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A common thread linking the design of guarantee and nonescalating payments of public annuities

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Abstract

Motivated by recent experiences in economies adopting the defined-contribution pension system, we study public annuities in the presence of survival probability heterogeneity. It is found that the difference of annuitization-weighted and unweighted averages of survival probabilities is a useful measure of the severity of adverse selection. We then examine public annuities with a guarantee feature which bundles annuity income and bequeathable wealth components. We show that when the heterogeneity in survival probability is limited, the magnitude of guarantee proportion is irrelevant. On the other hand, an increase in the guarantee proportion mitigates adverse selection when the extent of heterogeneity is sufficiently large, because the share of annuity purchase by retirees with lower (resp., higher) survival probabilities is increased (resp., decreased). We also obtain a similar set of results for public annuities with nonescalating payments. The results have useful implications regarding the design of public annuities.

KEYWORDS
asymmetric information, guarantee element, nonescalating payments, public annuity, severity of adverse selection

JEL CLASSIFICATION
H55, G52, D15

1 | INTRODUCTION

According to the World Bank, life expectancy at birth worldwide increased from 62.8 years in 1980 to 72.6 years in 2018. Among those who are 65 years old, 35% of women in the USA are expected to survive to 90 years old; the corresponding figures are 34% in the UK, 46% in France, and 52% in Japan. These increases in life expectancy bring benefits to most individuals, but they also cause problems, especially if individuals do not respond appropriately. One of these potential problems is that individuals may outlive their resources and face financial difficulties during old age.

In principle, people may insure against longevity risk (the risk of having inadequate financial resources when living longer than anticipated) by purchasing annuities in the private market. Alternatively, retirees in countries adopting the pay-as-you-go (PAYGO) retirement pension system may rely, at least partially, on pension benefits whenever they are alive. However, neither the PAYGO system nor a pure market solution seems to solve the problem satisfactorily. On the one hand, it is well known that many developed countries with an unfunded PAYGO system have suffered budgetary difficulties in the past few decades. In response, some countries have already extended the pension eligibility age and/or reduced the benefit. On the other hand, few retirees annuitize a substantial portion of their wealth (Benartzi et al., 2011; Modigliani, 1986). A lot of researchers have analyzed this phenomenon of "annuity puzzle," but no consensus has been reached regarding the underlying reasons, and their relative importance. As a result, it is hard for policymakers to propose appropriate market regulations to mitigate the possible efficiency loss due to low annuitization rates.

In this paper, we consider the public annuity (PA) scheme in the payout (or decumulation) phase under a fully funded pension system, an alternative to either the PAYGO system or private annuities. In recent decades, more countries have introduced the defined-contribution Individual Account system. An important policy debate for these countries is whether appropriate financial instruments are available for retirees to solve the problem of longevity risk. The focus on the PA scheme in this paper is motivated by two sets of factors. First, compared with private annuity providers, the government may have a lower cost structure (Diamond, 2004). Second, we have observed diverse practices in the PA plans in various economies, as summarized in Section 2.1. Even though these plans generally differ in various dimensions, there are also common features, such as the guarantee and nonescalating payments.

On the basis of these observed PA practices, we think it is interesting to study at least the following two aspects. First, it is not clear how to best organize and classify different public annuitization plans. The study of the pros and cons of mandatory versus voluntary PA plans in Lau and Zhang (2023) is an attempt in this direction. Second, the reasons of adopting several commonly observed features, including guarantee and nonescalating payments, by the PA providers are not well understood. This paper aims to study the implications of these two features by using a tractable analytical model of annuitization behavior in the presence of adverse selection.

We find that the difference of annuitization-weighted and unweighted averages of the retirees' survival probabilities is a useful measure of the severity of adverse selection. On the basis of this concept, we study whether introducing the guarantee or nonescalating payments affects the severity of adverse selection of the annuity market. Previous empirical and theoretical studies such as Finkelstein and Poterba (2002, 2004), Davidoff et al. (2005) and Cannon and Tonks (2016) examine whether these features affect annuity market efficiency through changing the severity of adverse selection. A key finding of this paper is that introducing either one of these two features leads to a decrease in the severity of adverse selection under some conditions regarding the degree of heterogeneity in survival probabilities, but has no effect on the severity of adverse selection when the conditions are not satisfied. (The intuition of these results will be provided in Sections 5 and 6.) Besides providing new results regarding these issues, our paper highlight the common underlying factors for the survival-contingent payments only versus the guarantee component on the one hand, and the escalating versus nonescalating payments on the other. To the best of our knowledge, previous researchers studied these questions separately, and have not linked them through a common lens. Moreover, the results in this paper have policy implications regarding the number and form of the PA plans provided.

The remaining sections of this paper are organized as follows. In Section 2, we provide the background material for this study, reviewing the PA plans in various economies, as well as related papers in the literature. Section 3 describes the model in detail. Section 4 analyzes the annuity buyers' behavior. We study how the buyer's annuitization behavior is related to her survival probability and the annuity payout level. We also show that the difference of annuitization-weighted and unweighted averages of survival probabilities is a useful measure of the severity of adverse selection of the PA plan. Both sets of results are useful for the subsequent analysis of policy issues. On the basis of the analysis in Section 4, we consider two policy issues relevant to the actual practices of the PA plans. Section 5 discusses how guarantee payment as a policy instrument could interact with adverse selection. Section 6 examines the issue of escalating versus nonescalating payments in the PA plan. Section 7 provides concluding remarks.

2 | ANNUITIZATION POLICIES AND LITERATURE REVIEW

In Section 2.1, we review annuitization policies in several economies, focusing particularly on those offering PA plans. We provide literature review in Section 2.2.

2.1 | Observed PA plans

Many governments adopt annuitization policies to help retirees deal with longevity risk. We observe a wide variety of these practices and there are differences in at least four major aspects. First, some economies (such as the UK before 2015 and Chile) simply require the retirees to purchase annuities from the private market, but some other economies provide public annuities directly.[1] Despite these differences, a common similarity among these policies is that the longevity risk is mainly shared by the same cohort of retirees, without relying on intergenerational transfer. These arrangements are less likely to suffer from budgetary problems as a result of population aging. Second, for those economies providing public annuities, some of them (such as Hong Kong and Singapore) adopt defined-contribution pension systems and provide public annuities for the payout phase, while others adopt defined-benefit pension systems but introduce a new and usually smaller defined-contribution plan to complement the original pension plan. Third, purchase of public annuities by the retirees may be mandatory or voluntary. Last, there are different features in the observed PA plans, including the trade-off between annuity income versus bequest level, and the presence or absence of escalating payments over time. Some of the above points are summarized in Table 1, in which we focus on six economies (Denmark, Hong Kong, India, Lithuania, Singapore, and Sweden) which provide public annuities.

At first glance, annuities supplied by the government or the private sector perform the same financial function of transforming lump sum payments at some specific time periods to a steady stream of income in the future (and possibly throughout the buyer's remaining lifetime). However, we think that several aspects of public annuities are quite unique. When the government provides public annuities, usually there is a specific objective in helping retirees insure against longevity risk. Compared with the private annuity providers whose other objectives such as profit maximization and/or risk management may be crucial, analyzing the simpler environment associated with public annuities is more likely to deliver sharper results. Moreover, the cost of annuities provided by the government may likely be lower because of economies of scale. This contrasts with the well-known evidence that the cost of annuities provided by the private sector is high (Friedman & Warshawsky, 1990). Another, and perhaps the most important, reason is that the design of PA policies may matter. To the extent that policy design of public annuities matters, it could be more direct for the government to deliver the result by implementing the PA plans, rather than relying on the private market to achieve the desired outcome. In particular, Lau and Zhang (2023) show that the problem of adverse selection is less severe when there is a restriction on the maximum level of PA purchase.[2] On the other hand, such a reduction in the severity of adverse selection could not be achieved in the private annuity market even if the insurance companies impose maximum purchase restriction to minimize financial risk.[3]

[1]We define public annuities as those provided by the government or a statutory body. For example, the ATP Lifelong Pension scheme in Denmark is administrated by the Danish Labor Market Supplementary Pension Scheme (ATP), which was established by the Danish Parliament in 1964.

[2]It is empirically interesting to examine whether annuity buyers' purchase amounts are restricted by the maximum level allowed by the PA plan. However, the maximum purchase levels in some PA plans have been changed frequently. For example, the maximum purchase level of the Hong Kong PA plan has been changed three times, from HK$1m to HK$2m, to HK$3m, and then to HK$5m, from 2018 to 2022. While these changes seem to indicate that some annuity buyers' purchase amounts have reached the maximum level (at some point in time), it may be challenging to provide rigorous empirical evidence regarding the proportion of annuity buyers whose purchase amounts are restricted by the maximum level, because of these policy changes in a short period of time, besides issues related to annuity purchase data confidentiality.

[3]In a competitive market where different insurance companies offer similar annuity products, if a company sets a purchase ceiling, a retiree with good health can buy the maximum amount from this company and then go to others to satisfy her unfulfilled demand. It is not likely (and usually not legal) that different companies share their customer lists and restrict the customers from purchasing annuities at other companies. (This well-known difficulty in private insurance markets [such as the annuity market] has been mentioned by Chiappori, 2000, p. 369: "agents can always "linearize" the schedule by buying a large number of small contracts from different insurers.") Since the overall severity of adverse selection is determined by the total amount of private annuities purchased by all retirees, the effect of imposing purchase restriction on the severity of adverse selection in the PA sector is absent in the private market.

TABLE 1 Annuities provided by governments or statutory bodies in six economies.

Economies	Denmark	Hong Kong	India	Lithuania	Singapore	Sweden
Annuity provider	ATP	HKMCA	NPS LIC	SODRA	CPF LIFE	PPM
Annuitization	Mandatory	Voluntary	Mandatory	Mandatory	Mandatory	Mandatory
Purchase of annuities offered by the government	Mandatory	Voluntary	Voluntary	Mandatory	Mandatory	Mandatory
Choice of annuitization amount	No	Available	Available	Limited[a]	Available	Available
One product or a menu of products	One product	One product	More choices[b]	More choices[c]	More choices[d]	More choices[e]
Guarantee element	Available	Available	Available	Available	Available	Available
Escalating or nonescalating payments	Nonescalating	Nonescalating	Both	Nonescalating	Both	Nonescalating
Gender-based or gender-neutral pricing	Gender-neutral	Gender-based	Gender-neutral	Gender-neutral	Gender-based	Gender-neutral

Abbreviations: ATP, Labor Market Supplementary Pension; CPF, Central Provident Fund; HKMCA, Hong Kong Mortgage Corporation Annuity; LIC, Life Insurance Corporation; LIFE, Lifelong Income for the Elderly; NPS, National Pension System; PPM, Premium Pension Authority; SODRA, Social Insurance Fund Board.

[a]No annuitization choice for retirees with Tier II pension fund account balance less than Euro 60,000.

[b]Including annuity without guarantee element, annuity with guarantee element, annuity with escalating payments.

[c]Three products: Standard pension annuity, standard pension annuity with a guarantee, and deferred pension annuity.

[d]Three products: Standard plan, basic plan, and escalating plan.

[e]Including annuity without survivor benefit, annuity with survivor benefit.

On the basis of the observed PA plans in the six economies examined in Table 1, we think that two major issues worth further examining. First, rows 3–6 of Table 1 show that while different governments may adopt PA policies emphasizing either the mandatory or voluntary aspect, retirees are usually given some freedom to choose the annuitization amount.[4] Second, we observe the differences in the exact form of annuity contracts offered, as indicated in rows 7–9 of Table 1.

Lau and Zhang (2023) focus on the mandatory versus the voluntary aspect of PA purchase, and propose a common framework to understand the degree of flexibility of different PA plans. This paper, on the other hand, aims to understand the underlying reasons of two features (guarantee elements and nonescalating payments) of the PA plans when retirees are allowed to choose the amount of annuity purchase.[5]

2.2 | Related literature

Our study is related to the large and growing literature on retirement financing. Researchers have examined how different retirement protection systems perform. Some studies (such as Hosseini & Shourideh, 2019) examine a PAYGO system, while others (such as McGrattan & Prescott, 2017) focus on a fully funded saving-for-retirement system. Whether it is a PAYGO or fully funded system, there are two phases in retirement financing: accumulation and payout. In the first phase people make a contribution. Depending on the earlier contribution, people then receive benefit in the second phase and spend their resources.

We focus on the question about how retirees spend their resources in the payout phase, taking their early contribution choices as given. Moreover, we focus on the idea of public, as opposed to private, provision of annuities. Diamond (2004) suggests that the government bears a lower administrative cost than the private market does. (For the cost of private annuity providers, see, e.g., Friedman & Warshawsky, 1990.) Although the idea of public provision of annuities arises from his study of the US Social Security system, it is also relevant to the defined-contribution Individual Account system. Fong et al. (2011) estimate the value of the public annuities sold by Singaporean government. Different from the above studies, our study focuses on various policy features of public annuities.

Our paper is also related to the literature about annuity demand. In his seminal work, Yaari (1965) obtains the sharp prediction that consumers would annuitize all their wealth under some assumptions such as no bequest motive and the only uncertainty relevant to a consumer's decision is the date of her death. Davidoff et al. (2005) further prove that this result holds under weaker assumptions. However, this prediction contrasts sharply with the observation that only a small percentage of consumers purchase annuities. This discrepancy generates interest among researchers in examining the annuity puzzle (Benartzi et al., 2011). Adverse selection and bequest motive (such as in Abel, 1986; Brugiavini, 1993; Friedman & Warshawsky, 1990; Hosseini, 2015; Lockwood, 2012; Pashchenko, 2013; Villeneuve, 2003) are regarded as important factors in understanding the annuity puzzle.

[4]For example, even though the purchase of annuities administered by the Premium Pension Authority in Sweden is mandatory, members are allowed to annuitize 25%, 50%, 75%, or 100% of their pension entitlements.

