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Research article

Global dynamics and pattern formation for predator-prey system with density-dependent motion

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Abstract: In this paper, we concern with the predator-prey system with generalist predator and density-dependent prey-taxis in two-dimensional bounded domains. We derive the existence of classical solutions with uniform-in-time bound and global stability for steady states under suitable conditions through the Lyapunov functionals. In addition, by linear instability analysis and numerical simulations, we conclude that the prey density-dependent motility function can trigger the periodic pattern formation when it is monotone increasing.

Keywords: global existence; asymptotic stability; pattern formation

1. Introduction and main results

The dynamical relationship between predators and their prey is one of the dominant themes in ecology. The origin and theory of predator-prey model is due to pioneer work of Lotka and Volterra [1, 2]. There have been a lot of studies on the dynamics of this particular type of model through developing various modifications of mathematical models of prey-predator interactions (e.g., [3–8]).

The non-random foraging strategies in the predator-prey dynamics, prey-taxis allows predators to move towards regions of higher prey density and to search more actively for prey. Such a prey-taxis model was derived by Kareiva and Odell in [9], and they studied predator aggregation in high prey density areas. It can be described as:

$$\begin{cases} u_t = \nabla\cdot(d(v)\nabla u) - \nabla\cdot(u\chi(v)\nabla v) + H_1(u,v), \\ v_t = D\Delta v + H_2(u,v), \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ denote the population density of predators and preys at position x and time t respectively, and D is a positive constant standing for the diffusion rate of the prey. The terms $\nabla\cdot(d(v)\nabla u)$

and $-\nabla \cdot (u\chi(v)\nabla v)$ account for the diffusion of predators with coefficient $d(v)$ and the prey-taxis with coefficient $\chi(v)$ respectively. $H_1(u, v)$ and $H_2(u, v)$ denote the predator-prey interactions. There mainly are three kinds of typical interspecific interactions: predator-prey, competition and mutualism, which can be represented as

$$H_1(u, v) = f(u) + c_1 uF(v), \;\; H_2(u, v) = g(v) - c_2 uF(v),$$

where the functions $f(u)$ and $g(v)$ stand for the intra-specific interactions of predators and prey respectively. The parameters c_1 and c_2 are positive constants representing the coefficients of inter-specific interactions of predators and prey, and $F(v)$ is the so-called functional response function.

In particular, if $\chi(v) = -d'(v)$, the system (1.1) can be written as

$$\begin{cases} u_t = \Delta(d(v)u) + H_1(u, v), \\ v_t = D\Delta v + H_2(u, v), \end{cases} \tag{1.2}$$

the diffusion term $\Delta(d(v)u)$ with $d'(v) < 0$ is called the "density-suppressed motility" (see [10–15]), which can also characterize the incessant tumbling of cells at high concentration, resulting in a vanishing macroscopic motility. Here $d(v)$ is called the motility function, $d'(v) < 0$ means that the predator reduce its motility when encountering the prey. When $H_1(u, v) = 0$, $H_2(u, v) = u - v$ and $d(v) = c_0 v^{-k}$ decays algebraically in v, the solution may exist globally in two or higher dimensions. For example, Yoon and Kim in [16] proved that system (1.2) has a unique global bounded classical solution for any $k > 0$ under a smallness assumption on c_0 in any dimensions. The only global existence result without smallness assumptions was recently given by Ahn and Yoon [17]. Under the assumptions that there exist positive constants $\gamma_1, \gamma_2, \gamma_3$ such that $\gamma_1 \le d(v) \le \gamma_2$ and $|d'(v)| \le \gamma_3$, Tao and Winkler in [18] proved the existence of global classical solutions in the 2-dimensional case and global weak solutions in 3-dimensions. While if $d(v)$ decays exponentially, the solution may blow-up in two dimensions with a critical mass, see [19–21] and so on. For $H_1(u, v) \neq 0$ with logistic growth on $f(u) = \mu u(1 - u)$, there also are many interesting results. The global existence and asymptotic behavior of solutions was first established by Jin, Kim and Wang in [12] under certain conditions on $d(v)$ in two dimensions, which has been developed by many authors, please refer to [19, 22–28] and so on. In the above mentioned references, the authors assumed $d'(v) < 0$. While under normal circumstances, the predators will increase their motilities and keep chasing (such as wolves and sheep) when encountering the prey until they succeed. Therefore, it is meaningful for us to consider the case $d'(v) \ge 0$, and we shall consider general case for $d(v)$ without monotonicity assumptions in this paper.

In this paper, we consider the following density-dependent predator-prey system:

$$\begin{cases} u_t = \Delta(d(v)u) + u(a_1 - b_1 u) + \alpha uF(v), & x \in \Omega, t > 0, \\ v_t = \Delta v + v(a_2 - b_2 v) - uF(v), & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $a_1, a_2 > 0$ represent the intrinsic growth rates of species, $b_1, b_2 > 0$ are the death rates due to intra-specific competition and $\alpha > 0$ denotes the intrinsic predation rate. $F(v)$ is the so-called functional response function accounting for the intake rate of predators as a function of prey density, and $d(v)$ is the motility function as we mentioned above. The most common types $F(v)$ in the literature are

• $F(v) = v$ (Lotka-Volterra type or Holling type I);
• $F(v) = \frac{v}{\lambda+v}$ (Holling type II);
• $F(v) = \frac{v^m}{\lambda^m+v^m}$ (Holling type III) with constants $\lambda > 0$ and $m > 1$.

We make the following assumptions throughout the whole paper:

(H1) $d(v) \in C^3([0,\infty))$ and $d(v) > 0$ on $[0,\infty)$.

(H2) $F(v) \in C^1([0,\infty)), F(0) = 0, F(v) > 0$ in $(0,\infty)$ and $F'(v) > 0$.

In (1.3), the predator is called the specialist predator if $a_1 < 0$, Jin and Wang investigate the global boundedness, asymptotic stability and pattern formation of system (1.3) [29]. They study the dynamic behaviors of the predator and the prey under the condition $d'(v) < 0$. In this paper, we assume $a_1 > 0$ (the corresponding predator is called the generalist predator) and make no assumptions on the monotonicity of $d(v)$. Usually, a generalist species is able to thrive in a wide variety of environmental conditions and can make use of a variety of different resources (for example, a heterotroph with a varied diet). A specialist species can thrive only in a narrow range of environmental conditions or has a limited diet. Most organisms do not all fit neatly into either group. Some species are highly specialized (the most extreme case being monophagous, eating one specific type of food), others less so, and some can tolerate many different environments. In other words, there is a continuum from highly specialized to broadly generalist species. We mainly focus on exploring the global dynamics and spatial-temporal patterns for generalist predators with density-dependent motion for more general motility functions $d(v)$. We mention here Nakashima and Yamada in [30] studied the existence of positive solutions for boundary value problems of nonlinear elliptic systems which arise in the study of the Lotka-Volterra prey-predator models with cross-diffusion in the special case $d(v) = 1 + \alpha v$.

We first derive the global boundedness and existence results for the classical solutions to the system (1.3).

Theorem 1.1 (Global boundedness). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H1)-(H2) hold. Assume $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ with $p > 2$ and $u_0, v_0 \gneqq 0$. Then the problem* (1.3) *has a unique global classical solution $(u, v) \in [C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))]^2$ satisfying $u, v > 0$ for all $t > 0$. Furthermore there exists a constant $C > 0$ independent of t such that*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \le C.$$

For the global stability, except for the hypotheses (H1)-(H2), we also need the following hypothesis on the compound function

$$\phi(v) := \frac{v(a_2 - b_2 v)}{F(v)}. \tag{1.4}$$

(H3) The function $\phi(v)$ is continuously differentiable on $(0,\infty)$, $\phi(0) = \lim_{v\to 0} \phi(v) > 0$ and $\phi'(v) \le 0$ for any $v \ge 0$.

Remark 1.1. *We remark that the hypothesis (H3) is not stringent, and can be satisfied by many forms by imposing some conditions on the parameters if needed. For example, if $F(v)$ is of Holling type I or Holling type II with $a_2 \le b_2\lambda$, then (H3) is automatically satisfied. In general, if (H3) is violated, pattern formations such as periodic orbits or non-constant steady state may arise (see [40]).*

Another relevant question is whether the interacting predator-prey population will arrive at the coexistence, exclusion or extinction eventually, which is always an important topic in population dynamics.

