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Design of build-operate-transfer contract for integrated rail and property development with uncertainty in future urban population

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Abstract

This paper proposes a novel model of the design of a build-operate-transfer (BOT) contract for integrated rail and property (R + P) development when the size of future urban population is uncertain. A real-option approach is adopted to accurately capture the potential economic value of a BOT investment project under uncertainty and its externality effects on urban spatial structure. The proposed model is formulated as a two-stage problem. The first stage of the model optimizes the concession period and rail line parameters (including rail line length, and number and locations of stations) through a Nash bargaining game between a private investor and the government. The second stage determines the headways and fares during the private operation and after transferring the BOT project to the government. The private investor's objective is to maximize its own net profit received during the concession period, whereas the government aims to maximize social welfare over the whole life-cycle of the project. The proposed model is extended to explore the effects of future population jumps due to non-recurrent random events and station deployments with even and uneven station spacings. The results show that compared with the rail-only scheme, the R + P scheme can cause urban sprawl, early investment, and a win-win situation for the government and private investor. In the BOT contract design, ignoring the effects of population jumps and using an average (or even) station spacing as an estimate of actual station deployment can cause a large bias of the parameter values designed in the contract and an underestimate of project values in terms of expected net profit and expected social welfare.

Keywords: BOT contract; integrated rail and property development; real options; population uncertainty; Poisson jump; Nash bargaining game.

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1. Introduction

1.1. Background and motivation

The past two decades have witnessed the rapid development of urban rail transit systems in many large Chinese cities, such as Beijing, Shanghai and Hong Kong. The investment of rail transit development projects requires a huge capital cost, which imposes a heavy financial burden on the government. To broaden the range of fiscal sources for rail transit development, the Chinese government has recently been encouraging private investors to invest in massive rail transit projects through various franchising programs, e.g., build-operate-transfer (BOT) contracts. Under a BOT contract, the private investor negotiates with the authorities (the government) to finance, design, construct, and operate transportation infrastructure for a certain period, defined as a concession period. Upon the expiration of the concession period, the infrastructure is transferred to the government.

For instance, a BOT scheme was recently implemented in Phase II of the Shenzhen Metro Line 4 project. Line 4 connects Shenzhen's Futian district with Hong Kong's Lok Ma Chau district, with a total length of about 16 km. Under this BOT scheme, the Mass Transit Railway (MTR) Corporation of Hong Kong negotiates with the Shenzhen municipal government to determine the concession period and railway line parameters (rail line length, number and locations of stations). After signing the contract, the MTR Corporation is responsible for designing and constructing the rail line at its own expense, and then operating it for 30 years (from 2011 to 2040). After the 30-year concession period is over, the rail line will be transferred to the Shenzhen municipal government for free.

Obviously, the feasibility of a rail BOT project depends on its benefit for the investor, which depends heavily on the urban population size and thus the level of passenger demand for the project services. However, the future urban population size often fluctuates. The population fluctuation can be categorized as a recurrent fluctuation and a non-recurrent random jump in terms of the intensity of fluctuation. The former may be caused by recurrent events, such as long-term population migration due to urbanization and development of high-speed rail networks. The latter may be resulted from non-recurrent or sudden events. For example, every year the Chinese Spring Festival causes the largest scale short-term population migration in the world. The number of the migrants is almost 3.0 billion person trips during the Spring Festival of 2017. Another example is the recent European immigration crisis: a number of refugees from the Middle East and North Africa are packed into the EU countries in order to escape the violent civil wars. This number of the refugees reaches to 1.2 million in 2016. It is thus necessary to consider the stochastic dynamics (including a recurrent fluctuation and a huge jump) of urban population size over time in rail transit BOT investment decision models.

In addition, to attract private investors to invest in mass rail transit systems, the rail and property (R + P) development model has been successfully implemented in Hong Kong's MTR system (Tang et al., 2004; Cervero and Murakami, 2009; Li et al., 2012), and is being tested in some Chinese cities, e.g., Phase II of the Shenzhen Metro Line 4 project, as mentioned above. Under the R + P development model, the Shenzhen municipal government grants the development rights for properties above metro stations to the MTR Corporation as a form of indirect subsidy for its transit operations. The MTR Corporation can use the revenue received from property development to subsidize the investment cost of the rail transit project. With the use of the R + P development model, Phase II of the Shenzhen Metro Line 4 project has generated a positive profit, as has the MTR system in Hong Kong. The success of the pilot project in Shenzhen has a significant implication for adoption of the BOT-type R + P project in other Chinese cities. For example, Beijing has recently been introducing such rail projects, such as Beijing Metro Line 4 project.

On the basis of the above, this paper proposes a new model for designing a BOT contract for integrated R + P project under future urban population uncertainty. The design variables in the BOT contract, including the concession period, rail line length, and number and locations of stations (or station spacing), are determined based on the negotiation between the private investor and the government. The private investor aims to maximize its own net profit during the concession period, whereas the government seeks to maximize the social welfare of the system over the whole life-cycle of the project (Shen et al., 2002; Niu and Zhang, 2013). The train headway and fare during the concession period of the BOT project and the train headway and fare after transferring the BOT project to the government are determined by the private investor and the government, respectively. The effects of the R + P development scheme and the urban population uncertainty on the BOT contract design are also examined, together with the effects of rail station deployments along the rail line (i.e., even and uneven station spacings). It is anticipated that the proposed model can serve as a useful tool for designing a "win-win" BOT contract for rail investment projects and for assessing the benefit of a BOT-type R + P project.

1.2. Literature review

A number of studies have been conducted on the design issues of BOT contracts for transportation infrastructure investments (see Meng and Lu (2017) for a comprehensive review). For readers' convenience, we summarize in Table 1 some of the principal contributions to the research on BOT contract design, addressing the type of transportation infrastructure to invest, decision variables, objective functions, the externality effects of investment on urban form, modeling approaches (static or dynamic), the use of property development as a subsidy of BOT project, and sources of uncertainty. Table 1 shows that the previous studies in this area have mainly focused on the design of BOT contracts for highway or road investments. Little attention has been paid to rail line investment, for which not only the concession period, but also rail line length, station spacing, headway, and fare need to be determined (see Vuchic and Newell, 1968; Vuchic, 1969; Wirasinghe and Seneviratne, 1986; Wirasinghe et al., 2002; Li et al., 2012). Investment timing is another important variable to be determined in the rail line investment context. Investing too early may result in low revenue and low efficiency due to low travel demand (or population size), and investing too late may result in large social costs (e.g., increasing congestion) incurred by leaving demand unmet for too long (Li et al., 2015). Recently, the issue of selection bias was also raised (Eliasson and Fosgerau, 2013; Xu et al., 2015). In addition, the time to invest in a rail project is closely related to the population size of the city concerned and thus its passenger demand, which further affects the project's investment profit and social welfare and thus the concession period, which is determined by the negotiation between the private investor and the government. On the other hand, determination of the concession period is conditional on that of the investment time or the project launch time. Both should thus be jointly determined.

Table 1 also shows that previous studies have mainly focused on static (stationary-state) and deterministic problems. However, in reality, the size of the future urban population and thus future travel demand stochastically and dynamically fluctuate over time (Saez et al., 2012). This is particularly true of for fast-growing Chinese cities, in which future population size exhibits strong randomness due to various events, such as urbanization and major holidays (e.g. Spring Festival). The benefit generated by a rail transit investment project therefore also

Table 1 Recent contributions to BOT contract design in transportation fields.

Reference	Type of transportation infrastructure	Decision variable(s)	Objective	Considering effects of project investment on housing price	Modeling approach	Considering time dimension	Considering property development as a subsidy (i.e., R + P)	Source of uncertainty
Yang and Meng (2000)	Road network	Toll level and highway capacity	Max. social welfare or private profit	No	NPV	Static	No	Deterministic
Shen et al. (2002)	Highway corridor	Concession period	Max. expected profit	No	NPV	Static	No	Deterministic
Chen and Subprasom (2007)	Highway corridor	Toll level	Max. expected social welfare or private profit or min. inequality	No	NPV	Static	No	Demand uncertainty
Guo and Yang (2009)	Road corridor	Concession period, road capacity, and toll level	Max. social welfare	No	NPV	Static	No	Deterministic
Tan et al. (2010)	Road corridor	Concession period, road capacity, and toll level	Max. social welfare and private profit	No	NPV	Static	No	Deterministic
Tan and Yang (2012)	Road corridor	Concession period, road capacity, and toll level	Max. social welfare and private profit	No	NPV	Static	No	Demand uncertainty
Niu and Zhang (2013)	Road corridor	Concession period, road capacity, and toll level	Max. social welfare and private profit	No	NPV	Static	No	Demand uncertainty
Lu and Meng (2017)	Road corridor	Road capacity and toll level	Max. social welfare and private profit	No	NPV	Static	No	Demand uncertainty
This paper	Rail transit line corridor	Investment timing, concession period, rail line length, number and locations of stations, headway, and fare	Max. expected social welfare and expected private profit	Yes	RO	Time-dependent	Yes	Urban population uncertainty

changes stochastically and dynamically over time. It is thus necessary to consider the dynamics (including a recurrent fluctuation and a non-recurrent random jump) of urban population size over time in rail transit BOT investment models.

The previous related studies have also usually ignored the effects of transportation infrastructure investment on the urban form. However, some studies, such as Bowes and Ihlanfeldt (2001), Li et al. (2012, 2015), Peng et al. (2017), and Li and Wang (2018), have shown that investing in a new transit line can incur externalities related to housing prices and space, including changes in households' residential locations, property value, and the housing market, due to improvements in accessibility for travel. It is therefore especially necessary to take into account the effects of rail transit line investment on households' relocation behavior and housing externalities in the design of BOT contracts.

The success of the R + P development model in Hong Kong's MTR system has attracted considerable attention from local governments of mainland China. Beijing and Shenzhen have recently introduced such a development model. Li et al. (2012) showed that compared with the rail-only development scheme, under which the government builds and owns the rail transit project and grants an operation service concession to an agent rail transit company, the integrated R + P development can lead to significant differences in the design of the parameters of the rail transit line. It is more profitable for a benefit-driven private operator to adopt the R + P model than the rail-only model. The integrated R + P development model can thus motivate private rail operators to invest in rail transit services. It is thus vital to compare the effects of the R + P development model and the traditional rail-only development model on the design of a BOT contract.

In addition, existing literature on BOT contract design has usually adopted the standard cost-benefit analysis method of net present value (NPV) (Snell, 2011). However, studies have shown that the conventional NPV approach cannot properly capture the flexible value of BOT projects due to the postponement, abandonment, or expansion of investment opportunities, particularly in an irreversible and uncertain investment environment (Dixit and Pindyck, 1994; Zhao et al., 2004). The real options (RO) valuation approach provides an effective way of capturing the value of the flexibility that goes unrecognized in NPV analysis (McDonald and Siegel, 1986; Trigeorgis, 1996; de Neufville and Scholtes, 2011; Li et al., 2015). For example, Galera and Soliño (2010) and Lv et al. (2014) used the RO method to determine the concession periods of highway BOT contracts under demand uncertainty. Saphores and Boarnet (2006) explored the impact of population uncertainty on the socially optimum timing of a congestion relief project in a linear monocentric city using the RO method. Gao and Driouchi (2013) presented an RO model to examine the investment decision problems of rail transit infrastructure by treating the population size and the attitudes of social planners as sources of risk and ambiguity. Recently, Li et al. (2015) addressed transit technology investment timing and selection issues under urban population volatility using the RO approach. However, these studies did not concern the design issues of BOT contracts for the integrated R + P project, particularly in the environment of random jumps in urban population size.

