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When should Fuzzy Analytic Hierarchy Process be used instead of Analytic Hierarchy Process?

Abstract

The analytic hierarchy process (AHP) has been widely applied in the last four decades, and a fuzzy logic version, the fuzzy AHP (FAHP), has also been employed in many studies since its formulation. However, it is not clear whether the FAHP is better than the AHP in terms of quality of the solution. First, the study demonstrates that the FAHP can provide a different solution than the AHP, regardless of which solution is better. However, it is also shown that such a difference does not provide an additional advantage over the AHP when a decision is proposed. Although the FAHP is a well-proven method, this study provides an insight on its practical applicability.

Keywords: Multiple criteria analysis; fuzzy sets; analytic hierarchy process; fuzzy AHP.

1. Introduction

Decision-making is an inevitable process in human life; therefore, there is a pressing need to develop tools or methods to aid decision-makers in making good decisions. Nevertheless, this is not an easy task. First, a multi-criteria decision analysis (MCDA) is needed as it is very likely that more than one objective is involved [13]. Second, in many real-life situations there are problems that are unstructured and involve qualitative factors [17]. This makes it difficult, if not impossible, to apply the traditional approaches in operations research. In this connection, Professor Thomas Saaty engineered a major breakthrough in the 1970s by formulating the analytic hierarchy process (AHP) [24]. The AHP can utilize both qualitative and quantitative factors to form a hierarchical structure in the decision-making process, which can then help decision-makers to select the best option based on the selection criteria presented in the model [9]. As this is a well-established approach, its operations and mechanisms are not discussed here. Interested readers can refer to the works of Saaty (e.g., [22,25]).

Later, Van Laarhoven and Pedrycz in [30] extended the AHP approach by adding fuzzy logic to establish the fuzzy AHP (FAHP). In their approach, they used fuzzy triangular membership functions and demonstrated how decisions can be made in the presence of uncertainty during the pairwise comparison process (i.e., to determine the corresponding weights). Buckley [1] further refined this approach by introducing the geometric mean method to calculate the weights after the fuzzy numbers were assigned by the decision makers (i.e., the defuzzification process). These two papers laid the foundation for subsequent FAHP applications. The main rationale, or argument, for using fuzzy numbers to represent the ratios in the pairwise comparison is the fact that the assignment itself is imprecise. Therefore, the advocates of the FAHP aim to tackle this imprecision with the advantage of fuzzy logic.

Since then, the rate of adoption of the FAHP soared at an extremely high rate. Numerous studies applied the FAHP in different areas, such as supplier or partner selection, process selection, green supplier evaluation, construction, general operations management, quality management, supply chain design, corporate environmental evaluation, and patient prioritization [2,4, 6-8,10,11,16,19,21,29,32-35]. For instance, Dincer et al. in [4] proposed a triangular FAHP-based hybrid analytical multi-criterion decision making model to solve the ranking problem of industry alternatives for portfolio investment to address the financial risks in the capital market. Kilic et al. in [11] developed a hybrid methodology based on the triangular FAHP to solve the supplier selection problem, which was further applied to address a real selection problem in an enterprise resource planning (ERP) system for airlines. Readers can also refer to the comprehensive survey conducted by Kubler et al. [15], which provides a comprehensive review of different application areas of the FAHP. Furthermore, a simple search of the term "fuzzy AHP" in different scholarly databases can support this assertion. Table 1 presents the number of publications between 2006 and 2018 in four common scholarly databases. Only IEEE Explore shows a different trend. The search itself is not very systematic, and no screening was conducted; however, the values in Table 1 are very clear.

Table 1
Number of publications with the term "fuzzy AHP"

Year of Publication	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Google Scholar	815	1020	1160	1470	1780	2300	2750	2960	3470	3320	3530	4080	4600
IEEE Xplore	35	47	106	166	258	159	96	62	72	78	78	81	73
Science Direct	127	150	190	254	260	437	440	419	484	601	663	715	843
Taylor & Francis	36	52	68	76	88	95	120	141	159	148	200	207	219

Despite the vast collection of applications, the use of the FAHP has caused some controversy.

Raharjo et al. in [20] argued that the tradeoff between the solution quality and the complexity of the FAHP may not be favorable for its selection. The developer of the AHP [26] commented that there is no real need to include fuzzy logic because making decisions via the AHP is already sufficiently fuzzy. He actually referred to the nine-point scale for a pairwise comparison, which can resemble the linguistic variables used in many fuzzy logic applications. Therefore, imprecise information during the decision-making process has already been taken into account in that sense. Then, Saaty and Tran [27] explained that "using fuzzy numbers in decision making is inadvisable until precise conditions are given for when the process works well and when it does not." Saaty and Tran [28] further theorized their earlier proposition. These three studies concluded that the AHP solution is insensitive to perturbations if the problem size is small and if the true values of the pairwise comparisons are known by other means. Saaty and Tran [28] employed simulations and examples to illustrate their ideas.

Nevertheless, the existence of such *conditions* is still a *mystery* mathematically. More recently, Zhü in [36] adopted a relatively more ambitious approach in an attempt to refute the FAHP method and tried to claim that it was invalid and violated the principles of the AHP. Although his attempt is welcome, especially as it draws attention to the applicability of the FAHP, his arguments are not perfect and suffer from some shortcomings. Fedrizzi and Krejčí in [5] have elucidated it convincingly so we will not repeat the counter arguments here. In other words, the aforementioned *mystery* is still an interesting debate in the FAHP literature.

Some clues are actually present in other studies. Nevertheless, they do not exist in an organized manner as an investigation of the validity of the FAHP is normally not the main objective in those studies. For example, Chan et al. [3] applied a mixed-method approach to quantify social media data for later MCDA applications. It was not their intention to compare the AHP and FAHP, but the numerical results indicate that the outputs of the two methods are similar, and both methods are indifferent in terms of ranking alternatives. In a spatial MCDA application, Kordi and Brandt [12] discovered that the difference between the AHP and FAHP is a function of the level of fuzziness (i.e., uncertainty) in the FAHP. Increasing the fuzziness leads to a more significant difference. In a similar spatial application (although in a different location), Mosadeghi et al. [18], based on a sensitivity analysis, made the similar observation that the FAHP is more stable when subjected to changes in weight criteria. Krejčí et al. [14] considered that the FAHP is still necessary when the pairwise comparisons are vague; otherwise, the traditional AHP will lead to mistakes. The main implication of these studies is that if a simpler method is acceptable in the analysis process, there is no need to apply a more sophisticated method (e.g., the AHP compared with the FAHP). This makes perfect sense, but leaves a query: how to determine the level of fuzziness that makes the FAHP superior to the AHP in the analysis?

It is clear from the above review that the usefulness of the FAHP is still debatable, simply because it is unable to reach a conclusion analytically. Therefore, the research motivation of this study is to provide insights on this problem by investigating the *difference* between the AHP and FAHP. It is not our intention to prove which method is better, or to challenge the operations of the FAHP. However, if we can prove that the two methods are not different, there is definitely no need to select a complex method over a simpler one. If there is any *difference* between the AHP and FAHP, it is worth finding out the *conditions* where this occurs. Obviously, this will provide insight on when the FAHP should be employed. This study makes a significant contribution on when to (or not to, of course) use the FAHP over the AHP. Experimental results can help explain some of the observations discussed earlier in this section. The rest of the paper is organized as follows: In Section 2, the *difference* between the AHP and triangular FAHP is formulated from a quantitative perspective, as discussed above. The main reason for doing so is to enable the subsequent experiments to be undertaken. Section 3 describes the experimental results when the matrix size of the problem ranges from 2 to 9, and the qualitative difference induced by the application of triangular fuzzy numbers is further explored and analyzed. A real-world case proposed in the previous study [11] is applied here for detailed illustration. In Section 4, we further consider the trapezoidal FAHP and show the robustness of the results derived with the triangular FAHP cases in Section 3. Section 5 concludes the paper.

2. Difference between the triangular FAHP and classical AHP

In this section, a function representing the differences between the classical AHP and triangular FAHP subjected to the same set of pairwise comparisons and the corresponding fuzzy counterpart is established. The function basically sums up the differences of the weights of all criteria between the two methods, based on which, experiments can be then conducted. The details are discussed below.

2.1 Weights estimated by the classical AHP

The pairwise comparison matrix A was first defined by Saaty in [22] as a method to estimate ratio scales. The scale of measurement is based on the fundamental 9-point scale defined in [25].

$$A=\begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{21} & 1 & \alpha_{23} & \cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & 1 & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ 1/\alpha_{12} & 1 & \alpha_{23} & \cdots & \alpha_{2n} \\ 1/\alpha_{13} & 1/\alpha_{23} & 1 & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1/\alpha_{1n} & 1/\alpha_{2n} & 1/\alpha_{3n} & \cdots & 1 \end{bmatrix}$$

The matrix A = (α_{ij}), where $\alpha_{ij} = \frac{w_i}{w_j}$, $i,j = 1,2,\ldots n$. n is the matrix size, which indicates the number of criteria or alternatives to be considered. The estimated ratio scale $\mathrm{w} = [w_1 \;\; w_2 \ldots w_n]^T$ is the solution of the eigenvalue problem $\mathrm{Aw} = \lambda_{max} w$. λ_{max} is the principal eigenvalue of A. The final weights of the criteria (alternatives) w^{N} are obtained by normalizing w.

2.2 Weights estimated by the triangular FAHP

In 1985, Buckley proposed the fuzzy hierarchical analysis, who introduced a triangular membership function for fuzzification. Later, various fuzzy numbers, such as the trapezoidal, interval, and type-2 fuzzy numbers, have been proposed and used for different decision methods. Among these, the triangular fuzzy numbers are among the most widely used in decision making as they are suitable for association with linguistic terms and can be utilized to judge different certainty levels [31,37]. According to the original matrix A, its corresponding symmetric triangular fuzzy version $\bar{A} = (l_{ij}, m_{ij}, u_{ij})$ is

$$\begin{bmatrix} (1,1,1) & (\alpha_{12}-\beta, \alpha_{12}, \alpha_{12}+\beta) & (\alpha_{13}-\beta, \alpha_{13}, \alpha_{13}+\beta) & \cdots & (\alpha_{1n}-\beta, \alpha_{1n}, \alpha_{1n}+\beta) \\ (1/\alpha_{12}+\beta, 1/\alpha_{12}, 1/\alpha_{12}-\beta) & (1,1,1) & (\alpha_{23}-\beta, \alpha_{23}, \alpha_{23}+\beta) & \cdots & (\alpha_{2n}-\beta, \alpha_{2n}, \alpha_{2n}+\beta) \\ (1/\alpha_{13}+\beta, 1/\alpha_{13}, 1/\alpha_{13}-\beta) & (1/\alpha_{23}+\beta, 1/\alpha_{23}, 1/\alpha_{23}-\beta) & (1,1,1) & \cdots & (\alpha_{3n}-\beta, \alpha_{3n}, \alpha_{3n}+\beta) \\ \vdots & \vdots & \vdots & & \vdots \\ (1/\alpha_{1n}+\beta, 1/\alpha_{1n}, 1/\alpha_{1n}-\beta) & (1/\alpha_{2n}+\beta, 1/\alpha_{2n}, 1/\alpha_{2n}-\beta) & (1/\alpha_{3n}+\beta, 1/\alpha_{3n}, 1/\alpha_{3n}-\beta) & \cdots & (1,1,1) \end{bmatrix}$$