One can easily compute that the system (1.3) has four possible steady states:

$$(u_s, v_s) = (0,0) \text{ or } \left(0, \frac{a_2}{b_2}\right) \text{ or } \left(\frac{a_1}{b_1}, 0\right) \text{ or } (u_*, v_*),$$

where (u_*, v_*) satisfies

$$u_* = \frac{v_*(a_2 - b_2 v_*)}{F(v_*)}, \quad \alpha F(v_*) = b_1 u_* - a_1. \tag{1.5}$$

By constructing suitable Lyapunov functionals, we can obtain the following global stability of the coexistence steady state (u_*, v_*) and semi-trivial steady state $\left(\frac{a_1}{b_1}, 0\right)$. In the context, we define

$$K := \max\left\{\frac{a_2}{b_2}, \|v_0\|_{L^\infty}\right\}. \tag{1.6}$$

Theorem 1.2 (Global stability)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H1)-(H2) hold. Assume $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ with $p > 2$ and $u_0, v_0 \gneqq 0$ and (u, v) is the solution of* (1.3) *obtained in Theorem 1.1.*

(1) If (H3) holds and the parameters satisfy

$$\max_{0 \le v \le K} \frac{u_* F^2(v)|d'(v)|^2}{4\alpha F(v_*) F'(v) d(v)} \le 1,$$

then

$$\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \to 0 \ \text{ as } \ t \to \infty,$$

and (u_*, v_*) satisfies (1.5)*, where K is defined in* (1.6)*. Moreover, there exist some positive constants σ, T_0 and C independent of t such that*

$$\|u(\cdot,t) - u_*\|_{L^\infty} + \|v(\cdot,t) - v_*\|_{L^\infty} \le Ce^{-\sigma t}, \ \ t > T_0.$$

(2) If the parameters satisfy

$$\lim_{v \to 0} \frac{F(v)}{v} \text{ exists, } \min_{\bar{\Omega}} \frac{F(v)}{v} \ge \frac{a_2 b_1}{a_1}, \text{ and } \frac{a_1 a_2 |d'(v)|^2}{2 b_1 b_2 d(v)} \le \frac{1}{\xi_1}, \tag{1.7}$$

then

$$\|u - \frac{a_1}{b_1}\|_{L^\infty} + \|v\|_{L^\infty} \to 0 \ \text{ as } \ t \to \infty,$$

where ξ_1 and K_0 are defined by $\xi_1 = \frac{1}{\alpha} + \frac{b_1 b_2}{K_0^2 \alpha^2}$ and $K_0 = \max_{\bar{\Omega}} \frac{F(v)}{v}$ respectively. Moreover, there exist some positive constants T_1 and C independent of t such that

$$\|u(\cdot,t) - \frac{a_1}{b_1}\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \le \frac{C}{1+t}, \ \ t > T_1.$$

Remark 1.2. *For the semi-trivial steady state $\left(0, \frac{a_2}{b_2}\right)$, in the special case of $F(v) = v$ (Holling type I), we have showed that it is linear unstable in section 5. For more general F, the globally stability for $\left(0, \frac{a_2}{b_2}\right)$ is nontrivial and has to be left open in the current paper.*

In the proof of global existence, the method used in [31] was based on a priori estimates for the energy functional $\int_\Omega u \ln u dx + \int_\Omega |\nabla v|^2 dx$ to attain the L^2 estimates of the solutions. However, such a method of a priori estimates is only applicable for the case where the motility function $d(v)$ is constant. Therefore, the method in [31] is not adaptable to the model (1.3). In this paper, we first derive the L^2 estimates for $|\nabla v|$ and then directly establish the L^2 estimates for u by the Gagliardo-Nirenberg inequality and regularity lemmas, in which we also need to prove the boundedness of $\int_t^{t+\tau}\int_\Omega u^2 dxds$ and $\int_t^{t+\tau}\int_\Omega |\Delta v|^2 dxds$. In the proof, we do not use the property of the self-adjoint realisation of $-\Delta + \delta$ (see [29]). Finally, we derive the boundedness for u by the Moser iteration technique.

If $d(v)$ is constant, the system (1.3) has been studied from many aspects as we mentioned above. If $d(v)$ is non-constant as considered in this paper, we find that the system (1.3) can generate pattern formation as presented in section 5 under the condition $d'(v) > 0$. The pattern formation is obviously different from the ones in [29], in which the authors studied the case $d'(v) < 0$. In our case $a_1 > 0$, the corresponding predator is the generalist predator, the pattern formation may not occur when $d'(v) \le 0$ by linear instability analysis and numerical simulations.

The paper is organized as follows. In section 2, we present the local existence theorem with some preliminary results, and we derive the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ and $\|\nabla v(\cdot,t)\|_{L^2}$. In section 3, we derive the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ by the technique of Moser iteration. In section 4, we construct suitable Lyapunov functionals and use the LaSalle invariance principle to prove the global stability and convergence rate stated in Theorem 1.2. In section 5, we further explore time-periodic patterns by linear instability analysis and numerical simulations.

2. Local existence and preliminaries

In the sequel, we shall use C or C_i to denote a positive generic constant which may vary in the context. Without confusion, the integration variables x and t will be omitted, for instance $\int_0^a\int_\Omega f(x,t)dxdt$ will be abbreviated as $\int_0^a\int_\Omega f(x,t)$. Often $\|f\|_{L^p(\Omega)}$ will be written as $\|f\|_{L^p}$. The existence and uniqueness of local solutions to (1.3), which can be readily proved by the Amann theorem [32, 33] or the well-established fixed pointed argument together with the parabolic regularity theory [12].

Lemma 2.1 (Local existence)**.** *Let the assumptions in Theorem 1.1 hold. Then there exists a constant* $T_{max} \in (0,\infty]$ *such that the problem* (1.3) *admits a unique classical solution*

$$(u,v) \in [C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max}))]^2$$

satisfying $u, v > 0$ *for all* $t > 0$. *Moreover,*

$$if\ \ T_{max} < \infty,\ then\ \limsup_{t\nearrow T_{max}} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}) = \infty. \tag{2.1}$$

Proof. Denote $z = (u,v)$. Then the system (1.3) can be written as

$$\begin{cases} z_t = \nabla\cdot(P(z)\nabla z) + Q(z), & x\in\Omega,\ \ t>0,\\ \frac{\partial z}{\partial \nu} = 0, & x\in\partial\Omega,\ \ t>0,\\ z(\cdot,0) = (u_0,v_0), & x\in\Omega, \end{cases}$$

where

$$P(z) = \begin{pmatrix} d(v) & d'(v)u \\ 0 & 1 \end{pmatrix}, \quad Q(z) = \begin{pmatrix} u(a_1 - b_1 u + \alpha F(v)) \\ v(a_2 - b_2 v) - uF(v) \end{pmatrix}.$$

Since the given initial value of (u_0, v_0) are nonnegative and satisfy $0 \le (u_0, v_0) \in [W^{1,p}(\Omega)]^2$ with $p > 2$, and hence the matrix $P(z)$ is positive definite at $t = 0$. This means that the system (1.3) is uniformly parabolic. Then the application of [33, Theorem 7.3] yields a $T_{max} > 0$ such that the system (1.3) possesses a unique solution $(u, v) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^2$.

Next, we prove the positivity of u and v. To this end, we rewrite the first equation of (1.3) as

$$\begin{aligned} u_t =& d(v)\Delta u + (d'(v)\nabla v + d'(v)\nabla v) \cdot \nabla u + d''(v)u|\nabla v|^2 + d'(v)u\Delta v \\ &+ u(a_1 - b_1 u + \alpha F(v)). \end{aligned} \tag{2.2}$$

Applying the strong maximum principle to (2.2) with the Neumann boundary condition deduces that $u > 0$ for all $(x, t) \in \Omega \times (0, T_{max})$ due to the fact $u_0 \gneqq 0$. In a similar way, we can prove $v > 0$ for any $(x, t) \in \Omega \times (0, T_{max})$ by using the second equation in (1.3). In addition, since $P(z)$ is an upper triangular matrix, the blow-up criterion (2.1) follows from [35, Theorem 5.2] directly. Then the proof of Lemma 2.1 is completed. □

Lemma 2.2. *Let the assumptions in Theorem 1.1 hold. Then the solution of* (1.3) *satisfies*

$$\|v(\cdot, t)\|_{L^\infty} \le K \tag{2.3}$$

for all $t > 0$, *where* K *is defined in* (1.6)*, and*

$$\limsup_{t\to\infty} v(\cdot, t) \le \frac{a_2}{b_2} \text{ for all } x \in \bar{\Omega}. \tag{2.4}$$

