SENSITIVITY-BASED UNCERTAINTY ANALYSIS OF TRANSIT ASSIGNMENT MODEL

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ABSTRACT

Transit assignment models are particularly useful to systematically access the operation or planning of a transit network. From this perspective, the transit assignment model has been developed and applied for a long time, and various of formulations come out to fulfill different application purposes. In order to analyze the uncertainties in transit network which may be caused by probabilistic travel demand, congestion effect, or transit frequencies, a variational inequality (VI) formulation is employed in this study. It models the passengers' route choice subject to congestion from waiting and in-vehicle time. The hyperpath concept is used to incorporate the passengers' strategy at transit stops. Then, the analytical sensitivity analysis expression is derived for the transit assignment model. Some typical applications of sensitivity analysis results will be discussed in numerical example. Especially, the uncertainties from the travel demand are considered as the random variables with mean and variance. Based on the analytical sensitivity analysis, the uncertainties in the result of transit assignment model will be estimated.

Keywords: Transit assignment model; sensitivity analysis; uncertainty; variational inequality

1. INTRODUCTION

Sensitivity analysis is a useful tool for assessing how changes on the model inputs or parameters affect the outputs. It has received considerable attention in regional traffic control and network operations applications (see Yang et al., 2009, etc.). The primary objective of employing sensitivity analysis is to quantify the rate of change in link flows and/or route travel costs as a function of changes in the link cost and demand functions. The quantitative results from sensitivity analysis can be utilized it to quantify goal fulfillment, such as reducing traffic flows for access control, or decreasing travel times on congested links. However, there is few studies focus on the sensitivity analysis of the transit network models. Lee et al. (2005) conducted some sensitivity-based numerical experiments considering changes on demand, link travel times, and transfer penalties, but they did not derive the analytical sensitivity expressions. Gao et al. (2004) derived the sensitivity expressions of a transit assignment model with route-section representation. In their study, the sensitivity analysis is only used to generate the derivatives for the solution of a bi-level transit network design problem. With the route-section representation, the transit assignment problem can transform into a form, which is similar to the car network, so that the solution and sensitivity analysis methods for car networks can be adopted. Nevertheless, this transformed representation cannot avoid unreasonable transfer problems (Jiang and Szeto, 2016). Although Jiang and Szeto (2016) provided a modified representation to deal with such issue, it may expand the magnitude of the transit network a lot. Besides, to transform the network representation still need extra efforts. In contrast, the hyperpath representation is another typical method that has been widely used in transit network models. It gives a good solution to the "common line" issue in transit networks. Considering its merits on representing the arc cost interactions, our study derives the sensitivity analysis method for the transit assignment model based on the hyperpath representation.

The transit assignment problem has been developed for almost thirty years. de Cea and Fernández (1993) presented a user equilibrium (UE) assignment model for the transit assignment problem on congested systems, in which the "transit route" and the "effective frequency" were introduced. The discomfort cost function of each transit link used in their paper is assumed to be an increasing convex volume function for modeling of link congestion. Wu et al. (1994) proposed an approach to the formulation of the transit UE assignment problem. Their approach extended Spiess and Florian's (1989) nonlinear model to *asymmetric* cost function, when formulate the transit equilibrium assignment problem (TEAP). The model considers the waiting and in-vehicle travel costs as functions of the transit flows. An arc-hyperpath incidence matrix is used to connect the arc flows and hyperpath flows. Also, the optimal strategy (Spiess and Florian, 1989), by hyperpath, is employed to model the frequency-based transit assignment. In this study, the transit equilibrium model and its sensitivity analysis are presented based on Wu et al. (1994). The next section introduces the VI formulation and algorithm for the TEAP. Based on the VI formulation, the sensitivity analysis method is derived in section 3. Section 4 presents the sensitivity-based uncertainty analysis method. After that, numerical experiments are conducted in section 5 to demonstrate the equilibrium of hyperpath flows, the sensitivity analysis and uncertainty analysis for the TEAP. Section 6 includes conclusions of this study.