The geometric mean method is used to calculate the fuzzy weights $\bar{w} = [\bar{w}_1 \;\; \bar{w}_2 \ldots \bar{w}_n]^T$ for the fuzzy matrix. The weight $\bar{w}_i = (\frac{l_i}{u}, \frac{m_i}{m}, \frac{u_i}{l})$(*i=1,2,...,n*), where $l_i = (\prod_{j=1}^{n} l_{ij})^{\frac{1}{n}}$, $m_i = (\prod_{j=1}^{n} m_{ij})^{\frac{1}{n}}$, $u_i = (\prod_{j=1}^{n} u_{ij})^{\frac{1}{n}}$, $l = \sum_{i=1}^{n} l_i$, $m = \sum_{i=1}^{n} m_i$, and $u = \sum_{i=1}^{n} u_i$. The details of the triangular fuzzy numbers are as follows. In the matrix, β represents the one-side width of the fuzzy triangular membership functions. When $\beta = 0, \bar{w}$ degenerates into the weights determined by the logarithmic least squares method.

$$l_i = (\prod_{j=1}^{n} l_{ij})^{\frac{1}{n}} = (\prod_{j<i} l_{ij} \prod_{j>i} l_{ij})^{\frac{1}{n}} = \left(\prod_{j<i} \frac{1}{\alpha_{ji}+\beta} \prod_{j>i} (\alpha_{ij} - \beta)\right)^{\frac{1}{n}}$$

$$u_i = (\prod_{j=1}^{n} u_{ij})^{\frac{1}{n}} = (\prod_{j<i} u_{ij} \prod_{j>i} u_{ij})^{\frac{1}{n}} = \left(\prod_{j<i} \frac{1}{\alpha_{ji}-\beta} \prod_{j>i} (\alpha_{ij} + \beta)\right)^{\frac{1}{n}}$$

$m_i = (\prod_{j=1}^{n} m_{ij})^{\frac{1}{n}} = (\prod_{j<i} m_{ij} \prod_{j>i} m_{ij})^{\frac{1}{n}} = \left(\prod_{j<i} \frac{1}{\alpha_{ji}} \prod_{j>i} \alpha_{ij}\right)^{\frac{1}{n}}$

$l = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \left(\prod_{j<i} \frac{1}{\alpha_{ji}+\beta} \prod_{j>i} (\alpha_{ij} - \beta)\right)^{\frac{1}{n}}$

$u = \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} \left(\prod_{j<i} \frac{1}{\alpha_{ji}-\beta} \prod_{j>i} (\alpha_{ij} + \beta)\right)^{\frac{1}{n}}$

$m = \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \left(\prod_{j<i} \frac{1}{\alpha_{ji}} \prod_{j>i} \alpha_{ij}\right)^{\frac{1}{n}}$

2.3 Quantitative differences between the classical AHP and triangular FAHP

The defuzzified weights are set as $\widetilde{w} = [\widetilde{w}_1 \ \widetilde{w}_2 \ ... \ \widetilde{w}_n]^T$, where each element $\widetilde{w}_i$ is the center of area of $\overline{w}_i$, namely $\widetilde{w}_i = \frac{\frac{l_i}{u}+\frac{m_i}{m}+\frac{u_i}{l}}{3}$. Therefore, the normalized defuzzified weight $\widetilde{w}_i^N = \frac{\frac{l_i}{u}+\frac{m_i}{m}+\frac{u_i}{l}}{\frac{l}{u}+1+\frac{u}{l}}$. First, to measure the quantitative difference between the weights generated by the classical AHP and the triangular fuzzy AHP, we have the following definition:

Definition 1(Quantitative Difference[1]): the application of the triangular fuzzy numbers makes a quantitative difference **d** if and only if $\mathbf{d} = \sum_{i=1}^{n} d_i > 0$, where $d_i = |\widetilde{w}_i^N - w_i^N| (1 \le i \le \text{n})$.

d is in fact the sum of the absolute difference between the corresponding weights of the criteria obtained by the classical AHP and the triangular fuzzy AHP, respectively. Given the positive reciprocal matrix A = (α_{ij}), $i, j = 1,2, \dots n$, it is verified that **d** is a function of β $(0 < \beta < 1)$, namely, $\mathbf{d} = \text{F}(\beta)$. Here, we set β within the range of (0, 1) to make sure the feasibility of the fuzziness of the positive reciprocal matrix A as the smallest element in A is "1." The closed forms of **d** for the cases of matrix size equal to 2 and 3, respectively, are as follows.

Proposition 1. For the cases that matrix size $\text{n} = 2$, for any $\alpha_{12} \ge 1$, $0 < \beta < 1$. The quantitative difference **d** between the weights under the triangular fuzzy AHP and the classical AHP methods is $\text{F}(\beta) = d_1 + d_2$,

where, $d_1 = \left| \frac{\alpha_{12}(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}[\frac{-\beta^2}{\alpha_{12}+1}+2(1+\frac{1}{\alpha_{12}+1})(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}+\frac{\alpha_{12}^2+1}{\alpha_{12}+1}+2]}{(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}\left[-\beta^2+2(1+\alpha_{12})(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}+\alpha_{12}^2+4\alpha_{12}+1\right]+2\alpha_{12}} - \frac{\alpha_{12}}{\alpha_{12}+1} \right|$,

$$d_2 = \left| 1 - \frac{\alpha_{12}(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}[\frac{-\beta^2}{\alpha_{12}+1}+2(1+\frac{1}{\alpha_{12}+1})(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}+\frac{\alpha_{12}^2+1}{\alpha_{12}+1}+2]}{(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}\left[-\beta^2+2(1+\alpha_{12})(\alpha_{12}^2-\beta^2)^{\frac{1}{2}}+\alpha_{12}^2+4\alpha_{12}+1\right]+2\alpha_{12}} - \frac{1}{\alpha_{12}+1} \right|.$$

Proof: see Appendix A.

[1] Note that the qualitative difference is also defined and further analyzed in the following section.

By conversion between the two elements, the cases in which $\alpha_{12} < 1$ are the same as the cases in which $\alpha_{12} \geq 1$. Proposition 1 implies that, for the case of the matrix size or the number of criteria to be considered $\mathrm{n} = 2$, the difference between the two methods is always greater than zero. In other words, differences can always be found. However, the magnitude of **d** is examined in Section 3, which demonstrates that the quantitative difference **d** is not significant for most cases.

Proposition 2. For the case where matrix size $\mathrm{n} = 3$, given the scale value $\alpha_{12}, \alpha_{13}, \alpha_{23} \in \{1,2,3,4,5,6,7,8,9\}$, and $0 < \beta < 1$, the quantitative difference **d** between the weights generated by applying the triangular FAHP and classical AHP is $\mathrm{F}(\beta) = d_1 + d_2 + d_3 = \left|\widetilde{w}_1^N - w_1^N\right| + \left|\widetilde{w}_2^N - w_2^N\right| + \left|\widetilde{w}_3^N - w_3^N\right|$, where

$$\widetilde{w}_1^N = \frac{(\alpha_{12}+\beta)^{\frac{1}{3}}(\alpha_{13}+\beta)^{\frac{1}{3}}u+(\alpha_{12}-\beta)^{\frac{1}{3}}(\alpha_{13}-\beta)^{\frac{1}{3}}l+\frac{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}+(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}lu}{l^2+u^2+lu},$$

$$\widetilde{w}_2^N = \frac{(\alpha_{12}+\beta)^{\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}u+(\alpha_{12}-\beta)^{\frac{1}{3}}(\alpha_{23}-\beta)^{\frac{1}{3}}l+\frac{\left(\alpha_{12}^2-\beta^2\right)^{\frac{1}{3}}(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}+(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}lu}{\left(\alpha_{12}^2-\beta^2\right)^{\frac{1}{3}}(l^2+u^2+lu)},$$

$$\widetilde{w}_3^N = \frac{(\alpha_{13}+\beta)^{\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}u+(\alpha_{13}-\beta)^{\frac{1}{3}}(\alpha_{23}-\beta)^{\frac{1}{3}}l+\frac{\left(\alpha_{13}^2-\beta^2\right)^{\frac{1}{3}}\left(\alpha_{23}^2-\beta^2\right)^{\frac{1}{3}}(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}+(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}lu}{\left(\alpha_{13}^2-\beta^2\right)^{\frac{1}{3}}\left(\alpha_{23}^2-\beta^2\right)^{\frac{1}{3}}(l^2+u^2+lu)},$$

where $l = f(\beta) = (\alpha_{12}-\beta)^{\frac{1}{3}}(\alpha_{13}-\beta)^{\frac{1}{3}} + (\frac{\alpha_{23}-\beta}{\alpha_{12}+\beta})^{\frac{1}{3}} + \frac{1}{(\alpha_{13}+\beta)^{\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}}, u = f(-\beta).$

$$w_1^N = \frac{a\alpha_{12}\alpha_{13}+\alpha_{12}^2\alpha_{23}}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}},$$

$$w_2^N = \frac{\alpha_{13}+a\alpha_{12}\alpha_{23}}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}},$$

$$w_3^N = \frac{\alpha_{12}(a^2-1)}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}},$$

where, $a = (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} + (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}}.$

Proof: see Appendix B.

For the other scenarios, such as

i) $\alpha_{12} \in \{1,2,\dots,9\}; \alpha_{13} \in \left\{\frac{1}{9},\frac{1}{8},\dots,\frac{1}{2}\right\}; \alpha_{23} \in \{\frac{1}{9},\frac{1}{8},\dots,\frac{1}{2}\}$,

ii) $\alpha_{12} \in \left\{\frac{1}{9},\frac{1}{8},\dots,\frac{1}{2}\right\}; \alpha_{13} \in \{1,2,\dots,9\}; \alpha_{23} \in \left\{\frac{1}{9},\frac{1}{8},\dots,\frac{1}{2}\right\}$,

iii) $\alpha_{12} \in \{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}\}$; $\alpha_{13} \in \left\{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}\right\}$; $\alpha_{23} \in \{1, 2, \dots, 9\}$,

and the other 4 scenarios, the results and proofs are similar. Owing to the similarity, no proof is presented here for the other 7 cases. For the case when the matrix size or the number of criteria n = 3, Proposition 2 implies that the quantitative differences in terms of the ranking of criteria by the two methods are always greater than zero. However, the magnitude of the quantitative difference **d** is examined in Section 3, which in fact is not sufficiently significant for most randomly generated pairwise comparison matrices. Refer to Section 3 for more details.

Proposition 3. A closed-form expression cannot be found for the function of the quantitative difference **d** when the matrix size $n \geq 4$.

Proof: see Appendix C.