[5]There are many aspects of an annuity contract. The two aspects that we focus in this paper correspond to "guarantee features" and "form of regular payment" (level vs. escalating, both in nominal terms) in Table 1.1 of Mackenzie (2006, p. 22), which shows the principal forms of annuities.

Rather than studying the factors leading to the low participation in the private annuity market, our study focuses on the features of PA products, building on the insight of existing research work on the demand-side factors of the annuity puzzle.

3 | A MODEL OF PUBLIC ANNUITIES

We study PA policy design issues using a consistent economic framework. The accumulated wisdom in the literature of annuity puzzle provides useful guidance in choosing important factors to analyze (Benartzi et al., 2011; Munnell et al., 2022). The list includes high annuity price (which may arise from adverse selection, high administrative cost, etc.), intrafamily risk sharing (including bequest motive and support from children), illiquidity concern, medical and long-term care expenditure, and crowding-out effect of government pension. On the basis of these ideas and the observed PA practices, we emphasize two key features in this paper.

First, both demand-side and supply-side imperfections are generally relevant in explaining the annuity puzzle. Since the government is more willing to bear relevant risks (such as future mortality risk and interest rate risk) and the administrative cost of public annuities may be lower, we assume in this paper that there is no supply-side imperfection when the PA plans are introduced. Specifically, we assume that the PA provider takes a financially neutral position under the zero-profit condition.[6] Under this situation, if there is still any imperfection in the outcome, it is due to the demand side. This approach enables us to focus more sharply on the sources of any resulting imperfection, and to provide appropriate solutions.

Second, among the factors causing demand-side imperfection, we think that adverse selection due to asymmetric information on survival probability is a major factor, based on previous theoretical work (such as Abel, 1986; Hosseini, 2015) and empirical studies (such as Einav et al., 2010; Finkelstein & Poterba, 2004; Hagen, 2015).[7] Moreover, since we are interested in understanding the effect of different guarantee components, we also focus on the bequest motive, based on the analysis in Lockwood (2012), Pashchenko (2013), and Liu et al. (2020).

On the basis of the above idea, we consider a simple two-period model of public annuities. The two periods in our model, to be called Period 1 and Period 2, correspond to two stages of an individual's postretirement life cycle relevant to the annuitization decision: (a) early part of retirement (e.g., between the ages of 65 and 80), and (b) later part of retirement (e.g., 80 years old and above), respectively. In the model, there exists a continuum of retirees. Each retiree lives with certainty for the first period; however, some retirees are able to survive to Period 2 but others are not.

Our model emphasizes the heterogeneity in survival probability, and the information asymmetry between the PA provider and the retirees. We use parameter θ $(0 < \theta < 1)$ to

[6]The premium pension system in Sweden, which is a "defined-contribution, financially stable pension system," follows the practice that "every year net income is more or less equal to zero" (Swedish Pension Agency, 2020, p. 5).

[7]There is also evidence of adverse selection based on pension claiming age or pension credit transfer between spouses. Goda et al. (2018) show that in the United States, people with higher mortality rates tend to claim Social Security benefits earlier. Since claiming pension benefits earlier reduces their amount of public annuitization, the evidence indicates that adverse selection is present. On the other hand, for the premium pension system in Sweden, pension credit transfer between spouses is allowed. According to the Swedish Pension Agency (2020, p. 44), "more transfers are made to women than to men." Since women have longer life expectancy than men, this evidence is consistent with the presence of adverse selection. More generally, the importance of adverse selection has been well known for a long time. For example, it was mentioned in the novel Sense and Sensibility that "… people always live forever when there is any annuity to be paid them…" (Austen, 1811, Chap. 2.)

represent the survival probability to Period 2, and assume that $\theta \in [\underline{\theta}, \bar{\theta}]$ follows a cumulative distribution function $F(\theta)$.

On the supply side, we assume that there is a single PA provider, who has less information about the annuity buyers' survival probability prospects than the buyers themselves.[8] In Period 1, the PA provider offers a life annuity contract specified in payout only, which is the same to all buyers. The PA plan operates in the following manner. If an annuity buyer pays one dollar to purchase the annuity at the beginning of Period 1, she will receive A dollars in every period that she is alive, and her beneficiary will receive some fraction of the annuity income, gA dollars, in Period 2 if the annuity purchaser dies (at the end of Period 1). Our specification allows for the presence of guarantee element (which can also be interpreted as a bequest element) in the contract, when $g > 0$.[9] On the other hand, when $g = 0$, the PA is in a pure form with survival-contingent payments only. The PA provider also imposes a ceiling on the purchase amount by an individual buyer.

On the demand side, it is assumed that a retiree with private information about θ chooses the quantities of PA (α_θ) and risk-free bond (s_θ) at Period 1 to maximize the expected lifetime utility

$$u(c_{1\theta}) + \frac{\theta}{1+\rho}\left[u(c_{2\theta}) + \frac{1}{1+\rho}v(b_{3\theta})\right] + \frac{1-\theta}{1+\rho}v(b_{2\theta}), \tag{1}$$

subject to the following budget constraints:

$$c_{1\theta} = w - \alpha_\theta + A\alpha_\theta - s_\theta, \tag{2}$$

$$c_{2\theta} = s_\theta(1+r) + A\alpha_\theta - \frac{b_{3\theta}}{1+r}, \tag{3}$$

and

$$b_{2\theta} = s_\theta(1+r) + gA\alpha_\theta, \tag{4}$$

where

$$0 \le \alpha_\theta \le m < w, \tag{5}$$

$$0 \le g < 1, \tag{6}$$

w is the retiree's wealth at retirement, r is the interest rate for the risk-free bond, ρ is the subjective discount rate, m is the maximum amount of public annuities that a retiree is allowed to purchase, α_θ is the amount of PA purchase by a retiree with survival probability parameter

[8]It is well known that men and women have different survival probabilities (and thus, life expectancies). Gender is public information, and annuity providers may offer different annuity products for the two groups if this practice is allowed. However, it is forbidden by law to offer annuities with gender-based pricing in some societies, such as the EU countries. On the other hand, there are other factors (such as genetic and behavioral differences) that lead to different survival probabilities of the individuals, and our assumption is based on the existence of asymmetric information on these factors between buyers and sellers. Our model may be interpreted as a PA plan offered to the whole population in an EU country adopting gender-neutral pricing. Alternatively, in economies (such as Hong Kong and Singapore) adopting gender-based pricing, our model can be used to examine the behavior of a particular gender.

[9]In the PA plans of Singapore, the Standard Plan offers higher annuity income stream but lower bequest level to beneficiaries, and the Basic Plan offers lower annuity income stream but higher bequest level. They correspond to different values of g in our model.

θ, $c_{i\theta}$ is the level of consumption expenditure in Period i ($i = 1, 2$), s_θ is the expenditure on risk-free bond in Period 1, and $b_{i\theta}$ is the bequest level in Period i ($i = 2, 3$).[10] Since we study an economy with two different financial assets: risk-free bond and a PA (with guarantee payments), we impose the restriction in (6) that g is strictly less than 1 to ensure that the two financial assets are different.[11] Note that we assume a simple model, given by (1)–(6), in which there is no private annuity. Our results do not depend on whether a private annuity is present or not.[12]

In the above specification, a retiree's lifetime utility consists of two parts: $u(c)$ from consumption and $v(b)$ from bequest. The $u(\cdot)$ and $v(\cdot)$ functions have the standard properties that they are strictly concave, as well as $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{b\to 0} v'(b) = \infty$. The budget constraint (3) is applicable to retirees who survive to Period 2, but (4) is applicable to those who do not.

Compared with private annuity companies, the government may have economies of scale and better credit rating, leading to lower costs in providing annuities. Moreover, some governments are willing and capable to take up various long-term risks associated with issuing lifetime annuities. To reflect this financial capability, it is assumed that the PA provider takes a financially neutral position of zero profit in issuing annuities. It can be shown that the zero-profit condition leads to[13]

$$A = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta \, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \left(1 + \frac{\theta + (1-\theta)g}{1+r}\right) \alpha_\theta \, dF(\theta)}. \tag{7}$$

As shown in Section 4.2, an advantage of using the zero-profit condition is that the severity of adverse selection is reflected in the equilibrium payout level. In addition, when we compare various PA plans in subsequent sections, these plans are subject to the same constraint (7) of zero profit. This assumption allows a fair comparison of different PA policies.

Three aspects of the above specification are related to observed PA practices in various economies. First, these PA providers do not require the potential buyers to provide health information when purchasing annuities. All buyers with the same age and gender are offered the same annuity contract. Using the terminology in the literature, the PA provider offers nonexclusive annuity contracts with linear pricing (as in Abel, 1986; Brugiavini, 1993; Hosseini, 2015), rather than exclusive contracts with price convexity (as in Eckstein et al., 1985; Eichenbaum & Peled, 1987).

[10]We assume the beneficiary receives the bequest (either $b_{2\theta}$ or $b_{3\theta}$) at the beginning of the next period of the buyer's death, to maintain symmetry.

[11]Note that the annuity with $g = 1$ is the same as the risk-free bond. To ensure that the PA pays different amounts in the two states of nature in Period 2, we assume (6). On the basis of observed PA practices, the maximum level of g is substantially less than 1.

[12]Lau and Zhang (2023) consider an economy with both public and private annuities, and show that a change in the PA market has an indirect effect on the private market, but not the other way around. The same result would hold if we extend our model to include private annuities.

[13]Condition (7) is obtained by equating the premium received by the PA provider (at Period 1), given by $\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta \, dF(\theta)$, and the present discounted value of payment, given by $\int_{\underline{\theta}}^{\bar{\theta}} \left(A + \frac{\theta}{1+r}A + \frac{1-\theta}{1+r}gA\right)\alpha_\theta \, dF(\theta)$. When the payout term A is set "too high," the total payment is higher than the total annuity premium received, resulting in a PA budget deficit. When A is set "too low," there will be PA budget surplus.

Second, there is a restriction on the maximum amount of public annuities that a retiree can purchase. This feature is present in all the PA plans in Table 1, either explicitly as in Hong Kong or Singapore,[14] or implicitly given by the retirees' maximum accumulated contributions to the defined-contribution schemes in the other four economies. On the basis of this observed feature, we consider a PA plan specifying the same price for all buyers but having quantity restrictions, instead of a pure nonexclusive contract with linear pricing or exclusive contracts with price convexity.

Third, if a PA purchaser dies at an early age, her beneficiary is entitled to receive some amount of money, which is a proportion of the promised payment received by the buyer if she is still alive. This guarantee feature is observed in all six economies examined in Table 1.

The equilibrium of the above PA plan is attained when (a) each retiree chooses consumption, bequest, and the financial assets ($c_{1\theta}$, $c_{2\theta}$, $b_{2\theta}$, $b_{3\theta}$, α_θ, and s_θ) to maximize (1) subject to her budget constraints (2)–(4) and the PA payout level (A); and (b) anticipating the retirees' annuitization choices (α_θ), the government chooses A to satisfy the zero-profit condition (7). At the equilibrium, the PA payout level and the retirees' choices are mutually consistent.

We now consider the retiree's optimal choices. In this two-period model, all retirees make decisions in Period 1, and those who survive to Period 2 also make decisions in that period. In Period 1, after knowing her private information θ, a retiree chooses the two financial assets (s_θ and α_θ) to maximize (1), subject to various budget constraints. If the retiree survives to Period 2, then she chooses $b_{3\theta}$ to maximize

$$u(c_{2\theta}) + \frac{1}{1+\rho} v(b_{3\theta}), \tag{1a}$$

subject to (3).

We solve the retiree's optimization problem backward. The first-order condition in Period 2, if the retiree survives to, is given by

$$u'\left(c_{2\theta}^*\right) = \frac{1+r}{1+\rho} v'\left(b_{3\theta}^*\right), \tag{8}$$

where we use * to denote the optimal value of a variable. This first-order condition is standard. Suppose the retiree consumes one less unit in Period 2 and saves one more unit (by buying risk-free bond), the marginal cost (in terms of utility loss) is equal to $u'(c_{2\theta}^*)$. The reduction of one unit of consumption leads to $1 + r$ more units of resources next period, which will be bequeathed to her beneficiaries after she dies. Thus, the marginal benefit equals to $\frac{1+r}{1+\rho} v'(b_{3\theta}^*)$, after discounting this future utility benefit back to Period 2. At the optimum, the marginal benefit equals to the marginal cost, as given by (8).

[14]Starting from June 2022, Hong Kong citizens over 60 years old are allowed to purchase public annuities up to a maximum amount of HK$5 million (roughly US$641,000). In Singapore, under the Retirement Sum Topping-Up Scheme, retirees are able to buy public annuities up to the Enhanced Retirement Sum, which is SGD288,000 (about US$208,800) in 2022.

There are two choice variables in Period 1. Conditional on a particular value of α_θ, the first-order condition for s_θ is given by[15]

$$u'\left(c_{1\theta}^*\right) = \frac{1+r}{1+\rho}\left[\theta u'\left(c_{2\theta}^*\right) + (1-\theta)v'\left(b_{2\theta}^*\right)\right], \tag{9}$$

after canceling some terms according to the envelope theorem. The intuition is as follows. Suppose the retiree purchases one more unit of the risk-free bond in Period 1, her current consumption falls by one unit and thus her current utility decreases by $u'(c_{1\theta}^*)$. Thus, the marginal cost (in terms of utility level) is given by the left-hand side (LHS) term. The reduction of one unit of consumption would lead to an increase of $1 + r$ units of resources at the next period. There are two components of the marginal benefit, depending on whether she survives in Period 2 or not. The utility gain is given by $u'(c_{2\theta}^*)$ from consumption if she is alive (with probability θ), and $v'(b_{2\theta}^*)$ from bequeathing if she dies (with probability $1-\theta$). Thus, the marginal benefit equals to the right-hand side (RHS) term, after discounting the expected value of this future utility increase back to Period 1. At the optimum, the marginal benefit equals to the marginal cost, as given by (9).