Proof. The proof is the similar to [31, Lemma 2.2], but for readers' convenience, we list the proof here. Since u, v and $F(v)$ are nonnegative, we can derive from the second equation of (1.3) that

$$\begin{cases} v_t - \Delta v = -uF(v) + v(a_2 - b_2 v) \le v(a_2 - b_2 v), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Let $\bar{v}(t)$ be the solution of the following ODE problem

$$\begin{cases} \frac{d\bar{v}(t)}{dt} = \bar{v}(t)(a_2 - b_2 \bar{v}(t)), & t > 0, \\ \bar{v}(0) = \|v_0\|_{L^\infty}. \end{cases} \tag{2.6}$$

Then we obtain from (2.6) that $\bar{v}(t) \le K := \max\{\frac{a_2}{b_2}, \|v_0\|_{L^\infty}\}$. It is obvious that $\bar{v}(t)$ is one of the super-solution of the following PDE problem

$$\begin{cases} V_t - \Delta V = V(a_2 - b_2 V), & x \in \Omega, t > 0, \\ \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ V(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

and therefore, we have

$$0 < V(x,t) \le \bar{v}(t) \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty), \tag{2.8}$$

where we have used the strong maximum principle to derive $V > 0$. Combining (2.5), (2.7) with (2.8) and employing the comparison principle, one has

$$0 < v(x,t) \le V(x,t) \le \bar{v}(t) \le K \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty). \tag{2.9}$$

This proves (2.3).

In addition, since $v(a_2 - b_2 v) < 0$ for $v > K$, we can further deduce from (2.6) that

$$\limsup_{t\to\infty} \bar{v}(t) \le K \text{ for all } x \in \bar{\Omega},$$

which together with (2.9) gives (2.4). □

Lemma 2.3. *Let the assumptions in Theorem 1.1 hold. Then the solution of* (1.3) *satisfies*

$$\int_\Omega u dx \le C, \text{ for all } t \in (0, T_{max}). \tag{2.10}$$

Moreover, one has

$$\int_t^{t+\tau} \int_\Omega u^2 dxds \le C, \text{ for all } t \in (0, \tilde{T}_{max}), \tag{2.11}$$

where

$$\tau := \min\left\{1, \frac{T_{max}}{2}\right\} \text{ and } \tilde{T}_{max} := \begin{cases} T_{max} - \tau, & \text{if } T_{max} < \infty, \\ \infty, & \text{if } T_{max} = \infty. \end{cases} \tag{2.12}$$

Proof. Multiplying the second equation of (1.3) by α and adding the result into the first equation of (1.3), then integrating the result over Ω, we obtain

$$\frac{d}{dt}\int_\Omega (u + \alpha v) = \int_\Omega u(a_1 - b_1 u) + \int_\Omega \alpha v(a_2 - b_2 v),$$

which implies that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+\alpha v)dx + \int_\Omega (u+\alpha v)dx + \frac{b_1}{2}\int_\Omega u^2 dx \\ &= \int_\Omega u\left(a_1 + 1 - \frac{b_1}{2}u\right)dx + \int_\Omega \alpha v(a_2 + 1 - b_2 v)dx \\ &\le \left(\frac{(a_1+1)^2}{2b_1} + \frac{\alpha(a_2+1)^2}{4b_2}\right)|\Omega|. \end{aligned} \tag{2.13}$$

Applying the Grönwall inequality to (2.13) yields

$$\int_\Omega (u + \alpha v)dx \le C, \tag{2.14}$$

it follows that (2.10) is valid. Then integrating (2.13) over $(t, t+\tau)$, and using (2.14) to obtain

$$\frac{b_1}{2}\int_t^{t+\tau}\int_\Omega u^2 dxds \le \int_\Omega (u+\alpha v)dx + \left(\frac{(a_1+1)^2}{2b_1} + \frac{\alpha(a_2+1)^2}{4b_2}\right)|\Omega| \le C,$$

which implies (2.11).

□

By Lemma 2.3, we establish the estimates for $\|\nabla v\|_{L^2}$ and $\int_t^{t+\tau}\int_\Omega |\Delta v|^2 dxds$.

Lemma 2.4. *Let the assumptions in Theorem 1.1 hold. Then there exists a constant* $C > 0$ *independent of t such that*

$$\|\nabla v\|_{L^2} \le C, \text{ for all } t \in (0, T_{max}), \tag{2.15}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2 dxds \le C, \text{ for all } t \in (0, \tilde{T}_{max}), \tag{2.16}$$

where τ *and* $\tilde{T}_{max}$ *are defined in* (2.12).

Proof. Multiplying the second equation of (1.3) by $-\Delta v$, integrating the result in Ω, using the assumption (H2) and the boundedness of v (see (2.3)), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta v|^2 dx &= \int_\Omega uF(v)\Delta v dx - \int_\Omega v(a_2 - b_2 v)\Delta v dx,\\ &\le \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \int_\Omega u^2 F^2(v) dx + \int_\Omega v^2(a_2 - b_2 v)^2 dx\\ &\le \frac{1}{2}\int_\Omega |\Delta v|^2 dx + F^2(K)\int_\Omega u^2 dx + C_1,\end{aligned}$$

which implies

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta v|^2 dx \le 2F^2(K)\int_\Omega u^2 dx + 2C_1, \tag{2.17}$$

where K is defined in (1.6).

Applying the Gagliardo-Nirenberg inequality, [29, Lemma 2.5] and noting the fact $\|v\|_{L^2} \le K|\Omega|^{\frac{1}{2}}$ yields

$$\int_\Omega |\nabla v|^2 dx = \|\nabla v\|_{L^2}^2 \le C_2(\|\Delta v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2) \le \frac{1}{2}\|\Delta v\|_{L^2}^2 + C_3. \tag{2.18}$$

Substituting (2.18) into (2.17), one has

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |\Delta v|^2 dx \le 2F^2(K)\int_\Omega u^2 dx + C_4, \tag{2.19}$$

which together with (2.11) yields (2.15).

Then integrating (2.19) over $(t, t+\tau)$, we obtain (2.16). □

3. Boundedness of solutions

In this section, we prove the boundedness of $\|u\|_{L^\infty}$ by the technique of Moser iteration. To the end, we first prove the boundedness of $\|u\|_{L^2}$.

Lemma 3.1. *Let the assumptions in Theorem 1.1 hold. Then there exists a constant* $C > 0$ *independent of t such that*

$$\|u(\cdot, t)\|_{L^2} \le C \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.1}$$

Proof. Multiplying the first equation in (1.3) by $2u$ and integrating the result with respect to x in Ω, one has

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2 + 2\int_\Omega d(v)|\nabla u|^2dx + 2b_1\int_\Omega u^3\\ =&-2\int_\Omega ud'(v)\nabla u\cdot\nabla v + 2a_1\int_\Omega u^2 + 2\alpha\int_\Omega u^2F(v)dx.\end{aligned}\tag{3.2}$$

By the assumptions in (H1), (H2), and (2.3), we know that $d(v)\in C^3$ and $0 < v \le K$, where K is defined in (1.6). Therefore, one has $d(v)\ge C_1$, $0 < F(v)\le F(K)$ and $|d'(v)|\le C_2$, then it follows from (3.2) that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2 + 2C_1\int_\Omega |\nabla u|^2dx + 2b_1\int_\Omega u^3\\ \le&2C_2\int_\Omega u|\nabla u||\nabla v|dx + a_1\int_\Omega u^2dx + 2\alpha F(K)\int_\Omega u^2dx\\ \le&C_1\int_\Omega|\nabla u|^2dx + \frac{C_2^2}{C_1}\int_\Omega u^2|\nabla v|^2dx + 2(a_1+\alpha F(K))\int_\Omega u^2dx\\ \le&C_1\|\nabla u\|_{L^2}^2 + \frac{C_2^2}{C_1}\|u\|_{L^4}^2\|\nabla v\|_{L^4}^2 + 2(a_1+\alpha F(K))\|u\|_{L^2}^2.\end{aligned}\tag{3.3}$$

Applying the Galiardo-Nirenberg inequality, one has

$$\|u\|_{L^4}^2 \le C_3(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2)\tag{3.4}$$

and

$$\|\nabla v\|_{L^4}^2 \le C_4(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2) \le C_5(\|\Delta v\|_{L^2}+1),\tag{3.5}$$

where we have used the boundedness of $\|\nabla v\|_{L^2}$ (see (2.15)) and [29, Lemma 2.5]. Combining (3.4) with (3.5) and applying the Young inequality yields

$$\begin{aligned}&\frac{C_2^2}{C_1}\|u\|_{L^4}^2\|\nabla v\|_{L^4}^2\\ \le&C_6(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2)(\|\Delta v\|_{L^2}+1)\\ \le&C_6\|\nabla u\|_{L^2}\|u\|_{L^2}\|\Delta v\|_{L^2} + C_6\|\nabla u\|_{L^2}\|u\|_{L^2} + C_6\|u\|_{L^2}^2\|\Delta v\|_{L^2} + C_6\|u\|_{L^2}^2\\ \le&C_1\|\nabla u\|_{L^2}^2 + \frac{C_6^2}{C_1}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2 + C_7\|u\|_{L^2}^2.\end{aligned}\tag{3.6}$$