3. Numerical Experiments

Numerical experiments have been carried out under the measurement of consistency to present the properties of $F(\beta)$ in terms of β $(0 < \beta < 1)$, which reflects the level of fuzziness and the size of the matrices. We first test the situations when the matrix size is 2, 3, and 4, respectively. The scale take the values $\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

For the measurement of consistency, we refer to the consistency index ($\mathrm{CI} = \frac{\lambda_{max} - n}{n-1}$) and consistency ratio ($\mathrm{CR} = \frac{CI}{critical\ point\ (n)}$) defined by Saaty [23] (see Table 2). We only consider the comparison matrices whose calculated CRs are less than 10%, in which case the consistency of the pairwise judgment can be accepted [16]. It is unnecessary to apply the measurement of consistency, CI or CR, in the situation when the matrix size is 2. As n = 2, $\mathrm{A} = \begin{bmatrix} 1 & \alpha_{12} \\ 1/\alpha_{12} & 1 \end{bmatrix}$, satisfying $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$, for i, j, k =1,2; thus, A is consistent for any α_{12} when $n = 2$.

Table 2
Critical points.

Matrix size (n)	2	3	4	5	6	7	8	9
Critical point	N/A	0.609	0.961	1.182	1.322	1.394	1.475	1.493

3.1 Comparison between the weights by the triangular FAHP and classical AHP

In this subsection, we first show the results after an exhaustive search of all the consistent cases for the matrix sizes from 2 to 4. To make the sensitivity analysis comprehensive while tractable, we further present the results for the cases when the matrix size varies from 5 to 9 by randomly selecting 100,000 to 2,000,000 consistent pairwise comparison matrices.

First, we analyze the impact of the fuzziness level on the quantitative difference between the weights obtained by the classical AHP and triangular AHP. Figs. 1 to 8 show the curves of the quantitative difference **d** in terms of fuzziness level β $(0 < \beta < 1)$ for all the arbitrarily generated consistent pairwise comparison matrices A when the matrix sizes n are 2, 3, 4, …, 9, respectively. For any matrix size, the difference between the weights generated by the triangular FAHP and AHP increases along with β. It is verified that the difference between the AHP and the triangular FAHP exists, but the differences become notable when β is sufficiently large (e.g., $\beta \geq 0.5$).

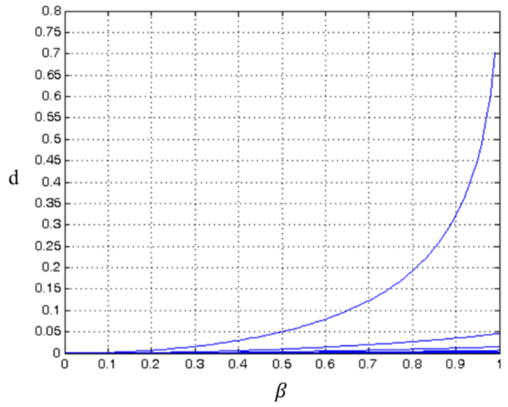

Fig. 1. Comparison between the weights by FAHP and AHP when n = 2.

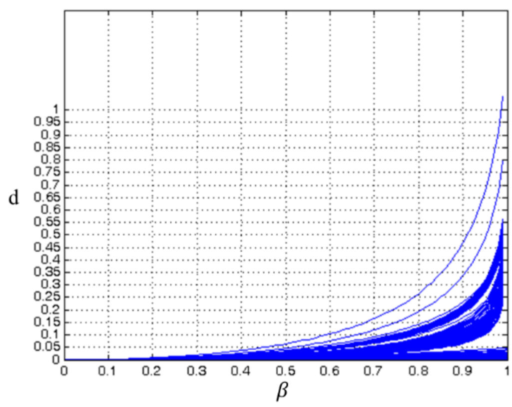

Fig. 2. Comparison between the weights by FAHP and AHP when n = 3.

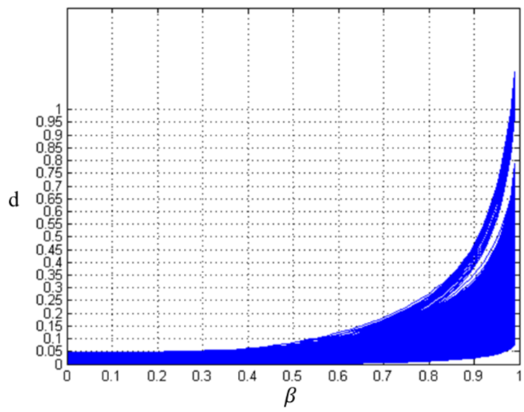

Fig. 3. Comparison between weights by FAHP and AHP when n = 4.

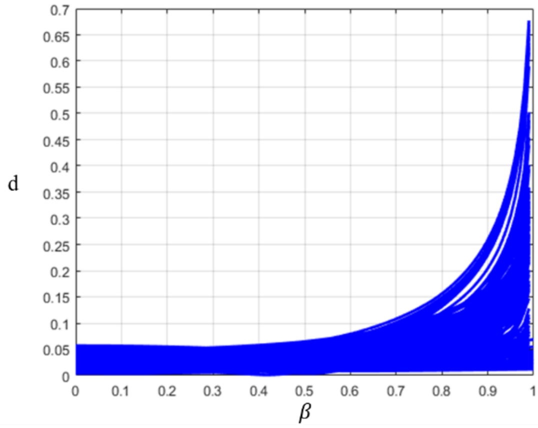

Fig. 4. Comparison between the weights by FAHP and AHP when n = 5.

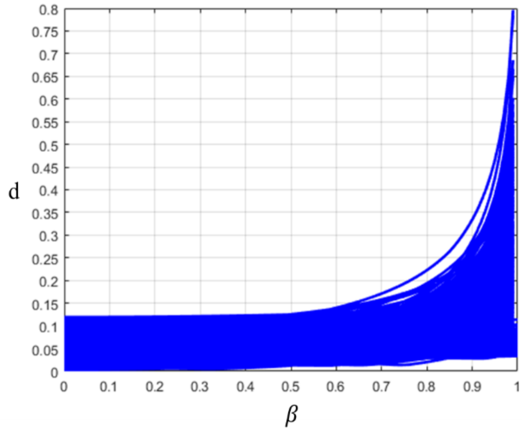

Fig. 5. Comparison between the weights by FAHP and AHP when n = 6.

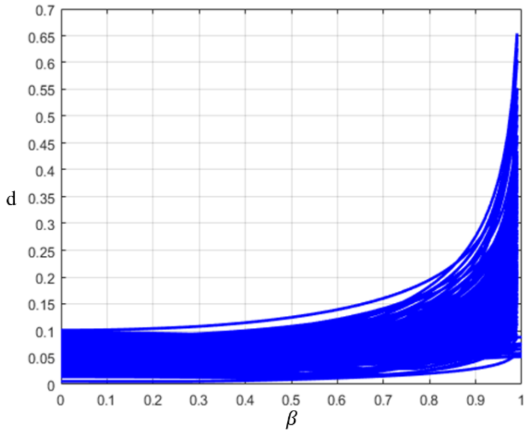

Fig. 6. Comparison between the weights by FAHP and AHP when n = 7.

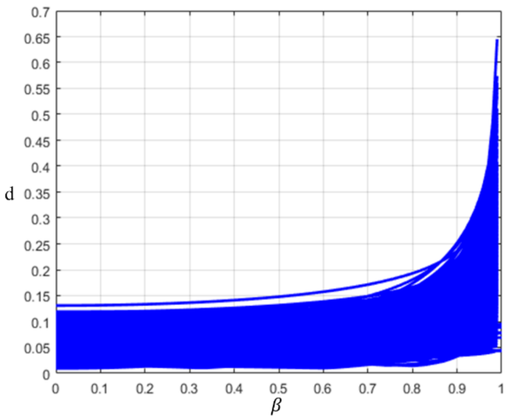

Fig. 7. Comparison between the weights by FAHP and AHP when n = 8.

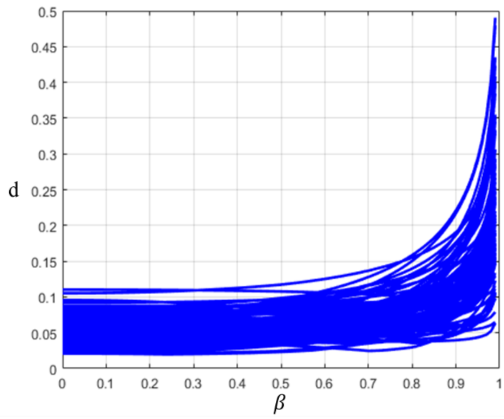

Fig. 8. Comparison between the weights by FAHP and AHP when n = 9.

Accordingly, we have the following observations:

Observation 1: The *difference* between the triangular FAHP and AHP increases with the matrix size.

Observation 2: The *difference* between the triangular FAHP and AHP increases with the fuzziness level of the membership function, which is measured by β.

These two observations actually confirm the previous studies outlined in Section 1. However, the findings in this study provide quantitative evidence to support the observations. More importantly,

Observation 3: an obvious *difference* between the triangular FAHP and the AHP (e.g., over 0.05 with respect to the 9-point scale) can be achieved only if β is sufficiently large (e.g., the difference reaches 0.1 when β is larger than 0.7, but not for all cases).

Observation 3 in fact leads to the conclusion that the triangular FAHP is unable to introduce much *difference*, in general. Although the usefulness of the triangular FAHP is quite limited, it is still worth understanding the circumstances when the triangular FAHP would be able to lead to different solutions compared to the AHP. This is addressed in the next section.

Observation 4: the *difference* **d** becomes large with the increase in matrix size, and the *difference* is mainly generated by the difference in terms of the weight calculation methodology between the two methods, i.e., the logarithmic least squares method and the eigenvalue method (when $\beta = 0$, $\mathbf{d}$ is much greater than 0, especially for the case of $n \geq 4$).

Observation 4 implies that, except for the impact generated by the two different calculation methods for the triangular FAHP and AHP, the matrix size in fact has limited impacts on the triangular FAHP and AHP *difference* **d**.

3.2 When to use the triangular FAHP?

Effect of matrix size

In this part, as the characteristics of the cases with larger size matrices ($n > 4$) are similar to the case when the matrix size $n = 4$, we focus on the discussion of the results when the matrix size n is 2, 3, and 4, respectively. As presented in Table 3, when the matrix size is 2 and $\beta = 0.5$, only in one case, i.e., $\alpha_{12} = 1$, the difference is greater than 5%. When β reaches 0.9, the difference between the weights greater than 5% is only obtained for the same case, i.e., $\alpha_{12} = 1$. For the case when the matrix size is 3 and $\beta = 0.5$, a 5% difference can only be obtained in less than 0.2% of all the consistent cases. The average standard deviation among the parameters is 0.4330. For the 99.8% consistent cases, the difference between the weights by the triangular FAHP and AHP is less than or equal to 5%. The average standard deviation among the parameters is 2.2767, much greater than those of matrices whose corresponding difference is larger than 5%. For the case when β reaches 0.9, the number of cases in which the difference of weights is larger than 5% is increased to 30%. For the cases when the matrix size is 4, 1.7% of the consistent matrices can make the weight difference greater than 5% when $\beta = 0.5$, and 43% when $\beta = 0.9$. Note that, although the percentage of the existence of *difference* (i.e., **d** > 5%) between the triangular FAHP and AHP increases, the difference is mainly caused by the calculation deviation between the logarithmic least squares method and the eigenvalue method for the large-size matrix, rather than the increase in matrix size. In addition, the matrix size *n* has limited correlated effects with the fuzziness level β on the increase of the quantitative difference **d** between the triangular FAHP and AHP.