Conditional on a particular value of s_θ, the first-order condition for α_θ is given by[16]

$$(1-A)u'\left(c_{1\theta}^*\right) = \frac{1}{1+\rho}\left[\theta A u'\left(c_{2\theta}^*\right) + (1-\theta)gAv'\left(b_{2\theta}^*\right)\right], \tag{10}$$

if $\alpha_\theta^* \in (0, m)$. The LHS term is the marginal cost of annuitizing one more unit of resources. Annuitizing one more unit reduces consumption level $c_{1\theta}$ by $1-A$ unit, resulting in direct utility loss of $(1-A)u'(c_{1\theta}^*)$.[17] The RHS term is the marginal benefit from two components: $Au'(c_{2\theta}^*)$ from annuity payout distributed with probability θ if she survives, and $gAv'(b_{2\theta}^*)$ from bequeathing due to the guarantee amount with probability $1-\theta$ if she dies.

Equation (10) is relevant for interior solutions of annuity purchase. If (10) does not holds for $\alpha_\theta^* \in (0, m)$, the optimal choice is a corner solution and there are two possibilities. If the marginal cost of annuitizing one more unit is strictly larger than the marginal benefit when $\alpha_\theta^* = 0$, as given by

$$(1-A)u'\left(w - s_\theta^*\right) > \frac{1}{1+\rho}\left[\theta A u'\left(s_\theta^*(1+r) - \frac{b_{3\theta}^*}{1+r}\right) + (1-\theta)gAv'\left(s_\theta^*(1+r)\right)\right], \tag{11}$$

then the retiree does not annuitize at all. On the other hand, if

[15]We express the first-order condition (9) in terms of $c_{1\theta}^*$, $c_{2\theta}^*$, and $b_{2\theta}^*$ because of its simple form. Equivalently, (9) can be expressed in terms of s_θ^* and α_θ^* explicitly, as $u'\left(w-(1-A)\alpha_\theta^* - s_\theta^*\right) = \frac{1+r}{1+\rho}\left[\theta u'\left(s_\theta^*(1+r) + A\alpha_\theta^* - \frac{b_{3\theta}^*}{1+r}\right) + (1-\theta)v'\left(s_\theta^*(1+r) + gA\alpha_\theta^*\right)\right]$.

[16]Similar to (9), it is possible to express the first-order condition (10) in terms of s_θ^* and α_θ^* explicitly, as $(1-A)u'\left(w-(1-A)\alpha_\theta^* - s_\theta^*\right) = \frac{1}{1+\rho}\left[\theta A u'\left(s_\theta^*(1+r) + A\alpha_\theta^* - \frac{b_{3\theta}^*}{1+r}\right) + (1-\theta)gAv'\left(s_\theta^*(1+r) + gA\alpha_\theta^*\right)\right]$.

[17]Because of the assumption of immediate annuity payment in Period 1, $1-A$ is the net decrease in the current consumption level. Note that our results regarding the effect of guarantee proportion (g) do not depend on whether annuity payments start in Period 1 or Period 2. (In the two-period model used in Lau & Zhang, 2023, annuity payments start in Period 2.) However, we need to consider a model with the annuity payments starting in Period 1 for the analysis in Section 6. We choose the current specification so that the models in this section and Section 6 are as similar as possible.

$$(1-A)u'\Big(w-(1-A)m-s_\theta^*\Big) < \frac{1}{1+\rho}\left[\theta Au'\left(s_\theta^*(1+r)+Am-\frac{b_{3\theta}^*}{1+r}\right)+(1-\theta)gAv'\Big(s_\theta^*(1+r)+gAm\Big)\right], \tag{12}$$

then the buyer annuitizes up to the maximum amount allowed (i.e., $\alpha_\theta^* = m$).

The presence of interior solutions for the retirees' PA purchases, corresponding to (10), and that of corner solutions, corresponding to (11) or (12), can be seen in Figure 1. We first focus on a given value of guarantee (g). There are several possible outcomes, depending on the parameter values. In Panel A, all retirees' PA purchases are interior solutions. In Panel B or C, there is one threshold such that the PA purchases of one group of retirees are interior solutions and the others are corner solutions. In Panel D, there are two thresholds such that the retirees' PA purchases are interior solutions when θ is in the middle range, but they are corner solutions for either high or low values of θ.

4 | ANNUITY BUYERS' BEHAVIOR AND SEVERITY OF ADVERSE SELECTION

In Section 4.1, we study how a retiree's annuitization decision (α_θ) varies with respect to survival probability θ, when other parameters are unchanged. We focus on interior solutions for α_θ in this section.[18] In Section 4.2, we suggest a measure of the severity of adverse selection of the PA plan.

4.1 | Effect of survival probability on annuitization

The two first-order conditions (9) and (10) can be expressed in several equivalent ways. In the following analysis, we choose to express in terms of two real variables $c_{2\theta}$ and $b_{2\theta}$, which provides a clearer interpretation of the results.[19] Moreover, the focus on real variables will be particularly useful for Propositions 1 and 3(a) in later sections. Combining the budget constraints (2)–(4) leads to

$$w = c_{1\theta} + \frac{\theta_w}{1+r}\left(c_{2\theta}+\frac{b_{3\theta}}{1+r}\right)+\frac{1-\theta_w}{1+r}b_{2\theta}, \tag{13}$$

where

$$\theta_w = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta^*\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^*\, dF(\theta)} = \frac{\left[1-\left(1+\frac{g}{1+r}\right)A\right](1+r)}{(1-g)A}. \tag{14}$$

[18]The corner solutions are not very interesting for the analysis in this section, but they will be relevant for the analysis in subsequent sections.

[19]Alternatively, we can analyze (9) and (10) in terms of the two financial variables α_θ and s_θ. While the main result, (21), can be obtained more directly using this method, the interpretation of the key results is easier to see based on the framework in this section.

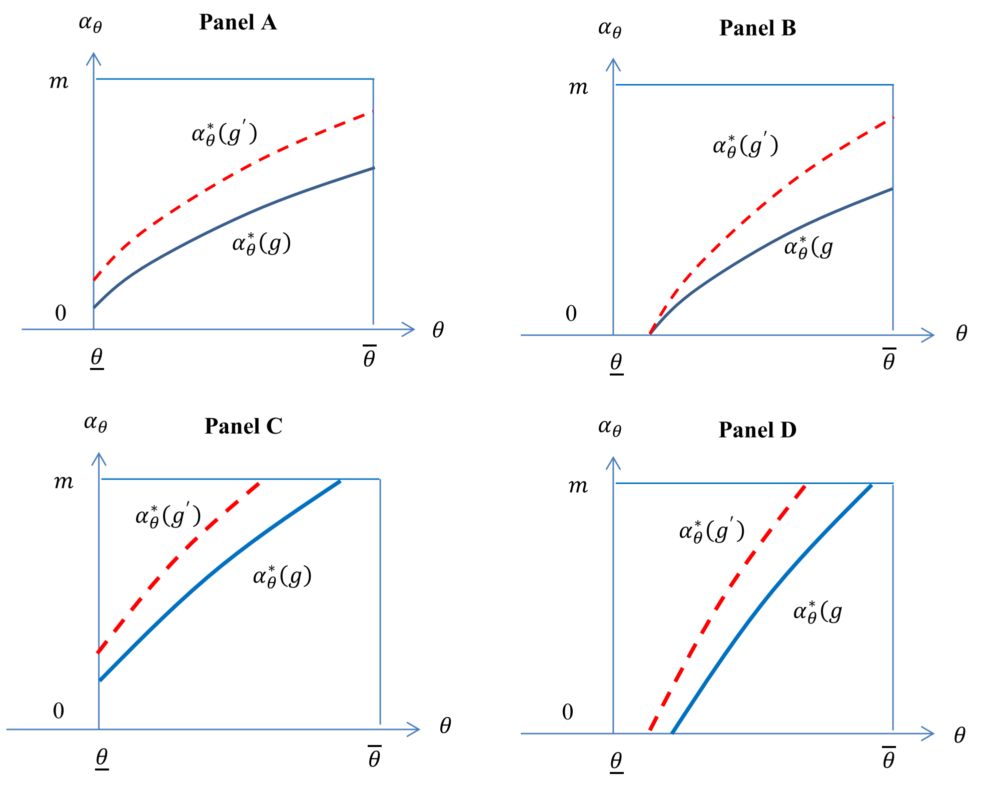

FIGURE 1 Shifting of the annuitization function (when g increases to g').

(See Appendix A.) The interpretation and importance of (14) will be discussed in Section 4.2. Substituting (13) into (9), we obtain

$$u'\left(w-\frac{\theta_w}{1+r}\left(c^*_{2\theta}+\frac{b^*_{3\theta}}{1+r}\right)-\frac{1-\theta_w}{1+r}b^*_{2\theta}\right)=\frac{1+r}{1+\rho}\left[\theta u'\left(c^*_{2\theta}\right)+(1-\theta)v'\left(b^*_{2\theta}\right)\right]. \tag{15}$$

Similarly, substituting (13) into (10) leads to

$$u'\left(w-\frac{\theta_w}{1+r}\left(c^*_{2\theta}+\frac{b^*_{3\theta}}{1+r}\right)-\frac{1-\theta_w}{1+r}b^*_{2\theta}\right)=\frac{1+r}{1+\rho}\frac{\left[\theta u'\left(c^*_{2\theta}\right)+(1-\theta)gv'\left(b^*_{2\theta}\right)\right]}{[\theta_w+(1-\theta_w)g]}, \tag{16}$$

after using

$$\frac{A}{1-A}=\frac{1+r}{[\theta_w+(1-\theta_w)g]}, \tag{14a}$$

which is obtained from (14).

Equations (15) and (16) constitute the bivariate system that determines $c^*_{2\theta}$ and $b^*_{2\theta}$ (the two choice variables in Period 1) simultaneously, with the expectation that $b^*_{3\theta}$ (to be chosen in Period 2 by those still alive) is related to $c^*_{2\theta}$ according to (8). To explore how these optimal values depend on the survival probability θ, we totally differentiate (15) and (16) with respect to θ and then solve these two simultaneous equations. In Appendix A, we report the standard (but tedious) analysis and show that

$$\frac{\partial c_{2\theta}^*}{\partial \theta} > 0 \tag{17}$$

and

$$\frac{\partial b_{2\theta}^*}{\partial \theta} < 0. \tag{18}$$

After using the budget constraints (3) and (4), the financial variable α_θ^* is related to the real variables ($c_{2\theta}^*$, $b_{2\theta}^*$, and $b_{3\theta}^*$) according to

$$\alpha_\theta^* = \frac{1}{(1-g)A}\left(c_{2\theta}^* + \frac{b_{3\theta}^*}{1+r} - b_{2\theta}^*\right). \tag{19}$$

The budget constraint (3) indicates that the annuity and risk-free bond are used to support $c_{2\theta}^*$ and $b_{3\theta}^*$ if the annuity buyer survives to Period 2. Similarly, the budget constraint (4) indicates that these two financial assets are used to provide the bequest level $b_{2\theta}^*$ if she dies at the end of Period 1. On the basis of the difference of these two equations, (19) shows that the difference in annuity payments in these two states of nature can be traced to the real variables $c_{2\theta}^*$, $b_{2\theta}^*$, and $b_{3\theta}^*$.

Using (8) and (17), it is easy to see that

$$\frac{\partial b_{3\theta}^*}{\partial \theta} > 0. \tag{20}$$

Combining (17)–(20), we conclude that

$$\frac{\partial \alpha_\theta^*}{\partial \theta} > 0. \tag{21}$$

The intuition of (21) is as follows. Compared with an annuity buyer with a particular value of survival probability parameter θ, another buyer with a higher θ chooses a higher level of $c_{2\theta}^*$ and $b_{3\theta}^*$ (as she puts higher weight on the components of lifetime utility associated with the state of survival in Period 2). On the basis of the same consideration, she chooses a lower level of $b_{2\theta}^*$. Since the annuity purchase (together with that of the risk-free bond) is used to support the real variables, (19) implies that a retiree with a higher survival probability θ chooses a larger value of α_θ^* to support a higher level of $c_{2\theta}^* + \frac{b_{3\theta}^*}{1+r} - b_{2\theta}^*$.

4.2 | A measure of the severity of adverse selection

We now show that the severity of adverse selection under the PA plan is related to the equilibrium annuity payout level (A). As seen in (14), there is a one-to-one relationship between A and θ_w. We find it more convenient to derive the results and obtain the interpretation in terms of θ_w, and we focus on this term in the subsequent analysis.

To understand the interpretation of the term θ_w in (14), it is helpful to rewrite the equation as[20]

$$\theta_w = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta^* \, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* \, dF(\theta)} = \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{\alpha_\theta^*}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* \, dF(\theta)} \right] dF(\theta). \tag{22}$$

According to (22), θ_w is a weighted average of survival probabilities of the retirees, with the weight given by the share of annuitization: $\frac{\alpha_\theta^*}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* dF(\theta)}$. As will be shown in Section 5, the interpretation of θ_w as the *annuitization-weighted average of survival probabilities* is helpful in the proof of Proposition 2.

Pursuing this line of reasoning, it can further be shown that

$$\theta_w = \frac{E\left(\theta\alpha_\theta^*\right)}{E\left(\alpha_\theta^*\right)} = \frac{E(\theta)E\left(\alpha_\theta^*\right) + cov\left(\theta, \alpha_\theta^*\right)}{E\left(\alpha_\theta^*\right)} = E(\theta) + \frac{cov\left(\theta, \alpha_\theta^*\right)}{E\left(\alpha_\theta^*\right)}, \tag{23}$$

where

$$E(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta \, dF(\theta), \tag{24}$$

the average survival probability of all the retirees. Since $\frac{\partial \alpha_\theta^*}{\partial \theta} > 0$ according to (21), we have

$$cov\left(\theta, \alpha_\theta^*\right) > 0. \tag{25}$$

Substituting (25) into (23) and using $E(\alpha_\theta^*) = \int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* \, dF(\theta) > 0$, we obtain

$$\theta_w - E(\theta) = \frac{cov\left(\theta, \alpha_\theta^*\right)}{E\left(\alpha_\theta^*\right)} > 0. \tag{26}$$

The result that θ_w is larger than $E(\theta)$ in (26) is traced to the positive dependence of α_θ^* on θ.

