fig.5(a). While, the singularity loci which is originated from calculating the determinant of the Jacobian matrix is shown in Fig.5(b). Comparing the loci in Fig.(a) and Fig.(b), it can be found that the two figures are identical. The result verifies the correctness of the geometric algebra approach to analyzing the singularity of 6-PUS parallel mechanism.

6. Conclusion

This paper shows that geometric algebra can be used to identifying the singular conditions for six-DOF parallel mechanisms. The approach is verified by two six-DOF parallel mechanisms 6-UPS and 6-PUS, which singularity loci are obtained and identical to the mothed of Jacobian matrix. The results indicate that the analytical expression of singularity loci only relate to linearly independent twists which represent all the passive joints of the mechanisms. As for the actuated limb own at least six-DOF, there are no dummy joints needed. For other types of parallel mechanisms, if there is common constraint in the wrenches acting on the moving platform, common constraint should be calculated only once. Comparing with Jacobian matrix method, neither differentiating the closed vector equation with respect to time, nor eliminating the variables which related to the passive joints, is needed. The geometric algebra approach is computationally advantageous as for only addition and multiplication are needed and can used to identifying the singular conditions of other types of parallel mechanisms, even the over-constrained parallel mechanisms. While the disadvantage is that a large amount of internal memory is required when the complete symbolic computation is performed in process of deriving the singular loci.

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