1.3. Problem statement and contributions

In view of the above discussions, this paper investigates the design issues of a BOT contract for an integrated R + P investment project under future population uncertainty. The BOT concession period (including investment timing), the rail line parameters (including rail line length, and number and locations of stations), and the train operating headways and fares over the life-cycle of the rail project are determined. The main contributions of this paper are as follows. (1) A bilateral bargaining game model between the government and the private investor is proposed for the design of the rail transit BOT contract, and is formulated as a two-stage problem. The first stage is to determine the duration of concession period, rail line length, and number and locations of stations through a Nash bargaining game between the private investor and the government. The second stage determines the train headway and fare for the private operation of the BOT project to maximize the private operator’s net profit, and the train headway and fare after transferring the BOT project to the government to maximize the social welfare of the system over the whole life-cycle of the project. The properties of the proposed model are analytically explored. (2) The effects of the integrated R + P development model on the design of the BOT contract are explicitly considered, and are compared with those of the traditional rail-only development model. (3) The externalities of the rail line investment on the urban spatial form in terms of households’ residential location choices and the housing market are incorporated into the design of the BOT contract. (4) The effects of future urban population uncertainty under recurrent fluctuation and non-recurrent random jump, which are investigated using the RO approach, are considered in the BOT contract design. The results with and without population jump occurring are compared, together with those with even and uneven station spacings.

The remainder of this paper is organized as follows. Section 2 describes some of the basic components of the proposed model. Section 3 presents the formulation of the R + P development model for rail BOT contract design, using a bargaining game approach. Section 4 extends the proposed model to consider the effects of population jumps. In Section 5, numerical examples are provided to illustrate the applications of the proposed model. Section 6 draws conclusions and offers recommendations for further studies.

2. Basic considerations

In this paper, we divide the whole life-cycle of a rail transit investment project into four phases (shown in Fig. 1): before project investment, during project construction, during project operation by the private investor during the concession period, and during project operation by the government after the completion of the concession period (i.e., after project transfer). For ease of presentation, these four phases are indexed by the subscripts "0", "1", "2", and "3", respectively. The public transit modes used in phases 0 and 1 are conventional/ordinary bus modes (e.g., regular bus or minibus), and those in phases 2 and 3 are mass public transit modes (e.g., metro). In the following, we first describe some basic assumptions of the model and then formulate the urban system equilibrium problem for a specific phase.

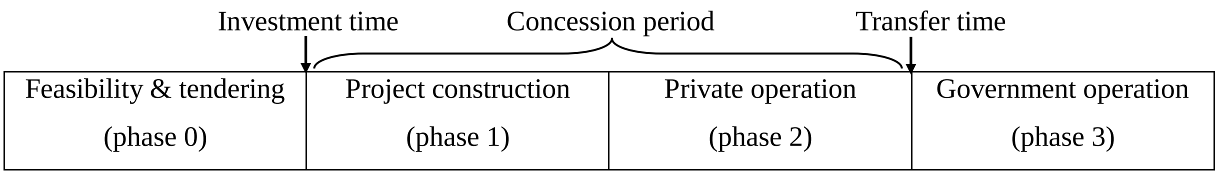


Fig. 1. Timeline of a typical BOT project.

2.1. Assumptions

To facilitate the presentation of essential ideas without loss of generality, the following basic assumptions are made.

A1. The city concerned in this paper is assumed to be linear, closed, and monocentric (Alonso, 1964; Mills, 1972; Pines and Sadka, 1986; Fujita, 1989; O'Sullivan et al., 2000; Kraus, 2006; Li et al., 2012, 2013). This means that all job opportunities are located in the central business district (CBD), and that the value of the land beyond the city boundary equals the agricultural rent or its opportunity cost. The urban population size is assumed to stochastically fluctuate over time and to follow a Geometric Brownian Motion (GBM), as assumed in Saphores and Boarnet (2006), Gao and Driouchi (2013), and Li et al. (2015).

A2. There exist five types of stakeholders in the urban economy: private investor, the government, property developers, households, and commuters. The relationships among them are described in Fig. 2. A two-stage modeling approach is adopted to determine such decision variables in the BOT contract as the concession period (including the investment time), the rail line parameters, and train headways and fares. Specifically, in the first stage, the concession period, rail line length, and number and locations of stations are determined through a negotiation between the government and the private investor. In the second stage, the private investor decides on its headway and fare during the concession period to maximize its own net profit, whereas the government decides on its headway and fare after the BOT project transfer to maximize the social welfare of the system over the life-cycle of the project.

A3. Each property developer is assumed to adopt a Cobb-Douglas housing production function (see Beckmann, 1974; Quigley, 1984; Li et al., 2013, 2015; Li and Peng, 2016; Peng et al., 2017). All the households in the city are assumed to be homogenous, meaning that their income and utility functions are identical. A household's income is spent on such areas as transportation, housing, and composite non-housing goods. Each household's goal is to maximize its own utility by choosing a residential location, amount of housing space, and number of composite goods within its income budget constraint (Beckmann, 1969, 1974; Solow, 1972, 1973; Fujita, 1989).

A4. Suppose that every day each worker makes a two-way commute trip between his/her residential location and workplace in the CBD. The average daily number of trips to the CBD per household is thus equal to its average number of workers. Following previous studies (Anas and Xu, 1999; Song and Zenou, 2006; Li et al., 2012, 2013, 2015), we assume that the average number of workers per household is 1.0. The worker by rail transit chooses an upstream or a downstream station to get on/off the train such that his/her travel cost is minimized.

A5. The quality of the rail transit service is measured by a generalized travel cost that is a weighted combination of the time taken to access the station, wait time at station, in-vehicle time, and fare. Passengers' responses to the rail transit service level are defined by a negative exponential elastic demand function (de Dios Ortuzar and Willumsen, 2011). Responses include the decision to switch to an alternative transportation mode (e.g., auto or bus) or the decision not to make the journey at all (Lam and Zhou, 2000; Li et al., 2012, 2015).

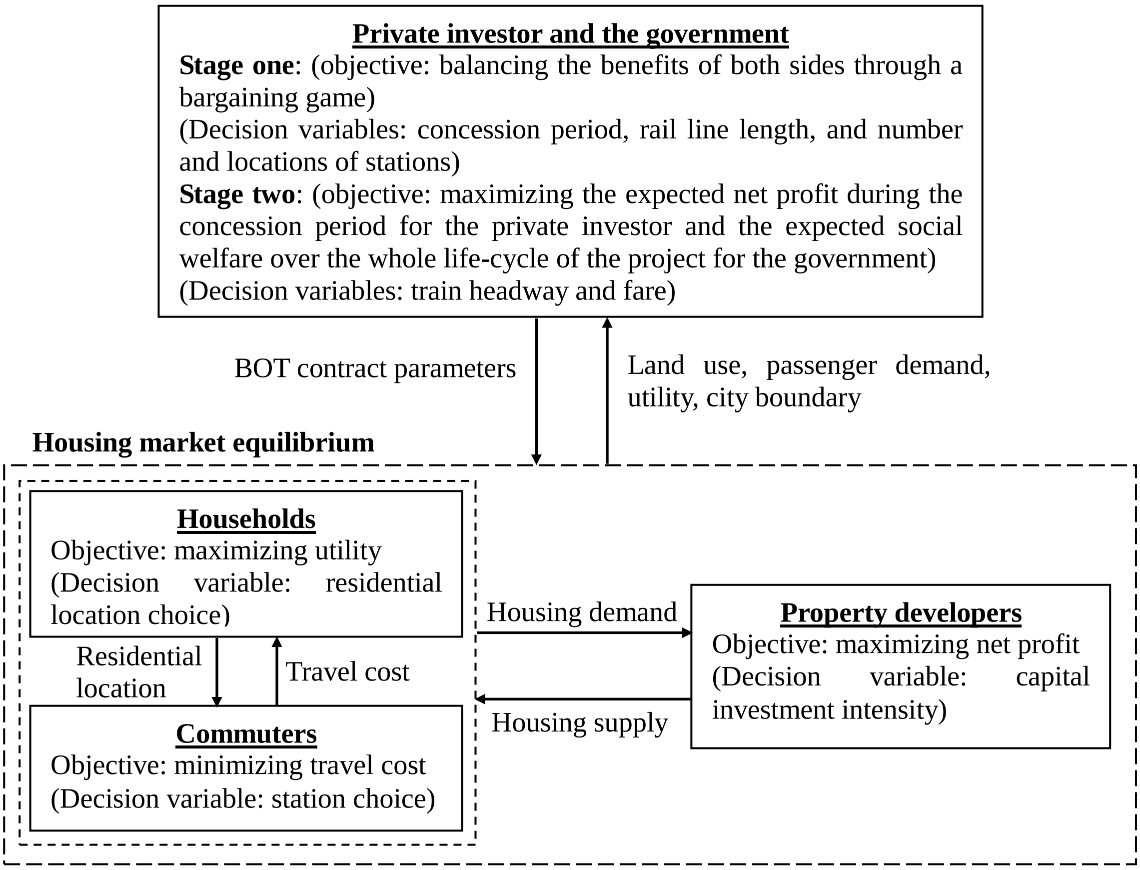


Fig. 2. Interactions among the stakeholders in the urban economy.

2.2. Urban system equilibrium

According to **A2**, there are five types of stakeholders in the urban system: private investor, the government, property developers, households, and commuters (or passengers). The interactions among these stakeholders lead to a number of interrelated equilibria: commuters' rail station choice equilibrium, household residential location choice equilibrium, housing demand-supply equilibrium, and the bargaining game equilibrium between the private investor and the government. In the following subsections, we formulate the equilibria among the property developers, the households, and the commuters. The bilateral bargaining game equilibrium between the private investor and the government will be described later.

2.2.1. Passenger travel cost

Consider a linear transportation corridor connecting the city's CBD with its boundary, as shown in Fig. 3. A new rail line is constructed in the linear corridor. Suppose that the stations on the rail line are given and numbered as $1, 2, \ldots, M+1$, where $M+1$ is the total number of stations. D_1 represents the rail line length and D_s represents the distance between railway station *s* and the CBD. We define the travel cost of a passenger traveling from any location *x* of the corridor to the CBD as follows.