The difference of θ_w and $E(\theta)$ is a useful measure of the severity of adverse selection in the annuity market. Since $E(\theta)$ is not affected by the behavior of the retirees, we can simply use θ_w as the measure in the analysis of PA policies. The term θ_w is helpful in understanding the buyers' low annuity purchase. As shown in (21), buyers with low survival probabilities would have low levels of annuity purchase. These choices, together with higher levels of annuity

[20]Similar expressions have appeared in earlier work, such as eq. (8) of Villeneuve (2003) and (1) in Fang et al. (2008). Villeneuve (2003), for example, labels the ratio $\frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* dF(\theta)}$ as "average clientele risk."

purchase by buyers with higher survival probabilities, would lead to a high degree of severity of adverse selection (θ_w) of the PA plan, which is equivalent to a low annuity payout A according to (14a). The low annuity payout further feedbacks to lower annuity demand for all buyers.

The above analysis is based on the simple case that all retirees' PA purchases are interior solutions (Panel A of Figure 1). However, it can easily be seen that as long as some retirees' PA purchases are interior solutions, all the results from (22) to (26) are also applicable to Panels B–D of Figure 1.

5 | GUARANTEE PAYMENT TO THE BENEFICIARIES

We have examined annuity buyers' behavior in Section 4. In particular, we have found that θ_w in (22) can be used to measure the severity of adverse selection. Since adverse selection is a major source of inefficiency in the annuity market, any policy reducing θ_w at a reasonable cost would lead to potential improvement. This idea guides our analysis of the PA policies in Sections 5 and 6, in which we consider the effects of different annuity policies on the behavior of the annuity buyers.

In this section, we compare annuities with different levels of guarantee proportion g, where g satisfies (6). For example, one may compare a pure life annuity with only survival-contingent payments ($g = 0$) versus another annuity with guarantee payments to the beneficiaries ($0 < g < 1$). Our investigation is motivated by the observed PA plans with the guarantee element in all the six economies mentioned in Table 1. In the literature, the importance of guarantee element has been mentioned in the empirical analysis in Finkelstein and Poterba (2002, 2004) and the theoretical analysis of Davidoff et al. (2005), in the context of private annuity market. Some of the insights in these papers will be relevant for the analysis of guarantee features of the PA plans, but our analysis reveals a new result that the major reason emphasized in the literature may not be effective under some conditions.

We show that the effect of the guarantee feature depends on the extent of heterogeneity in survival probability. We obtain various results when the heterogeneity appears differently. To see the underlying reasons more clearly, we conduct the analysis in order of increasing complexity, starting with the simplest case in Section 5.1 before moving to more complicated cases.

The term θ_w in (22) or (23), which measures the severity of adverse selection, is important in understanding the results in this section. In Section 4 when we compare the behavior of annuity buyers with different survival probabilities for a given annuity contract offered by the PA provider, it is natural to take θ_w as unchanged. However, θ_w *may change* when the PA provider offers *a different annuity plan*. To take advantage of the useful results derived in the previous sections, we will take a two-step procedure in the following analysis. First, we conduct the analysis conditional on a particular value of θ_w when we study individual behavior in response to a change in g. Our second step is to obtain the value of θ_w consistent with the behavior of all buyers under the new PA.[21]

[21]The above procedure is valid when the equilibrium is unique. According to all computational analyses that we have conducted and reported in Section 5.6 and the Supporting Information Appendix, the equilibrium is always unique. More generally, it is difficult to prove the uniqueness of equilibria when θ_w and the retirees' annuitization choices are mutually dependent; see, for example, Abel (1986, p. 1086) and Villeneuve (2003, p. 534). Nevertheless, even if multiple equilibria are present, it is reasonable to assume that the government chooses the one with the highest annuity payout level (and lowest θ_w), which would benefit all buyers of the PA plan. This equilibrium is consistent with the government's objectives of financial sustainability under the zero-profit condition and promoting social welfare.

5.1 | Limited heterogeneity, with all buyers annuitizing a positive amount less than the maximum level

We first consider the simplest case (Panel A in Figure 1), with all PA buyers choosing $\alpha_\theta^* \in (0, m)$.[22] Interpret the first-order conditions (15) and (16) as functions of the two endogenous variables $c_{2\theta}^*$ and $b_{2\theta}^*$ being affected by the exogenous change in the guarantee proportion (g). It can be shown that the effect of a change in g on each of the two endogenous variables is zero:

$$\frac{\partial c_{2\theta}^*}{\partial g} = \frac{\partial b_{2\theta}^*}{\partial g} = 0. \tag{27}$$

While all real variables remain unchanged when the guarantee proportion g changes, the annuitization level (α_θ^*) changes. It turns out that annuity buyers with different values of θ have the *same percentage change* in α_θ^*. This result is easier to see when we consider a discrete change in g, say, to a higher level g′ (>g). For this increase in g, we use (14a) and (19) to obtain

$$\frac{\alpha_\theta^*(g')}{\alpha_\theta^*(g)} = \frac{(1-g)(1+r+\theta_w+(1-\theta_w)g')}{(1-g')(1+r+\theta_w+(1-\theta_w)g)} \equiv R_{gg'} > 1, \tag{28}$$

where $R_{gg'}$ is the same for all buyers (i.e., $R_{gg'}$ is independent of θ).

The above analysis is conditional on a particular value of θ_w. The next step is to see whether the resulting value of θ_w changes when there is equal percentage change in α_θ^* according to (28). Substituting (28) into (14), we have

$$\theta_w(g') = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta^*(g')\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^*(g')\, dF(\theta)} = \frac{R_{gg'}\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta^*(g)\, dF(\theta)}{R_{gg'}\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^*(g)\, dF(\theta)} = \theta_w(g). \tag{29}$$

Therefore, the value of θ_w does not change when the guarantee proportion changes, because of an equal percentage change in the optimal annuitization choices for all annuity buyers.

We summarize these results in the following proposition.

Proposition 1. *When the heterogeneity in θ is limited such that all annuity buyers choose $\alpha_\theta^* \in (0, m)$ for any proportion of guarantee g satisfying (6), a change in g leads to*

(a) *no change in consumption and bequest variables, and thus no change in any annuity buyer's utility level;*
(b) *the same percentage change in the annuitization level for all buyers according to (28); and*
(c) *no change in the severity of adverse selection (θ_w) of the PA plan.*

[22]The monotonicity result stated in (21) suggests that when the support of the distribution $F(\theta)$ is narrow, it is more likely to observe that all PA buyers purchase a positive but not the maximum amount. We label this case with only one type of annuity buyers (with $0 < \alpha_\theta^* < m$) as "limited heterogeneity." Note that we do not consider the other two cases with only one type of annuity buyers ($\alpha_\theta^* = 0$ or $\alpha_\theta^* = m$), which are less relevant and less interesting.

The intuition of Proposition 1 is as follows. The guarantee proportion g does not appear in (15). On the other hand, a rise in g has two effects on the marginal benefit term in the first-order condition (16). It increases the amount received by the beneficiaries if the buyer does not survive to Period 2, and thus tends to increase the second term on the RHS of (16). However, a higher g also leads to a lower level of $\frac{1}{[\theta_w + (1-\theta_w)g]}$, which is related to a lower payout A according to (14a), and thus tends to decrease both components on the RHS of (16). It turns out that the two effects exactly cancel out.[23] Thus, we have $m_{2g} = 0$ according to (A7d) in Appendix A, which further leads to (27).[24]

At first glance, it seems fortuitous that the two effects in (A7d) cancel exactly. A further investigation suggests that as long as a PA buyer purchases a positive but not the maximum amount, she can adjust her portfolio (between annuity and bond) freely to reach her original optimal levels of consumption and bequest, making the magnitude of guarantee payment irrelevant in affecting the levels of the real variables (i.e., Equation 27 holds). As a result, there is no change in the PA buyer's utility level. Furthermore, when the guarantee proportion increases from g to g', the government has to reduce the payout term A according to the zero-profit condition (7). In response to the smaller payment (for each unit of the PA) in the survival state of Period 2, the buyer has to increase her PA purchase to support her desired level of consumption. It is found in (28) that the proportional changes in annuitization for all annuity buyers are the same. The equal proportional change in annuitization by all buyers neutralizes the effect of a change in the guarantee proportion (g), leading to an unchanged level in the severity of adverse selection (θ_w) according to (29).

Proposition 1, together with the analysis in Section 5.3, helps understand the possible benefit of guarantee element in annuity contracts. We will discuss these issues in Section 5.5.

In Section 7, we will discuss the policy implications of this and the remaining Propositions.

5.2 | Greater heterogeneity, with some buyers not annuitizing at all

In this case (Panel B of Figure 1), there are two groups of annuity buyers: some buyers choose $\alpha_\theta^* \in (0, m)$, and some do not annuitize at all ($\alpha_\theta^* = 0$). For the second group of buyers, (11) is relevant, instead of (10). Since the analysis in this case is similar to those in Section 5.1, we just summarize the results briefly.

Starting with a PA plan with a guarantee fraction g, suppose now the government introduces a new PA with higher guarantee fraction ($g' > g$). It can be shown that (i) for those annuity buyers who choose a positive amount but not the maximum level, they keep the real variables unchanged and change the annuitization level proportionally according to (28); (ii) the threshold in survival probability determining no annuitization remains the same; and (iii) for those who do not annuitize originally, they still do not annuitize. As a consequence, parts (a) and (c) of Proposition 1 will also hold in this case.

[23]Combining (15) and (16), we obtain $\theta(1-\theta_w)u'(c_{2\theta}^*) = (1-\theta)\theta_w v'(b_{2\theta}^*)$. Using this equation, it can easily be shown that the two effects exactly offset each other.

[24]The effects of a change in g on the optimal values of $c_{2\theta}^*$ and $b_{2\theta}^*$ are determined by solving (A6) and (A7) simultaneously, with $d\theta = 0$. This leads to $\frac{\partial c_{2\theta}^*}{\partial g} = \frac{-m_{12}m_{2g}}{m_{11}m_{22} - m_{21}m_{12}}$ and $\frac{\partial b_{2\theta}^*}{\partial g} = \frac{m_{11}m_{2g}}{m_{11}m_{22} - m_{21}m_{12}}$. Using (A7d), we obtain (27).

5.3 | Greater heterogeneity, with some buyers annuitizing up to the maximum amount

In this case (Panel C of Figure 1), there are two groups of annuity buyers: some buyers annuitize less than the maximum amount ($0 < \alpha_\theta^* < m$), and some annuitize up to the maximum amount ($\alpha_\theta^* = m$).[25] For the second group of buyers, (12) is relevant, instead of (10).

The key element in the proof of Proposition 1 for the case of limited heterogeneity is that in response to a rise in the guarantee proportion (g) of the PA, the resulting proportional increases in annuitization for all retirees are consistent with an unchanged level in the severity of adverse selection (θ_w). In the current case when some buyers have already annuitized up to the maximum level, this group of buyers cannot annuitize further. As a result, the proportional changes in annuitization are not uniform for all buyers. The relatively straightforward proof for Proposition 1 is not valid in this case.

Nevertheless, the underlying idea behind the proof of Proposition 1 is still useful for the current analysis. Following the above approach, it is helpful to define the function

$$J(\theta_w, g) = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta(\theta_w, g)\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w, g)\, dF(\theta)}, \tag{30}$$

where

$$\alpha_\theta(\theta_w, g) = \begin{cases} \alpha_\theta^* & \text{if (10) holds,} \\ m & \text{if (12) holds} \end{cases} \tag{31}$$

with α_θ^* (together with s_θ^* and $b_{3\theta}^*$) determined by solving (8)–(10) simultaneously.[26] In the definition in (31), we express the annuitization amount α_θ as a function of both θ_w and g explicitly.[27]

The definition of $J(\theta_w, g)$ in (30) is useful because comparing (30) with (14), it can be concluded that the equilibrium value of $\theta_w(g)$, which is consistent with the optimal annuitization choices for a given level of g and the zero-profit condition (7), is defined according to[28]

[25]The two special cases in Sections 5.2 and 5.3 are covered by the general case in Section 5.4. An initial conjecture is that each of these two special cases is some kind of "mirror image" of the other: a given PA plan with a low guarantee proportion (such as a PA plan with $g = 0$) is preferred in one case but another plan with a high guarantee proportion is preferred in the other case. It turns out that this conjecture is not correct, and the following analysis highlights the interesting asymmetry between them.

[26]There are only two groups of annuity buyers in the definition of (31), because of the focus here (See Panel C of Figure 1 also.) If we use this method for the general case in Section 5.4, we have to include the third group (such that (11) holds) in the definition. Note also that the first-order conditions (8) and (9) are also relevant in (31). For example, when (12) holds, it holds with s_θ^* and $b_{3\theta}^*$ taking the value obtained by solving (8) and (9) with $\alpha_\theta^* = m$. We assume that this point is well understood and do not express it explicitly in (31) to keep the equation simple.

[27]Note that g, A, and θ_w are related according to (14), and the payout term A will adjust when g or θ_w changes. We find that the proof is easier to construct by focusing on θ_w and g, with A implicitly determined based on (14).