Table 3
Difference between the weights by the classical AHP and triangular FAHP.

Matrix size (n)	**d**	No. of consistent cases $\beta = 0.5$	Percentage	Ave.Sta.	No. of consistent cases $\beta = 0.9$	Percentage	Ave.Sta.*
2	>5%	1	5.9%	0	1	5.9%	0
	<=5%	16	94.1%	0	16	94.1%	0
3	>5%	2	0.2%	0.4330	331	30.0%	2.0644
	<=5%	1103	99.8%	2.2767	774	70.0%	2.2767
4	>5%	14616	1.7%	2.3252	376099	43.0%	2.2622
	<=5%	858641	98.3%	2.3840	497158	57.0%	2.4744

* Average standard deviation among the parameters in the pairwise comparison matrix A

Effect of number of equally important criteria

Table 4 presents a more detailed analysis for those 3×3 consistent pairwise comparison matrices in which the *difference* between the weights of the triangular FAHP and AHP can be more than 5% when $\beta = 0.9$. It shows that when the *difference* becomes larger, the average standard deviation among the parameters in the pairwise comparison matrix A becomes smaller. At the same time, the number of "1" among the parameters increases, which indicates that the more equally important elements we have in the matrix, the greater the *difference* that the triangular FAHP can make. This is also consistent with the average standard deviation. Additionally, for the cases when the difference **d** is less than 5%, the number of value "1" among the parameters is 0, which verifies that when the intensity of importance among the criteria that is judged by experts is distinct from each other, the significance generated by the triangular FAHP becomes smaller.

Table 4
Details of the case where the difference is significant for *n* = 3.

Matrix size (n)	**d**	No. of consistent cases	Percentage	No. of "1"	Ave.Sta.
3	<=5%	774	70.0%	0	2.2767
	5%–15%	145	13.1%	>=1	2.2418
	15%–25%	183	16.6%	>=1	1.953
	>25%	3	0.3%	>=2	0.2887

When matrix size $n = 4$, all the consistent cases are displayed by Figs. 11–27 in Appendix D, which correspond to 17 exclusive cases. Fig. 3 presents one of the 17 exclusive cases where all the other matrix elements, given that element $\alpha_{12} = 1$, are exhaustively tested. Table 5 presents the detailed analysis corresponding to Fig. 3. From this result, it is found that the number of consistent cases decreases as the difference "**d**" increases. In 56.93% of the cases, the difference of the fuzzy weights and eigenvector weights is less than or equal to 5% when $\beta = 0.9$. In less than 10% of all the consistent cases, the corresponding difference is greater than 0.2. When the difference between the weights obtained by the triangular FAHP and AHP is significant, the number of equally important elements increases and the average standard deviation decreases, and this is the same for the cases when the matrix size is 3.

Consistency level

To further verify the conditions where the application of the fuzzy numbers makes a qualitative difference on the ranking of alternatives, we have the following definition. It is easy to understand that when the triangular FAHP makes no difference regarding the ranking of alternatives, the sign of $(\widetilde{w}_i^N - \widetilde{w}_j^N)$ is the same as that of $\left(w_i^N - w_j^N\right)$. Hence, we have

Definition 2 (Qualitative Difference): The application of fuzzy numbers to the AHP makes a qualitative difference $d^{qua} = \left(\widetilde{w}_i^N - \widetilde{w}_j^N\right)\left(w_i^N - w_j^N\right)$ in terms of ranking of alternatives if and only if there exist *i, j* that let $d^{qua} < 0$ [2] $(\mathrm{i} \neq \mathrm{j}, 1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n})$. In other words, there is no qualitative difference if and only if $d^{qua} > 0$ for any i, j $(i \neq j, 1 \leq i, j \leq n)$.

According to Definition 2, we exhaustively test the cases when the matrix sizes are 3 and 4, respectively. Based on the previous results, we find that similar results can be obtained under different scenarios. To avoid duplication, in this section we only conduct the computational test under the scenario when the elements of the randomly generated consistent pairwise comparison matrix are within the scale of {1, 2, 3, … , 9}, i.e., $a_{ij(i<j)} \in \{1,2,3,\ldots,9\}$.

Table 6 reports the results in a statistical way, which corresponds to the cases when the application of fuzzy numbers makes no qualitative difference regarding the ranking of alternatives by the AHP. It is presented that in more than 90% of all the consistent cases, the application of fuzzy numbers has no influence on the ranking. Even though the fuzziness level β is extremely large (i.e., $\beta \geq 0.9$), the application of fuzzy numbers makes no qualitative difference for 72.6% and 65% of all the consistent matrices, corresponding to the situations when the matrix sizes are 3 and 4, respectively. In addition, it is interesting to note that there exists a positive relationship between the fuzziness level β and the consistent ratio (CR) of the group. CR reflects the consistency level of the pairwise comparison matrix proposed by the experts. If the transitivity rule $a_{ik} = a_{ij} \times a_{jk}$ $(\forall \mathrm{i}, \mathrm{k} \in \{1,2,\ldots,\mathrm{n}\}, \mathrm{i} \neq \mathrm{k}, \mathrm{j} \in \{1,2,\ldots,\mathrm{n}\} \backslash \{\mathrm{i}, \mathrm{k}\})$ is perfectly satisfied, then CR equals *n*. For instance, the perfect CR for the case where the matrix size n is 3. The larger the CR is, the fuzzier the pairwise comparisons among the different alternatives proposed by the experts are. This indicates that it is unnecessary to apply fuzzy numbers into the AHP if the consistency level of the pairwise comparison matrices is not satisfactory. In other words, when the judgment made by the experts is already fuzzy, especially when there exist pairs of equal importance (i.e., $a_{ij} = 1, i \neq j$), further consideration of fuzzy numbers is useless.

[2] Note that the cases where $d^{qua} = 0, i.e., \widetilde{w}_i^N = \widetilde{w}_j^N (or\ \ w_i^N = w_j^N)$ are exclusive of this study, and it is not the common case from the computational perspective.

The implications of the above findings actually are in consensus with the results in the previous section regarding the discussion of the quantitative difference. It is demonstrated that the AHP is useful in assisting the decision-making process, especially when the problem is complex and the judgment on the criteria is already a little fuzzy. The larger the matrix size is, the probability to obtain fuzzy judgments becomes larger, and in that case, the application of fuzzy numbers to the AHP becomes less useful. However, when the judgments on the criteria are highly consistent, to avoid subjectivity from a small group of experts, the application of fuzzy numbers suggested by other experts may be necessary to obtain different rankings for the references.

Table 5

Details of the case where the difference is significant for $n = 4$

Matrix size (n)	**d**	No. of consistent cases	Percentage	No. of "1"	Ave.Sta.
	<=5%	497158	56.93%	<=1	2.474437868
	5%–10%	123988	14.17%	>=1(98.8%)	2.2952
4	10%–20%	235425	26.90%	>=1	2.30785
	20%–40%	16508	9.56%	>=2	1.8278
	>40%	178	0.02%	>=3	0.8621

Table 6

Situations when the application of fuzzy numbers makes no qualitative differences

β	n	No. of consistent matrices	Percentage	Avg.CR	n	No. of consistent matrices	Percentage	Avg.CR
(0,0.1]		0	0%	N/A		1	0	4.005935
(0,0.2]		0	0%	N/A		91	0.2%	4.111753
(0,0.3]		0	0%	N/A		293	0.5%	4.117781
(0,0.4]		7	2.9%	3.001542		1344	2.3%	4.139364
(0,0.5]	3	7	2.9%	3.002597	4	1864	3.1%	4.144563
(0,0.6]		11	4.6%	3.013293		2855	4.8%	4.146537
(0,0.7]		13	5.4%	3.016717		4528	7.6%	4.152964
(0,0.8]		11	4.6%	3.039688		6668	11.2%	4.169719
(0,0.9]		175	72.6%	3.044325		38702	65.0%	4.183363
Overall		224(241)	92.9%			56346(59509)	94.7%	

According to the comprehensive analysis above, we can summarize the necessary conditions to apply triangular fuzzy numbers in the AHP as follows, and they are displayed in Fig. 9.

1) $CR \leq CR^*(n)$, where $CR^*(n)$ is a threshold of the consistency ratio given the matrix size n;

2) Judgments with dominant preference and equally important elements.

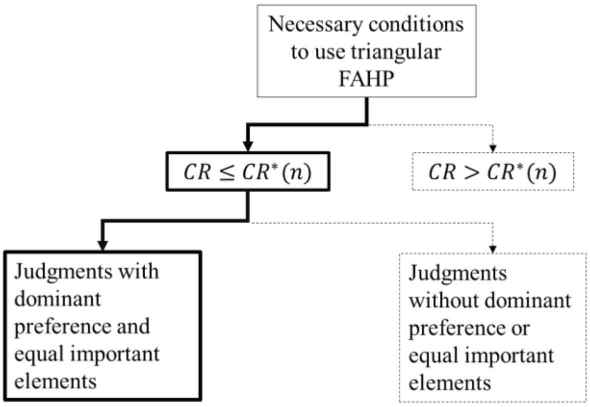

Fig. 9. Necessary conditions for the application of triangular fuzzy numbers in AHP.

3.3 Real-world illustrative case

Here, we apply both the triangular fuzzy AHP and the classical AHP into a real-world case to further illustrate the proposed research implication. It is a problem of ERP selection faced by Turkish Airlines [11]. The objective is to determine the best ERP system to choose. A series of weights are supposed to be determined among different corporate criteria (i.e., references, adequacy, after sales, and know-how). Table 7 shows the original pairwise comparison among those four criteria based on the triangular fuzzy version proposed in [11]. Accordingly, we have the corresponding CR under the pairwise comparison and the weights by applying both the AHP and fuzzy AHP as follows:

CR = 4.1213,

$(w_1^N, w_2^N, w_3^N, w_4^N) = (0.1953, 0.2761, 0.3905, 0.1381)$,

$(\widetilde{w}_1^N, \widetilde{w}_2^N, \widetilde{w}_3^N, \widetilde{w}_4^N) = (0.2046, 0.2744, 0.3684, 0.1527)$.

Obviously, there is no qualitative difference between the rankings under the classical AHP and the triangular fuzzy AHP. To demonstrate our proposed research implication, we change the original a_{12}, which is equal to 1/2 in Table 7, into 1, i.e., a_{12} = 1. In this way, the consistency level of the revised pairwise comparison matrix is improved to 4.0606. We maintain the other numbers the same and recalculate the weights by applying the AHP and fuzzy AHP again. We have

$(w_1^N, w_2^N, w_3^N, w_4^N) = (0.2322, 0.2322, 0.3952, 0.1404)$,

$(\widetilde{w}_1^N, \widetilde{w}_2^N, \widetilde{w}_3^N, \widetilde{w}_4^N) = (0.2146, 0.2630, 0.3691, 0.1529)$.