2. EQUILIBRIUM ASSIGNMENT IN TRANSIT NETWORK

2.1 Notations and Relationships

This section describes the basic ideas of transit assignment. A transit network is denoted by a graph $G(N, A)$, where N is a set of nodes, and A is a set of arcs. The nodes include the stop nodes, intermediate line nodes, and centroids, while the arcs include the transit line segments (each segment is associated with an in-vehicle arc, a wait arc, and an alight arc) and walk arcs. Each arc $a \in A$ has two attributes, t_a and f_a, representing the travel time and the frequency of arc a, respectively. Figure 1 illustrates a subgraph of one node which contains all four types of transit arcs: (1) in-vehicle arcs denote the segment of the transit lines between two sequential nodes along the transit routes; (2) wait arcs model the waiting of passengers for the associated transit lines; (3) alight arcs connect transit line segments to the stop nodes where passengers can approach to destination centroid or transfer to other lines. (4) walk arcs are from zone centroids to the stop nodes of the transit network.

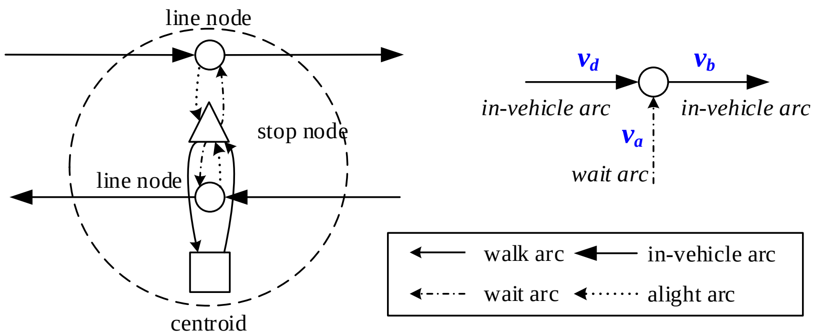

Figure 1. Representation of transit network with decision variables

Furthermore, to consider the congestion effect, the travel time on a transit arc is denoted as the function of the flows on related arcs. Here, following Wu et al. (1994), let $s_a(v)$ be the (general) travel cost on arc a. The arc cost is not only dependent on the flow on itself but also affected by the flows of other related arcs. Thus, corresponding the four types of transit arcs, the following asymmetric functions were defined (Wu et al., 1994):

- Walk arcs: $s_a = \alpha_1 t_a$
- Wait arcs: $s_a(v_a, v_d) = \alpha_2 \left[\frac{v_a+\beta_2 v_d}{K_b}\right]^{\rho}$
- In-vehicle arcs: $s_b(v_a, v_b) = \alpha_3 t_b + \beta_3 \left[\frac{v_b+(\gamma_3-1)v_a}{K_b}\right]^{\rho}$
- Transfer/alight arcs: $s_a = \alpha_4 t_a$

where t_a can be referred as the travel time without congestions, i.e., free-flow time. α, β, γ and ρ are parameters for calibration. K_b is the capacity of the corresponding line, which can be equal to "time period considered $\times$ bus frequency $\times$ bus capacity". Note that the costs of wait arcs and in-vehicle arcs are also dependent on the flows of the adjacent transit arcs. The notations v_a, v_b and v_d are respectively the boarding flow, the upstream in-vehicle flow and the downstream in-vehicle flow, as Figure 1 shows. Also, the flow conservation is given by $v_b = v_a + v_d$.

Unlike to car network, same interstation sections may be served by more than one transit lines, which is referred as the "common line" problem in transit assignments (Spiess and Florian, 1989). To deal with this problem, the concept of hyperpath is employed. A hyperpath is a directed acyclic graph with a flow distribution rule (Nyuyen et al, 1988). It allows the passengers make decisions as any combination of available transit routes, which can minimize the passengers' travel cost. To formulate the TEAP, the following notations are defined in advance.

S_k the cost of a hyperpath *k*;
h_k flow along hyperpath *k*;
$F^k_{tail(a)}$ all the arcs of the hyperpath *k* that have *tail*(*a*) as a tail node, where *tail*(*a*) is the tail node of arc *a*;
w_a the time that the passenger waits until boarding the line *a*, and $w_k = 1/f_a$;
w_{ak} the time that the passenger waits until boarding a bus, and $w_{ak} = 1/\sum_{a' \in F^k_{tail(a')}} f_{a'}$;
p^k_a the probability that the passenger uses arc *a*: $p^k_a = \frac{f_a}{\sum_{a' \in F^k_{tail(a')}} f_{a'}}$;
$\alpha^k_{t(a)}$ the proportion of the flow arriving from *p* at the tail node *t*(*a*) of arc *a*;
δ^k_a the proportion of the flow of the hyperpath *k* assigned to the arc *a*;
δ is the arc-hyperpath incidence matrix with element δ^k_a;
Λ is the OD-hyperpath incidence matrix.