Then substituting (3.6) into (3.3), we obtain

$$\frac{d}{dt}\|u\|_{L^2}^2 \le \frac{C_6^2}{C_1}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2 + (C_7+2\alpha F(K))\|u\|_{L^2}^2 \le C_8\|u\|_{L^2}^2(\|\Delta v\|_{L^2}^2+1).\tag{3.7}$$

For any $t\in(0,T_{max})$, by (2.10), there exists a nonnegative $t_0\in((t-\tau)_+,t)$ with $\tau=\min\{1,\frac{1}{2}T_{max}\}$ such that

$$\int_\Omega u^2(x,t_0)dx \le C_9\tag{3.8}$$

and by (2.16), one has

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta v(x,s)|^2 dxds \le C_{10}, \quad \text{for all} \quad t_0 \in (0, \tilde{T}_{max}). \tag{3.9}$$

Then integrating (3.7) on (t_0, t), and applying (3.8), (3.9) and the fact $t \le t_0 + \tau \le t_0 + 1$, we have

$$\|u(\cdot,t)\|_{L^2}^2 \le \|u(\cdot,t_0)\|_{L^2}^2 e^{C_8 \int_{t_0}^t (\|\Delta v(\cdot,s)\|_{L^2}^2 + 1)ds} \le C_{11},$$

which indicates (3.1) and thus completes the proof. □

To derive the boundedness for $\|u\|_{L^\infty}$, we need the following regularity lemma.

Lemma 3.2. *([37]) Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Suppose that $y(x,t) \in C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ is the solution of*

$$\begin{cases} y_t = \Delta y - y + \phi(x,t), & x \in \Omega, \quad t \in (0, T_{max}), \\ \frac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T_{max}), \\ y(x,0) = y_0(x) \in C^0(\bar{\Omega}), \end{cases}$$

where $\phi(x,t) \in L^\infty((0,T_{max}); L^p(\Omega))$. Then there exists a constant $C > 0$ such that

$$\|y(\cdot,t)\|_{W^{1,q}} \le C \text{ for all } t \in (0, T_{max})$$

with

$$q \in \begin{cases} [1, \frac{np}{n-p}), & \text{if } p \le n, \\ [1, \infty], & \text{if } p > n. \end{cases}$$

We will prove the boundedness of u through the Moser iteration procedure.

Lemma 3.3. *Let the assumptions in Theorem 1.1 hold. Then there exists a positive constant C independent of t such that*

$$\|u(\cdot,t)\|_{L^\infty} + \|\nabla v(\cdot,t)\|_{L^\infty} \le C \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.10}$$

Proof. Multiplying the first equation of the system (1.3) by u^{p-1} with $p \ge 2$, and integrating the resulting equation by parts, we derive

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + (p-1)\int_\Omega d(v)u^{p-2}|\nabla u|^2 dx + b_1\int_\Omega u^{p+1}dx - a_1\int_\Omega u^p dx \\ =& -(p-1)\int_\Omega d'(v)u^{p-1}\nabla u\cdot\nabla v dx + \alpha\int_\Omega u^p F(v)dx. \end{aligned} \tag{3.11}$$

Noting the fact $d(v) \ge C_1$, $0 < F(v) \le F(K)$ and $|d'(v)| \le C_2$, and using the Young inequality, we obtain from (3.11) that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + (p-1)C_1\int_\Omega u^{p-2}|\nabla u|^2 dx + \int_\Omega u^p dx \\ \le&(p-1)C_2\int_\Omega u^{p-1}|\nabla u||\nabla v|dx + (a_1 + 1 + \alpha F(K))\int_\Omega u^p dx \\ \le&\frac{(p-1)C_1}{2}\int_\Omega u^{p-2}|\nabla u|^2 dx + \frac{C_2^2(p-1)}{2C_1}\int_\Omega u^p|\nabla v|^2 dx + (a_1 + 1 + \alpha F(K))\int_\Omega u^p dx, \end{aligned}$$

which gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p dx + p\int_\Omega u^p dx + \frac{2(p-1)C_1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx\\ \leq&\frac{C_2^2 p(p-1)}{2C_1}\int_\Omega u^p|\nabla v|^2 dx + p(a_1+1+\alpha F(K))\int_\Omega u^p dx\end{aligned}\tag{3.12}$$

for all $t\in(0,T_{max})$ and $p\geq 2$. By Lemma 3.1, we have $\|u(\cdot,t)\|_{L^2}\leq C_3$, and thus we derive $\|\nabla v(\cdot,t)\|_{L^4}\leq C_4$ from Lemma 3.2. Then applying the Gagliardo-Nirenberg inequality and the Hölder inequality yields

$$\begin{aligned}&\frac{C_2^2 p(p-1)}{2C_1}\int_\Omega u^p|\nabla v|^2 dx\\ \leq&\frac{C_2^2 p(p-1)}{2C_1}\left(\int_\Omega u^{2p}dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4 dx\right)^{\frac{1}{2}}\\ \leq&\frac{C_2^2C_4^2 p(p-1)}{2C_1}\|u^{\frac{p}{2}}\|_{L^4}^2\\ \leq&C_5\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})}\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{2}{p}}+\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2\right)\\ \leq&C_3C_5\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{1}{p})}+C_3^pC_5\\ \leq&\frac{(p-1)C_1}{p}\|\nabla u^{\frac{p}{2}}\|_{L^2}^2+\frac{C_1}{p}\left(\frac{C_3C_5}{C_1}\right)^p+C_3^pC_5,\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}p(a_1+1+\alpha F(K))\int_\Omega u^p dx =&p(a_1+1+\alpha F(K))\|u^{\frac{p}{2}}\|_{L^2}^2\\ \leq&p(a_1+1+\alpha F(K))\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{2}{p})}\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^{\frac{4}{p}}+\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}}^2\right)\\ \leq&C_3^2C_6\|\nabla u^{\frac{p}{2}}\|_{L^2}^{2(1-\frac{2}{p})}+C_3^pC_6\\ \leq&\frac{(p-1)C_1}{p}\|\nabla u^{\frac{p}{2}}\|_{L^2}^2+\frac{C_1}{p}\left(\frac{C_3^2C_6}{d(0)}\right)^p+C_3^pC_6.\end{aligned}\tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we obtain

$$\frac{d}{dt}\int_\Omega u^p dx + p\int_\Omega u^p dx \leq C_7,$$

by the Grönwall inequality, we derive

$$\|u(\cdot,t)\|_{L^p}^p\leq e^{-pt}\|u_0\|_{L^p}^p+\frac{C_7}{p}(1-e^{-pt})\leq\|u_0\|_{L^p}^p+\frac{C_7}{p}.$$

Let $p=4$ in the above inequality and apply Lemma 3.2 again. We obtain that there exists a constant $C_7>0$ such that

$$\|\nabla v(\cdot,t)\|_{L^\infty}\leq C_7.$$

Then using the Moser iteration procedure (see [34]), one derives (3.10) and thus proves Lemma 3.3. □

Proof of Theorem 1.1. Theorem 1.1 is a consequence of Lemma 2.1, Lemma 2.2 and Lemma 3.3. □

4. Globally stability

In this section, we will investigate the asymptotical behavior of solutions solving system (1.3) and prove Theorem 1.2 based on Lyapunoval functional method along with Barbălat's lemma.

Let us first recall the basic Barbălat's lemma.

Lemma 4.1. *(Barbălat's Lemma [36]) Suppose that* $h : [1, \infty) \to \mathbb{R}$ *is a uniformly continuous function such that* $\lim_{t\to\infty} \int_1^t h(s)ds$ *exists, then* $\lim_{t\to\infty} h(t) = 0$.

Now we give the uniform estimates of the global solution.