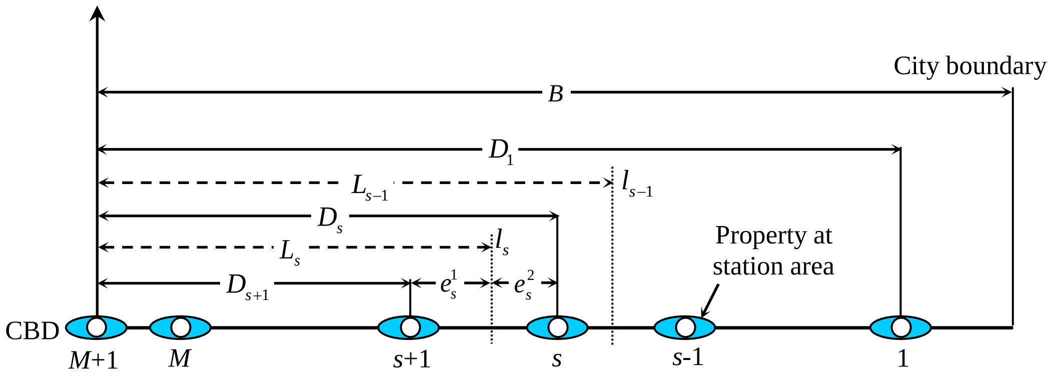


Fig. 3. Rail line along an urban corridor.

Note that the improvement of public transportation services due to rail line investment can lead to a change in the travel cost. Thus, the passenger travel cost may differ among phases. Let *x* be the distance of a location from the CBD and $C_{s,i}(x)$ be the (one-way) travel cost of passengers who travel from location *x* to the CBD and get on at station *s* in phase *i*. It consists of time taken to access the railway station, wait time at the station, in-vehicle travel time, and fare. For a given rail line configuration (i.e., given $D_s, s = 1, 2, \ldots, M$), $C_{s,i}(x)$ is defined as

$$C_{s,i}(x) = \tau_a \frac{|D_s - x|}{\bar{V}} + \tau_w (\gamma H_i) + \tau_t \left(\frac{D_s}{V_i} + t_0 (M + 1 - s) \right) + f_{s,i}, \forall s, i , \tag{1}$$

where τ_a, τ_w, and τ_t are the values of access time, wait time, and in-vehicle travel time, respectively. $|D_s - x|$ is the average time taken to access station *s* from location *x*, and $\bar{V}$ is the average passenger walk speed, which is assumed to be given and fixed. Thus $|D_s - x|/\bar{V}$ is the average time taken for a passenger at location *x* to access station *s*. H_i is the train headway in phase *i* and γH_i is the average passenger wait time in phase *i*. γ is a parameter that relies on the distributions of both train headway and passenger arrivals. The γ-value is 0.5, based on the assumptions of a constant train headway and a uniform passenger arrival distribution. $D_s/V_i + t_0(M+1-s)$ is the in-vehicle time from station *s* to the CBD in phase *i*, consisting of two parts: non-stop line-haul travel time D_s/V_i, and total train dwell time $t_0(M+1-s)$ between station *s* and the CBD. The line-haul travel time D_s/V_i in phase *i* is equal to the in-vehicle trip length divided by the average train speed V_i in that phase. t_0 is the average train dwell time at a station, and can be calibrated by survey data (see e.g., Wirasinghe and Szplett, 1984; Lam et al., 1998). $f_{s,i}$ is the rail fare from station *s* to the CBD in phase *i*.

The rail fare, $f_{s,i}$, from station *s* to the CBD in phase *i* is assumed to be a linear function of the distance D_s traveled from station *s* to the CBD (Li et al., 2012), i.e.,

$$f_{s,i} = f_f + f_{v,i} D_s, \forall s, i, \tag{2}$$

where f_f is a fixed component of the rail fare and $f_{v,i}$ is its variable component in phase *i*.

2.2.2. Passengers' station choice behavior

Any two adjacent railway stations on the rail line compete for passengers between those two stations. There is thus a passenger watershed line that partitions the line segment between two adjacent stations into two sub-segments (Vuchic and Newell, 1968; Vuchic, 1969; Li et al., 2012, 2013). The passengers in these two sub-segments use the upstream and downstream stations of the line segment, respectively. Obviously, the location of the passenger watershed line and thus the lengths of the sub-segments change with improvements in the public transit service. We denote $l_{s,i}$ as the passenger watershed line between stations *s* and $s+1$ in phase *i*, and $e_{s,i}^1$ and $e_{s,i}^2$ as the two associated sub-segments in that phase. Obviously, the relationship $e^1_{s,i} + e^2_{s,i} = D_s - D_{s+1}$ holds.

The passenger watershed line $l_{s,i}$ is located such that the walking time from the watershed line to the downstream station s + 1 is equal to the walking time from the watershed line to the upstream station s plus the riding time from station s to s + 1, i.e.,

$$\frac{e^1_{s,i}}{\bar{V}} = \frac{e^2_{s,i}}{\bar{V}} + \frac{D_s - D_{s+1}}{V_i}, \forall s, i . \tag{3}$$

From Eq. (3) and $e^1_{s,i} + e^2_{s,i} = D_s - D_{s+1}$, we can derive

$$\begin{cases} e^1_{s,i} = \dfrac{V_i + \bar{V}}{2V_i}\left(D_s - D_{s+1}\right), \\ e^2_{s,i} = \dfrac{V_i - \bar{V}}{2V_i}\left(D_s - D_{s+1}\right). \end{cases} \tag{4}$$

Let $L_{s,i}$ be the distance of watershed line $l_{s,i}$ from the CBD. From Fig. 3, we can obtain

$$L_{s,i} = \frac{V_i + \bar{V}}{2V_i} D_s + \frac{V_i - \bar{V}}{2V_i} D_{s+1}, \forall s, i , \tag{5}$$

where D_{M+1} equals zero. By the above definition of the passenger watershed line, the catchment or coverage area of station s is $[L_{s,i}, L_{s-1,i}],\ s = 1, 2, \ldots, M+1$.

In order to define the passenger demand at each station of the rail line, we denote η as the average number of daily trips per household, and $n_i(x)$ as the household residential density (i.e., the number of households per unit of land area) at location x in phase i, implying that $\eta n_i(x)$ is the daily passenger demand per unit of distance at location x in phase i. Note that the passenger demand for the rail line service is sensitive to service and fare level, and is thus elastic. In this paper, we adopt an exponential elastic demand density function to model the elasticity of rail passenger demand. Let $q_{s,i}(x)$ be the daily density of actual rail passenger demand originating at location x and boarding at station s in phase i, given as

$$q_{s,i}(x) = \eta n_i(x) \exp\left(-\pi C_{s,i}(x)\right), \tag{6}$$

where π is a parameter for reflecting demand sensitivity to travel cost.

The daily rail passenger demand at station s in phase i, denoted as $Q_{s,i}$, can then be calculated as follows:

$$Q_{s,i} = \int_{L_s}^{L_{s-1}} q_{s,i}(x)dx = \int_{L_s}^{L_{s-1}} \eta n_i(x)\exp\left(-\pi C_{s,i}(x)\right)dx\,. \tag{7}$$

2.2.3. Households' residential location choice behavior

According to **A3**, each household in the urban system chooses its residential location to maximize its own utility, which is defined as a quasi-linear function with regard to housing consumption and non-housing goods consumption. The household's utility maximization problem with an income budget constraint is expressed as

$$\max_{z_i, g_i} \ U_i(x) = z_i(x) + \alpha \log g_i(x)\,, \tag{8}$$

$$\text{s.t.} \ \ z_i(x) + p_i(x)g_i(x) = Y_i - \varphi_i(x), \forall x \in \left[0, B_i\right], \tag{9}$$

where $U_i(x)$ is the household utility function at location x in phase i; $z_i(x)$ is the non-housing goods consumption at location x in phase i, with price normalized to 1; α is a positive parameter; $p_i(x)$ is the average annual rental price per unit of housing at location x in phase i; $g_i(x)$ is the housing consumption at location x in phase i, measured in square meters of floor space; Y_i is the annual household income in phase i; $\varphi_i(x)$ is the average annual travel cost from location x to the CBD in phase i; and B_i is the distance between the city boundary and the CBD in phase i (i.e., city size). Eq. (9) represents the household's income budget constraint for phase i, which indicates that the household's income is spent on transportation, and housing and non-housing goods consumptions.

The average annual travel cost $\varphi_i(x)$ from location x to the CBD through station s in phase i can be expressed as

$$\varphi_i(x) = 2\rho C_{s,i}(x), \forall x \in [L_{s,i}, L_{s-1,i}], s = 1, 2, ..., M+1\,, \tag{10}$$

where "2" denotes a round trip between the CBD and location x, and $C_{s,i}(x)$ is given by Eq. (1). ρ is the average annual number of trips to the CBD per household.

Based on the first-order optimality conditions of maximization problems (8) and (9), the following can be obtained:

$$p_i(x) = p_i(0)\exp\left(-\frac{\varphi_i(x)}{\alpha}\right), \tag{11}$$

$$g_i(x) = \frac{\alpha}{p_i(0)}\exp\left(\frac{\varphi_i(x)}{\alpha}\right), \tag{12}$$

$$z_i(x) = Y_i - \varphi_i(x) - \alpha, \tag{13}$$

$$U_i = Y_i - \alpha + \alpha\log\left(\frac{\alpha}{p_i(0)}\right), \tag{14}$$

where $p_i(0)$ is the average housing rental price in the CBD in phase *i*, which is defined later. Once $p_i(0)$ is determined, one can then determine the equilibrium housing rental price per unit of housing area, equilibrium amount of housing floor space per household, and equilibrium consumption of non-housing goods per household at any location in any phase, by Eqs. (11)-(13), respectively. Eq. (14) represents the resultant equilibrium household utility.

2.2.4. Property developer's housing production behavior

Let $S_i(x)$ be the capital investment per unit of land area at location *x* in phase *i*, which is also referred to as "capital investment intensity" in this paper. According to **A3**, the property developer's housing production follows a Cobb-Douglas function, as follows:

$$h\left(S_i(x)\right) = a\left(S_i(x)\right)^{\theta}, \quad 0 < \theta < 1, \tag{15}$$

where $h\left(S_i(x)\right)$ is the housing supply per unit of land area at location *x* in phase *i*, and a and θ are positive parameters.

For a given phase, each property developer is assumed to maximize its own net profit (denoted as NP) by determining the capital investment intensity at any location, given as

$$\max_{S_i(x)} \; NP_i(x) = p_i(x)h\left(S_i(x)\right) - \left(r_i(x) + kS_i(x)\right), \tag{16}$$

where *k* is the price of the capital (i.e., the riskless interest rate), and $r_i(x)$ is the land value at location *x* in phase *i*. The two terms on the right-hand side of Eq. (16) represent the total revenue from the housing rents and the sum of the land rent cost plus the capital investment cost, respectively.