In this case, the ranking of the criteria (i.e., references, adequacy, after sales, and know-how) by the classical AHP is 2, 2, 1, 4[3], and their ranking by applying the triangular fuzzy AHP is 3, 2, 1, 4. The result is consistent with our findings and indicates that when the judgment of the criteria by the experts is already fuzzy under the AHP, there is no need to apply fuzzy numbers again. However, when the judgment on the criteria by the experts is clear and there exist criteria of equal importance, the application of fuzzy numbers may induce qualitatively different priorities, which provides the decision maker with more references.

Table 7

Pairwise comparison matrix of the corporate criteria based on the triangular fuzzy version.

Criteria	References	Adequacy	After sales	Know-how
References	(1,1,1)	(1/3,1/2,1)	(1/3,1/2.1)	(1,2,3)
Adequacy	(1,2,3)	(1,1,1)	(1/3,1/2,1)	(1,2,3)
After sales	(1,2,3)	(1,2,3)	(1,1,1)	(1,2,3)
Know-how	(1/3,1/2,1)	(1/3,1/2,1)	(1/3,1/2,1)	(1,1,1)

4. Extension to trapezoidal fuzzy numbers

In this section, the general case of fuzzy numbers, i.e., trapezoidal fuzzy numbers, is further discussed to show the robustness of our results derived from the case of triangular fuzzy numbers in Section 3. As shown in Fig. 10, the mode of a trapezoidal fuzzy number is a flat line instead of a point. Accordingly, the corresponding symmetric[4] trapezoidal fuzzy version $\hat{A} = (l_{ij}, k_{ij}, o_{ij}, u_{ij})$ is

$$\begin{bmatrix} (1,1,1,1) & (\alpha_{12}-\beta, \alpha_{12}-\beta_1, \alpha_{12}+\beta_1, \alpha_{12}+\beta) & (\alpha_{13}-\beta, \alpha_{13}-\beta_1, \alpha_{13}+\beta_1, \alpha_{13}+\beta) & \cdots (\alpha_{1n}-\beta, \alpha_{1n}-\beta_1, \alpha_{1n}+\beta_1, \alpha_{1n}+\beta) \\ (1/\alpha_{12}+\beta, 1/\alpha_{12}+\beta_1, 1/\alpha_{12}-\beta_1, 1/\alpha_{12}-\beta) & (1,1,1,1) & (\alpha_{23}-\beta, \alpha_{23}-\beta_1, \alpha_{23}+\beta_1,, \alpha_{23}+\beta) & \cdots (\alpha_{2n}-\beta, \alpha_{2n}-\beta_1, \alpha_{2n}+\beta_1, \alpha_{2n}+\beta) \\ (1/\alpha_{13}+\beta, 1/\alpha_{13}+\beta_1, 1/\alpha_{13}-\beta_1, 1/\alpha_{13}-\beta) & (1/\alpha_{23}+\beta, 1/\alpha_{23}+\beta_1, 1/\alpha_{23}-\beta_1, 1/\alpha_{23}-\beta) & (1,1,1,1) & \cdots (\alpha_{3n}-\beta, \alpha_{3n}-\beta_1, \alpha_{3n}+\beta_1, \alpha_{3n}+\beta) \\ \vdots & \vdots & \vdots & \vdots \\ (1/\alpha_{1n}+\beta, 1/\alpha_{1n}+\beta_1, 1/\alpha_{1n}-\beta_1, 1/\alpha_{1n}-\beta) & (1/\alpha_{2n}+\beta, 1/\alpha_{2n}+\beta_1, 1/\alpha_{2n}-\beta_1, 1/\alpha_{2n}-\beta) & (1/\alpha_{3n}+\beta, 1/\alpha_{3n}+\beta_1, 1/\alpha_{3n}-\beta_1, 1/\alpha_{3n}-\beta) \cdots & (1,1,1,1) \end{bmatrix}$$

, where $\beta_1 < \beta$. The geometric mean method is used to calculate the fuzzy weights $\widehat{w} =$

[3] Here, the weights of the criteria references and adequacy are the same, and both are ranked as second.

[4] Note that, we continue utilizing the symmetric version of fuzzy numbers to make the content consistent and focusing on the study of the impact of the magnitude of fuzziness on the ranking (i.e., weights).

$[\widehat{w}_1, \widehat{w}_2, \dots, \widehat{w}_n]^T$ under the trapezoidal fuzzy pairwise comparison matrix. In this case, the fuzzy weight $\widehat{w}_i = (\frac{l_i}{u}, \frac{k_i}{o}, \frac{o_i}{k}, \frac{u_i}{l})(i = 1,2, \dots, n)$, where $l_i = (\prod_{j=1}^{n} l_{ij})^{\frac{1}{n}}$, $k_i = (\prod_{j=1}^{n} k_{ij})^{\frac{1}{n}}$, $o_i = (\prod_{j=1}^{n} o_{ij})^{\frac{1}{n}}$, $u_i = (\prod_{j=1}^{n} u_{ij})^{\frac{1}{n}}$, and $l = \sum_{i=1}^{n} l_i$, $k = \sum_{i=1}^{n} k_i$, $o = \sum_{i=1}^{n} o_i$, $u = \sum_{i=1}^{n} u_i$.

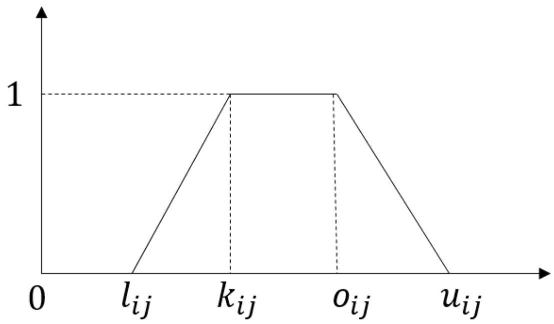

Fig. 10. Trapezoidal membership function.

Accordingly, the normalized defuzzified weight $\widehat{w}_i^N = \frac{\frac{l_i}{u}+\frac{k_i}{o}+\frac{o_i}{k}+\frac{u_i}{l}}{\frac{l}{u}+\frac{k}{o}+\frac{o}{k}+\frac{u}{l}}$. To simplify the expression of the defuzzified weight, we define $\widehat{w}_i^N = G(\beta, \beta_1)$, where *G* is a continuous function of β, β_1. Owing to the complexity of the close form of Function *f*, in the following we will conduct numerical experiments to show the impact of the relationship between β and β_1 on the magnitude of *difference* **d** between the triangular FAHP and AHP.

We let $\beta_1 = 0$, $\frac{1}{4}\beta$, $\frac{1}{3}\beta$, $\frac{1}{2}\beta$, $\frac{3}{4}\beta$, respectively. $\beta_1 = 0$ refers to the triangular FAHP. With the increase in β_1, the flat line of the trapezoidal membership function of each fuzzy number in matrix $\hat{A}$ becomes wider, which implies higher fuzziness in terms of the decision maker's judgment. As demonstrated in Section 3, an obvious difference **d** between the triangular FAHP and AHP, e.g., **d** = 5%, can only be attained if β is sufficiently large. Hence, here we compare the performance in the case of $\beta = 0.9$.

As presented in Table 8, given the matrix size, the percentage of a trapezoidal fuzzy number-based pairwise comparison matrices to attain 5% difference (i.e., total deviation from the weights by the non-fuzzy pairwise comparison matrix) is increased with β_1. This is reasonable as the fuzziness of the judgment becomes significant. However, the percentage to attain 5% difference is no larger than that of the triangular fuzzy number-based pairwise comparison matrix. It is demonstrated that the results derived in the case of triangular fuzzy numbers is robust to the case of trapezoidal fuzzy numbers. The impact of the fuzziness on the weights under the trapezoidal FAHP might be not significant compared to the triangular FAHP unless the level of fuzziness β_1 is close to β and β is sufficiently large.

Table 8
Comparison between triangular and trapezoidal fuzzy numbers

n \ β_1	0	$\frac{1}{4}\beta$	$\frac{1}{3}\beta$	$\frac{1}{2}\beta$	$\frac{3}{4}\beta$
2	5.9%	5.9%	5.9%	5.9%	5.9%
3	30.4%	25.5%	29.9%	26.7%	30.4%
4	43.1%	34.0%	35.7%	37.7%	42.3%

5. Conclusions

In this study, a comprehensive analysis has been carried out to provide insights on the conditions where there is difference between the fuzzy AHP and classical AHP from both the quantitative and qualitative perspectives. The closed forms of the difference between the fuzzy AHP and classical AHP are proven and presented for small matrix scales. We further verified the conditions where it is necessary to apply the fuzzy AHP.

First, the main theoretical contribution of this study is that, according to the numerical experiments, it is verified that a quantitative difference between the triangular fuzzy AHP and classical AHP exists, even for small matrix scales, and such difference increases with the matrix size and the level of fuzziness. However, a significant difference cannot be observed in most cases, unless the fuzziness level is sufficiently large (e.g., a 0.1 difference occurs only when β reaches 0.7, where $0 < \beta < 1$). It is also presented that in more than 65% of all the consistent cases, the application of fuzzy numbers has no influence on the ranking of criteria, even though the fuzziness level β is extremely large (i.e., β is approaching 1). Thus, in general, the triangular FAHP is unable to introduce much difference from both the quantitative and qualitative perspectives. The triangular FAHP may become useful when the pairwise comparison matrix is highly consistent. It provides different criteria rankings for references to avoid the subjectivity from a small group of experts when the judgment on the criteria made by them is highly consistent. The matrix size does not make the influence of the application of triangular fuzzy numbers on the AHP more significant. With the increase in matrix size, the quantitative difference between the triangular FAHP and classical AHP is mainly generated by the distinction in terms of the calculation techniques between the two classical methods, i.e., the logarithmic least squares method and the eigenvalue method. In addition, the probability to obtain a high consistency level regarding the pairwise comparison becomes lower with the increase in the number of criteria to be considered. That is also one of the reasons why the application of triangular fuzzy numbers in the AHP is unnecessary when the problem becomes complex and the judgments are already fuzzy for large size problems.

The other condition in which the application of the triangular fuzzy numbers in the AHP is useful is for the cases when the number of equally important (i.e., 1 or small numbers in the matrix) elements is sufficiently high. In other words, the triangular FAHP might work better when the relative importance of the multiple criteria is close to each other. In other words, there is one dominating criterion (or a few of them) in the group. In such a case, the applicability of the FAHP is in fact weakened because the final decision is highly dependent on the dominating criterion (or a few criteria). In the extension analysis, we verify the robustness of the results derived from Section 3 for the case of trapezoidal fuzzy numbers, and further demonstrates that the difference attained by the trapezoidal FAHP may not be greater than that obtained by the triangular FAHP.

This study provides insights on the usefulness of FAHP, analytically, and describes the conditions when fuzzy AHP can introduce differences over classical AHP. However, by examining the conditions when such differences arise, this study infers that FAHP is in fact not a favorable method over classical AHP. A sophisticated method is not necessarily better than a simple method!