Lemma 4.2. *Let* (u, v) *be the unique global bounded classical solution of* (1.3) *given by Theorem 1.1. Then for any given* $0 < \alpha < 1$, *there exists a constant* $C(\alpha) > 0$ *such that*

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \le C(\alpha). \tag{4.1}$$

Proof. This proof is based on the standard regularity for parabolic equations. For readers' convenience, we sketch the proof here. Due to the boundedness of (u, v), applying the interior L^p estimate [39] to (1.3), we derive that

$$\|u\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{4},i+3])} + \|v\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{4},i+3])} \le C_1, \ \forall i \ge 0. \tag{4.2}$$

Using the Sobolev embedding theorem and (4.2), we derive

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\times[\frac{1}{4},\infty))} + \|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\times[\frac{1}{4},\infty))} \le C_2. \tag{4.3}$$

Applying (4.3) and the Schauder estimate [38] to the second equation of (1.3), we obtain

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[i+\frac{1}{3},i+3])} \le C_3, \ \forall i \ge 0, \tag{4.4}$$

which implies

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[\frac{1}{3},+\infty))} \le C_4. \tag{4.5}$$

Rewrite the first equation in (1.3) as

$$u_t - d(v)\Delta u - 2d'(v)\nabla v \cdot \nabla u = G(x, t), \ x \in \Omega, \ t > 0, \tag{4.6}$$

where

$$G(x, t) = (d''(v)|\nabla v|^2 + d'(v)\Delta v)u + u(a_1 - b_1 u + \alpha F(v)).$$

Due to (4.3) and (4.4), we see that

$$\|G\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[i+\frac{1}{3},i+3])} \le C_5, \ \forall i \ge 0.$$

Applying the Schauder estimate to (4.6) gives $\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[i+1,i+3])} \le C_6$ for all $i \ge 0$. Thus

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,+\infty))} \le C_7. \tag{4.7}$$

Then (4.1) follows from (4.5) and (4.7). This completes the proof of Lemma 4.2. □

Next we shall prove the global stability of the coexistence steady state (u_*, v_*) by constructing the following Lyapunov functional:

$$V(u(t), v(t)) = V(t) = \frac{1}{\alpha}\int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right)dx + \int_\Omega \int_{v_*}^{v} \frac{F(s) - F(v_*)}{F(s)} ds dx, \tag{4.8}$$

where (u_*, v_*) satisfies (1.5).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H1)-(H3) hold. Assume $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ with $p > 2$ and $u_0, v_0 \gneqq 0$. If (u, v) is the solution of* (1.3) *obtained in Theorem 1.1, and the parameters satisfy*

$$\max_{0 \le v \le K} \frac{u_* F^2(v)|d'(v)|^2}{4\alpha F(v_*)F'(v)d(v)} \le 1,$$

then

$$\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} \to 0 \ \text{ as } \ t \to \infty, \tag{4.9}$$

where (u_*, v_*) satisfies (1.5)*, and K is defined in* (1.6).

Proof. First by a similar argument as [31, Lemma 4.3], we deduce that

$$V(t) \ge 0 \quad \text{for all} \quad u,\ v \ge 0.$$

We compute the derivative of $V(t)$ by using the system (1.3) to derive

$$\begin{aligned}\frac{d}{dt}V(t) =& \frac{1}{\alpha}\int_\Omega \left(1 - \frac{u_*}{u}\right)u_t dx + \int_\Omega \frac{F(v) - F(v_*)}{F(v)} v_t dx \\ =& \underbrace{-\frac{u_*}{\alpha}\int_\Omega \frac{d(v)|\nabla u|^2}{u^2}dx - \frac{u_*}{\alpha}\int_\Omega \frac{d'(v)\nabla u \cdot \nabla v}{u}dx - F(v_*)\int_\Omega F'(v)\left|\frac{\nabla v}{F(v)}\right|^2 dx}_{I_1} \\ & \underbrace{+ \frac{1}{\alpha}\int_\Omega (u - u_*)(a_1 - b_1 u + \alpha F(v))dx + \int_\Omega (F(v) - F(v_*))\left(\frac{v(a_2 - b_2 v)}{F(v)} - u\right)}_{I_2}.\end{aligned} \tag{4.10}$$

Then I_1 can be rewritten as

$$I_1 = -\int_\Omega XAX^T,$$

where X^T denotes the transpose of $X := (\nabla u, \nabla v)$, and

$$A = \begin{pmatrix} \frac{u_* d(v)}{\alpha u^2} & \frac{d'(v)u_*}{2\alpha u} \\ \frac{d'(v)u_*}{2\alpha u} & \frac{F(v_*)F'(v)}{|F(v)|^2} \end{pmatrix}.$$

It is easy to check that the matrix A is nonnegative definite if and only if

$$\frac{u_* F^2(v)|d'(v)|^2}{4\alpha F(v_*)F'(v)d(v)} \le 1,$$

and thus

$$I_1 \leq 0. \tag{4.11}$$

Next, we compute I_2. By (1.5), we see that

$$a_1 - b_1 u_* + \alpha F(v_*) = 0 \ \text{ and } \ \frac{v_*(a_2 - b_2 v_*)}{F(v_*)} - u_* = 0,$$

we obtain from (H2) and (H3) that there exists a constant $T_1 > 0$ such that

$$0 < \frac{1}{2}F'(v_*) \leq F'(\xi_1) \leq 2F'(v_*), \ \ \phi'(v_*) \leq \phi'(\xi_2) \leq \frac{1}{2}\phi'(v_*) < 0.$$

Therefore, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} I_2 &= -\frac{b_1}{\alpha}\int_\Omega (u-u_*)^2 dx + \int_\Omega (F(v) - F(v_*))(\phi(v) - \phi(v_*))dx \\ &\leq -\frac{b_1}{\alpha}\int_\Omega (u-u_*)^2 dx + \int_\Omega F'(\xi_1)\phi'(\xi_2)(v-v_*)^2 dx \\ &\leq -C\int_\Omega [(u-u_*)^2 + (v-v_*)^2]dx := -C_1 E(t), \end{aligned} \tag{4.12}$$

where $\phi(v) = \frac{v(a_2 - b_2 v)}{F(v)}$ is defined in (1.4), ξ_1 and ξ_2 are lying between v and v_*. Combining (4.10), (4.11) with (4.12) gives

$$\frac{d}{dt}V(t) \leq -C_1 E(t) \tag{4.13}$$

with $E(t) = \int_\Omega [(u-u_*)^2 + (v-v_*)^2]dx.$

Since $V(t) \geq 0$, we have

$$\int_1^\infty E(t)dt \leq \frac{1}{C_1}V(1) < \infty.$$

It follows from the regularity of u, v that $E(t)$ is uniformly continuous in $[1, \infty)$. An application of Lemma 4.1 yields

$$E(t) = \int_\Omega [(u-u_*)^2 + (v-v_*)^2]dx \to 0 \text{ as } t \to \infty. \tag{4.14}$$

By Lemma 4.2, we derive that $u(\cdot, t)$ and $v(\cdot, t)$ are bounded for $t > 1$ in the space $W^{1,\infty}(\Omega)$. Applying the Gagliardo-Nirenberg inequality

$$\|\phi\|_\infty \leq c\|\phi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\phi\|_2^{\frac{2}{n+2}}, \ \forall \phi \in W^{1,\infty}(\Omega)$$

to $u - u_*$ and $v - v_*$, respectively, we can obtain (4.9) from (4.14). This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that the conditions of Lemma 4.3 hold. Then there exist some positive constants σ, T_0 and C independent of t such that*

$$\|u(\cdot, t) - u_*\|_{L^\infty} + \|v(\cdot, t) - v_*\|_{L^\infty} \leq Ce^{-\sigma t}, \ \ t > T_0. \tag{4.15}$$

Proof. By Lemma 4.3, we have

$$\|u-u_*\|_{L^\infty}\to 0 \text{ as } t\to\infty,$$

then applying L'Hôpital's rule to derive

$$\lim_{u\to u_*}\frac{u-u_*-u_*\ln\frac{u}{u_*}}{(u-u_*)^2}=\frac{1}{2u_*}.$$

Therefore, we obtain from the continuity that there exists a constant $t_1>0$ such that

$$\frac{1}{4u_*}\int_\Omega (u-u_*)^2\le\int_\Omega\Big(u-u_*-u_*\ln\frac{u}{u_*}\Big)\le\frac{1}{u_*}\int_\Omega (u-u_*)^2,\ \ t>t_1. \tag{4.16}$$

In addition, by Lemma 4.1 in [31], we find that there exist some constants $C_1,C_2>0$ and $t_2>0$ such that

$$C_1\int_\Omega (v-v_*)^2\le\int_\Omega\int_{v_*}^{v}\frac{F(s)-F(v_*)}{F(s)}ds\le C_2\int_\Omega (v-v_*)^2,\ \ t>t_2. \tag{4.17}$$