From the first-order optimality condition of maximization problem (16), we have

$$S_i(x) = \left(p_i(x)a\theta k^{-1}\right)^{\frac{1}{1-\theta}} = \left(a\theta k^{-1}p_i(0)\exp\left(-\frac{\varphi_i(x)}{\alpha}\right)\right)^{\frac{1}{1-\theta}}. \tag{17}$$

For a perfect competitive real estate market, the property developers earn zero profit, i.e., $NP_i(x) = 0$. From Eqs. (11) and (17), and $NP_i(x) = 0$, one obtains the equilibrium land value at location *x* in phase *i* as

$$r_i(x) = (1-\theta)\left(a\theta^{\theta}k^{-\theta}p_i(0)\exp\left(-\frac{\varphi_i(x)}{\alpha}\right)\right)^{\frac{1}{1-\theta}}. \tag{18}$$

2.2.5. Housing market equilibrium

At the housing market equilibrium, for a given phase, the total housing supply at any location equals the total housing demand at that location. Therefore, $h\left(S_i(x)\right) = g_i(x)n_i(x)$. The household residential density $n_i(x)$ at location *x* in phase *i* can thus be obtained as follows:

$$n_i(x) = \frac{h\left(S_i(x)\right)}{g_i(x)} = \alpha^{-1}(\theta k^{-1})^{\frac{\theta}{1-\theta}}\left(ap_i(0)\exp\left(-\frac{\varphi_i(x)}{\alpha}\right)\right)^{\frac{1}{1-\theta}}. \tag{19}$$

The housing market equilibrium satisfies two conditions. First, for a given phase, all households are within the city boundary, i.e.,

$$\int_0^{B_i} n_i(x)dx = N, \tag{20}$$

where *N* is the total number of households in the city concerned.

Second, the equilibrium rent per unit of land area at the city's fringe is equal to the agricultural rent or opportunity cost of the land in terms of **A1**, i.e.,

$$r_i(B_i) = r_A, \tag{21}$$

where r_A is the agricultural rent, which is assumed to be a constant.

Eqs. (20) and (21) contain two unknown parameters: housing rental price $p_i(0)$ in the CBD in phase *i* and city boundary B_i in phase *i*. Their values can be determined by jointly solving the system of the two equations, expressed as

$$p_i(0) = a^{-1}\left(k\theta^{-1}\right)^{\theta}\left(\frac{r_A}{1-\theta}\right)^{1-\theta}\left(\frac{1}{\Omega_i}\left(\frac{2\rho\tau_a N}{r_A\bar{V}}+1\right)\right)^{1-\theta}, \tag{22}$$

$$B_i = \frac{\bar{V}}{\tau_a}\left(\frac{\alpha}{2\rho}\log\left(\frac{1}{\Omega_i}\left(\frac{2\rho\tau_a N}{r_A\bar{V}}+1\right)\right)-C_{1,i}\left(D_1\right)\right)+D_1, \tag{23}$$

where

$$\Omega_i = 1-\exp\left(-\frac{2\rho\tau_a L_{M,i}}{\alpha(1-\theta)\bar{V}}\right)+\sum_{s=1}^{M}\left(2\exp\left(-\frac{2\rho}{\alpha(1-\theta)}C_{s,i}(D_s)\right)-\exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(D_s-L_{s,i})+C_{s,i}(D_s)\right)\right)\right)$$
$$-\sum_{s=2}^{M}\exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(L_{s-1,i}-D_s)+C_{s,i}(D_s)\right)\right), \text{ and} \tag{24}$$

$$C_{s,i}(D_s) = \tau_w\gamma H_i+\tau_t\left(\frac{D_s}{V_i}+t_0(M+1-s)\right)+f_f+f_{v,i}D_s. \tag{25}$$

From Eqs. (6), (7), (19), and (22), one can derive the daily passenger demand $Q_{s,i}$ at station s as

$$Q_{s,i} = \frac{\eta(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}\Phi_{s,i}}{2\rho\tau_a+\alpha\pi(1-\theta)\tau_a}, \tag{26}$$

where

$$\Phi_{s,i} = 2\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)C_{s,i}(D_s)\right)-\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)\left(\frac{\tau_a}{\bar{V}}(D_s-L_{s,i})+C_{s,i}(D_s)\right)\right)$$
$$-\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)\left(\frac{\tau_a}{\bar{V}}(L_{s-1,i}-D_s)+C_{s,i}(D_s)\right)\right). \tag{27}$$

The detailed derivations of Eqs. (22), (23) and (26) are provided in Appendix A.

The following proposition shows the comparative static results of $p_i(0)$, B_i, and $Q_{s,i}$. Its proof is also provided in Appendix A.

Proposition 1. Housing rental price $p_i(0)$ in the CBD increases with urban population size N and agricultural rent r_A. The city boundary B_i increases with N, but decreases with r_A. Daily transit passenger demand $Q_{s,i}$ increases with N and r_A.

3. Model formulation

3.1. Investment cost of transit project

The investment cost of a rail transit project is incurred by train operations, rail line, and rail stations (Li et al., 2012, 2015), defined as follows. Train operating cost C_i^O during phase *i* consists of fixed operating cost and variable operating cost during that phase, denoted as

$$C_i^O = C_f^O + C_v^O \frac{\Theta_i}{H_i}, \tag{28}$$

where C_f^O denotes the fixed operating cost, C_v^O denotes the marginal operating cost per train per year, Θ_i denotes the vehicle (two-way) journey time, and $\frac{\Theta_i}{H_i}$ denotes the number of vehicles (or fleet size) on the rail line. The two-way journey time Θ_i consists of the terminal time, the non-stop line-haul travel time, and the train dwell time at stations, expressed as

$$\Theta_i = \zeta t_c + 2(t_{t,i} + t_d), \tag{29}$$

where t_c denotes the constant terminal time on the circular line and ζ denotes the number of terminal times on the line. $t_{t,i}$ is the total line-haul travel time from station 1 to the CBD, i.e., $t_{t,i} = \frac{D_1}{V_i}$. t_d is the total train dwell time through this rail line, i.e., $t_d = t_0 M$.

The annual rail line cost C^L is the sum of the fixed costs C_f^L (e.g., line overhead cost) and the variable costs $C_v^L D_1$ (e.g., line construction, maintenance, and labor costs), proportional to rail line length D_1, i.e.

$$C^L = C_f^L + C_v^L D_1, \tag{30}$$

where C_v^L is the marginal operating cost of rail line per kilometer per year.

The annual railway station cost C^S comprises fixed costs (e.g., station overhead cost) and variable costs (e.g., construction, operating, and maintenance costs), represented as

$$C^S = C_f^S + C_v^S (M+1), \tag{31}$$

where C_f^S denotes the fixed cost for station operations and C_v^S denotes the marginal operating cost per station per year.

3.2. Investment revenue of transit project

In the integrated R + P development model, the rail operator has two sources of revenue: passenger fares and above-station property development (Li et al., 2012). The fare revenue of the rail operator is the sum of the number of passengers boarding at each station multiplied by the corresponding fare. The property revenue is the sum of the housing rental price per unit of housing space at each station multiplied by the corresponding housing space.

Let FR_i represent the annual fare revenue of the rail operator during phase *i*, expressed as

$$FR_i = 2\rho \sum_{s=1}^{M} \left(f_f + f_{v,i} D_s \right) Q_{s,i} , \tag{32}$$

where the daily passenger demand $Q_{s,i}$ at station *s* during phase *i* is determined by Eq. (26).

Let PR_i be the annual net property revenue of the rail operator during phase *i*. It can be represented as

$$PR_i = \sum_{s=1}^{M+1} \int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0} \left(p_i(x) h\left(S_i(x)\right) - r_i(x) \right) dx , \tag{33}$$

where the integral term on the right-hand side of Eq. (33) represents the annual net property revenue at station *s*, which is equal to the property gross revenue minus the associated cost for the land development above station *s*. Δ_s^0 is the radius of the property development above station *s*, which is assumed to be a constant. $[D_s - \Delta_s^0, D_s + \Delta_s^0]$ is the coverage of property development above station *s*.

3.3. Project values for private investor and the government

We now define the values of a BOT-type R + P project under the future urban population uncertainty for the private investor and the government. To do so, we assume that the number of urban residents at time *t*, denoted by N_t, fluctuates stochastically and continuously (i.e., no jump) over time and follows a Geometric Brownian Motion (GBM), as follows:

$$dN_t = \mu N_t dt + \sigma N_t dw_t , \tag{34}$$

where μ and σ are the growth rate and volatility rate of the urban population size, respectively; dt is an infinitesimal time increment; and dw_t is an increment of a standard Wiener process. For any given time *t*, dw_t satisfies the equation $dw_t = w_t\sqrt{t}$, where w_t is a normally distributed random variable with a mean of 0 and a standard deviation of 1.

The private investor cares about the net profit that it can obtain from the rail investment project. The net profit of the private investor during the concession period equals its total revenue minus its total investment cost. Let $\Lambda_p(N)$ represent the expected net profit of the rail line investment project during the concession period (i.e., phase 2) for the private investor, defined as

$$\Lambda_p(N) = E_N\left[\int_\Delta^T FR_2 e^{-kt}dt + \int_\Delta^T PR_2 e^{-kt}dt - \int_\Delta^T C_2^O e^{-kt}dt - \int_0^\Delta (C^L + C^S)e^{-kt}dt\right], \tag{35}$$

where the subscript "*p*" denotes the private investor. FR_2 and PR_2 are the annual fare revenue and annual net property revenue of the private operator during phase 2, which can be determined by Eqs. (32) and (33), respectively. E_N is the expectation operator with regard to population size *N*, *k* is the riskless interest rate or discount rate, T is the concession period, and Δ is the construction duration of the rail project, which is assumed to be a constant in this paper. The first term on the right-hand side of Eq. (35) represents the total fare revenue. The second term represents the total net property revenue at all stations. The final two terms represent the total investment cost of the rail project. It should be noted that for the rail-only model, the second term should be removed because fare revenue is the only source of revenue for the private operator in the rail-only scheme. The R + P scheme and the rail-only scheme will be compared later.

The government is concerned about the total social welfare generated by the rail investment project over its whole life-cycle. This consists of welfare generated during project construction, during private operation, and during government operation, which is the sum of consumer surplus and project profit over the project life-cycle. Let $W_{g,1}(N)$, $W_{g,2}(N)$, and $W_{g,3}(N)$ be the expected social welfare during project construction, during private operation, and during government operation, respectively. The subscript "*g*" denotes the government. $W_{g,1}(N)$, $W_{g,2}(N)$, and $W_{g,3}(N)$ can thus be expressed as

$$W_{g,1}(N) = E_N\left[\int_0^{\Delta} N_t U_1\left(N_t\right)e^{-kt}dt\right], \tag{36}$$

$$W_{g,2}(N) = E_N\left[\int_{\Delta}^{T} N_t U_2\left(N_t\right)e^{-kt}dt\right] + \Lambda_p(N), \tag{37}$$

$$W_{g,3}(N) = E_N\left[\int_T^{+\infty} N_t U_3\left(N_t\right)e^{-kt}dt + \int_T^{+\infty} FR_3 e^{-kt}dt + \int_T^{+\infty} PR_3 e^{-kt}dt - \int_T^{+\infty} C_3^O e^{-kt}dt\right], \tag{38}$$

where FR_3 and PR_3 are the annual fare revenue and annual net property revenue of the government during phase 3, which are determined by Eqs. (32) and (33), respectively. C_3^O is the train operating cost during phase 3, which is determined by Eq. (28).