[28]The reasons of defining (30)–(32) are based on combining the idea behind the proof of Proposition 1 and the zero-profit condition (7). At given values of θ_w and g, the annuitization rate $\alpha_\theta(\theta_w, g)$ is determined according to (31). Combining all retirees' annuitization choices leads to $J(\theta_w, g)$ in (30). If $J(\theta_w, g) < \theta_w$, the PA plan is in surplus. If $J(\theta_w, g) > \theta_w$, then the PA plan is in deficit. When $J(\theta_w, g) = \theta_w$, the zero-profit condition (7) holds and the equilibrium severity of adverse selection of the PA sector is determined according to (32).

$$J(\theta_w(g), g) \equiv \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta(\theta_w(g), g)\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g)\, dF(\theta)} = \theta_w(g). \tag{32}$$

We now consider an increase of the guarantee proportion (from g to g'). To see the underlying reasons clearly, we consider the situation such that the two groups of buyers (those with $\alpha_\theta^* \in (0, m)$ and those with $\alpha_\theta^* = m$) are both present, both before and after the change in g.[29] We show in Proposition 2 that when g increases to g', the severity of adverse selection is reduced in the new equilibrium. The proof is presented in Appendix B.

Proposition 2. *Consider an increase in the guarantee proportion (from g to g') when the heterogeneity in θ is such that some annuity buyers choose $\alpha_\theta^* \in (0, m)$ and others choose $\alpha_\theta^* = m$. There exists a new equilibrium value of $\theta_w(g')$, which satisfies*

$$E(\theta) < \theta_w\left(g'\right) = J(\theta_w\left(g'\right), g') < \theta_w(g). \tag{33}$$

As seen in Appendix B, the proof of Proposition 2, which focuses on the $J(\theta_w, g')$ function over $\theta_w \in [E(\theta), \theta_w(g)]$ where $\theta_w(g)$ is given by (32), has three components. First, it is straightforward to show that at the beginning point $\theta_w = E(\theta)$, the PA provider is in deficit. Second, at the endpoint $\theta_w = \theta_w(g)$, the PA plan is in surplus. Neither of them is the new equilibrium at g'. Third, combining these two results with the continuity of the $J(\theta_w, g')$ function, we obtain Proposition 2. (The graph of $J(\theta_w, g')$ against θ_w is plotted in Panel (a) of Figure 2.)

The most important aspect of the proof is the second component, which is related to the different responses of the two groups of buyers (those with $\alpha_\theta^* \in (0, m)$ and those with $\alpha_\theta^* = m$) to a change in the guarantee proportion of the PA plan. We now provide the intuition of the result. If θ_w remains unchanged at $\theta_w(g)$ when the guarantee proportion increases (to g'), annuity buyers who annuitize less than the maximum amount initially will increase their annuity purchase. For this group of buyers, there are two possible outcomes. The new annuitization level is either given by $\alpha_\theta(\theta_w(g), g') = R_{gg'}\alpha_\theta(\theta_w(g), g)$ if it is less than m where $R_{gg'}$ is defined in (28), or $\alpha_\theta(\theta_w(g), g') = m$ otherwise. For the second group of buyers who annuitize up to the maximum amount initially, they cannot annuitize more and the new annuitization amount is still m. (See Panel (b) of Figure 2.) In Appendix B, we show that the distribution function induced by $\alpha_\theta(\theta_w(g), g)$ before the change in the guarantee proportion first-order stochastically dominates the distribution function induced by $\alpha_\theta(\theta_w(g), g')$ after the change.[30] As a result, the PA provider's budget is in surplus if $\theta_w(g)$ remains unchanged. To maintain the zero-profit condition, the new equilibrium value of θ_w has to decrease. Combining with the result that the PA plan is in deficit if θ_w is equal to $E(\theta)$, we conclude that the new equilibrium severity of adverse selection is between $E(\theta)$ and $\theta_w(g)$, according to (33).

[29]The other possibility that we exclude is one in which all buyers annuitize up to the maximum amount after the increase in g. However, this case, which is straightforward to analyze, is not very interesting.

[30]A distribution function $P(\theta)$ first-order stochastically dominates another distribution function $Q(\theta)$ if $P(\theta) \le Q(\theta)$ for all θ, with a strict inequality over some interval.

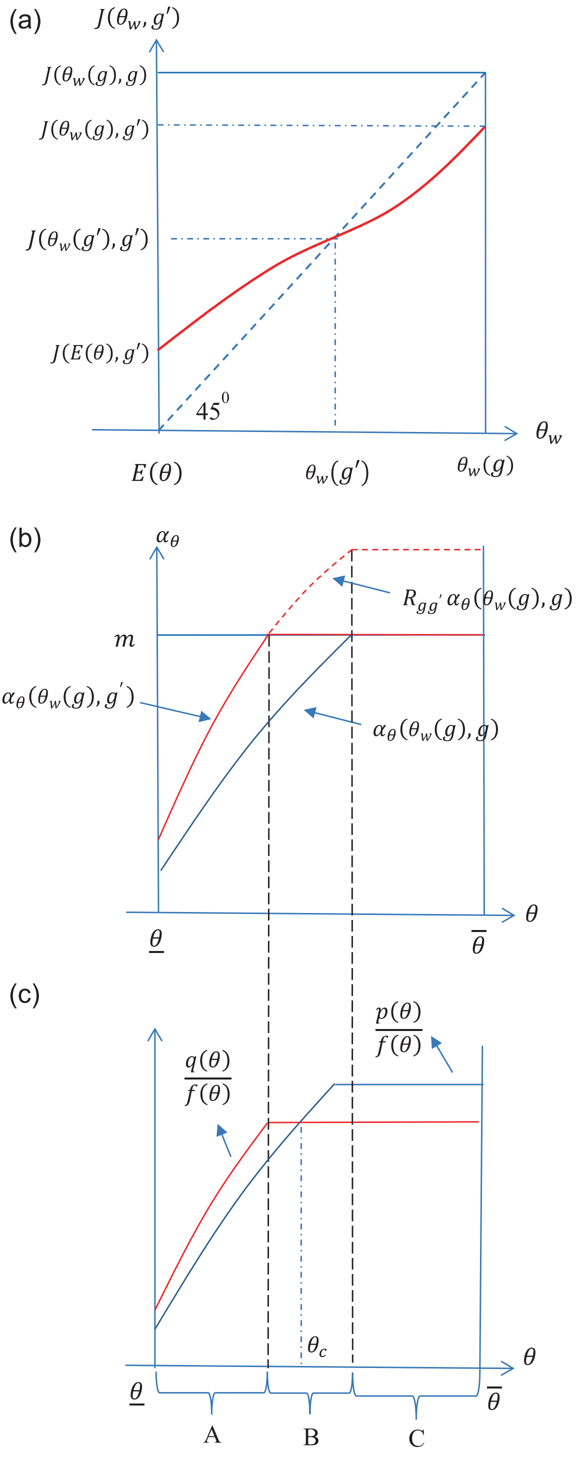

FIGURE 2 Guarantee payment and the severity of adverse selection. (a) Function $J(\theta_w, g')$, (b) functions $\alpha_\theta(\theta_w(g), g)$ and $\alpha_\theta(\theta_w(g), g')$, and (c) single crossing.

The interpretation of θ_w in (22) as the annuitization-weighted average of survival probabilities is helpful to understand Proposition 2. Since the amount of annuity purchase is positively related to the survival probability if they annuitize less than the maximum amount but is constant if they annuitize up to the maximum amount, these systematically different responses of retirees with different survival probabilities lead to a change in the severity of adverse selection when the guarantee proportion changes. When g increases, more buyers with high θ (i.e., the high-risk group from the perspective of the PA provider) are constrained at the maximum purchase of $\alpha_\theta^* = m$, and thus, the weight of this group of buyers is reduced. At the same time, the weight of buyers with low θ becomes larger. As a result, even though adverse selection (due to the positive correlation of θ and α_θ^*) is still present in the new PA plan, its severity is reduced.

5.4 | Substantial heterogeneity, with all three types of annuity buyers

We only comment on this case briefly, because it is very similar to the case in Section 5.3. Due to the same mechanism described in Proposition 2, an increase in the guarantee proportion g reduces the severity of adverse selection, thereby resulting in a higher value of the payout term A (under the zero-profit condition). The higher payout value shifts up the marginal benefit of annuitization. Accordingly the annuitization function shifts upward (except for the region of θ such that the buyers have already annuitized up to the maximum amount).

Besides this similarity regarding the intensive margin, there is a new element regarding the annuitization behavior on the extensive margin. The increase in guarantee proportion not only encourages some annuity buyers to purchase more public annuities, but also encourages some original nonbuyers to purchase the PA (see Panel D of Figure 1). The effect of the guarantee element on annuity buyers' behavior is more salient for all types of buyers when the heterogeneity in θ is substantial.

5.5 | Possible benefit of the guarantee element

The results in Propositions 1 and 2 are useful in understanding the possible benefit of guarantee element in annuity contracts, as suggested in Finkelstein and Poterba (2002, 2004) and Davidoff et al. (2005). The guarantee (or bequest element) is useful to the annuity buyer only in the nonsurvival state in Period 2, but the retiree can also use risk-free bond to achieve this objective. Since the state-contingent outcomes resulting from buying the annuity with a guarantee can be achieved by buying a portfolio of risk-free bond and pure annuity with survival-contingent payments only, one may wonder why a PA bundling life annuity and bequeathable wealth components, instead of a pure PA with only survival-continent payments, is offered.

Two results of this paper are relevant to this question. First, we show in Proposition 1 that under some conditions (that the heterogeneity in survival probability is limited such that the annuity choices of all buyers are interior solutions), the retirees are indifferent to the guarantee proportion of the PA (including the pure PA with $g = 0$). Comparing with a pure PA, if the government provides a new PA which guarantees that some amount of the payment will be received by the beneficiaries if the buyer does not survive to Period 2, then the PA provider has to reduce the level of survival-contingent payments to maintain its budget (the zero-profit assumption). Because of the smaller survival-contingent payment level, the retirees want to buy more public annuities (and less bond) to support their original targeted level of consumption. Proposition 1 shows that as a result of the equal proportional change in annuitization according to (28), the equilibrium severity of adverse selection in (29) is unchanged. Essentially, when the heterogeneity in survival probability is limited, the retirees' adjustment of annuity and bond purchases neutralize the PA provider's change in guarantee proportion.

The essence of the irrelevance result (of the guarantee proportion of the PA) in Proposition 1 is very similar to that of the famous result in Modigliani and Miller (1958, Proposition 1). They show that in a world of perfect capital market and no distortionary tax, the market value of a firm is independent of its capital structure. In this environment, if a firm changes to rely more on, for example, debt financing, arbitrage opportunity would be present. Investors could respond by adjusting their portfolio of shares and bonds, undoing the effect of the firm's action of more debt financing on the firm's market value.

While Proposition 1 shows that the benefit of guarantee in annuity contracts is absent under some conditions, Proposition 2 shows that it is present under other conditions. Once again, if the PA provider supplies a new PA with a higher guarantee proportion, then the level of survival-contingent payments has to be reduced to maintain its budget balance. In response, the retirees want to buy more public annuities. When the heterogeneity in survival probability is more substantial that some retirees with high survival probability reach the level of maximum purchase restriction, they cannot take full advantage of their private information to buy more annuities. As a result, the severity of adverse selection of the PA sector is reduced, according to (33). Compared with a PA with guarantee which bundles annuity income provision and bequeathable asset, Proposition 2 suggests that the pure PA which provides only survival-contingent payments is less preferred under appropriate conditions.

5.6 | Heterogeneities in health and wealth: A brief computational analysis

This paper focuses on the PA policy design when adverse selection is present. In this environment, the most crucial element is health heterogeneity, not wealth heterogeneity. For modeling purpose, we use the simplifying assumption of wealth homogeneity in the above analysis. The simpler environment with only health heterogeneity helps deliver sharp results.

The logical next step is to examine whether the results will continue to hold if we incorporate wealth heterogeneity, which is observed in many economies. We now extend the above model to include wealth heterogeneity and provide a brief computational analysis. In particular, we allow for the well-known feature that health and wealth are positively correlated (Meer et al., 2003; Michaud & Van Soest, 2008). We represent the two random variables (survival probability and wealth) by a bivariate normal distribution, with a nonnegative correlation coefficient, as follows:

$$\begin{pmatrix} \theta \\ w \end{pmatrix} \backsim N\left(\begin{pmatrix} \mu_\theta \\ \mu_w \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \psi\sigma_\theta\sigma_w \\ \psi\sigma_\theta\sigma_w & \sigma_w^2 \end{pmatrix} \right), \tag{34}$$

where μ_θ and μ_w are the expected values of survival probability and wealth, respectively; σ_θ^2 and σ_w^2 are their variances; and ψ $(0 \leq \psi < 1)$ is the correlation coefficient.

When health and wealth are heterogeneous, the retirees' optimal choices remain the same as those in Section 3. The equilibrium PA payout is now given by

$$A = \frac{\int_{\underline{w}}^{\bar{w}} \int_{\underline{\theta}}^{\bar{\theta}} \alpha_{\theta w}^{*} f(\theta, w)\, d\theta\, dw}{\int_{\underline{w}}^{\bar{w}} \int_{\underline{\theta}}^{\bar{\theta}} \left(1 + \frac{\theta + (1-\theta)g}{1+r}\right) \alpha_{\theta w}^{*} f(\theta, w)\, d\theta\, dw}, \tag{35}$$

where $\alpha_{\theta w}^{*}$ is used to express the dependence of the amount of PA purchase on θ and w explicitly, $w \in [\underline{w}, \overline{w}]$, and $f(\theta, w)$ is the joint probability density function of survival probability and wealth, given in (34).

Following Hosseini (2015), we use the CRRA utility function: $u(c) = \frac{c^{1-\phi} - 1}{1-\phi}$ and $v(b) = \xi \frac{b^{1-\phi} - 1}{1-\phi}$, where $\phi = 2$ and $\xi = 0.9$, which measures the strength of the bequest motive relative to consumption. We also assume that $\underline{w} = 2.5, \overline{w} = 3.5, \mu_w = 3, \sigma_w = 0.15, r = 0.2, \rho = 0.15$, and $m = 3.15$.[31]

We vary the values of the remaining parameters in the following exercises. We use two values in the guarantee element ($g = 0$ or 0.2) and select a wide range of values in the correlation coefficient between health and wealth ($\psi = 0, 0.1, 0.3, 0.5, 0.7$, or 0.9). First, consider the limited heterogeneity case (in which all retirees annuitize some positive amounts less than the maximum level m, as in Panel A of Figure 1) by setting $\underline{\theta} = 0.2, \bar{\theta} = 0.3, \mu_\theta = 0.25, \sigma_\theta = 0.0141$. As seen in Panel (a) of Table 2, with the value of ψ fixed, the values of Θ_w are the same when g increases from 0 to 0.2. When heterogeneity in survival probability is limited such that all retirees annuitize below the maximum level, an increase in the guarantee payment does not change the severity of adverse selection of the PA plan, whether the health-wealth correlation is low or high.