Combining (4.16), (4.17), and the definition of $V(t)$ in (4.8), we conclude that there exist some positive constants C_3,C_4 and t_3 such that for all $t>t_3$

$$C_3E(t)\le V(t)\le C_4E(t).$$

This, together with (4.13) yields a constant $C_5>0$ such that

$$\frac{d}{dt}V(t)\le -C_5V(t),\ \ t>t_3,$$

therefore, we derive that there exist some constants $C_6>0$ and $C_7>0$ such that

$$\|u-u_*\|_{L^2}^2+\|v-v_*\|_{L^2}^2\le C_6e^{-C_7t}. \tag{4.18}$$

To finish the proof, we need the L^∞ estimates for $u-u_*$ and $v-v_*$. By Lemma 4.2 and the Gagliardo-Nirenberg inequality, we have

$$\|u-u_*\|_{L^\infty}\le C_8(\|\nabla u\|_{L^4}^{\frac{2}{3}}\|u-u_*\|_{L^2}^{\frac{1}{3}}+\|u-u_*\|_{L^2})\le C_9\|u-u_*\|_{L^2}^{\frac{1}{3}}, \tag{4.19}$$

by a similar argument, one has

$$\|v-v_*\|_{L^\infty}\le C_{10}\|u-u_*\|_{L^2}^{\frac{1}{3}},$$

which, combined with (4.18) and (4.19), gives (4.15). Therefore, we complete the proof of Lemma 4.4. □

Lemma 4.5. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H1)-(H2) hold. Assume $(u_0,v_0)\in[W^{1,p}(\Omega)]^2$ with $p>2$ and $u_0,v_0\gneqq 0$. If (u,v) is the solution of* (1.3) *obtained in Theorem 1.1, and the parameters satisfies*

$$\lim_{v\to 0}\frac{F(v)}{v}\ \text{exists},\ \min_{\bar\Omega}\frac{F(v)}{v}\ge\frac{a_2b_1}{a_1},\ \text{and}\ \frac{a_1a_2(d'(v))^2}{2b_1b_2d(v)}\le\frac{1}{\xi_1}, \tag{4.20}$$

then

$$\|u-\frac{a_1}{b_1}\|_{L^\infty}+\|v\|_{L^\infty}\to 0\ \text{ as }\ t\to\infty, \tag{4.21}$$

where K is defined in (1.6)*, ξ_1 and K_0 are defined by $\xi_1=\frac{1}{\alpha}+\frac{b_1b_2}{K_0^2\alpha^2}$ and $K_0=\max_{\bar\Omega}\frac{F(v)}{v}$ respectively.*

Next we prove the global stability of the generalist predator only steady state $(\frac{a_1}{b_1}, 0)$ by constructing the following Lyapunov functional:

$$V_1(u(t), v(t)) = V_1(t) = \xi_1 \int_\Omega \left(u - \frac{a_1}{b_1} - \frac{a_1}{b_1} \ln \frac{u}{\frac{a_1}{b_1}} \right) dx + \int_\Omega v dx + \frac{b_2}{4a_2} \int_\Omega v^2 dx.$$

Proof. It is obvious that $V_1(t) \geq 0$ from the definition. Next we compute the derivative of $V_1(t)$ by using the system (1.3) and (4.20) to obtain

$$\begin{aligned}
\frac{d}{dt} V_1(t) =& \xi_1 \int_\Omega \left(1 - \frac{a_1}{b_1 u}\right) u_t dx + \int_\Omega v_t dx + \frac{b_2}{2a_2} \int_\Omega v v_t dx \\
=& -\xi_1 \frac{a_1}{b_1} \int_\Omega d(v) \frac{|\nabla u|^2}{u^2} - \xi_1 \frac{a_1}{b_1} \int_\Omega \frac{d'(v) \nabla u \cdot \nabla v}{u} - \xi_1 b_1 \int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 dx \\
&+ \alpha \xi_1 \int_\Omega \left(u - \frac{a_1}{b_1}\right) F(v) dx + a_2 \int_\Omega v dx - \frac{b_2}{2} \int_\Omega v^2 dx - \int_\Omega \left(u - \frac{a_1}{b_1}\right) F(v) dx \\
&- \frac{a_1}{b_1} \int_\Omega F(v) dx - \frac{b_2}{2a_2} \int_\Omega |\nabla v|^2 dx - \frac{b_2^2}{2a_2} \int_\Omega v^3 dx - \frac{b_2}{2a_2} \int_\Omega u v F(v) dx \qquad (4.22)\\
\leq& \underbrace{-\xi_1 \frac{a_1}{b_1} \int_\Omega d(v) \frac{|\nabla u|^2}{u^2} - \xi_1 \frac{a_1}{b_1} \int_\Omega \frac{d'(v) \nabla u \cdot \nabla v}{u} - \frac{b_2}{2a_2} \int_\Omega |\nabla v|^2 dx}_{J_1} \\
& \underbrace{-\xi_1 b_1 \int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 dx - \frac{b_2}{2} \int_\Omega v^2 dx + K_0 |\alpha \xi_1 - 1| \int_\Omega \left(u - \frac{a_1}{b_1}\right) v dx}_{J_2}.
\end{aligned}$$

Then J_1 and J_2 can be rewritten as

$$J_1 = -\int_\Omega X_1 A_1 X_1^T dx, \quad J_2 = -\int_\Omega Y_1 B_1 Y_1^T dx,$$

where X_1^T and Y_1^T denotes the transpose of $X_1 := \left(\frac{\nabla u}{u}, \nabla v\right)$ and $Y_1 := \left(u - \frac{a_1}{b_1}, v\right)$, and

$$A_1 = \begin{pmatrix} \frac{a_1 \xi_1 d(v)}{b_1} & \frac{a_1 \xi_1 d'(v)}{2b_1} \\ \frac{a_1 \xi_1 d'(v)}{2b_1} & \frac{b_2}{2a_2} \end{pmatrix},$$

and

$$B_1 = \begin{pmatrix} b_1 \xi_1 & -\frac{K_0 |\alpha \xi_1 - 1|}{2} \\ -\frac{K_0 |\alpha \xi_1 - 1|}{2} & \frac{b_2}{2} \end{pmatrix},$$

By the definition of ξ_1 and K_0 defined in Lemma 4.5, and (4.20), we derive that

$$J_1 \leq 0$$

and

$$J_2 \leq -C \left(\int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 dx + \int_\Omega v^2 dx \right) := -C E_1(t).$$

It follows that

$$\frac{d}{dt}V_1(t) \le -CE_1(t) \tag{4.23}$$

with $E_1(t) = \int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 dx + \int_\Omega v^2 dx$.

By a similar argument as Lemma 4.3, we obtain (4.21) and thus completes the proof of Lemma 4.5. □

By a similar argument as Lemma 4.4, we can derive the following decay rate estimates.

Lemma 4.6. *Suppose that the conditions in Lemma 4.5 hold. Then there exist some positive constants T_1 and C independent of t such that*

$$\|u(\cdot,t) - \frac{a_1}{b_1}\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \le \frac{C}{1+t}, \ t > T_1. \tag{4.24}$$

Proof. Since

$$V_1(t) \le CE_1^{\frac{1}{2}}(t),$$

then combining this with (4.23), we obtain

$$\frac{d}{dt}V_1(t) \le -CV_1^2(t).$$

Solving this ordinary differential inequality, for sufficiently large t, we arrive at

$$V_1(t) \le \frac{C}{1+t}.$$

Using the same argument as in the proof of Lemma 4.4, we readily get (4.24) and thus complete the proof of Lemma 4.6. □

5. Applications and spatio-temporal patterns

In this section, we shall study the possible pattern formation generated by the system (1.3). As a typical example, we consider the case Holling type I of $F(v) = v$.