3.4. Investment timing and feasible concession period

3.4.1. Investment timings for private investor and the government

The rail line investment timing problem is actually an optimal stopping problem. Both the private investor and the government consider their expected present value as an optimal investment rule to equal the value of their investment opportunities (that is, the value of the option to invest) at the future time at which the investment is made. In this paper, we denote N_p^* and N_g^* as the optimal trigger population thresholds for the private investor and the government, respectively. When $N \geq N_p^*$ ($N \geq N_g^*$), investing immediately in the rail line is the best decision for the private investor (for the government). Otherwise, waiting is the best strategy.

Note that at the trigger population thresholds N_p^* and N_g^*, the private investor and the government have no preference for either “no investment” or “immediate investment”. Specifically, for the private investor, at N_p^*, the value of continuing to wait is equal to that of immediate investment. The former (i.e., the value of continuing to wait) equals the option value of waiting to invest in the rail transit project. The latter (i.e., the value of making the investment immediately) is the expected net profit of the rail line investment project during the concession period, i.e., $\Lambda_p(N)$. Let $F_p(N)$ be the option value for the private investor to invest in the rail transit project at population size N. We then have

$$F_p(N_p^*) = \Lambda_p(N_p^*). \tag{39}$$

For the government, at N_g^*, the value of continuing to wait is equal to the value of immediate investment, in terms of the expected social welfare. The value of continuing to wait is the sum of the expected social welfare without the transit project and the option value of waiting to invest in the transit project. The value of immediate investment is the total expected social welfare with the transit investment project, which is the sum of the expected social welfares during project construction, during private operation, and during government operation. Let $F_g(N)$ be the option value for the government to invest in the rail project. Let $W_{g,0}(N)$ be the expected social welfare without the rail investment project. At the population threshold N_g^*, we have

$$F_g(N_g^*) + W_{g,0}(N_g^*) = W_{g,1}(N_g^*) + W_{g,2}(N_g^*) + W_{g,3}(N_g^*)\,, \tag{40}$$

where $W_{g,1}(N)$, $W_{g,2}(N)$, and $W_{g,3}(N)$ are given by Eqs. (36)-(38), respectively. $W_{g,0}(N)$ is defined as

$$W_{g,0}(N) = E_N\left[\int_0^{+\infty} N_t U_0(N_t) e^{-kt} dt\right]. \tag{41}$$

From Eqs. (36), (37), (38), (40) and (41), we have

$$F_g(N_g^*) = W_g(N_g^*)\,, \tag{42}$$

where $W_g(N)$ is given by

$$\begin{aligned} W_g(N) = E_N\Big[&\int_0^{+\infty} N_t\left(U_1(N_t) - U_0(N_t)\right)e^{-kt}dt + \int_\Delta^T N_t\left(U_2(N_t) - U_1(N_t)\right)e^{-kt}dt + \Lambda_p(N) \\ &+ \int_T^{+\infty} N_t\left(U_3(N_t) - U_1(N_t)\right)e^{-kt}dt + \int_T^{+\infty} FR_3 e^{-kt}dt \\ &+ \int_T^{+\infty} PR_3 e^{-kt}dt - \int_T^{+\infty} C_3^O e^{-kt}dt\Big]. \end{aligned} \tag{43}$$

In Eq. (43), $W_g(N)$ represents the change in the expected social welfare (or called the expected welfare gain) due to the introduction of the rail investment project.

It should be pointed out that in order to guarantee the existence of the optimal trigger population threshold N_p^* and N_g^*, the expected net profit $\Lambda_p(N)$ and the expected welfare gain $W_g(N)$ should be monotonic with regard to population N, according to the RO theory (see Appendix B of Chapter 4, Dixit and Pindyck, 1994). To ensure the monotonicity of

$\Lambda_p(N)$ and $W_g(N)$, we assume that $(1+kdt)^{-1}\int \Lambda_p(N+dN)d\Pi_p(N+dN)-\Lambda_p(N)$ and $(1+kdt)^{-1}\int W_g(N+dN)d\Pi_g(N+dN)-W_g(N)$ are monotonically decreasing functions of N so that the waiting and investing regions are separated. Herein, $\Pi_p(N)$ and $\Pi_g(N)$ are the cumulative probability distribution functions of Λ_p and W_g, respectively. These assumptions are directly taken from Dixit and Pindyck (1994).

In the following, we derive $F_j(N_t), j=p,g$ in Eqs. (39) and (42). According to dynamic programming, if time changes by a small increment dt, the state variable N_t will move to (N_t+dN_t) and the expected present value $F_j(N_t)$ will change to $F_j(N_t+dN_t)$. To express this in equivalent units of time t, we must discount it by a factor of e^{-kdt}. Further, dN_t is a random increment, we must thus take an expectation. Therefore, one obtains

$$F_j(N)=e^{-kdt}E_t\left[F_j(N_t+dN_t)\right]. \tag{44}$$

Expanding the right-hand side of Eq. (44) by Ito's lemma (see Dixit and Pindyck, 1994) and omitting the terms that go to zero faster than dt as $dt\to 0$, one obtains

$$\begin{aligned}
&e^{-kdt}E_t\left[F_j(N_t+dN_t)\right]\\
&=(1-kdt)E_t\left(F_j(N_t)+\frac{\partial F_j(N_t)}{\partial t}dt+\mu N\frac{\partial F_j(N_t)}{\partial N}dt+\frac{1}{2}\sigma^2N^2\frac{\partial^2 F_j(N_t)}{\partial N^2}dt\right)\\
&=E_t\left(F_j(N_t)+\frac{1}{2}\sigma^2N^2\frac{\partial^2 F_j(N_t)}{\partial N^2}dt+\mu N\frac{\partial F_j(N_t)}{\partial N}dt+\frac{\partial F_j(N_t)}{\partial t}dt-kF_j(N_t)dt\right).
\end{aligned} \tag{45}$$

From Eqs. (44) and (45), one can obtain the following partial differential equation:

$$\frac{1}{2}\sigma^2N^2\frac{\partial^2 F_j(N_t)}{\partial N^2}+\mu N\frac{\partial F_j(N_t)}{\partial N}-kF_j(N_t)=0,\ j=p,\mathrm{g}. \tag{46}$$

This is a dynamic programming equation, satisfying the following boundary conditions:

$$F_j(0)=0,\ j=p,\mathrm{g}, \tag{47}$$

$$F_p(N_p^*)=\Lambda_p(N_p^*),\text{ and }\ F_g(N_g^*)=W_g(N_g^*), \tag{48}$$

$$\left.\frac{dF_p(N^*)}{dN^*}\right|_{N^*=N_p^*}=\left.\frac{d\Lambda_p(N^*)}{dN^*}\right|_{N^*=N_p^*},\text{ and }\ \left.\frac{dF_g(N^*)}{dN^*}\right|_{N^*=N_g^*}=\left.\frac{dW_g(N^*)}{dN^*}\right|_{N^*=N_g^*}. \tag{49}$$

Eq. (47) means that if N goes to zero, the project value is zero. Eq. (48) is the "value-matching" condition, which states that upon investing, the private investor (or the government) receives a net payoff $\Lambda_p(N_p^*)$ (or $W_g(N_g^*)$). Eq. (49) is the "smooth-pasting" condition, which requires not just the values but also the derivatives or slopes of the two functions to match at the boundary.

Solving Eq. (46) subject to boundary conditions (47)-(49) yields

$$F_j(N_t) = A_j(N_t)^{\beta},\ j = p, \mathrm{g}, \tag{50}$$

where parameters β and A_j are given by

$$\begin{cases} \beta = \dfrac{1}{2} - \dfrac{\mu}{\sigma^2} + \sqrt{\left(\dfrac{\mu}{\sigma^2} - \dfrac{1}{2}\right)^2 + \dfrac{2k}{\sigma^2}}, \\ A_p = \left(\dfrac{e^{-(k-\mu)\Delta} - e^{-(k-\mu)T}}{\beta(k-\mu)}\left(\xi_2 - \alpha(1-\theta)\log\left(\dfrac{\Omega_2}{\Omega_1}\right)\right)\right)^{\beta}\left(\dfrac{k(\beta-1)}{C_2^O(e^{-k\Delta} - e^{-kT}) + (C^L + C^S)(1-e^{-k\Delta})}\right)^{\beta-1}, \\ A_g = \left(\dfrac{\xi_1 + (e^{-(k-\mu)\Delta} - e^{-(k-\mu)T})\xi_2 + e^{-(k-\mu)T}\xi_3}{\beta(k-\mu)}\right)^{\beta}\left(\dfrac{k(\beta-1)}{C_2^O(e^{-k\Delta} - e^{-kT}) + C_3^O e^{-kT} + (C^L + C^S)(1-e^{-k\Delta})}\right)^{\beta-1}. \end{cases} \tag{51}$$

The following proposition provides the solutions for the expected net profit $\Lambda_p(N)$ of the private investor during the concession period and for the expected welfare gain $W_g(N)$ of the government over the project life-cycle. Its proof is given in Appendix B.

Proposition 2. (i) Suppose that $r_A\bar{V} << 2\rho\tau_a N$ holds. Then, the expected net profit $\Lambda_p(N)$ and the expected welfare gain $W_g(N)$ can, respectively, be given by

$$\Lambda_p(N) = \frac{e^{-(k-\mu)\Delta} - e^{-(k-\mu)T}}{k-\mu}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2} + \frac{\alpha\theta\Psi_2}{\Omega_2}\right)N - \frac{1}{k}\left(C_2^O(e^{-k\Delta} - e^{-kT}) + (C^L + C^S)(1-e^{-k\Delta})\right), \tag{52}$$

$$\begin{aligned} W_g(N) = {} & \frac{\alpha(1-\theta)\log\left(\dfrac{\Omega_1}{\Omega_0}\right)N}{k-\mu} + \frac{e^{-(k-\mu)\Delta} - e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2} + \frac{\alpha\theta\Psi_2}{\Omega_2}\right)N \\ & + \frac{e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,3}D_s\right)\Phi_{s,3}}{(2\rho+(1-\theta)\alpha\pi)\Omega_3} + \frac{\alpha\theta\Psi_3}{\Omega_3}\right)N - \frac{1}{k}\left(C_2^O(e^{-k\Delta} - e^{-kT}) + C_3^O e^{-kT} + (C^L + C^S)(1-e^{-k\Delta})\right). \end{aligned} \tag{53}$$

(ii) For a given concession period T, the trigger population thresholds N_p^* and N_g^* for the private investor and the government are given, respectively, by

$$N_p^* = \frac{\beta(k-\mu)}{k(\beta-1)} \frac{C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})}{(e^{-(k-\mu)\Delta}-e^{-(k-\mu)T})\left(\xi_2-\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)}, \text{ and} \tag{54}$$

$$N_g^* = \frac{\beta(k-\mu)}{k(\beta-1)} \frac{C_2^O(e^{-k\Delta}-e^{-kT})+C_3^O e^{-kT}+(C^L+C^S)(1-e^{-k\Delta})}{\xi_1+(e^{-(k-\mu)\Delta}-e^{-(k-\mu)T})\xi_2+e^{-(k-\mu)T}\xi_3}, \tag{55}$$

where $\xi_1 = \alpha(1-\theta)\log\left(\frac{\Omega_1}{\Omega_0}\right)$, $\xi_2 = \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2} + \frac{\alpha\theta\Psi_2}{\Omega_2}$, and

$\xi_3 = \alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,3}D_s\right)\Phi_{s,3}}{(2\rho+(1-\theta)\alpha\pi)\Omega_3} + \frac{\alpha\theta\Psi_3}{\Omega_3}$. Ω_1, Ω_2 and Ω_3 are given by Eq. (24).