Second, we consider different parameter values ($\underline{\theta} = 0.5, \bar{\theta} = 0.99, \mu_\theta = 0.7$, and $\sigma_\theta = 0.1$) such that all retirees either annuitize at the maximum level or at some positive amounts less than the maximum level, as in Panel C of Figure 1. The results in the new setting are presented in Panel (b) of Table 2. Whether ψ is low or high, we observe that when g increases from 0 to 0.2, the value of Θ_w decreases. When heterogeneity in survival probability is more substantial such that some retirees are constrained by the maximum purchase level, an increase in the guarantee payment reduces the severity of adverse selection, regardless of the level of health-wealth correlation.

To summarize, we prove Propositions 1 and 2 in a model with homogeneous wealth in Sections 5.1 and 5.3. On the basis of our computational results in this section, these two propositions are also likely to hold when health and wealth are heterogeneous and possibly correlated.[32]

[31]Strictly speaking, we use a truncated bivariate normal distribution, since the values of survival probability and wealth outside the specified upper and lower limits are truncated. These limits are imposed to eliminate variables with inappropriate values (such as a negative value in survival probability or wealth). On the basis of the bivariate normal distribution with the above parameter values, less than 1% of the distribution (Panel [a]) and less than 3% of the distribution (Panel [b]) are truncated. Note also that in our two-period model, a period corresponds to 15 calendar years. The value of $r = 0.2$ corresponds roughly to an annual interest rate of 1.22%.

[32]We have also shown that the above results hold when there are small changes in the parameter values (such as g, ϕ, and ξ). These results are reported in the Supporting Information Appendix.

TABLE 2 The severity of adverse selection when health and wealth are heterogeneous.

(a) With $\underline{\theta} = 0.20, \bar{\theta} = 0.30, \mu_\theta = 0.25$, and $\sigma_\theta = 0.0141$			
ψ	**$g = 0$**	**$g = 0.2$**	**Difference[a]**
$\psi = 0$	0.2509	0.2509	0
$\psi = 0.1$	0.2510	0.2510	0
$\psi = 0.3$	0.2512	0.2512	0
$\psi = 0.5$	0.2513	0.2513	0
$\psi = 0.7$	0.2514	0.2514	0
$\psi = 0.9$	0.2516	0.2516	0
(b) With $\underline{\theta} = 0.50, \bar{\theta} = 0.99, \mu_\theta = 0.70$, and $\sigma_\theta = 0.1000$			
ψ	**$g = 0$**	**$g = 0.2$**	**Difference[a]**
$\psi = 0$	0.7378	0.7344	−0.0034
$\psi = 0.1$	0.7382	0.7347	−0.0036
$\psi = 0.3$	0.7390	0.7352	−0.0039
$\psi = 0.5$	0.7398	0.7357	−0.0041
$\psi = 0.7$	0.7406	0.7362	−0.0044
$\psi = 0.9$	0.7414	0.7367	−0.0047

[a]Each value in this column represents $\theta_w(g = 0.2) - \theta_w(g = 0)$ when ψ is unchanged.

6 | ESCALATING VERSUS NONESCALATING PAYMENTS

In this section, we examine issues regarding whether the PA payments are escalating or not. Once again, the main motivation of this investigation is the observed PA practices in different economies. As seen in Table 1, both escalating and nonescalating PA plans are provided in Singapore and India, but the PA plans in the other four economies are nonescalating in nominal terms. In the Escalating plan provided by the Central Provident Fund LIFE (Lifelong Income for the Elderly) scheme in Singapore, the initial payment level is lower but the payment is increasing by 2% each year. In one of the annuities offered by the Life Insurance Corporation (LIC) in India, the payment is increasing by 3%/year.

A major purpose of annuity purchase is to support the retiree's expenses at old age if she lives longer than anticipated, and it is natural to expect that the payments received from public annuities purchased are used to cover the expenses in this scenario. Since the inflation rates for most developed countries in the coming years are likely to be positive, though hopefully at a relatively low level, the retirees generally prefer that the future payments received from the PA are not eroded by inflation. In considering whether to purchase the PA or not, and by how much, there are at least two sources of risks involved: longevity risk and inflation risk.[33] While the PA enables the buyers to make better choice to hedge against the longevity risk, one wonders whether it is also able to help them deal effectively with the inflation risk. One way to

[33]Other risks, such as insolvency risk (Li et al., 2021) and interest rate risk, are also present in annuity contracts. However, we want to stay focused and do not examine these risks in this paper.

implement the idea of ensuing that people's future financial resources are less affected by inflation is to offer an escalating PA, in which the payment level is increasing in nominal value over time.[34] Our purpose in this section is to understand the pros and cons of escalating versus nonescalating payments (in nominal terms) in the PA plans.[35]

To understand this issue in the simple two-period model, we modify the model in Section 3 in two aspects: the model is restricted such that the PA plan provides survival-contingent payments only, but it is relaxed such that the payments in Period 1 and Period 2 can be different. Specifically, if the retiree buys one unit of this PA, she will receive a payment of B in Period 1 (with certainty under the assumption that she will not die in Period 1), and a payment of hB in Period 2 if she survives, where

$$0 < h \leq 1. \tag{36}$$

On the other hand, a buyer's beneficiaries will not receive any payment from the PA if she does not survive to Period 2. In this environment, the budget constraints of a retiree with θ become

$$c_{1\theta} = w - \alpha_\theta + B\alpha_\theta - s_\theta \tag{37}$$

in Period 1,

$$c_{2\theta} = s_\theta(1 + r) + hB\alpha_\theta - \frac{b_{3\theta}}{1 + r} \tag{38}$$

if she survives to Period 2, and

$$b_{2\theta} = s_\theta(1 + r) \tag{39}$$

if she does not survive to Period 2, where α_θ $(0 \leq \alpha_\theta \leq m)$ is the amount of her purchase of this PA, and the other terms have the same definition as in Section 3.[36]

Since the consumption and bequest variables in this standard two-period model are expressed in real instead of nominal terms, we use this model to compare the escalating and nonescalating public annuities based on the following interpretation. We interpret the PA with escalating nominal payments over time as one in which the payments in real terms are constant; this PA plan corresponds to $h = 1$ in (36). We call it "escalating PA." On the other hand, when the future inflation rate is expected to be on average positive, rather than

[34]A more rigorous idea is to offer a PA with payments indexed to the inflation rate. (Mackenzie, 2006, Table 1.1, provides a classification of different annuity contracts in the financial markets according to the form of regular payment. Annuity with indexed payment is one of them.) Since this PA involves extra issues such as how the inflation risk is shared between the seller and buyers and our model is not developed to address it, we only focus on the escalating PA as a close alternative in providing payment levels that are less affected, but not totally eliminated, by anticipated inflation in the future.

[35]The issue of escalating versus nonescalating annuities has been examined in Finkelstein and Poterba (2002, 2004), who consider various aspects of the private annuity contracts. They perform empirical analysis and find that the buyers who are longer-lived tend to choose escalating annuities. However, Cannon and Tonks (2016) argue that this empirical result, while consistent with the presence of adverse selection, may also be caused by the annuity suppliers' concern of cohort mortality risk.

[36]Note that it is straightforward to specify the model more generally to cover the versions in Section 3 and here, as follows. If the individual buys one unit of this PA, she will receive a payment of A in Period 1, and a payment of hA in Period 2 if she survives, and her beneficiaries will receive a payment of ghA if the buyer dies before Period 2. While all the main results in Section 3 and this section continue to hold in the more general model, it can be shown that parameter h is irrelevant for the issues related to the magnitude of guarantee payment to the buyer's beneficiaries, and parameter g is irrelevant for the issue of escalating versus nonescalating PA. Thus, we choose different specifications in these two sections, so that the models are simpler and the analysis is more focused.

negative, a PA with constant payment in nominal terms over time is represented as one with decreasing payments (in real terms) over time in our two-period model; that is, with $h < 1$ in (36). We call this case "nonescalating PA."

Following a similar analysis as in earlier sections, the zero-profit condition leads to

$$B = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta \, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \left(1 + \frac{\theta h}{1+r}\right) \alpha_\theta \, dF(\theta)}. \tag{40}$$

The first-order condition of the retiree's optimal choice in Period 2 is the same as (8). In Period 1, the first-order conditions of her optimal choices of the two financial assets (s_θ and α_θ) are given by (9) and one of the following three mutually-exclusive conditions. Specifically, if

$$(1-B)u'\left(w - (1-B)\alpha_\theta^* - s_\theta^*\right) = \frac{1}{1+\rho}\theta h B u'\left(s_\theta^*(1+r) + hB\alpha_\theta^* - \frac{b_{3\theta}^*}{1+r}\right), \tag{41}$$

then an interior solution for annuity purchase (i.e., $0 < \alpha_\theta^* < m$) appears. If

$$(1-B)u'\left(w - s_\theta^*\right) > \frac{1}{1+\rho}\theta h B u'\left(s_\theta^*(1+r) - \frac{b_{3\theta}^*}{1+r}\right), \tag{42}$$

then the retiree does not annuitize at all (i.e., $\alpha_\theta^* = 0$). If

$$(1-B)u'\left(w - (1-B)m - s_\theta^*\right) < \frac{1}{1+\rho}\theta h B u'\left(s_\theta^*(1+r) + hBm - \frac{b_{3\theta}^*}{1+r}\right), \tag{43}$$

then the retiree annuitizes up to the ceiling level (i.e., $\alpha_\theta^* = m$). The interpretations of the first-order conditions for the two financial assets (s_θ and α_θ) are essentially the same as those in Section 3.

It turns out that the analysis of the escalating PA versus nonescalating PA in this section and the analysis of guarantee elements versus only survival-contingent payments in earlier sections are mathematically very similar. As a result, we only present the key results in the paper and leave the detailed analysis in a Supporting Information Appendix. Our main emphasis is to explain the intuition of these results, and in particular, to link the intuition of the analysis in this section with those emphasized in earlier sections in which the public annuities with different guarantee proportions to the buyer's beneficiaries are analyzed.

For subsequent analysis, we rewrite (40) to express θ_w as

$$\theta_w = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta^* \, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^* \, dF(\theta)} = \frac{(1-B)(1+r)}{hB}. \tag{44}$$

Consider the behavior of buyers whose annuitization choices are given by the interior solution according to (41). Substituting (13) into (9), we obtain (15), where θ_w is given by (44), instead of (14), in the model of this section. Similarly, substituting (13) into (41) leads to

$$u'\left(w - \frac{\theta_w}{1+r}\left(c_{2\theta}^* + \frac{b_{3\theta}^*}{1+r}\right) - \frac{1-\theta_w}{1+r}b_{2\theta}^*\right) = \left(\frac{1+r}{1+\rho}\right)\frac{\theta}{\theta_w}u'\left(c_{2\theta}^*\right), \tag{45}$$

after using (44).

It is easy to conclude from (38) and (39) that

$$\alpha_\theta^* = \frac{1}{hB}\left(c_{2\theta}^* + \frac{b_{3\theta}^*}{1+r} - b_{2\theta}^*\right). \tag{46}$$

Moreover, if interior solution for annuity purchase holds (i.e., $0 < \alpha_\theta^* < m$), the buyer's annuitization amount is increasing in survival probability, as in (21). As a result, the value of θ_w defined in (44) is larger than $E(\theta)$. Once again, the deviation of θ_w from $E(\theta)$ represents the severity of adverse selection of this PA plan with possible time-varying payments.

With the above results regarding the retirees' annuitization choices, we compare the escalating versus nonescalating PA plans by conducting comparative static analysis regarding parameter h. We are particularly interested in the comparison when parameter h decreases from the initial value of $h = 1$ (the escalating PA) to a new value of $h' \in (0, 1)$ (the nonescalating PA). As in Section 5, the results depend on the extent of heterogeneity of survival probability. We consider two major cases in Sections 6.1 and 6.2.

6.1 | When all annuity buyers choose less than the maximum amount

In the first case when all buyers choose $\alpha_\theta^* \in (0, m)$, the relevant first-order conditions are (9) and (41). Following the approach in earlier sections, we conduct the analysis in two steps: first, analyzing the change in annuitization (α_θ^*) when θ_w is fixed; second, checking whether the change in annuitization behavior is consistent with an unchanged value of θ_w or not.