Based on the above notations, the travel cost of a hyper path can be represented as:

$$S_k = \sum_{a \in A} \delta^k_a [s_a(v) + w_{ak}]. \tag{1}$$

In vector form,

$$S_k = (\delta^k)^T [s(v) + w^k]. \tag{2}$$

Let $W_k = \sum_{a \in A} \delta_{ak} w_{ak} = (\delta^k)^{\mathrm{T}} w^k$, which is the waiting time for hyperpath *k*. Besides, the flow distribution rule among the hyperpaths between a same section is frequency-based, of which the probability that choosing a transit line is given by p^k_a.

2.2 VI formulation of the TEAP

According to equations (1) and (2), the cost of hyperpath is flow-dependent. This indicates that as the travel demand increasing in the transit network, the original optimal hyperpath may become non-optimal. Therefore, the network equilibrium can be used to model the route choice in transit system for consideration of the congestion effect. By assuming the passenger flows on the transit network satisfy Wardrop's user equilibrium conditions, all passengers will travel along the minimum hyperpaths connected the origin-destination (O-D) pairs. According to Wu et al. (1994), the user equilibrium conditions in transit network can be stated as the following variational inequality (VI) problem:

VI(1): $S(h^*)^T(h - h^*) \geq 0, \forall h \in \Omega$
where $\Omega = \{h | \Lambda h = g, h \geq 0\}$
or
VI(2): $s(v^*)^T(v - v^*) + W^T(h - h^*) \geq 0, \forall v \in \Omega_v \; and \; \forall h \in \Omega$
where $\Omega_v = \{v | v = \delta h, for \; h \in \Omega\}$

Also, from Theorem 2 in Wu et al. (1994), $v^* \; and \; W^T h^* \; are \; unique \; if \; s(v)$ is strictly monotone. *W* is a vector of $\{\ldots, W_k, \ldots\}$. Similar to Tobin and Friesz (1988), the equilibrium hyperpath flows are not unique and are contained in the convex polytope $\Gamma^* = \{h | \delta h = v^*, \Lambda h = g, h \geq 0\}$, where v^* is the unique solution of the VI problem (2) above. Note that the TEAP has one special characteristic which differentiates it from other VI problems: it cannot be transformed into a VI problem in the space of arc flows only.

2.3 Solution Algorithm for TEAP

From Wu et al. (1994), the solution algorithm for TEAP are provided as follows.

Step 1 Initialization.
Let ϵ be a tolerance parameter. Let K_w^0 be a subset of K_w for $w \in W$.
Let h^0 and v^0 be an initial feasible solution. Set *n* := 1.
Step 2 Compute Shortest Hyperpath.
For each OD pair w, based on the current cost $s(v^{n-1})$, compute a shortest hyperpath k_w, and compute $GAP(h^{n-1})$. Let $K_w^n = K_w^{n-1} \cup k_w$.
Step 3 Convergence Test.
If $GAP(h^{n-1}) \leq \epsilon$ then stop.
***Step 4 Compute* S(h^{n-1}) *and* B(h^{n-1}).**
Based on $s(v^{n-1})$, find $S_k(h^{n-1})$ and $B_k(h^{n-1})$ for $k \in K_w^n$.
B(h^{n-1}) is the diagonal of the Jacobian matrix of **S** evaluated at h^{n-1}.
Step 5 Projection.
Solve the following quadratic program and let h^n be its solution
$$\min_{h \in \Omega}(h - h^n)^T S(h^n) + \frac{1}{2\alpha}(h - h^n)^T B(h^n)(h - h^n),$$
where $\Omega = \{h | \Lambda h = g, h \geq 0\}$, and α a positive parameter.
Step 6 Update.
Compute $v^n = \delta h^n$ and update $s(v^n)$. Set *n* := *n* + 1 and return to step 2.