There are still some limitations in this study. The comprehensive experimental analysis is conducted for relatively small size problems. To generalize the research findings, big size problems need to be tested in a more comprehensive way. Another limitation is that we focus on the discussion of the triangular and trapezoidal fuzzy AHP. The other extensions of fuzzy AHP such as Intuitionistic Fuzzy AHP and Hesitant Fuzzy AHP methods can be the possible future directions. Their further comparisons with the classical AHP and the exploration of corresponding conditions to make differences regarding the criteria/alternative rankings are interesting and challenging.

Appendix A

Proof of Proposition 1:

Assume A=$\begin{bmatrix} 1 & \alpha_{12} \\ \frac{1}{\alpha_{12}} & 1 \end{bmatrix}$. To solve $|A - \lambda I| = \begin{vmatrix} 1-\lambda & \alpha_{12} \\ \frac{1}{\alpha_{12}} & 1-\lambda \end{vmatrix} = 0$, equivalently to solve

$(1-\lambda)^2 - 1 = 0. \lambda = 0 \;\; or \;\; \lambda = 2.$ Thus $\lambda_{max} = 2.$

Let $(A - \lambda_{max}I)X = \begin{bmatrix} -1 & \alpha_{12} \\ \frac{1}{\alpha_{12}} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Thus, $x_1 = \alpha_{12} x_2$. So, the principle eigenvector is $k \begin{bmatrix} \alpha_{12} \\ 1 \end{bmatrix} (k \in R)$. To normalize it, $w^N = \begin{bmatrix} \frac{\alpha_{12}}{\alpha_{12}+1} \\ \frac{1}{\alpha_{12}+1} \end{bmatrix}$.

As $l_1 = (\alpha_{12} - \beta)^{\frac{1}{2}}, l_2 = \left(\frac{1}{\alpha_{12}+\beta}\right)^{\frac{1}{2}}, u_1 = (\alpha_{12} + \beta)^{\frac{1}{2}}, u_2 = \left(\frac{1}{\alpha_{12}-\beta}\right)^{\frac{1}{2}}, m_1 = \alpha_{12}{}^{\frac{1}{2}}, m_2 = (\frac{1}{\alpha_{12}})^{\frac{1}{2}}$.

$l = l_1 + l_2, u = u_1 + u_2, m = m_1 + m_2$.

$$\widetilde{w}_1^N = \frac{\frac{l_1}{u} + \frac{m_1}{m} + \frac{u_1}{l}}{\frac{l}{u} + 1 + \frac{u}{l}} = \frac{\frac{(\alpha_{12}-\beta)^{\frac{1}{2}}}{(\alpha_{12}+\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}-\beta}\right)^{\frac{1}{2}}} + \frac{(\alpha_{12}+\beta)^{\frac{1}{2}}}{(\alpha_{12}-\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}+\beta}\right)^{\frac{1}{2}}} + \frac{\alpha_{12}{}^{\frac{1}{2}}}{\alpha_{12}{}^{\frac{1}{2}} + (\frac{1}{\alpha_{12}})^{\frac{1}{2}}}}{\frac{(\alpha_{12}-\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}+\beta}\right)^{\frac{1}{2}}}{(\alpha_{12}+\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}-\beta}\right)^{\frac{1}{2}}} + \frac{(\alpha_{12}+\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}-\beta}\right)^{\frac{1}{2}}}{(\alpha_{12}-\beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}+\beta}\right)^{\frac{1}{2}}} + 1}$$

By multiplying both denominator and nominator by $(\alpha_{12} + \beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}-\beta}\right)^{\frac{1}{2}}$, $(\alpha_{12} - \beta)^{\frac{1}{2}} + \left(\frac{1}{\alpha_{12}+\beta}\right)^{\frac{1}{2}}$, $\alpha_{12} + \beta$, $\alpha_{12} - \beta$ and simplification,

$$\widetilde{w}_1^N = \frac{\alpha_{12}(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}[\frac{-\beta^2}{\alpha_{12}+1} + 2(1 + \frac{1}{\alpha_{12}+1})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \frac{\alpha_{12}^2+1}{\alpha_{12}+1} + 2]}{(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}\left[-\beta^2 + 2(1 + \alpha_{12})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \alpha_{12}^2 + 4\alpha_{12} + 1\right] + 2\alpha_{12}}, \ \widetilde{w}_2^N = 1 - \widetilde{w}_1^N.$$

Therefore,

$$d_1 = \left| \frac{\alpha_{12}(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}[\frac{-\beta^2}{\alpha_{12}+1} + 2(1 + \frac{1}{\alpha_{12}+1})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \frac{\alpha_{12}^2+1}{\alpha_{12}+1} + 2]}{(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}\left[-\beta^2 + 2(1 + \alpha_{12})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \alpha_{12}^2 + 4\alpha_{12} + 1\right] + 2\alpha_{12}} - \frac{\alpha_{12}}{\alpha_{12}+1} \right|,$$

$$d_2 = \left| 1 - \frac{\alpha_{12}(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}[\frac{-\beta^2}{\alpha_{12}+1} + 2(1 + \frac{1}{\alpha_{12}+1})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \frac{\alpha_{12}^2+1}{\alpha_{12}+1} + 2]}{(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}}\left[-\beta^2 + 2(1 + \alpha_{12})(\alpha_{12}^2 - \beta^2)^{\frac{1}{2}} + \alpha_{12}^2 + 4\alpha_{12} + 1\right] + 2\alpha_{12}} - \frac{1}{\alpha_{12}+1} \right|.$$

■

Appendix B

Proof of Proposition 2:

Assume A is a 3×3 positive reciprocal matrix $\begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ \frac{1}{\alpha_{12}} & 1 & \alpha_{23} \\ \frac{1}{\alpha_{13}} & \frac{1}{\alpha_{23}} & 1 \end{bmatrix}$, $\alpha_{12}, \alpha_{13}, \alpha_{23} \geq 1$. Its corresponding symmetric triangular fuzzy version $\bar{A} = (l_{ij}, m_{ij}, u_{ij})$ is $\begin{bmatrix} (1,1,1) & (\alpha_{12} - \beta, \alpha_{12}, \alpha_{12} + \beta) & (\alpha_{13} - \beta, \alpha_{13}, \alpha_{13} + \beta) \\ (\frac{1}{\alpha_{12}+\beta}, \frac{1}{\alpha_{12}}, \frac{1}{\alpha_{12}-\beta}) & (1,1,1) & (\alpha_{23} - \beta, \alpha_{23}, \alpha_{23} + \beta) \\ (\frac{1}{\alpha_{13}+\beta}, \frac{1}{\alpha_{13}}, \frac{1}{\alpha_{13}-\beta}) & (\frac{1}{\alpha_{23}+\beta}, \frac{1}{\alpha_{23}}, \frac{1}{\alpha_{23}-\beta}) & (1,1,1) \end{bmatrix}$. To solve $|A - \lambda I| = 0$, equivalently to solve the polynomial equation $(1-\lambda)^3 - 3(1-\lambda) + \frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} + \frac{\alpha_{13}}{\alpha_{12}\alpha_{23}} = 0.$

$\lambda_1 = (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} + (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}} + 1, \lambda_2 = -\frac{(1-i\sqrt{3})}{2}\left(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}\right)^{\frac{1}{3}} - \frac{(1+i\sqrt{3})}{2}\left(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}\right)^{-\frac{1}{3}} + 1,$

$\lambda_3 = -\frac{(1+i\sqrt{3})}{2}\left(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}\right)^{\frac{1}{3}} - \frac{(1-i\sqrt{3})}{2}\left(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}\right)^{-\frac{1}{3}} + 1.$ Thus $\lambda_{max} = (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} + (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}} + 1.$

The normalized weight w^N satisfies $(A - \lambda_{max}I)w^N = 0$, namely,

$$\begin{bmatrix} -(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} - (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}} & \alpha_{12} & \alpha_{13} \\ \frac{1}{\alpha_{12}} & -(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} - (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}} & \alpha_{23} \\ \frac{1}{\alpha_{13}} & \frac{1}{\alpha_{23}} & -(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} - (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}} \end{bmatrix} \begin{bmatrix} w_1^N \\ w_2^N \\ w_3^N \end{bmatrix} = 0.$$

Let a = $(\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{\frac{1}{3}} + (\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}})^{-\frac{1}{3}}$, then $w_1^N = \frac{a\alpha_{13}+\alpha_{12}\alpha_{23}}{a^2-1} w_3^N$, $w_2^N = \frac{a\alpha_{13}+\alpha_{12}\alpha_{23}+(a^2-1)\alpha_{12}\alpha_{23}}{\alpha_{12}a(a^2-1)} w_3^N$.

Because $w_1^N + w_2^N + w_3^N = 1$,

$\left(\frac{a\alpha_{13}+\alpha_{12}\alpha_{23}}{a^2-1} + \frac{a\alpha_{13}+\alpha_{12}\alpha_{23}+(a^2-1)\alpha_{12}\alpha_{23}}{\alpha_{12}a(a^2-1)} + 1\right) w_3^N = 1,$

$w_3^N = \frac{\alpha_{12}a(a^2-1)}{a^2\alpha_{12}\alpha_{13}+a\alpha_{12}^2\alpha_{23}+a\alpha_{13}+\alpha_{12}\alpha_{23}+(a^2-1)\alpha_{12}\alpha_{23}+\alpha_{12}a(a^2-1)} =$

$\frac{\alpha_{12}(a^2-1)}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}}$, thus,

$w_1^N = \frac{a\alpha_{12}\alpha_{13}+\alpha_{12}^2\alpha_{23}}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}},$

$w_2^N = \frac{\alpha_{13}+a\alpha_{12}\alpha_{23}}{\alpha_{12}a^2+(\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{23})a+\alpha_{12}^2\alpha_{23}+\alpha_{13}-\alpha_{12}}.$

As $l_1 = ((\alpha_{12}-\beta)(\alpha_{13}-\beta))^{\frac{1}{3}}$, $l_2 = \left(\left(\frac{1}{\alpha_{12}+\beta}\right)(\alpha_{23}-\beta)\right)^{\frac{1}{3}}$, $l_3 = \left(\left(\frac{1}{\alpha_{13}+\beta}\right)\left(\frac{1}{\alpha_{23}+\beta}\right)\right)^{\frac{1}{3}}$,

$u_1 = ((\alpha_{12}+\beta)(\alpha_{13}+\beta))^{\frac{1}{3}}$, $u_2 = \left(\left(\frac{1}{\alpha_{12}-\beta}\right)(\alpha_{23}+\beta)\right)^{\frac{1}{3}}$, $u_3 = \left(\left(\frac{1}{\alpha_{13}-\beta}\right)\left(\frac{1}{\alpha_{23}-\beta}\right)\right)^{\frac{1}{3}}$,

$m_1 = (\alpha_{12}\alpha_{13})^{\frac{1}{3}}$, $m_2 = \left(\frac{\alpha_{23}}{\alpha_{12}}\right)^{\frac{1}{3}}$, $m_3 = \left(\frac{1}{\alpha_{13}\alpha_{23}}\right)^{\frac{1}{3}}$,

$l = l_1 + l_2 + l_3$, $u = u_1 + u_2 + u_3$, $m = m_1 + m_2 + m_3$.