Since $c_{2\theta}^*$ and $b_{2\theta}^*$ are determined from (15) and (45) simultaneously, it is easy to see from these two equations that conditional on an unchanged value of θ_w, a change in h does not affect the optimal values of $c_{2\theta}^*$ and $b_{2\theta}^*$. This leads to

$$\frac{\partial c_{2\theta}^*}{\partial h} = \frac{\partial b_{2\theta}^*}{\partial h} = 0. \tag{47}$$

Using (44), (46), and (47), we further show that if h changes to h', then

$$\frac{\alpha_\theta^*(h')}{\alpha_\theta^*(h)} = \frac{h(1+r) + hh'\theta_w}{h'(1+r) + hh'\theta_w} \equiv R_{hh'}. \tag{48}$$

In particular, if there is a decrease in parameter h (with $h' < h$), then $R_{hh'} > 1$ and is the same for all buyers. Moreover, when all annuity buyers choose $\alpha_\theta^* \in (0, m)$ before and after the change in h, the proportional change result in (48) implies that θ_w is unchanged, as follows:

$$\theta_w(h') = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta^*(h')\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^*(h')\, dF(\theta)} = \frac{R_{hh'}\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta^*(h)\, dF(\theta)}{R_{hh'}\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta^*(h)\, dF(\theta)} = \theta_w(h). \tag{49}$$

6.2 | When some annuity buyers choose the maximum amount and some choose less

We now consider the second case when some buyers choose $\alpha_\theta^* \in (0, m)$ and some choose $\alpha_\theta^* = m$. As in Section 5, it is helpful to define the function

$$K(\theta_w, h) = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta(\theta_w, h)\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w, h)\, dF(\theta)}, \tag{50}$$

where

$$\alpha_\theta(\theta_w, h) = \begin{cases} \alpha_\theta^* & \text{if (41) holds,} \\ m & \text{if (43) holds,} \end{cases} \tag{51}$$

with α_θ^* (together with s_θ^* and $b_{3\theta}^*$) determined by solving (8), (9), and (41) simultaneously.[37]

The definition of $K(\theta_w, h)$ in (50) is useful because the equilibrium value of $\theta_w(h)$ is defined according to

$$K(\theta_w(h), h) \equiv \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta\alpha_\theta(\theta_w(h), h)\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(h), h)\, dF(\theta)} = \theta_w(h). \tag{52}$$

It can be shown that when we move from an escalating PA (with $h = 1$) to a nonescalating PA (with $0 < h' < 1$), the severity of adverse selection of the new PA plan, $\theta_w(h')$, is reduced, as given by

$$E(\theta) < \theta_w(h') = K(\theta_w(h'), h') < \theta_w(h). \tag{53}$$

The proof, which is similar to that of Appendix B, is presented in the Supporting Information Appendix. The main ingredient of the proof is that the distribution induced by

[37]Similar to the definition in (31), the payout term B does not appear explicitly in the definition of $\alpha_\theta(\theta_w, h)$ in (51), but is present through parameters h and θ_w according to (44).

$\alpha_\theta(\theta_w(h), h)$ before the decrease in the escalating parameter h first-order stochastically dominates the distribution induced by $\alpha_\theta(\theta_w(h), h')$ after the decrease in h (to h').

Similar to the analysis in Section 5, the intuition of (53) is that when compared with an escalating PA ($h = 1$), more buyers reach the maximum purchase limit under the new PA plan with nonescalating payment ($0 < h' < 1$). As a result, the weight of the high-risk group (with high θ) is reduced, leading to a fall in the severity of adverse selection, θ_w.

6.3 | A summary

We now summarize the results in the above two cases in the following proposition.[38]

Proposition 3. *Consider a PA plan in the two-period model given by* (1) *and* (36)–(39).

(a) *When the heterogeneity in θ is limited such that all annuity buyers choose $\alpha_\theta^* \in (0, m)$ for any h satisfying* (36), *a change in h leads to the same percentage change in annuitization for all buyers according to* (48), *and no change in the severity of adverse selection (θ_w) of the PA plan, according to* (49).

(b) *When the heterogeneity in θ is such that some buyers choose $\alpha_\theta^* \in (0, m)$ and some choose $\alpha_\theta^* = m$ in an escalating PA plan (with $h = 1$), a change to a nonescalating PA plan (where $0 < h' < 1$) leads to a reduction in the equilibrium value of $\theta_w(h')$, according to* (53).

7 | CONCLUSION

Motivated by the idea in Diamond (2004) and the PA experiences in various economies, this paper studies two commonly observed features of the PA plan: guarantee element and nonescalating payments. We find in Proposition 1 that when the extent of heterogeneity is limited, the guarantee payment is irrelevant because the buyers change their annuitization choices proportionally, leading to no change in the severity of adverse selection. On the other hand, by making use of the idea of first-order stochastic dominance, it is shown in Proposition 2 that the guarantee payment reduces the severity of adverse selection in (22) when the extent of heterogeneity is more substantial. A parallel set of results, stated in Proposition 3, applies to the issues related to escalating versus nonescalating payments of the PA plan. In particular, Proposition 3(b) shows that under some conditions, the feature of nonescalating payments can also reduce the severity of adverse selection of the PA plan.

Our paper contributes to the studies of annuity demand. Using the difference of annuitization-weighted and unweighted averages of the retirees' survival probabilities to measure the severity of adverse selection, we find interesting results regarding the possible benefit of public annuities with guarantee payment. The empirical analysis in Finkelstein and Poterba (2002, 2004) and the theoretical analysis of Davidoff et al. (2005) suggest that

[38]Similar to Section 5, the results for the case of greater heterogeneity with two types of retirees (some retirees choose $\alpha_\theta^* \in (0, m)$ and some choose $\alpha_\theta^* = 0$) are similar to the analysis in Section 6.1, and the results for the case of substantial heterogeneity with all three types of retirees are similar to the analysis in Section 6.2.

the guarantee element in an annuity contract helps mitigate the severity of adverse selection. In the context of public provision of annuity contracts, our analysis substantiates their claim by showing that this mechanism works when some conditions (as given in Proposition 2) hold. In addition, we show that this mechanism does not work under other conditions (as given in Proposition 1). In this case, the retirees adjust their portfolio choices to undo the effect of the change in guarantee proportion of the PA plan. As a result, the severity of adverse selection of the PA sector is not affected by a change in the guarantee proportion of the PA plan.

Our analysis also contributes to the policy design regarding the number and form of government-provided annuities. Propositions 1 and 3(a) have policy implications regarding the number of PA plans. The introduction of public annuities to the retirees has a beneficial effect, even if there is only one plan, because it provides benefits to the retirees through the sharing of a "mortality premium," with the resources of those who die earlier being transferred to those who live longer. The exact form of the PA plan is immaterial, at least when the heterogeneity of survival probability is limited. These results suggest that the decision about introducing public annuities or not (i.e., no PA plan vs. one plan) is likely to be a more important policy issue than the precise number of PA plans introduced. Moreover, Propositions 2 and 3(b) show that the guarantee element or nonescalating payments of the PA plan would reduce the severity of adverse selection under some other circumstances. These results provide support that pure PA plans (with survival-contingent payments only or with escalating feature to deal with anticipated inflation) may be less preferred when compared with weaker alternatives (with guarantee element or with nonescalating payments). These results provide useful inputs in designing the form of PA plans.

This paper analyzes two key aspects of public annuities. The results in this paper are useful to the countries that have launched PA plans, as well as to those considering to introduce national insurance against longevity risk. Nevertheless, our study is not able to address all the important questions, and more in-depth studies are warranted. We end this paper by suggesting three interesting ideas to follow up.

First, although the literature suggests that survival probability and bequest motive are important in explaining annuitization behavior, it is also useful to include other relevant factors, such as wealth heterogeneity. If health and wealth are positively correlated, then less healthy retirees are more likely to be wealth-constrained, and this constraint may cause them not to purchase annuity, leading to more severe adverse selection. The computational analysis in Section 5.6 focuses on one issue: whether the guarantee element of the PA plan could still be useful when adverse selection is more severe under health-wealth correlation.[39] A follow-up topic is to investigate whether other policies would offer a better solution to deal with longevity risk when health and wealth are correlated. Retirees with a low level of wealth are less likely to purchase public annuities because of low survival probability (when health and wealth are correlated) as well as a lack of resource (particularly after setting aside precautionary savings for possible medical expenses). Targeted policies such as means-tested subsidies to poor retirees may help them more effectively. In future work, it is interesting to study whether policies targeting certain groups of retirees, or the PA plans with guarantee element or nonescalating payments which apply uniformly to all retirees, or some

[39]The conjecture that a higher health-wealth correlation leads to higher severity of adverse selection (θ_w) is consistent with the computational results in Table 2, regardless of the value of the guarantee proportion (g).

combination of both, would be more effective in insuring against longevity risk for different groups of retirees.

Second, an interesting follow-up study is the "optimal policy design" of the PA plan. Our results suggest that the pure PA plans (with survival-contingent payments only or with escalating payments) may be less preferred under some conditions. Using the notation of our two-period model, $g = 0$ or $h = 1$ is not optimal. Some follow-up questions include what guarantee proportion or what degree of escalating payments of the PA plan is optimal, as well as whether the PA plan with optimal guarantee proportion or the plan with optimal degree of escalating payments is more effective. However, welfare analysis in a more realistic multiperiod model would be a better framework to deliver quantitative results useful to policymakers.[40] This contrasts to the approach used in this paper, which relies on theoretical analysis based on a simple two-period model to obtain useful economic insights. We leave this complementary follow-up study to future work.

Third, it is helpful to examine not only conventional economic factors, but also behavioral factors. Our study focuses on conventional factors and shows that a PA plan with guarantee element reduces the severity of adverse selection when the survival probability heterogeneity is sufficiently large. While this result is useful, we do not think that examining the severity of adverse selection is the only important perspective in understanding the design of PA plans. In the literature of behavioral economics, Hu and Scott's (2007) study based on the behavioral factor of loss aversion suggests that the annuity contract with guarantee element is preferred by retirees who are loss averse.[41] In future work, it will be helpful to compare the two different approaches to see which one is better, and to what extent, in explaining the guarantee element in the PA plan.

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[40]For this broader question, some researchers have suggested that annuitization leads to efficiency improvement (see Einav et al., 2010; Hosseini, 2015) while some others have found welfare loss (see Feigenbaum et al., 2013; Heijdra et al., 2014).

[41]Other behavioral factors are also important for annuity studies. In Brown et al. (2008), for example, the framing of annuity product could affect annuitization choice. They show that when an annuity is presented in a consumption frame, by using words such as "spend" or "payment," many individuals would like to buy. However, when the annuity is framed as an investment, by using words such as "invest" and "earnings," only a small fraction of individuals tends to buy.

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SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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APPENDIX A: ANNUITY BUYERS' BEHAVIOR

If the buyer survives to Period 2, her ex post lifetime budget is obtained from (2) and (3):

$$c_{1\theta} + \frac{1}{1+r}\left(c_{2\theta} + \frac{b_{3\theta}}{1+r}\right) = w - (1-A)\alpha_\theta + \frac{1}{1+r}A\alpha_\theta. \tag{A1}$$

If the buyer dies in Period 1, her ex post lifetime budget is obtained from (2) and (4):

$$c_{1\theta} + \frac{1}{1+r}b_{2\theta} = w - (1-A)\alpha_\theta + \frac{g}{1+r}A\alpha_\theta. \tag{A2}$$

Adding up (A1) multiplied by θ and (A2) multiplied by $1-\theta$, we obtain

$$\begin{aligned} c_{1\theta} + \tfrac{\theta}{1+r}\left(c_{2\theta} + \tfrac{b_{3\theta}}{1+r}\right) + \tfrac{1-\theta}{1+r}b_{2\theta} &= w - (1-A)\alpha_\theta + \frac{\theta}{1+r}A\alpha_\theta + \frac{(1-\theta)g}{1+r}A\alpha_\theta \\ &= w + (\theta - \theta_w)\frac{1-g}{1+r}A\alpha_\theta, \end{aligned} \tag{A3}$$

where (14) has been used in the last step.

The LHS of (A3) can be rewritten as

$$c_{1\theta} + \frac{\theta}{1+r}\left(c_{2\theta} + \frac{b_{3\theta}}{1+r}\right) + \frac{1-\theta}{1+r}b_{2\theta} = c_{1\theta} + \frac{\theta_w}{1+r}\left(c_{2\theta} + \frac{b_{3\theta}}{1+r}\right) + \frac{1-\theta_w}{1+r}b_{2\theta} + \frac{(\theta-\theta_w)}{1+r}\left(c_{2\theta} + \frac{b_{3\theta}}{1+r} - b_{2\theta}\right)$$
$$= c_{1\theta} + \frac{\theta_w}{1+r}\left(c_{2\theta} + \frac{b_{3\theta}}{1+r}\right) + \frac{1-\theta_w}{1+r}b_{2\theta} + (\theta-\theta_w)\frac{1-g}{1+r}A\alpha_\theta, \tag{A4}$$

where (19) has been used in the last step. Combining (A3) and (A4) gives (13).