3. DERIVATION OF SENSITIVITY ANALYSIS FOR TRANSIT ASSIGNMENT

As the Kuhn-Tucker condition, for any optimal solution h^*, let the perturbations $\epsilon = 0$ and obtain:
$$S(h^*, 0) - \pi - \Lambda^T \mu = 0$$
$$\pi^T h^* = 0$$
$$\Lambda h^* - g(0) = 0$$
$$\pi \geq 0$$
When the problem is restricted to the hyperpath with only positive flows, i.e., the corresponding Lagrange multiplier $\pi = 0$. Thus, we obtain
$$S(h^*, 0) - \Lambda^T \mu = 0$$
$$\Lambda h^* - g(0) = 0$$
where $S = \delta^T[s(v) + w]$; δ is the incidence matrix of arcs and hyperpaths. Reduce the solutions of the hyperpath flows by setting the non-basic variables of the above equation system. Note that the non-basic variables can be selected according to a maximum set of linearly independent columns of the incidence matrix $\begin{bmatrix}\delta \\ \Lambda\end{bmatrix}$. By selecting the independent hyperpaths only, we have
$$S^0(h^*, 0) - \Lambda^{0T}\mu = 0,$$
$$\Lambda^0 h^{0*} - g(0) = 0.$$
Thus, to this extent, we can apply the *implicit function theorem* to the above restricted equation system. Taking further derivatives with respect to (h, μ) and ϵ, respectively, we obtain
$$J_h = \begin{bmatrix}\nabla_h S^0(h^*, 0) & -\Lambda^{0T} \\ \Lambda^0 & 0\end{bmatrix}, \text{ and } J_\epsilon = \begin{bmatrix}\nabla_\epsilon S^0(h^*, 0) \\ -\nabla_\epsilon g(0)\end{bmatrix},$$
where $\nabla_h S^0(h^*, 0) = \delta^{0T} \nabla_v s(v) \delta^0$, and $\nabla_\epsilon S^0(h^*, 0) = \delta^{0T}(\nabla_\epsilon s(v, \epsilon) + \nabla_\epsilon w(\epsilon))$.
Thus,
$$\begin{bmatrix}\nabla_\epsilon h^0 \\ \nabla_\epsilon \mu^0\end{bmatrix} = [J_h]^{-1} \cdot [-J_\epsilon].$$
The above equation provides the derivatives of solution variables (h, μ) with respect to the perturbations, only if the Jacobian matrix J_h is invertible. The invertibility of J_h can be proved as Du et al. (2012). Moreover, the derivatives of arc flows can be given by
$$\nabla_\epsilon v = \delta \cdot \nabla_\epsilon h^0 \tag{3}$$

4. SENSITIVITY-BASED UNCERTAINTY ANALYSIS

The aims of modelling the TEAP is to forecast the transit network conditions given the value of model inputs and parameters. In practice, these inputs and parameters are measured or calibrated with errors. Traditional forecasting methods may use the mean of the inputs and parameters to produce a point estimate of the outputs. However, if considering the errors associated with a model, it is reasonable to consider how to achieve a proper level of precision in forecasts. Therefore, rather than the mean of the outputs, the variability of outputs (i.e., standard deviation) is also needed.

Based on the sensitivity analysis results, it is possible to estimate the variance of the model outputs, which is known as the sensitivity-based uncertainty analysis. The first-order Taylor's approximation is used to derive the variance of the outputs for a given set of input distributions. Consider an input error ϵ with a standard deviation σ_ϵ, to find the output error derived from the propagation of input errors in a function $y(\epsilon)$, the below equation will be used

$$\sigma_y^2 = \nabla_\epsilon y \cdot \sigma_\epsilon^2 \cdot \nabla_\epsilon y^T \tag{4}$$

where σ_y^2 is the variance of output error; and $\nabla_\epsilon y$ can be obtained from sensitivity analysis for the travel demand model.