$$\widetilde{w}_1^N = \frac{\frac{l_1}{u}+\frac{m_1}{m}+\frac{u_1}{l}}{\frac{l}{u}+1+\frac{u}{l}} = \frac{l_1 l+u_1 u+\frac{m_1}{m}lu}{l^2+u^2+lu} = \frac{(\alpha_{12}-\beta)^{\frac{1}{3}}(\alpha_{13}-\beta)^{\frac{1}{3}}l+(\alpha_{12}+\beta)^{\frac{1}{3}}(\alpha_{13}+\beta)^{\frac{1}{3}}u+\frac{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{\alpha_{23}}{\alpha_{12}})^{\frac{1}{3}}+(\frac{1}{\alpha_{13}\alpha_{23}})^{\frac{1}{3}}}lu}{l^2+u^2+lu},$$

$$\widetilde{w}_2^N = \frac{\frac{l_2}{u}+\frac{m_2}{m}+\frac{u_2}{l}}{\frac{l}{u}+1+\frac{u}{l}} = \frac{l_2 l+u_2 u+\frac{m_2}{m}lu}{l^2+u^2+lu} = \frac{(\alpha_{12}+\beta)^{-\frac{1}{3}}(\alpha_{23}-\beta)^{\frac{1}{3}}l+(\alpha_{12}-\beta)^{-\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}u+\frac{(\frac{\alpha_{23}}{\alpha_{12}})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{\alpha_{23}}{\alpha_{12}})^{\frac{1}{3}}+(\frac{1}{\alpha_{13}\alpha_{23}})^{\frac{1}{3}}}lu}{l^2+u^2+lu}$$

Multiplying both denominator and nominator by $(\alpha_{12}+\beta)^{\frac{1}{3}}(\alpha_{12}-\beta)^{\frac{1}{3}}$,

$$\widetilde{w}_2^N=\frac{(\alpha_{12}-\beta)^{\frac{1}{3}}(\alpha_{23}-\beta)^{\frac{1}{3}}l+(\alpha_{12}+\beta)^{\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}u+\frac{\left(\alpha_{12}^2-\beta^2\right)^{\frac{1}{3}}(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}+(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}lu}{(\alpha_{12}^2-\beta^2)^{\frac{1}{3}}(l^2+u^2+lu)}.$$

Similarly,

$$\widetilde{w}_3^N=\frac{\frac{l_3}{u}+\frac{m_3}{m}+\frac{u_3}{l}}{\frac{l}{u}+1+\frac{u}{l}}=\frac{l_3l+u_3u+\frac{m_3}{m}lu}{l^2+u^2+lu}=\frac{(\alpha_{13}-\beta)^{\frac{1}{3}}(\alpha_{23}-\beta)^{\frac{1}{3}}l+(\alpha_{13}+\beta)^{\frac{1}{3}}(\alpha_{23}+\beta)^{\frac{1}{3}}u+\frac{\left(\alpha_{13}^2-\beta^2\right)^{\frac{1}{3}}\left(\alpha_{23}^2-\beta^2\right)^{\frac{1}{3}}(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}{(\alpha_{12}\alpha_{13})^{\frac{1}{3}}+(\frac{a_{23}}{a_{12}})^{\frac{1}{3}}+(\frac{1}{a_{13}a_{23}})^{\frac{1}{3}}}lu}{(\alpha_{13}^2-\beta^2)^{\frac{1}{3}}(\alpha_{23}^2-\beta^2)^{\frac{1}{3}}(l^2+u^2+lu)}.$$

■

Appendix C

Proof of Proposition 3:

For n=4, assume any 4×4 positive reciprocal matrix A=(α_{ij}). Let $|A-\lambda I|=0$, a 4 power polynomial equation for λ is obtained: $(1-\lambda)^4-4(1-\lambda)^2+\left(\frac{\alpha_{34}\alpha_{23}}{\alpha_{24}}+\frac{\alpha_{24}}{\alpha_{34}\alpha_{23}}+\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}+\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}+\frac{\alpha_{13}\alpha_{34}}{\alpha_{14}}+\frac{\alpha_{14}}{\alpha_{12}\alpha_{24}}\right)(1-\lambda)-\frac{\alpha_{12}\alpha_{23}\alpha_{34}}{\alpha_{14}}-\frac{\alpha_{13}\alpha_{34}}{\alpha_{12}\alpha_{24}}-\frac{\alpha_{24}\alpha_{13}}{\alpha_{23}\alpha_{14}}-\frac{\alpha_{14}}{\alpha_{12}\alpha_{23}\alpha_{34}}+2=0$. Solved by Mathematica 9, we have: $\lambda_{1,2}=1-\frac{1}{2}\sqrt{a}\pm\frac{1}{2}\sqrt{8-a-\frac{2b}{\sqrt{a}}}$, $\lambda_{3,4}=1+\frac{1}{2}\sqrt{a}\pm\frac{1}{2}\sqrt{8-a+\frac{2b}{\sqrt{a}}}$, where,

$\text{a}=\frac{8}{3}+\frac{42^{1/3}(4+3c)}{3(-128+27b^2+288c+3\sqrt{3}\sqrt{-256b^2+27b^4-4096c+576b^2c+2048c^2-256c^3})^{1/3}}+\frac{(-128+27b^2+288c+3\sqrt{3}\sqrt{-256b^2+27b^4-4096c+576b^2c+2048c^2-256c^3})^{1/3}}{32^{1/3}}$,

where, $\text{b}=\frac{\alpha_{34}\alpha_{23}}{\alpha_{24}}+\frac{\alpha_{24}}{\alpha_{34}\alpha_{23}}+\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}}+\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}+\frac{\alpha_{13}\alpha_{34}}{\alpha_{14}}+\frac{\alpha_{14}}{\alpha_{12}\alpha_{24}}$, $\text{c}=-\frac{\alpha_{12}\alpha_{23}\alpha_{34}}{\alpha_{14}}-\frac{\alpha_{13}\alpha_{34}}{\alpha_{12}\alpha_{24}}-\frac{\alpha_{24}\alpha_{13}}{\alpha_{23}\alpha_{14}}-\frac{\alpha_{14}}{\alpha_{12}\alpha_{23}\alpha_{34}}+2$.

Because λ_{max} cannot be decided in this stage. Thus w^N cannot be expressed in closed-form. Therefore, $\text{F}(\beta)$ cannot be expressed in closed form.

For the cases $\text{n}\geq 5$, no closed form can be found for λ_s satisfying $|\text{A}-\lambda\text{I}|=0$, whose highest power is greater than equal to 5 (Galois theory).