5.1. Linear instability analysis

To begin with, we first consider the linear stability of the system (1.3). We linearise the system (1.3) at an equilibrium (u_s, v_s) and write the linearised system as

$$\begin{cases} \Phi_t = \mathcal{A}\Delta\Phi + \mathcal{B}\Phi, & x \in \Omega, \ t > 0, \\ (\nu \cdot \nabla)\Phi = 0, & x \in \partial\Omega, \ t > 0, \\ \Phi(x,0) = (u_0 - u_s, v_0 - v_s)^T, & x \in \Omega, \end{cases} \tag{5.1}$$

where T denotes the transpose and

$$\Phi = \begin{pmatrix} u - u_s \\ v - v_s \end{pmatrix}, \ \mathcal{A} = \begin{pmatrix} d(v_s) & u_s d'(v_s) \\ 0 & 1 \end{pmatrix},$$

as well as

$$\mathcal{B} = \begin{pmatrix} a_1 - 2b_1u_s + \alpha v_s & \alpha u_s \\ -v_s & -u_s + a_2 - 2b_2v_s \end{pmatrix}.$$

Let $W_k(x)$ be the eigenfunction of the following eigenvalue problem:

$$\begin{cases} \Delta W_k(x) + k^2 W_k(x) = 0, \\ \frac{\partial W_k}{\partial \nu}(x) = 0, \end{cases}$$

the allowable wavenumbers k are discrete in a bounded domain, for instance, if $\Omega = (0, l)$, then $k = \frac{n\pi}{l}$, $n = 0, 1, 2, \cdots$. Since the system (5.1) is linear, the solution of $\Phi(x, t)$ has the form of

$$\Phi(x, t) = \sum_{k \geq 0} c_k e^{\rho t} W_k(x), \tag{5.2}$$

where ρ is the temporal eigenvalue and $c_k (k \geq 0)$ are determined by the Fourier expansion of the initial conditions in terms of $W_k(x)$. Substituting (5.2) into (5.1) yields

$$\rho W_k(x) = -k^2 \mathcal{A} W_k(x) + \mathcal{B} W_k(x),$$

which indicates that ρ is the eigenvalue of the matrix M_k with

$$M_k = \begin{pmatrix} -k^2 d(v_s) + a_1 - 2b_1u_s + \alpha v_s & -k^2 u_s d'(v_s) + \alpha u_s \\ -v_s & -k^2 - u_s + a_2 - 2b_2v_s \end{pmatrix}.$$

Then the eigenvalue $\rho(k^2)$ satisfies

$$\rho^2 + a(k^2)\rho + b(k^2) = 0,$$

where

$$a(k^2) = (d(v_s) + 1)k^2 - a_1 + 2b_1u_s - \alpha v_s + u_s - a_2 + 2b_2v_s$$

and

$$b(k^2) = (-k^2 d(v_s) + a_1 - 2b_1u_s + \alpha v_s)(-k^2 - u_s + a_2 - 2b_2v_s) - v_s(k^2 u_s d'(v_s) - \alpha u_s).$$

Next we proceed to consider the stability of equilibria $(0, 0)$, $\left(\frac{a_1}{b_1}, 0\right)$, $\left(0, \frac{a_2}{b_2}\right)$ and $\left(\frac{a_2\alpha + a_1b_2}{\alpha + b_1b_2}, \frac{a_2b_1 - a_1}{\alpha + b_1b_2}\right)$ in the presence of spatial structure.

• For the steady state $(0, 0)$, it can be easily check that ρ satisfies

$$(\rho - a_1)(\rho + k^2 - a_2) = 0.$$

At least one of the roots is positive, therefore, $(0, 0)$ is always linearly unstable.

• For the steady state $\left(\frac{a_1}{b_1}, 0\right)$, it can be easily check that ρ satisfies

$$(\rho + k^2 d(0) + a_1)(\rho + k^2 + u_s - a_2) = 0.$$

Therefore if $a_2 = u_s$, $\left(\frac{a_1}{b_1}, 0\right)$ is marginally stable. If $a_2 > u_s$, then there exists a k such that one of the root is negative and the other is positive, and $\left(\frac{a_1}{b_1}, 0\right)$ is linearly unstable.

• For the steady state $\left(0, \frac{a_2}{b_2}\right)$, ρ satisfies

$$(\rho + k^2 d(v_s) - a_1 - \alpha v_s)(\rho + k^2 + a_2) = 0.$$

Therefore there exists a k such that one of the root is negative and the other is positive, and $\left(0, \frac{a_2}{b_2}\right)$ is linearly unstable.

• For the coexistence steady state $(u_s, v_s) = \left(\frac{a_2\alpha + a_1 b_2}{\alpha + b_1 b_2}, \frac{a_2 b_1 - a_1}{\alpha + b_1 b_2}\right)$, ρ satisfies

$$\rho^2 - (M_1 + M_4)\rho + M_1 M_4 - M_2 M_3 = 0,$$

with $M_1 = -k^2 d(v_s) - b_1 u_s$, $M_2 = -k^2 u_s d'(v_s) + \alpha u_s$, $M_3 = -v_s$ and $M_4 = -k^2 - b_2 v_s$, where we have used the identity

$$a_1 - b_1 u_s + \alpha v_s = 0 \text{ and } a_2 - b_2 v_s - u_s = 0.$$

Since $M_1 + M_4 < 0$, (u_s, v_s) is linearly unstable iff

$$M_1 M_4 - M_2 M_3 < 0.$$

One root is negative and the other is positive. That is, we need the condition

$$(k^2 d(v_s) + b_1 u_s)(k^2 + b_2 v_s) + v_s(\alpha u_s - k^2 u_s d'(v_s)) < 0.$$

which indicates that

$$d(v_s)k^4 + (b_2 v_s d(v_s) + b_1 u_s - u_s v_s d'(v_s))k^2 + (b_1 b_2 + \alpha) u_s v_s < 0. \tag{5.3}$$

We conclude that a steady-state bifurcation may occur if

$$u_s v_s d'(v_s) - b_2 v_s d(v_s) - b_1 u_s > 2\sqrt{(b_1 b_2 + \alpha) d(v_s) u_s v_s}$$

and there are allowable wavenumbers k such that

$$k_1^- < k^2 < k_1^+,$$

where $k_1^\pm = \frac{u_s v_s d'(v_s) - b_2 v_s d(v_s) - b_1 u_s) \pm \sqrt{(b_2 v_s d(v_s) + b_1 u_s - u_s v_s d'(v_s))^2 - 4(b_1 b_2 + \alpha) d(v_s) u_s v_s}}{2d(v_s)}$.

We remark here that when $d'(v) \leq 0$, the inequality (5.3) can not be hold for any k. Therefore, the coexistence steady state (u_s, v_s) is linearly stable if $d'(v) \leq 0$.

5.2. *Spatio-temporal patterns*

In this subsection, we take some examples to present the periodic patterns.

According to the condition (5.3) and the linear stability analysis in subsection 5.1, we fix the value of the parameters in all simulations as follows:

$$a_1 = b_1 = 1, \ a_2 = b_2 = \alpha = 2, \ d(v) = e^{20v} \text{ and } \ l = 10. \tag{5.4}$$

Then we obtain the possible steady states are $(u_s, v_s) = \left(\frac{3}{2}, \frac{1}{4}\right)$ or $(1, 0)$ or $(0, 1)$.

The numerical simulations of patterns are then shown in the following Figures 1–3.

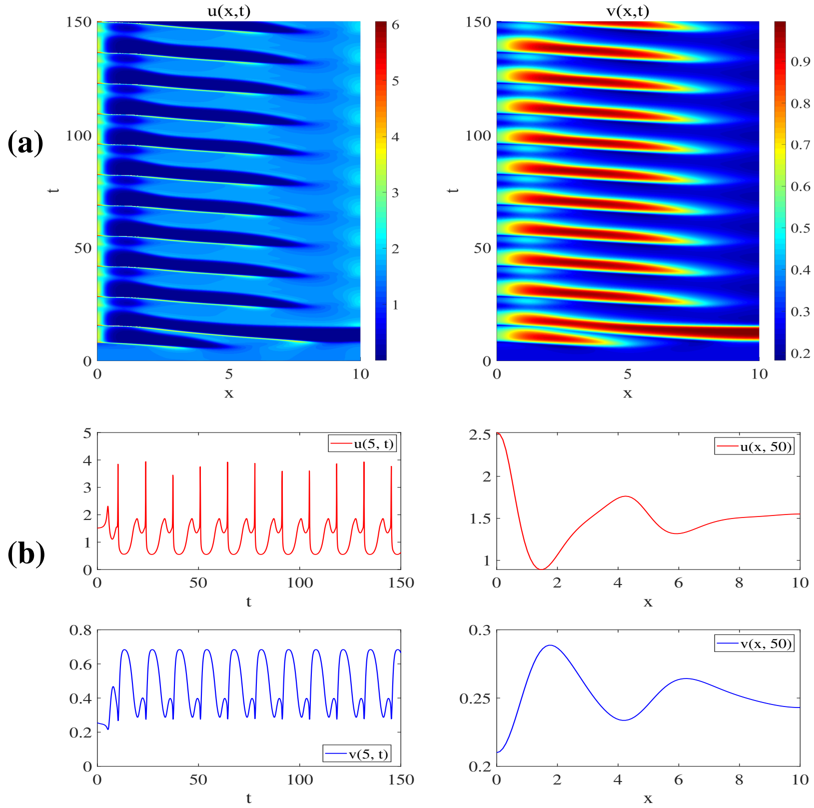

Figure 1. Numerical simulation of time-periodic patterns generated by (1.3) with $d(v) = e^{20v}$ in the interval $[0, 10]$, and $a_1 = b_1 = 1,\ a_2 = b_2 = \alpha = 2$. The initial datum (u_0, v_0) is setted as a small random perturbation of the homogeneous semi-trivial steady state $(\frac{3}{2}, \frac{1}{4})$.