5. NUMERICAL EXPERIMENTS

5.1 Equilibrium Results

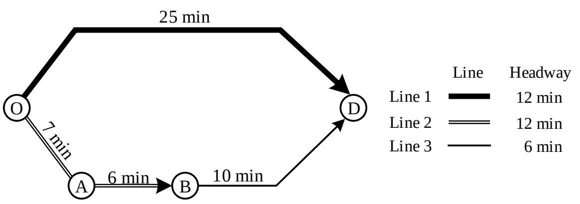

Figure 2. Three lines transit network

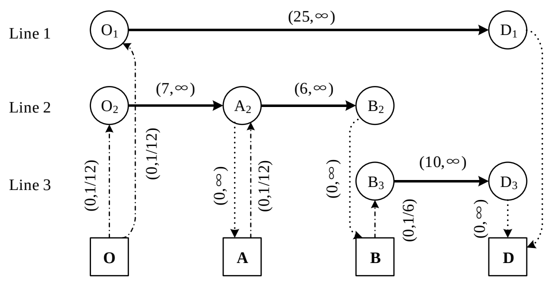

Figure 3. Arc representation of the example network

In this section, a simple transit network with three transit lines are considered as Figure 2. There is only one O-D pair from node O to node D. The original network can be represented as the augmented transit network in Figure 3, where all types of transit arcs are illustrated. The characteristics of these arcs are listed in Table 1. In this example, we specified the arc cost functions as follows (as there is no walk arc in this example).

- Wait arcs: $s_a(v_a, v_d) = \left[\frac{v_a + 0.2 v_d}{K_h}\right]^2$, where d is the downstream in-vehicle arc;
- In-vehicle arcs: $s_b(v_a, v_b) = t_b + \left[\frac{v_b + 0.2 v_a}{K_h}\right]^2$, where a is the upstream wait arc;
- Alight arcs: $s_4 = 0$.

The capacity of an in-vehicle arc is given by $K_b = 1 \times \bar{C}_{bus} \times f_a$, where '1' is the period of one hour; $\bar{C}_{bus}$ is the average capacity of a bus vehicle.

Table 1. Characteristics of Transit Arcs

Type	Arcs	(Tail node, Head node)	Free Flow Time t_a	Frequency f_a
Wait arc	1	(O, O_1)	0	1/12
	2	(O, O_2)	0	1/12
	3	(B, B_3)	0	1/6
In-vehicle arc	4	(O_1, D_1)	25	∞
	5	(O_2, A_2)	7	∞
	6	(A_2, B_2)	6	∞
	7	(B_3, D_3)	10	∞
Alight arc	8	(B_2, B)	0	∞
	9	(D_1, D)	0	∞
	10	(D_3, D)	0	∞

In this example, three hyperpaths can be identified from node O to node D:

- Hyperpath 1: O→O_1→D_1→D
- Hyperpath 2: O→O_2→A_2→B_2→B→B_3→D_4→D
- Hyperpath 3: {Hyperpath 1 and 2}

Thus, the arc-hyperpath incidence matrix δ can be given by

$$\underbrace{3\ hyperpath\left\{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}^{\mathrm{T}}\right.}_{10\ arcs}$$

By using the solution algorithm presented in section 2.3, the equilibrium flows can be obtained. When the O-D travel demand is 600 passenger/hour, the equilibrium hyperpath flows are h^* = (33.56, 0, 566.44), and the equilibrium arc flows are v^* = (316.78, 283.22, 283.22, 316.78, 283.22, 283.22, 283.22, 283.22, 316.78, 283.22).

Since the equilibrium of hyperpath flows is more complex than the equilibrium of simple path flows in car network, we demonstrate how the travel demand distributes among the hyperpaths in transit networks. When the travel demand from node O to node D is set to 600 passenger/hour, the equilibrium hyperpath solution is obtained as h^* = (33.56, 0, 566.44). The costs of the three hyperpaths are 61.49, 73.49, and 61.49. In this case, the Hyperpath 1 and 3 are the minimum hyperpaths and are used by travelers. The Hyperpath 2 is unused as its cost is larger than the others'. Hence, the solution follows the Wardrop's first principle. Figure 4 illustrates how the costs of three hyperpaths change by increasing the O-D demand from 0 to 800. When the demand is lower than 445.8, the cost of Hyperpath 3 is the only optimal strategy between O-D. When the demand is greater than 445.8, the costs of Hyperpath 1 and 3 will be same, and Hyperpath 1 will take flows. Since Hyperpath 3 is essentially a combination of Hyperpath 1 and 2, the cost of Hyperpath 2 should be always larger that that of Hyperpath 3, which means the Hyperpath 2 will not take flows in any case.