■

Appendix D

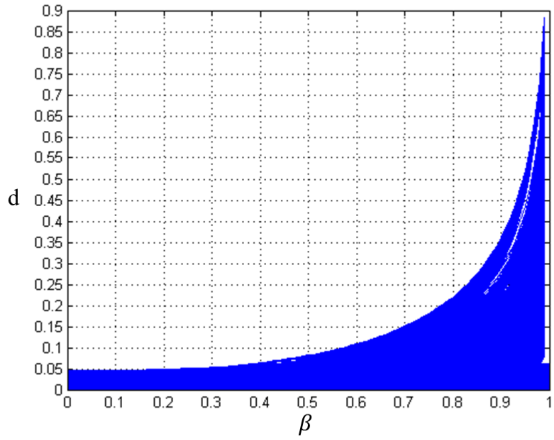

Fig. 11. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{9}$

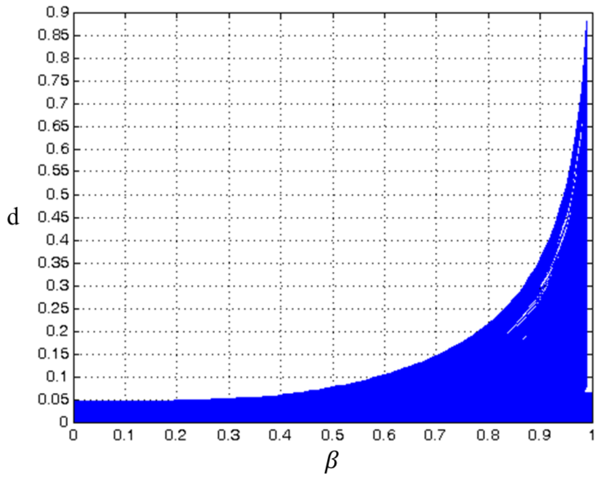

Fig. 12. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{8}$

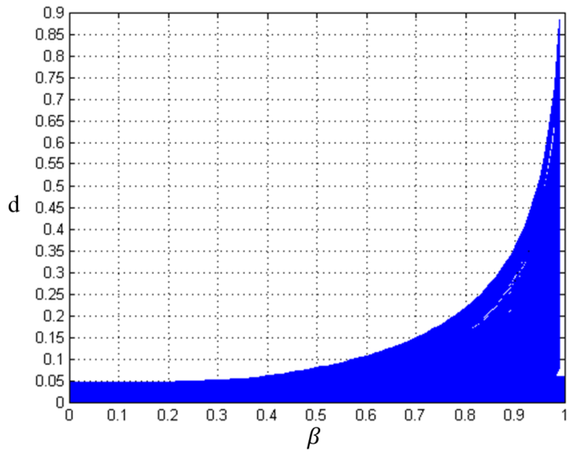

Fig. 13. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{7}$

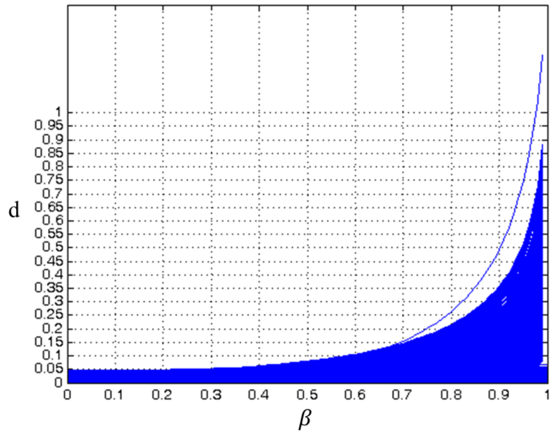

Fig. 14. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{6}$

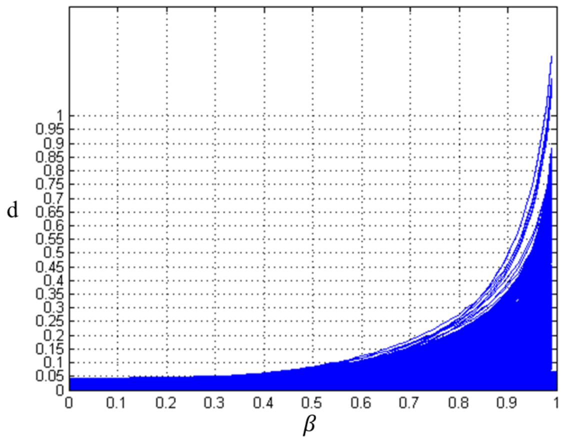

Fig. 15. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{5}$

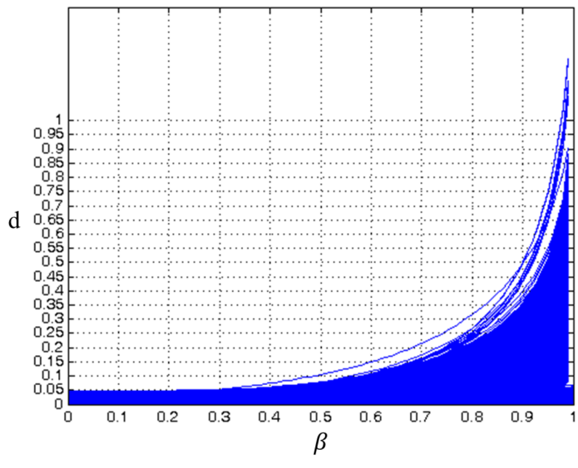

Fig. 16. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{4}$

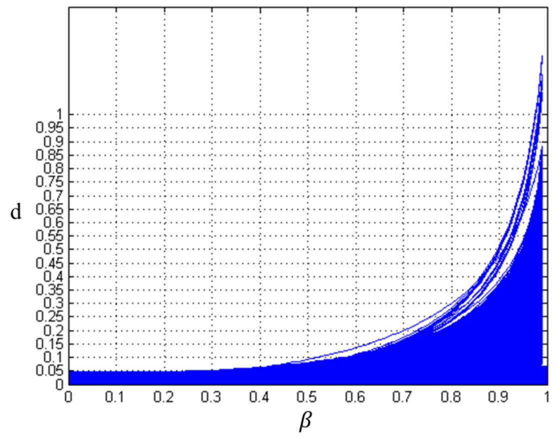

Fig. 17. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{3}$

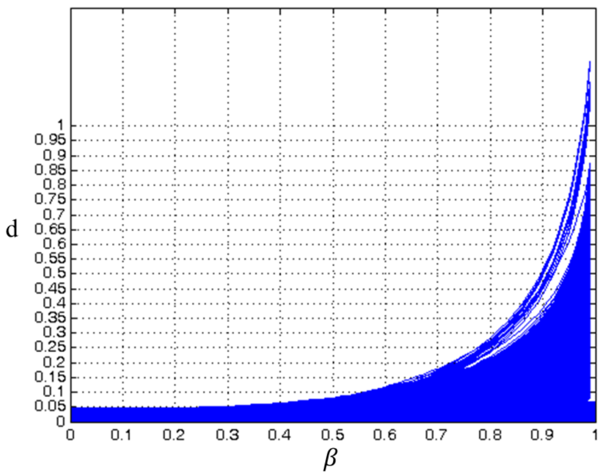

Fig. 18. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = \frac{1}{2}$

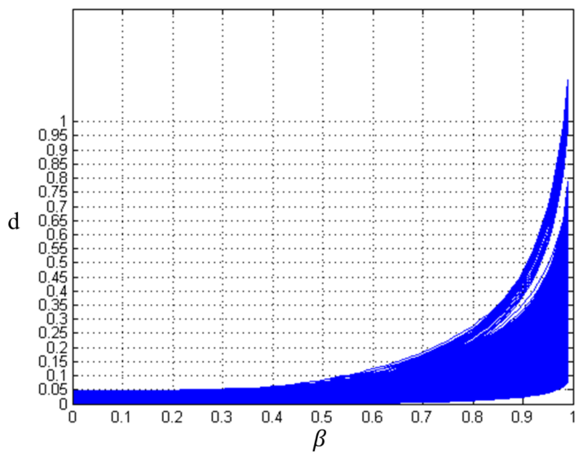

Fig. 19. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 1$

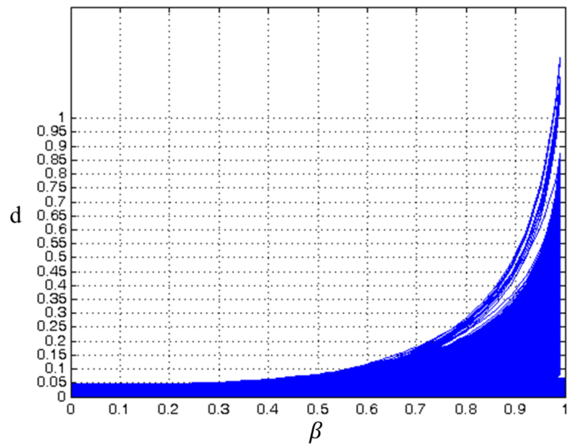

Fig. 20. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 2$

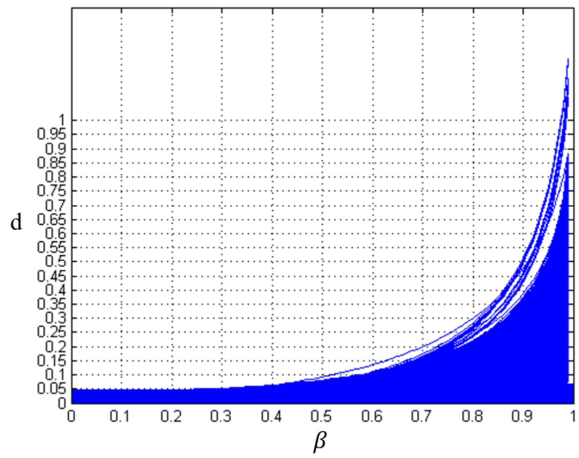

Fig. 21. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 3$

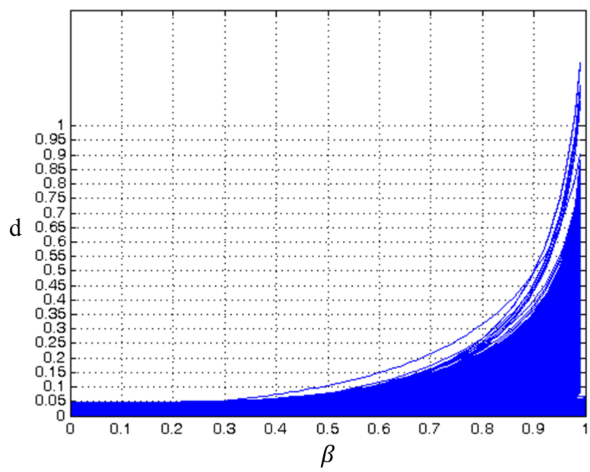

Fig. 22. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 4$

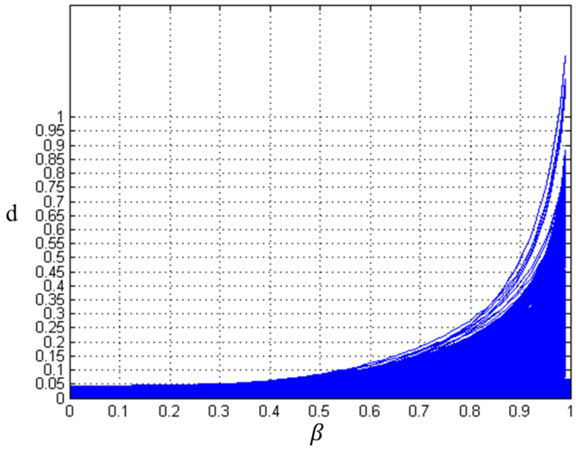

Fig. 23. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 5$

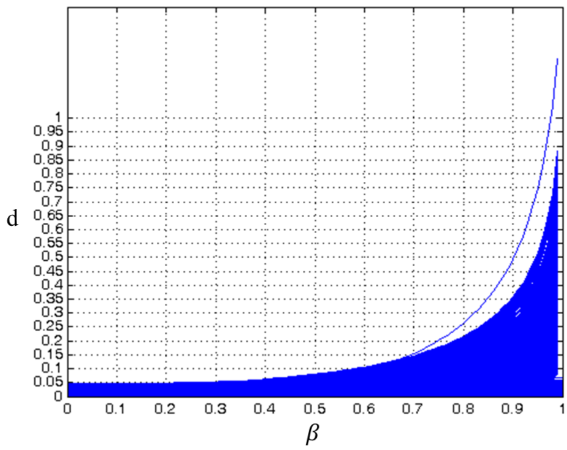

Fig. 24. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 6$

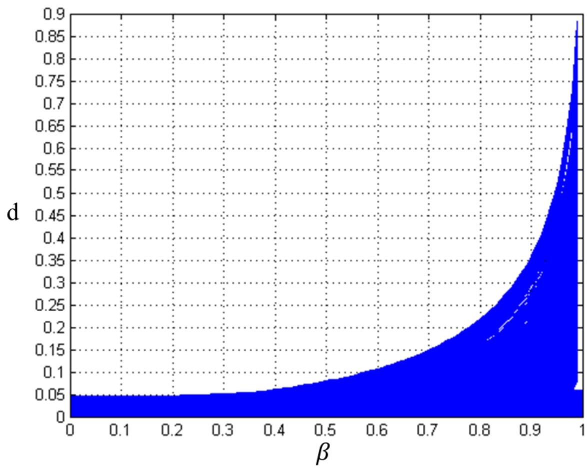

Fig. 25. Comparison between the weights generated the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 7$

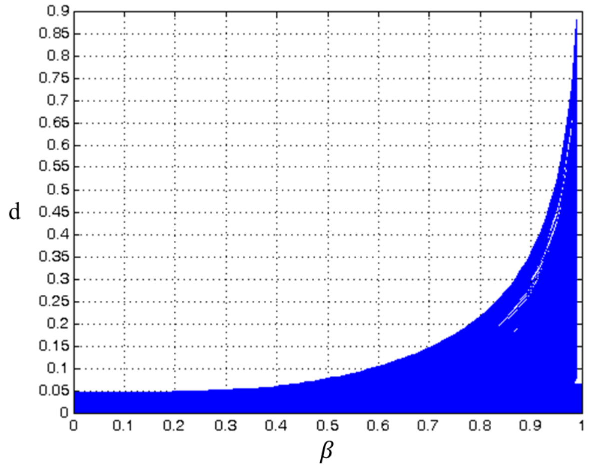

Fig. 26. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 8$

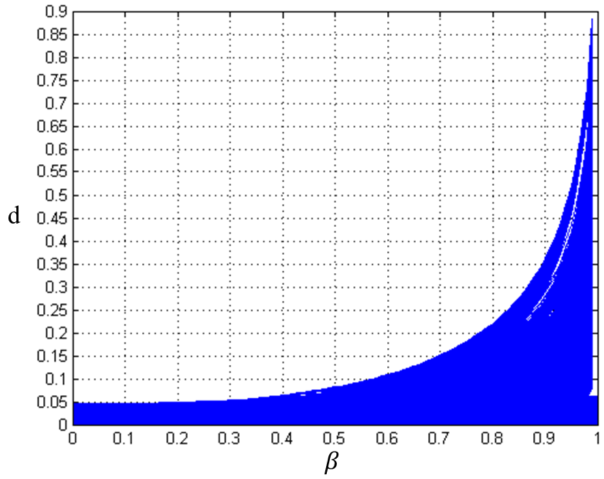

Fig. 27. Comparison between the weights generated by the FAHP and the AHP when matrix size n = 4 for the consistent cases where all the parameters are exhaustively tested except $\alpha_{12} = 9$

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