(iii) The trigger population thresholds for the private investor and the government under the RO and NPV approaches, respectively, satisfy the following relationships:

$$N_p^* = \frac{\beta}{\beta-1}N_{p(NPV)}^*, \text{ and } N_g^* = \frac{\beta}{\beta-1}N_{g(NPV)}^*, \tag{56}$$

where the subscript “NPV” denotes the solutions obtained by the NPV approach.

Eq. (56) means that the trigger population threshold generated by the RO approach is always greater than that by the NPV approach, regardless of the private investor or the government. This is because the RO approach incorporates the value of flexibility through the option to wait and defer investment.

3.4.2. Feasible region of concession period

When the population size $N < \min(N_p^*, N_g^*)$, neither the private investor nor the government has an incentive to launch the rail investment project. When $N_g^* \le N < N_p^*$, the government wishes to invest, whereas the private investor does not. When $N \ge \max(N_p^*, N_g^*)$, both the private investor and the government wish to invest. Using Eqs. (54) and (55), one can easily determine the acceptable (or feasible) concession periods for the private investor and the government. Specifically, for any given population size *N* larger than or equal to N_p^* (i.e., $N \ge N_p^*$), a minimum acceptable concession period (denoted as T_{MIN}) exists for the private investor, i.e., $T_{MIN} = \left\{ T \middle| \Lambda_p(N(T)) = F_p(N(T)), N \geq N_p^* \right\}$. For any given population size $N \geq N_g^*$, the maximum acceptable concession period (denoted as T_{MAX}) exists for the government, i.e., $T_{MAX} = \left\{ T \middle| W_g(N(T)) = F_g(N(T)), N \geq N_g^* \right\}$. According to these conditions, one can obtain T_{MIN} and T_{MAX} which, respectively, satisfy the following equations:

$$\frac{\beta(k-\mu)}{k(\beta-1)} \frac{C_2^O\left(e^{-k\Delta} - e^{-kT_{MIN}}\right) + \left(C^L + C^S\right)\left(1 - e^{-k\Delta}\right)}{\left(e^{-(k-\mu)\Delta} - e^{-(k-\mu)T_{MIN}}\right)\left(\xi_2 - \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)} - N = 0 \text{, and} \tag{57}$$

$$\frac{\beta(k-\mu)}{k(\beta-1)} \frac{C_2^O\left(e^{-k\Delta} - e^{-kT_{MAX}}\right) + C_3^O e^{-kT_{MAX}} + \left(C^L + C^S\right)\left(1 - e^{-k\Delta}\right)}{\xi_1 + \left(e^{-(k-\mu)\Delta} - e^{-(k-\mu)T_{MAX}}\right)\xi_2 + e^{-(k-\mu)T_{MAX}}\xi_3} - N = 0. \tag{58}$$

3.5. Design of BOT contract

A BOT contract is designed by balancing the benefits of the government and the private investor. The government usually aims to maximize the expected social welfare over the whole life-cycle of the project, whereas the private investor cares only about its own expected net profit during the concession period. It is vital to coordinate the private investor's and the government's desires when designing the parameters of a rail BOT contract, such as the concession period and rail service parameters (rail line length, number and locations of stations, headway and fare). To do so, a two-stage approach is adopted, in which the first stage determines the concession period, rail line length, and number and spacing of stations, and the second stage determines the train headways and fares for both the private operator and the government.

In stage one, the concession period *T*, rail line length D_1, station number *M* and station locations D_s (*s* = 2, …, *M*) are the results of a bilateral bargaining game between the private investor and the government, given the values of train headways and fares. In the process of reaching an agreement on these decision variables, the private investor and the government may have imbalanced bargaining powers. Such effect may be captured by a Nash bargaining (or negotiating) model, which was proposed by Nash (1951) and has been widely applied and extended in the literature (see, e.g., Harsanyi and Selten, 1972; Chen and Woolley, 2001; Hu et al., 2013; Lv et al., 2014; Yao et al., 2017). Consequently, for a city with population size *N*, the Nash bargaining solutions for the concession period, rail line length, and the number and locations of stations can be determined by the following maximization problem:

$$\max_{T,M,\mathbf{D}} W(T,M,\mathbf{D}) = \left(W_g(T,M,\mathbf{D}) - W_g(T_{N_g^*}, M_{N_g^*}, \mathbf{D}_{N_g^*})\right)^{\omega} \left(\Lambda_p(T,M,\mathbf{D}) - \Lambda_p(T_{N_p^*}, M_{N_p^*}, \mathbf{D}_{N_p^*})\right)^{1-\omega}, \quad (59)$$

$$\text{s.t. } T_{MIN} \le T \le T_{MAX}, \quad (60)$$

$$\delta_{MIN} \le D_s - D_{s+1} \le \delta_{MAX}, \ s = 1, 2, ..., M, \quad (61)$$

where $W(T,M,\mathbf{D})$ is called negotiation value in this paper. The bolded symbol "**D**" represents the vector of station locations, i.e., $\mathbf{D} = (D_s, s = 1, 2, ..., M)$. $W_g(T,M,\mathbf{D}) - W_g(T_{N_g^*}, M_{N_g^*}, \mathbf{D}_{N_g^*})$ and $\Lambda_p(T,M,\mathbf{D}) - \Lambda_p(T_{N_p^*}, M_{N_p^*}, \mathbf{D}_{N_p^*})$ are the relative expected welfare of the government and the relative expected net profit of the private investor, respectively. $W_g(T_{N_g^*}, M_{N_g^*}, \mathbf{D}_{N_g^*})$ is the threat point expected welfare with population threshold N_g^* for the government and $\Lambda_p(T_{N_p^*}, M_{N_p^*}, \mathbf{D}_{N_p^*})$ is the threat point expected net profit with population threshold N_p^* for the private investor.

In Eq. (59), parameter ω measures the relative bargaining power of the government and the private investor, $0 \le \omega \le 1$. If $\omega = 0.5$, then both parties have an equal bargaining power; otherwise, they have imbalanced bargaining powers. If $\omega = 0$ or $\omega = 1$, then the decisions are made by a single side. Eq. (60) represents the negotiating space or the bound constraint of the concession period. T_{MIN} and T_{MAX} can be determined by Eqs. (57) and (58), respectively. δ_{MIN} and δ_{MAX} are the minimum and maximum station spacings (i.e., the lower and upper bounds of station spacings), respectively. Eq. (61) represents the station spacing constraint.

From Eqs. (52)-(55), one can obtain

$$W_g(T,M,\mathbf{D}) - W_g(T_{N_g^*}, M_{N_g^*}, \mathbf{D}_{N_g^*}) = \frac{\left(\xi_1 + (e^{-(k-\mu)\Delta} - e^{-(k-\mu)T})\xi_2 + e^{-(k-\mu)T}\xi_3\right)N}{k-\mu} - \frac{\beta}{\beta-1}\frac{C_2^O(e^{-k\Delta} - e^{-kT}) + C_3^O e^{-kT} + (C^L + C^S)(1 - e^{-k\Delta})}{k}, \quad (62)$$

$$\Lambda_p(T,M,\mathbf{D}) - \Lambda_p(T_{N_p^*}, M_{N_p^*}, \mathbf{D}_{N_p^*}) = \frac{e^{-(k-\mu)\Delta} - e^{-(k-\mu)T}}{k-\mu}\left(\xi_2 - \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)N - \frac{\beta}{\beta-1}\frac{C_2^O(e^{-k\Delta} - e^{-kT}) + (C^L + C^S)(1 - e^{-k\Delta})}{k}. \quad (63)$$

The maximization problem (59)-(61) is a mixed integer programming problem with bound constraints, which is difficult to find its global solution. Note that the number of stations on the rail line is usually a finite number, and can thus be determined by an enumeration method. Given a specific station number *M*, one can then determine the optimal concession period and station locations (including the rail line length). To do so, one can first look at its relaxed problem with removed bound constraints, and then set its solution at the corresponding bound once some bound constraint is violated. From the first-order optimality conditions of the relaxed problem, one can derive the expressions for the optimal solution of the concession period, as shown in Appendix C. The solutions for the locations of stations can be determined by a heuristic solution approach (e.g., cyclic coordinate method, see Bazaraa et al., 2006, pages 365 and 366), in which one determines the location of one station at a time while holding the locations of other stations fixed.

In stage two, the private investor sets the train headway and fare to maximize its own expected net profit during the given concession period (i.e., phase 2). The expected profit maximization problem of the private investor can be expressed as

$$\max \Lambda_p(H_2, f_{v,2}), \tag{64}$$

$$\text{s.t.} \quad \sum_{s=1}^{M} R_{ph} Q_{s,2} \le \frac{K_{cap}}{H_2}, \tag{65}$$

$$f_{v,2} \le f_{MAX}, \tag{66}$$

where R_{ph} is the peak-hour factor, i.e. the ratio of peak-hour flow to daily flow. It is used to convert traffic volume from a daily basis to an hourly basis. $R_{ph}Q_{s,2}$ is the peak-hour passenger demand at station s during private operation (i.e., phase 2). K_{cap} denotes the vehicle capacity. Eq. (65) represents the vehicle capacity constraint. f_{MAX} is the upper bound of the rail fare. Eq. (66) represents the fare cap constraint.

After the concession period is over, the rail line is transferred to the government (i.e., phase 3), and the government sets the headway and fare to maximize the expected social welfare over the whole project life-cycle. The expected social welfare maximization problem for the government can be represented as

$$\max W_g(H_3, f_{v,3}), \tag{67}$$

$$\text{s.t.} \quad \sum_{s=1}^{M} R_{ph} Q_{s,3} \le \frac{K_{cap}}{H_3}, \tag{68}$$

$$f_{v,3} \le f_{MAX}. \tag{69}$$

From the first-order optimality conditions of the expected profit maximization problem (64) and the expected social welfare maximization problem (67), one can derive the optimal headway and fare solutions for the private operator and for the government, as shown in Appendix C.

The BOT-type R + P project contract design problem described above considers the feedback between the R + P project investment and household residential relocation due to the project investment. It is non-linear and non-convex, and thus is difficult to find its global optimal solution. In this paper, a heuristic solution algorithm is proposed to solve this problem. As mentioned before, the number of stations on the rail line can be determined by an enumeration method. For a given number of stations *M*, one can solve the two-stage model using an iterative method, in which the concession period and station location solutions are first determined in one module, and the headway and fare solutions are then determined in the other module. The step-by-step procedure of the heuristic solution algorithm is as follows.

Step 1. Outer loop operation. Determine the optimal number of stations. Set the station number counter to $M=1$.

Step 2. Inner loop operation. Determine the concession period, station locations (including the rail line length), headways and fares based on the two-stage model.

Step 2.0. Initialization of decision variables: concession period $T^{(0)}$, station locations $\mathbf{D}^{(0)}$, headways $H_2^{(0)}$ and $H_3^{(0)}$, and fares $f_{v,2}^{(0)}$ and $f_{v,3}^{(0)}$. Set the inner loop iteration counter to $\mathrm{k}=1$.