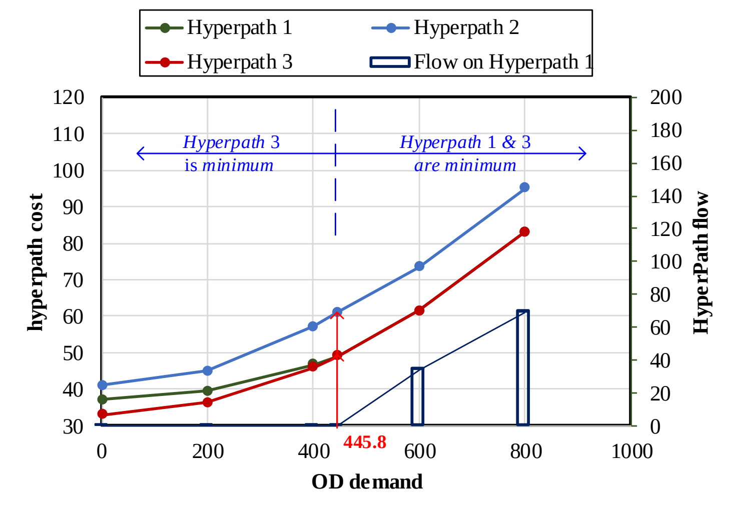

Figure 4. The hyperpath equilibrium costs versus OD demand

5.2 Sensitivity Analysis Results

In this section, the sensitivity analysis of TEAP are derived based on the equilibrium solution at which the O-D demand is equal to 600. According to the equilibrium solution, only Hyperpath 1 and 3 are the used hyperpaths, so we only preserve these two hyperpath in the reduced problem. Thus, only the first and the third columns of δ are kept. In addition, as the matrix $\begin{bmatrix}\delta \\ \Lambda\end{bmatrix}$ is of full column rank, all the used hyperpaths are preserved in the restricted problem. Then, follows the sensitivity analysis procedures in section 3, the derivatives of outputs with respect to the inputs and parameters can be obtained. The outputs of the TEAP are the hyperpath flows, **h**, and the Lagrange multiplier **μ**, which are denoted as **y** = (**h, μ**). In this experiment, we select the free flow time t_a^0, the frequency of bus service f_a, and the O-D travel demand g_{od} as the model inputs with perturbation; select $\beta_2, \beta_3, \gamma_3$ and $\bar{C}_{bus}$ as the parameters with perturbation. The entire sensitivity analysis results are reported in Table 2 and 3. The derivatives of arc flows are computed by equation (3).

Table 2. Derivatives of Outputs with respect to Free Flow Times, Frequencies and Demand.

Outputs	$\partial(\cdot)/\partial t_4^0$	$\partial(\cdot)/\partial t_5^0$	$\partial(\cdot)/\partial t_6^0$	$\partial(\cdot)/\partial t_7^0$	$\partial(\cdot)/\partial f_1$	$\partial(\cdot)/\partial f_2$	$\partial(\cdot)/\partial f_3$	$\partial(\cdot)/\partial g_{od}$
h_1	-0.0500	0.0500	0.0500	0.0500	0.2044	-0.0875	-0.0324	0.2908
h_3	0.0500	-0.0500	-0.0500	-0.0500	-0.2044	0.0875	0.0324	0.7092
μ	0.6454	0.3546	0.3546	0.3546	-1.4395	-0.6212	-0.2301	9.1593
v_1	-0.0250	0.0250	0.0250	0.0250	0.1022	-0.0438	-0.0162	0.6454
v_2	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_3	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_4	-0.0250	0.0250	0.0250	0.0250	0.1022	-0.0438	-0.0162	0.6454
v_5	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_6	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_7	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_8	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546
v_9	-0.0250	0.0250	0.0250	0.0250	0.1022	-0.0438	-0.0162	0.6454
v_{10}	0.0250	-0.0250	-0.0250	-0.0250	-0.1022	0.0438	0.0162	0.3546

Table 3. Derivatives of Outputs with respect to Parameters.

Outputs	$\partial(\cdot)/\partial\beta_3$	$\partial(\cdot)/\partial\gamma_3$	$\partial(\cdot)/\partial\rho$	$\partial(\cdot)/\partial\bar{C}_{bus}$
h_1	0.4003	104.1228	0.1096	-0.0056
h_3	-0.4003	-104.1228	-0.1096	0.0056
μ	17.2905	3147.2316	31.6451	-0.5291
v_1	0.2001	52.0614	0.0548	-0.0028
v_2	-0.2001	-52.0614	-0.0548	0.0028
v_3	-0.2001	-52.0614	-0.0548	0.0028
v_4	0.2001	52.0614	0.0548	-0.0028
v_5	-0.2001	-52.0614	-0.0548	0.0028
v_6	-0.2001	-52.0614	-0.0548	0.0028
v_7	-0.2001	-52.0614	-0.0548	0.0028
v_8	-0.2001	-52.0614	-0.0548	0.0028
v_9	0.2001	52.0614	0.0548	-0.0028
v_{10}	-0.2001	-52.0614	-0.0548	0.0028