Case 1. $(u_s, v_s) = (\frac{3}{2}, \frac{1}{4})$. In this case, $k = \frac{n\pi}{10}, n = 0, 1, 2, \cdots$, the condition (5.3) turns into

$$e^5 k^4 + (\frac{3}{2} - 7e^5)k^2 + \frac{3}{2} < 0,$$

it can be checked that the above inequality is valid iff $1 \leq n \leq 8$.

The numerical spatial-temporal patterns generated by the model (1.3) under the condition (5.4) are plotted in Figure 1(a), where we observe the spatially inhomogeneous temporal periodic patterns arising from the vicinity of equilibrium $(\frac{3}{2}, \frac{1}{4})$. The time distributions of predators and prey at a fixed space position plotted in Figure 1(b) show that predators and prey are periodic. We also plot the spatial distributions of predators and prey at a fixed time in Figure 1(b).

Case 2. $(u_s, v_s) = (1, 0)$. In this case, from the above linear analysis, we know that ρ satisfies

$$(\rho + k^2 + 1)(\rho + k^2 - 1) = 0,$$

therefore, then one root is positive and the other is negative for the above equation iff $n = 0, 1, 2, 3$.

Similarly, we observe the spatially inhomogeneous temporal periodic patterns arising from the vicinity of equilibrium $(1, 0)$. The time distributions of predators and prey at a fixed space position plotted in Figure 2(b) show that predators and prey are periodic. We also plot the spatial distributions of predators and prey at a fixed time in Figure 2(b).

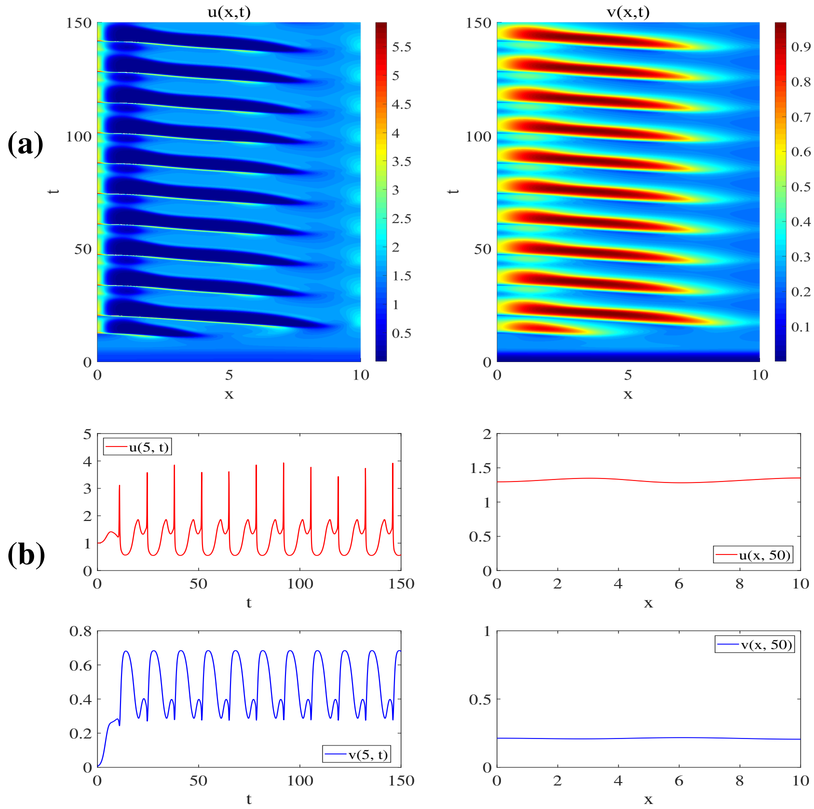

Figure 2. Numerical simulation of time-periodic patterns generated by (1.3) with $d(v) = e^{20v}$ in the interval $[0, 10]$, and $a_1 = b_1 = 1,\ a_2 = b_2 = \alpha = 2$. The initial datum (u_0, v_0) is set as a small random perturbation of the homogeneous semi-trivial steady state $(1, 0)$.

Case 3. $(u_s, v_s) = (0, 1)$. In this case, we know that ρ satisfies

$$(\rho + e^{20}k^2 - 3)(\rho + k^2 + 2) = 0,$$

therefore, then one root is positive and the other is negative for the above equation iff $n = 0$.

We also can observe the spatially inhomogeneous temporal periodic patterns arising from the vicinity of equilibrium $(0, 1)$. The time distributions of predators and prey at a fixed space position plotted in Figure 3(b) show that predators and prey are periodic. We also plot the spatial distributions of predators and prey at a fixed time in Figure 3(b).

In summary, we conclude that the time-periodic patterns have been obtained when $d(v)$ is monotone increasing and satisfies some conditions. While from the linear stability analysis, we know that the coexistence steady state and the semi-trivial steady states are both linearly stable. These results indicate that the motility function $d(v)$ can trigger pattern formation and is a factor inducing the spatial heterogeneity of populations. In addition, if $a_1 < 0$, (u represents the specialist predator) Jin and Wang in [29] have proved that the semi-trivial steady state $\left(0, \frac{a_2}{b_2}\right)$ is linearly stable if $\alpha\frac{a_2}{b_2} \leq -a_1$. While our results indicate that when u is a generalist predator ($a_1 > 0$), the semi-trivial steady state $\left(0, \frac{a_2}{b_2}\right)$ is linearly unstable, because the generalist predator u can gain food from other preys.

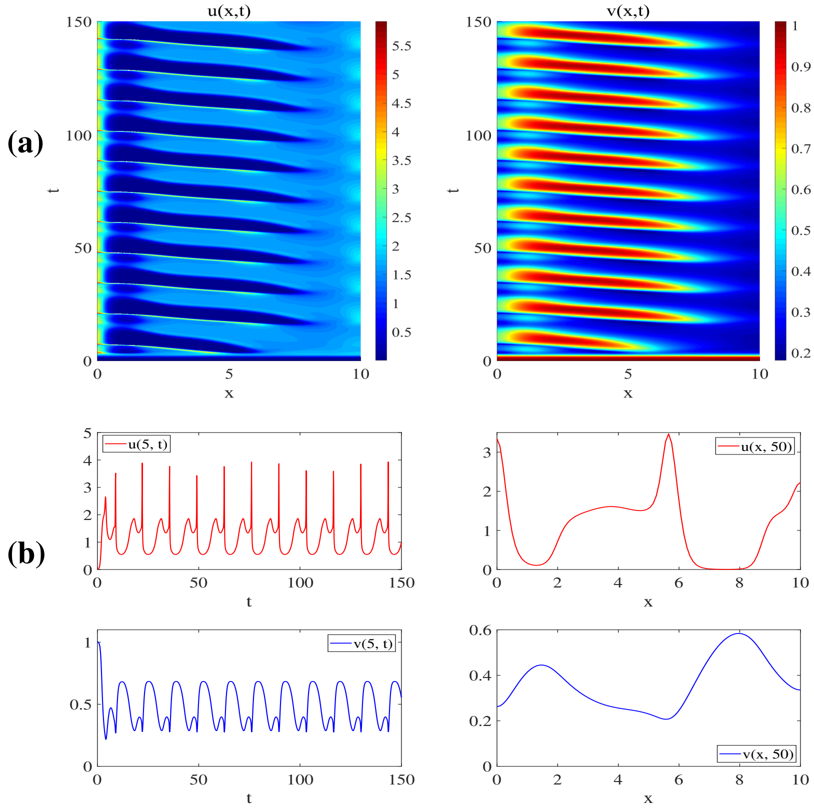

Figure 3. Numerical simulation of time-periodic patterns generated by (1.3) with $d(v) = e^{20v}$ in the interval $[0, 10]$, and $a_1 = b_1 = 1,\ a_2 = b_2 = \alpha = 2$. The initial datum (u_0, v_0) is set as a small random perturbation of the homogeneous semi-trivial steady state $(0, 1)$.

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Conflict of interest

The authors declare there is no conflict of interest.

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