Step 2.1. Determination of trigger population thresholds: Calculate the values of the following parameters: $p(0)^{(\kappa)}$, $B^{(\kappa)}$, $g^{(\kappa)}$, $S^{(\kappa)}$, $r^{(\kappa)}$, and $n^{(\kappa)}$ using Eqs. (22), (23), (12), (17), (18), and (19), respectively, and then the annual fare revenue, the annual property revenue, and the expressions for U_1-U_0, U_2-U_1, and U_3-U_1 (see Appendix B). Calculate the rail project investment cost (train operating cost, rail line cost, and rail station cost) by Eqs. (28)-(31). Calculate the passenger demand for each station $Q_s^{(\kappa)}$ by Eq. (26). Calculate the trigger population thresholds N_p^* and N_g^* for the private investor and the government by Eqs. (54) and (55).

Step 2.2. Solving the two-stage model for a given population size $N \geq \max(N_p^*, N_g^*)$.

Step 2.2.1. Solving the first-stage problem. For fixed values of $H_2^{(\kappa-1)}$, $H_3^{(\kappa-1)}$, $f_{v,2}^{(\kappa-1)}$, and

$f_{v,3}^{(\kappa-1)}$, update the concession period $T^{(\kappa)}$ by Eq. (C.1). Calculate $T_{MIN}^{(\kappa)}$ and $T_{MAX}^{(\kappa)}$ by Eqs. (57) and (58), respectively. Check whether the resultant concession period $T^{(\kappa)}$ satisfies bound constraint (60). If it is outside $[T_{MIN}^{(\kappa)}, T_{MAX}^{(\kappa)}]$, then set it at the corresponding bound. Sequentially update the station locations $\mathbf{D}^{(\kappa)}$ using the cyclic coordinate method, i.e. update the location of one station at a time while holding the locations of other stations fixed, until the negotiation value function Eq. (59) cannot be improved. If the station location obtained is outside $[\delta_{MIN}^{(\kappa)}, \delta_{MAX}^{(\kappa)}]$, then set it at the corresponding bound.

Step 2.2.2. Solving the second-stage problem. Update the headways $H_2^{(\kappa)}$ and $H_3^{(\kappa)}$ and the fares $f_{v,2}^{(\kappa)}$ and $f_{v,3}^{(\kappa)}$ by Eqs. (C.2)-(C.5), with fixed values of concession period $T^{(\kappa)}$ and station locations $\mathbf{D}^{(\kappa)}$. Check whether the resultant headways $H_2^{(\kappa)}$ and $H_3^{(\kappa)}$ satisfy constraints (65) and (68). If no, set $H_2^{(\kappa)}$ and/or $H_3^{(\kappa)}$ at the corresponding bound. Check whether the resultant fares $f_{v,2}^{(\kappa)}$ and $f_{v,3}^{(\kappa)}$ satisfy fare constraints (66) and (69). If $f_{v,2}^{(\kappa)}$ or $f_{v,3}^{(\kappa)}$ exceeds the maximum bound, let $f_{v,2}^{(\kappa)}$ or $f_{v,3}^{(\kappa)}$ be equal to the bound.

Step 2.2.3. Convergence check for the inner loop operation. Calculate the values of $W^{(\kappa)}$ (Eq. (59)), $\Lambda_p^{(\kappa)}$ (Eq. (64)) and $W_g^{(\kappa)}$ (Eq. (67)) based on the updated variables. If their values for successive iterations are sufficiently close, then terminate the iteration and output the optimal values $W^{(M)}$, $\Lambda_p^{(M)}$ and $W_g^{(M)}$ and the corresponding variables $\left\{ T^{(M)}, \mathbf{D}^{(M)}, H_2^{(M)}, H_3^{(M)}, f_{v,2}^{(M)}, f_{v,3}^{(M)} \right\}$. Otherwise, set $\kappa = \kappa + 1$ and go to Step 2.1.

Step 3. Convergence check for the outer loop operation. Check the value of $W^{(M)}$ and $W^{(M+1)}$ (Eq. (59)): if their values are sufficiently close, then stop and output the optimal number of stations M^*, objective values (W^*, Λ_p^* and W_g^*) and the corresponding variables $\left\{ T^*, \mathbf{D}^*, H_2^*, H_3^*, f_{v,2}^*, f_{v,3}^* \right\}$. Otherwise, set $M = M + 1$ and go to Step 2.

In Step 2.1, the trigger population thresholds N_p^* and N_g^* for the private investor and the government are determined. After that, one should check whether the population size N in the city concerned has reached the threshold $\max(N_p^*, N_g^*)$. If it is yes, the rail investment project is thus feasible for both the government and the private investor. Otherwise, it should wait till the urban population size grows to the threshold $\max(N_p^*, N_g^*)$.

4. Extension: Incorporating the effects of random jumps in population size

In the previous section, the urban population size is assumed to fluctuate continuously over the whole horizon due to recurrent random events. However, in reality, sudden changes (i.e., jumps or falls) in population size may take place at some time points due to non-recurrent major events, such as the Chinese Spring Festival, and the recent European immigration crisis due to the wars in the Middle East and North Africa. In order to consider the effects of the random jumps in urban population size, we assume that the urban population size, denoted by $N_{t(J)}$, follows a mixed geometric Brownian motion with a Poisson jump upward, which is expressed as

$$dN_{t(J)} = \mu N_{t(J)} dt + \sigma N_{t(J)} dw_t + N_{t(J)} dJ_t, \tag{70}$$

where the subscript "*J*" represents the case with population jumps. dJ_t is the increment of a Poisson process. We assume that the mean arrival rate of dJ_t is λ, and dJ_t and dw_t are independent such that $E\left(dw_t dJ_t\right) = 0$. Suppose that if an event occurs, J_t increases by a fixed percentage ϕ (with $0 \le \phi \le 1$) with a probability of 1. In other words, $N_{t(J)}$ fluctuates as a GBM, but over each time interval dt there is a small probability of λdt that it increases to $(1+\phi)$ times of its original value, and then continues fluctuating until another event occurs. Note that the expected percentage rate of $N_{t(J)}$ is $E\left(dN_{t(J)}\right) / N_{t(J)} = \left(\mu + \lambda\phi\right) dt$.

We now solve the BOT investment problem with a random jump in the population size using the dynamic programming, similar to the previous section. Define $F_j(N_{t(J)})$ as the value of the investment opportunity with jump process. It satisfies

$$F_j(N_{t(J)}) = e^{-kdt} E_t\left[F_j(N_{t(J)} + dN_{t(J)})\right]. \tag{71}$$

Expanding the right-hand side of Eq. (71) by Ito's lemma for the combined Brownian and Poisson processes (see Dixit and Pindyck, 1994), and omitting the terms that go to zero faster than dt as $dt \to 0$ yield the following:

$$e^{-kdt} E_t \left[F_j (N_{t(J)} + dN_{t(J)}) \right]$$

$$= (1-kdt) E_t \left(F_j(N_{t(J)}) + \frac{\partial F_j(N_{t(J)})}{\partial t} dt + \mu N_J \frac{\partial F_j(N_{t(J)})}{\partial N_J} dt + \frac{1}{2} \sigma^2 N_J{}^2 \frac{\partial^2 F_j(N_{t(J)})}{\partial N_J{}^2} dt + \lambda \left(F_j\left((1+\phi) N_{t(J)}\right) - F_j(N_{t(J)}) \right) dt \right)$$

$$= E_t \left(F_j(N_{t(J)}) + \frac{1}{2} \sigma^2 N^2 \frac{\partial^2 F_j(N_{t(J)})}{\partial N_J{}^2} dt + \mu N_J \frac{\partial F_j(N_{t(J)})}{\partial N_J} dt + \frac{\partial F_j(N_{t(J)})}{\partial t} dt - k F_j(N_{t(J)}) dt + \lambda \left(F_j\left((1+\phi) N_{t(J)}\right) - F_j(N_{t(J)}) \right) dt \right). \quad (72)$$

From Eqs. (71) and (72), one can obtain the following partial differential equation:

$$\frac{1}{2} \sigma^2 N_J{}^2 \frac{\partial^2 F_j(N_{t(J)})}{\partial N_J{}^2} + \mu N_J \frac{\partial F_j(N_{t(J)})}{\partial N_J} + \frac{\partial F_j(N_{t(J)})}{\partial t} - (k+\lambda) F_j(N_{t(J)}) + \lambda F_j\left((1+\phi) N_{t(J)}\right) = 0 . \quad (73)$$

According to the boundary conditions (47)-(49), the solution of Eq. (73) satisfies the form

$$F_j(N_{t(J)}) = A_{j(J)} (N_{t(J)})^{\beta_J} , \quad (74)$$

where β_J is a positive solution of the following equation

$$\frac{1}{2} \sigma^2 \beta_J (\beta_J - 1) + \mu \beta_J - (k+\lambda) + \lambda (1+\phi)^{\beta_J} = 0 . \quad (75)$$

The parameter $A_{j(J)}$ is similar to A_j in Eq. (51), and can be obtained by replacing μ and β in Eq. (51) by $\mu + \lambda\phi$ and β_J, respectively.

Similarly, one can derive the project values (i.e., expected net profit $\Lambda_{p(J)}(N_J)$ and expected welfare gain $W_{g(J)}(N_J)$), investment timing (i.e., $N^*_{p(J)}$ for the private investor and $N^*_{g(J)}$ for the government), and feasible concession period (i.e., minimum acceptable concession period $T_{MIN(J)}$ for the private investor and maximum acceptable concession period $T_{MAX(J)}$ for the government) with population random jumps, by replacing μ and β in Section 3.4.1 by $\mu + \lambda\phi$ and β_J, respectively. For convenience of readers, we summarize their results in Appendix D.

5. Model applications

In this section, numerical examples are used to illustrate the applications of the proposed model and the contributions of this paper. Specifically, we examine the effects of the R + P scheme, population volatility level, population jumps, and negotiation power on the rail BOT contract design and the effects of the R + P project on the urban spatial structure. We also examine and compare the optimal BOT contract solutions with different station deployment schemes (even and uneven station spacings). Table 2 defines all of the model parameters and their baseline values as used in the numerical examples.

Table 2 Values of input parameters.