Differentiating (8) totally with respect to $c_{2\theta}^*$ and $b_{3\theta}^*$, we obtain

$$db_{3\theta}^* = \frac{(1+\rho)}{(1+r)}\frac{u''\left(c_{2\theta}^*\right)}{v''\left(b_{3\theta}^*\right)}dc_{2\theta}^*. \tag{A5}$$

Differentiating (15) totally with respect to $c_{2\theta}^*$, $b_{2\theta}^*$, θ, and g, and using (A5), we obtain

$$m_{11}dc_{2\theta}^* + m_{12}db_{2\theta}^* = m_{1\theta}d\theta, \tag{A6}$$

where

$$m_{11} = \frac{\theta_w}{1+r}\left[1 + \frac{(1+\rho)}{(1+r)^2}\frac{u''(c_{2\theta^*})}{v''\left(b_{3\theta}^*\right)}\right]u''\left(c_{1\theta}^*\right) + \frac{1+r}{1+\rho}\theta u''\left(c_{2\theta}^*\right) < 0, \tag{A6a}$$

$$m_{12} = \frac{1-\theta_w}{1+r}u''\left(c_{1\theta}^*\right) + \frac{1+r}{1+\rho}(1-\theta)v''\left(b_{2\theta}^*\right) < 0, \tag{A6b}$$

and

$$m_{1\theta} = -\frac{1+r}{1+\rho}\left[u'\left(c_{2\theta}^*\right) - v'\left(b_{2\theta}^*\right)\right]. \tag{A6c}$$

Differentiating (16) totally with respect to $c_{2\theta}^*$, $b_{2\theta}^*$, θ, and g, and using (A5), we obtain

$$m_{21}dc_{2\theta}^* + m_{22}db_{2\theta}^* = m_{2\theta}d\theta + m_{2g}dg, \tag{A7}$$

where

$$m_{21} = \frac{\theta_w}{1+r}\left[1 + \frac{(1+\rho)}{(1+r)^2}\frac{u''(c_{2\theta^*})}{v''\left(b_{3\theta}^*\right)}\right]u''\left(c_{1\theta}^*\right) + \frac{(1+r)}{(1+\rho)}\frac{1}{[\theta_w + (1-\theta_w)g]}\theta u''\left(c_{2\theta}^*\right) < 0, \tag{A7a}$$

$$m_{22} = \frac{1-\theta_w}{1+r}u''\left(c_{1\theta}^*\right) + \frac{1+r}{1+\rho}\frac{1}{[\theta_w + (1-\theta_w)g]}(1-\theta)gv''\left(b_{2\theta}^*\right) < 0, \tag{A7b}$$

$$m_{2\theta} = -\frac{(1+r)}{(1+\rho)}\frac{1}{[\theta_w + (1-\theta_w)g]}\left[u'\left(c_{2\theta}^*\right) - gv'\left(b_{2\theta}^*\right)\right], \tag{A7c}$$

and

$$m_{2g} = \frac{1+r}{1+\rho}\left\{\frac{(1-\theta_w)\left[\theta u'\left(c_{2\theta}^*\right) + (1-\theta)gv'\left(b_{2\theta}^*\right)\right]}{[\theta_w + (1-\theta_w)g]^2} - \frac{(1-\theta)v'\left(b_{2\theta}^*\right)}{\theta_w + (1-\theta_w)g}\right\} = 0. \tag{A7d}$$

The effects of a change in θ on the optimal values of $c_{2\theta}^*$ and $b_{2\theta}^*$ ($\frac{\partial c_{2\theta}^*}{\partial\theta}$ and $\frac{\partial b_{2\theta}^*}{\partial\theta}$) are determined by solving (A6) and (A7) simultaneously, with $dg = 0$. This leads to

$$\frac{\partial c_{2\theta}^*}{\partial\theta} = \frac{m_{22}m_{1\theta} - m_{12}m_{2\theta}}{m_{11}m_{22} - m_{21}m_{12}} \tag{A8}$$

and

$$\frac{\partial b_{2\theta}^*}{\partial\theta} = \frac{m_{11}m_{2\theta} - m_{21}m_{1\theta}}{m_{11}m_{22} - m_{21}m_{12}}. \tag{A9}$$

It can be shown that

$$\begin{aligned} m_{11}m_{22} - m_{21}m_{12} \;=\; & -\frac{\theta_w}{1+r}\left[1 + \frac{(1+\rho)}{(1+r)^2}\frac{u''(c_{2\theta^*})}{v''\left(b_{3\theta}^*\right)}\right]u''\left(c_{1\theta}^*\right)\left[\frac{1+r}{1+\rho}(1-\theta)v''\left(b_{2\theta}^*\right)\right] \\ & \times\frac{\theta_w(1-g)}{[\theta_w + (1-\theta_w)g]} \\ & -\left[\frac{1-\theta_w}{1+r}u''\left(c_{1\theta}^*\right)\right]\left[\frac{1+r}{1+\rho}\theta u''\left(c_{2\theta}^*\right)\right]\frac{(1-\theta_w)(1-g)}{[\theta_w + (1-\theta_w)g]} \\ & -\left[\frac{1+r}{1+\rho}\theta u''\left(c_{2\theta}^*\right)\right]\left[\frac{1+r}{1+\rho}(1-\theta)v''\left(b_{2\theta}^*\right)\right]\frac{(1-g)}{[\theta_w + (1-\theta_w)g]} < 0, \end{aligned} \tag{A10}$$

$$\begin{aligned} m_{22}m_{1\theta} - m_{12}m_{2\theta} \;=\; & \left[\frac{1-\theta_w}{1+r}u''\left(c_{1\theta}^*\right)\right]\left[\frac{1+r}{1+\rho}u'\left(c_{2\theta}^*\right)\right]\frac{(1-\theta_w)(1-g)}{[\theta_w + (1-\theta_w)g]} \\ & +\left[\frac{1-\theta_w}{1+r}u''\left(c_{1\theta}^*\right)\right]\left[\frac{1+r}{1+\rho}v'(b_{2\theta})\right]\frac{\theta_w(1-g)}{[\theta_w + (1-\theta_w)g]} \\ & +\left[\frac{1+r}{1+\rho}u'\left(c_{2\theta}^*\right)\right]\left[\frac{1+r}{1+\rho}(1-\theta)v''\left(b_{2\theta}^*\right)\right]\frac{(1-g)}{[\theta_w + (1-\theta_w)g]} < 0, \end{aligned} \tag{A11}$$

and

$$m_{11}m_{2\theta} - m_{21}m_{1\theta} = -\frac{\theta_w}{1+r}\left[1 + \frac{(1+\rho)}{(1+r)^2}\frac{u''(c_{2\theta^*})}{v''\left(b_{3\theta}^*\right)}\right]u''\left(c_{1\theta}^*\right)\left[\frac{1+r}{1+\rho}u'\left(c_{2\theta}^*\right)\right] \times \frac{(1-\theta_w)(1-g)}{[\theta_w + (1-\theta_w)g]} - \frac{\theta_w}{1+r}\left[1 + \frac{(1+\rho)}{(1+r)^2}\frac{u''(c_{2\theta^*})}{v''\left(b_{3\theta}^*\right)}\right]u''\left(c_{1\theta}^*\right)\left[\frac{1+r}{1+\rho}v'\left(b_{2\theta}^*\right)\right] \times \frac{\theta_w(1-g)}{[\theta_w + (1-\theta_w)g]} - \left[\frac{1+r}{1+\rho}\theta u''\left(c_{2\theta}^*\right)\right]\left[\frac{(1+r)}{(1+\rho)}v'\left(b_{2\theta}^*\right)\right]\frac{(1-g)}{[\theta_w + (1-\theta_w)g]} > 0. \tag{A12}$$

Substituting (A10) and (A11) into (A8), we obtain (17). Substituting (A10) and (A12) into (A9), we obtain (18). Combining (17)–(20), we obtain (21).

APPENDIX B: PROOF OF PROPOSITION 2

On the basis of the definition of the equilibrium severity of adverse selection in (32), we look for the intersection of the $J(\theta_w, g')$ function at the new guarantee proportion (g') and the 45° line, as shown in Panel (a) of Figure 2. We plot $J(\theta_w, g')$ as a function of θ_w when $\theta_w \in [E(\theta), \theta_w(g)]$.[42]

Consider the beginning point when $\theta_w = E(\theta)$. When the heterogeneity in θ is such that some annuity buyers (those with relatively low θ) still annuitize less than the maximum amount, it can be shown from (26) that

$$J(E(\theta), g') > E(\theta). \tag{B1}$$

Consider the endpoint when $\theta_w = \theta_w(g)$, the severity of adverse selection of the original PA (with g). When the guarantee proportion increases from g to g' and θ_w remains unchanged at $\theta_w(g)$, (28) indicates that for any buyer with $\alpha_\theta^* \in (0, m)$, her annuitization level increases proportionally if the new level is still below m; otherwise, the new annuitization level is m. On the other hand, those with $\alpha_\theta^* = m$ are unable to increase the PA purchase beyond the ceiling level.

As seen in Panel (b) of Figure 2, there are three distinct regions of θ in which the buyers' annuity purchases before and after the change in g differ qualitatively. Define Region A as the region of θ such that the buyers' annuity purchases do not reach the maximum limit for both g and g', Region B as the region of θ such that the buyers do not reach the purchase ceiling at g but reach the ceiling at g', and Region C as the region of θ such that the buyers purchase the maximum amount of public annuities for both g and g'. Using (28), it can be shown that $\alpha_\theta(\theta_w(g), g') = R_{gg'}\alpha_\theta(\theta_w(g), g) < m$ in region A, and $\alpha_\theta(\theta_w(g), g') = m$ in regions B and C.

[42]We focus on $\theta_w \in [E(\theta), \theta_w(g)]$ because of the following reasons. First, $\theta_w > E(\theta)$ according to (26) when some retirees choose $\alpha_\theta^* \in (0, m)$. Second, we want to compare $\theta_w(g')$ with $\theta_w(g)$, the severity of adverse selection of the original PA plan.

On the basis of the annuitization-weighted average interpretation of $\theta_w(g)$ in (22), it is helpful to rewrite (32) as

$$\theta_w(g) = J(\theta_w(g), g) = \int_{\underline{\theta}}^{\bar{\theta}} \theta p(\theta)\, d\theta, \tag{B2}$$

where

$$p(\theta) = \frac{\alpha_\theta(\theta_w(g), g) f(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g) f(\theta) d\theta}, \tag{B3}$$

and $f(\theta)$ is the probability density function corresponding to the cumulative distribution function $F(\theta)$. Similarly, we rewrite

$$J(\theta_w(g), g') = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta \alpha_\theta(\theta_w(g), g')\, dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g')\, dF(\theta)} = \int_{\underline{\theta}}^{\bar{\theta}} \theta q(\theta)\, d\theta, \tag{B4}$$

where

$$q(\theta) = \frac{\alpha_\theta(\theta_w(g), g') f(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g') f(\theta)\, d\theta}. \tag{B5}$$

It can be seen from (B3) and (B5) that

$$\int_{\underline{\theta}}^{\bar{\theta}} p(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\, d\theta = 1. \tag{B6}$$

Therefore, $p(\theta)$ and $q(\theta)$ can be interpreted as probability density functions, induced by the annuity choices $\alpha_\theta(\theta_w(g), g)$ and $\alpha_\theta(\theta_w(g), g')$, respectively.

Our proof makes use of the idea of stochastic dominance. We state the following well-known result useful for our proof. (See, e.g., Hadar & Russell, 1969, Theorem 1').

Lemma 1. *If a distribution $P(\theta)$ first-order stochastically dominates another distribution $Q(\theta)$, then the mean of distribution $P(\theta)$ is higher than the mean of distribution $Q(\theta)$. That is, $\int_{\underline{\theta}}^{\bar{\theta}} \theta\, dP(\theta) > \int_{\underline{\theta}}^{\bar{\theta}} \theta\, dQ(\theta)$.*

To compare (B2) with (B4), we first show that

$$\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g') f(\theta)\ d\theta < \int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g') f(\theta)\ d\theta < R_{gg'} \int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g) f(\theta)\ d\theta. \tag{B7}$$

Combining (B3), (B5), and (B7), we obtain (a) $p(\theta) < q(\theta)$ in region A, (b) $p(\theta) > q(\theta)$ in region C, and (c) $\frac{q(\theta)}{f(\theta)}$ is constant but $\frac{p(\theta)}{f(\theta)}$ is increasing in θ in region B. (See Panel (c) of Figure 2.) Combining these results, we conclude that $p(\theta)$ and $q(\theta)$ cross once, and it happens in region B.

On the basis of the single-crossing feature, define the critical value θ_c such that

$$p(\theta)\begin{cases} < q(\theta), & \theta \in [\underline{\theta}, \theta_c), \\ = q(\theta), & \theta = \theta_c, \\ > q(\theta), & \theta \in (\theta_c, \bar{\theta}]. \end{cases} \tag{B8}$$

Define $P(\theta)$ (resp., $Q(\theta)$) as the distribution function corresponding to $p(\theta)$ (resp., $q(\theta)$). When $\theta \in [\underline{\theta}, \theta_c]$, we use (B8) to obtain

$$P(\theta) - Q(\theta) = \int_{\underline{\theta}}^{\theta} p(\xi)\,d\xi - \int_{\underline{\theta}}^{\theta} q(\xi)\,d\xi = \int_{\underline{\theta}}^{\theta} [p(\xi) - q(\xi)]\,d\xi < 0, \tag{B9}$$

where ξ is an index of survival probability. When $\theta \in (\theta_c, \bar{\theta}]$, we use (B6) and (B8) to obtain

$$\begin{aligned} P(\theta) - Q(\theta) &= \int_{\underline{\theta}}^{\theta} [p(\xi) - q(\xi)]\,d\xi = \int_{\underline{\theta}}^{\bar{\theta}} [p(\xi) - q(\xi)]\,d\xi - \int_{\theta}^{\bar{\theta}} [p(\xi) - q(\xi)]\,d\xi \\ &= -\int_{\theta}^{\bar{\theta}} [p(\xi) - q(\xi)]\,d\xi < 0. \end{aligned} \tag{B10}$$

Combining (B9) and (B10), we conclude that the distribution $P(\theta)$ first-order stochastically dominates $Q(\theta)$. Thus, the mean of $P(\theta)$ is higher than the mean of $Q(\theta)$, according to Lemma 1. Since the mean of $P(\theta)$ is given by (B2) and the mean of $Q(\theta)$ is given by (B4), we obtain[43]

$$\begin{aligned} J(\theta_w(g), g') &= \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{\alpha_\theta(\theta_w(g), g')f(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g')f(\theta)\,d\theta} \right] d(\theta) \\ &< \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{\alpha_\theta(\theta_w(g), g)f(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \alpha_\theta(\theta_w(g), g)f(\theta)\,d\theta} \right] d(\theta) = \theta_w(g). \end{aligned} \tag{B11}$$

[43]Note that our ultimate objective is to compare $J(\theta_w(g), g)$ with $J(\theta_w(g'), g')$, but it is difficult to compare them directly because both arguments of the $J(\theta_w, g)$ function change. By using the result of $\alpha_\theta(\theta_w(g), g') = R_{gg'}\alpha_\theta(\theta_w(g), g) < m$ in region A, we are able to achieve the objective through an intermediate step of comparing $J(\theta_w(g), g)$ with $J(\theta_w(g), g')$ according to (B11). (See Panel [a] of Figure 2 also).

Combining (B1) and (B11) with the continuity of the $J(\theta_w, g')$ function, we conclude that there exists a value of $\theta_w(g')$ between $E(\theta)$ and $\theta_w(g)$ that satisfies the equilibrium condition $J(\theta_w(g'), g') = \theta_w(g')$. Moreover, the two inequalities in (33) follow immediately, based on the intersection of the $J(\theta_w, g')$ function and the 45° line, as seen in Panel (a) of Figure 2.