One popular usage of the sensitivity information is to estimate the equilibrium solutions without the need to resolve the transportation forecasting model. One can use the Taylor series approximation to estimate the equilibrium solutions under small perturbations on the input variables and/or parameters. For demonstration purpose, perturbations on the O-D demand are selected to demonstrate that using the sensitivity results to estimate perturbed solution. Table 4 shows the exact and estimated solutions for the perturbations of $\delta g_{od} = 30$.

5.3 Uncertainty Analysis

Following Yang et al. (2009), in this numerical example, we assume the demand $\tilde{g}_{od}$ is associated with an error term ξ, and $\tilde{g}_{od} = g_{od} + \xi$. If $\xi \sim N(0, \sigma_\xi^2)$ and $\sigma_\xi^2 = 25$, then $\tilde{g}_{od} \sim N(600, 25)$. By using the linear approximation based on sensitivity analysis, the TEAP is solved only once using the mean value of $\tilde{g}_{od}$. Sensitivity analysis is conducted to obtain the gradient of link flows with respect to g_{od} (see results in Table 2). Using Eq. (4), the standard deviations of arc flows can be calculated. By assuming normality, the confidence levels are calculated. Table 5 lists the 90% confidence intervals of arc flows using the sensitivity-based uncertainty analysis. Considering the uncertainty in model outputs due to parameter uncertainty, it is not appropriate to use only the expected values of outputs to evaluate a network improvement scheme. With the standard deviations and confidence intervals of outputs, we can conduct hypothesis tests to obtain a statistically significant evaluation of a network improvement scheme.

Table 4. Estimated and Exact Solutions for Perturbed O-D Demand

	Unperturbed solution	δg_{od} = 30 (i.e. 600×5%)		
		Exact	Estimated	Difference
v_1	316.78	338.87	336.14	2.73 (0.81%)
v_2	283.22	298.22	293.86	4.36 (1.48%)
v_3	283.22	298.22	293.86	4.36 (1.48%)
v_4	316.78	338.87	336.14	2.73 (0.81%)
v_5	283.22	298.22	293.86	4.36 (1.48%)
v_6	283.22	298.22	293.86	4.36 (1.48%)
v_7	283.22	298.22	293.86	4.36 (1.48%)
v_8	283.22	298.22	293.86	4.36 (1.48%)
v_9	316.78	338.87	336.14	2.73 (0.81%)
v_{10}	283.22	298.22	293.86	4.36 (1.48%)

Table 5. Variance and Confidence Interval of Arc Flows

	Variances	5% percentile	95% percentile
v_1	260.34	285.16	348.40
v_2	78.59	265.84	300.60
v_3	78.59	265.84	300.60
v_4	260.34	285.16	348.40
v_5	78.59	265.84	300.60
v_6	78.59	265.84	300.60
v_7	78.59	265.84	300.60
v_8	78.59	265.84	300.60
v_9	260.34	285.16	348.40
v_{10}	78.59	265.84	300.60

6. CONCLUSIONS

This study developed an analytical sensitivity analysis approach for the transit equilibrium model. The sensitivity results are further used to analyze the uncertainties in transit network from probabilistic travel demand, congestion effect, or transit frequencies. A variational inequality (VI) formulation is employed, by which the influences from related arcs are modeled in the cost of each transit arc. The hyperpath representation is used to incorporate the passengers' route choices. From the numerical experiments, the hyperpath equilibrium is illustrated by changing the O-D demand. The applications of sensitivity analysis are discussed. Especially, the uncertainties from the travel demand are considered as the random variables, and then the confidence interval are estimated for the solutions of the model.

ACKNOWLEDMENTS

The work was jointly supported by research grants from the Research Grants Council of the Hong Kong Special Administrative Region (Project No. PolyU 15267116; Project No. PolyU 15212217) and the Research Committee of the Hong Kong Polytechnic University (Project No. 1-ZVJV).

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