Symbol	Definition	Baseline value
f_f	Fixed component of fare (RMB)	3
f_{MAX}	Upper bound of variable rail fare (RMB/km)	1.0
$f_{v,0}, f_{v,1}$	Fare before and during project construction (RMB)	0, 0
K_{cap}	Capacity of vehicles (passengers/vehicle)	1500
$\bar{V}$	Average walking speed of passengers (km/h)	3.0
V_0, V_1	Average journey speed before and during rail construction (km/h)	20, 18
H_0, H_1	Transit service headway (hour/vehicle)	0.3, 0.3
V_2, V_3	Average vehicle speed during private and government operations (km/h)	60, 55
Y	Average annual income of households (RMB)	200,000
ρ	Average annual number of trips to CBD per household	365
τ_a	Value of access time (RMB/h)	80
τ_w	Value of wait time (RMB/h)	80
τ_t	Value of in-vehicle time (RMB/h)	30
α	Parameters in household utility function	40,000
γ	Parameter in passenger wait time function	0.5
a, θ	Parameters in housing production function	0.05, 0.7
π	Sensitivity parameter in elastic demand function	0.003
t_0	Average train dwell time at a rail station (h)	0.01
ζ	Number of terminal times on rail line	2.0
t_c	Constant terminal time on the line (h)	0.1
Δ_s^0	Radius of property development above stations (km)	0.1
η	Average daily number of trips to CBD per household	1.0
R_{ph}	Peak-hour factor, i.e., ratio of peak-hour flow to daily average flow	0.1
δ_{MIN}	Minimum station spacing requirement (km)	0.8
δ_{MAX}	Maximum station spacing requirement (km)	3.0
μ	Annual population growth rate	1.1%
σ	Population volatility rate	0.1
ω	Relative bargaining power	0.5
λ	Mean arrival rate	0.1
ϕ	A parameter in geometric Brownian motion with a Poisson jump	0.1
k	Discount rate	6%
C_f^O, C_v^O	Fixed and variable components of annual train operating cost (million RMB/year)	15, 30
C_f^L, C_v^L	Fixed and variable components of annual rail line cost (million RMB/year)	33, 46
C_f^S, C_v^S	Fixed and variable components of annual rail station cost (million RMB/year)	38, 57
Δ	Construction duration of a rail line project (years)	5

Source: please refer to Li et al. (2012, 2015) and Peng et al. (2017).

5.1. Comparison of investment timings under R + P and rail-only schemes with and without jumps

Fig. 4 shows the option value curves and the NPV curves under the R + P and rail-only schemes with and without jump occurring. For ease of presentation, we adopt the superscript "′" to represent the results with the rail-only scheme and the subscript "*J*" to represent the results with jump occurring, as shown in Fig. 4(b), (a′) and (b′). Fig. 4(a) depicts the option value curves and the NPV curves under the R + P scheme without jump occurring, which can be calculated by Eqs. (50), (52) and (53). In Fig. 4(a), the intersection points D_1, E_1, D_2, and E_2 between the option value curves and the NPV curves represent the population thresholds N_g^* and N_p^* of the government (shown in dotted lines) and the private investor (shown in solid lines) under the RO approach in terms of Eqs. (39) and (42). The intersection points G_1 and G_2 between the NPV curves and the horizontal line denote the population thresholds $N_{g(NPV)}^*$ and $N_{p(NPV)}^*$ of the government and the private investor under the NPV approach. In Fig. 4(b), the intersection points (D_1', E_1', D_2', and E_2') and (G_1' and G_2') are, respectively, the population threshold solutions with the RO approach and the NPV approach for the rail-only scheme. Fig. 4(a′) and (b′) shows the associated results for the R + P and rail-only schemes with jump occurring, respectively.

From Fig. 4(a), (b), (a′) and (b′), we obtain the following insights. First, compared with the RO approach, the NPV approach underestimates the value of the rail transit investment project regardless of whether the R + P scheme or the rail-only scheme is used and/or whether the population jump occurs, causing premature investment and thus a loss in the project value. For example, in Fig. 4(a), for the R + P scheme with $\sigma = 0.1$, the losses in the project value due to the adoption of the NPV approach for the government and the private investor are the differences between the total option value and the total NPV accrued from G_1 (the NPV solution for the government) to D_1 (the RO solution for the government) and from G_2 (the NPV solution for the private investor) to D_2 (the RO solution for the private investor), respectively. They are associated with the areas of $D_1G_1F_1$ and $D_2G_2F_2$, respectively. Similar observations can also be found in Fig. 4(b), (a′) and (b′).

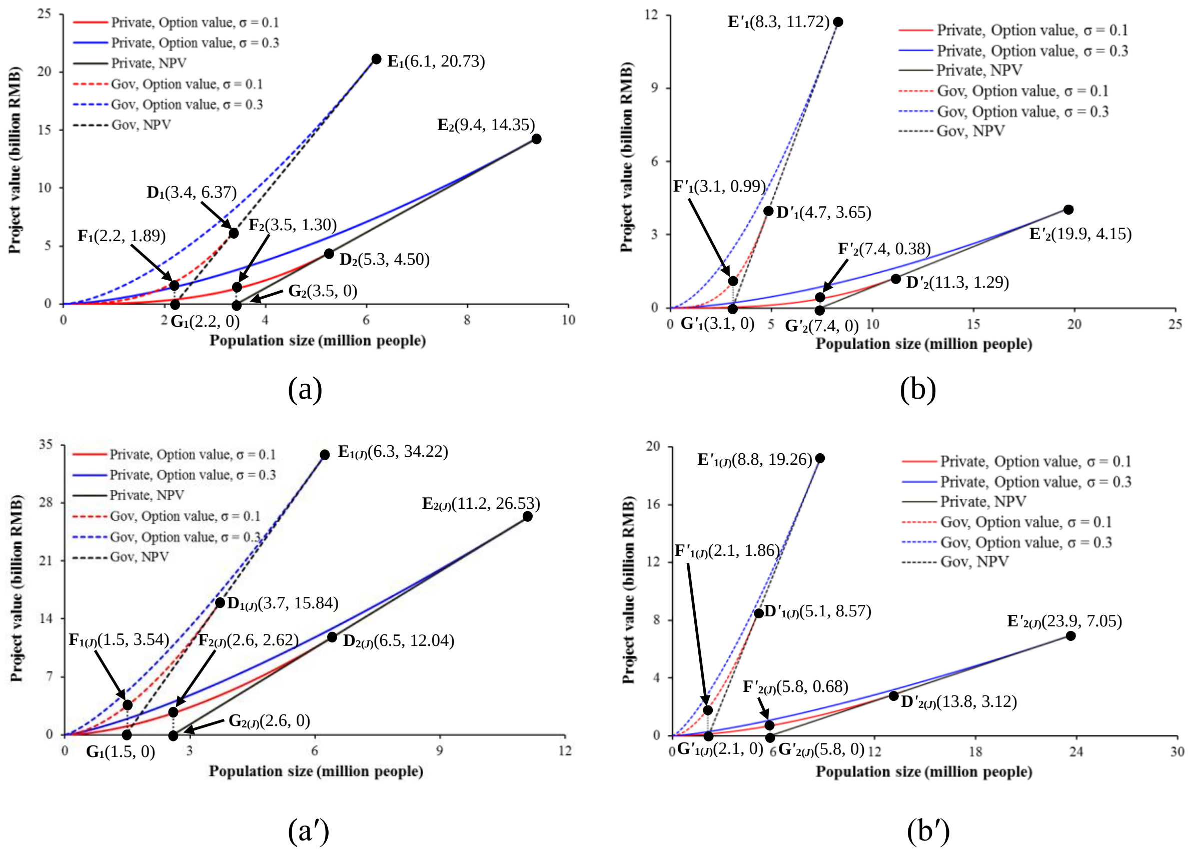


Fig. 4. Option value curves and NPV curves with and without jump occurring: (a) R + P scheme without jump occurring; (b) rail-only scheme without jump occurring; (a′) R + P scheme with jump occurring; (b′) rail-only scheme with jump occurring. The dotted and solid lines represent the results associated with the government and private investor, respectively.

Second, the government wishes to invest earlier than the private investor regardless of whether the R + P model or the rail-only model is used and/or whether the population jump occurs. Furthermore, both the government and the private investor choose to invest earlier with the R + P model than with the rail-only model regardless of whether the population jump occurs. For example, for the R + P scheme with $\sigma = 0.1$ and no-jump occurring (see Fig. 4(a)), the trigger population thresholds N_g^* and N_p^* for the government and the private investor are 3.4 and 5.3 million households, corresponding to points D_1 and D_2 in Fig. 4(a), respectively. The corresponding values of $N_g'^*$ and $N_p'^*$ with the rail-only scheme are 4.7 and 11.3 million households, shown as points D_1' and D_2' in Fig. 4(b). For the R + P scheme with $\sigma = 0.1$ and jump occurring (see Fig. 4(a′)), the trigger population thresholds $N_{g(J)}^*$ and $N_{p(J)}^*$ for the government and the private investor are 3.7 and 6.5 million households, associated with points $D_{1(J)}$ and $D_{2(J)}$, respectively. The corresponding values of $N'^{*}_{g(J)}$ and $N'^{*}_{p(J)}$ with the rail-only scheme are 5.1 and 13.8 million households, shown as points $D'_{1(J)}$ and $D'_{2(J)}$ in Fig. 4(b′). The results are consistent for the population volatility rate of $\sigma = 0.3$.

Third, the option value curve $F(N)$ with a large σ is located above that with a small σ for the government or the private investor, regardless of whether the population jump occurs. This means that a larger σ leads to a later investment. Accordingly, with greater population uncertainty, the investor must wait longer to invest to allow the urban population to reach the level of the trigger population threshold. For example, under the R + P scheme without jump occurring (see Fig. 4(a)), as σ increases from 0.1 to 0.3, the population thresholds N^{*}_{g} and N^{*}_{p} for the government and the private investor increase from 3.4 (D_1) to 6.1 (E_1) million households and from 5.3 (D_2) to 9.4 (E_2) million households, respectively. The option values for the government and the private investor increase from RMB6.37 (D_1) to 20.73 (E_1) billion and from RMB4.50 (D_2) to 14.35 (E_2) billion, respectively. Similar observations can also be made for the rail-only scheme and for the jump occurring case, as shown in Fig. 4(b), (a′) and (b′).

Fourth, population jumps upward lead to a late investment for the government or the private investor, compared to no jump case. For example, comparing Fig. 4(a) and (a′), it can be seen that for the R + P scheme with $\sigma = 0.1$ and jump occurring (see Fig. 4(a′)), the trigger population thresholds $N^{*}_{g(J)}$ and $N^{*}_{p(J)}$ for the government and the private investor are 3.7 and 6.5 million households, associated with points $D_{1(J)}$ and $D_{2(J)}$, respectively. However, the corresponding trigger population thresholds N^{*}_{g} and N^{*}_{p} without jump occurring are 3.4 and 5.3 million households, as shown in Fig. 4(a).

5.2. Comparison of BOT contracts under R + P and rail-only schemes with and without jumps

Table 3 shows the optimal solutions for the concession period, rail line length, number of stations, headway, and fare under the R + P and rail-only schemes with and without jump occurring. We first look at the effects of introducing the R + P scheme for the no jump case. It can be seen from Columns 2 and 4 of Table 3 that the number of stations under the R + P scheme (21 stations) is far more than that under the rail-only scheme (13 stations). Based on the optimal number of stations, the optimal rail line length is 41.50 km for the R + P scheme and 24.70 km for the rail-only scheme. The optimal durations of the concession periods for the R + P and rail-only schemes are 30.39 and 37.28 years, respectively. These results show that the R + P scheme leads to a longer rail line and more stations but a shorter concession period. In addition, the R + P scheme leads to an increase in the headway during the private and government operations by 0.04 and 0.02 h, but a decrease in the fare during the private and government operations by RMB0.26 and RMB0.22, respectively.

Table 3 Optimal solutions under R + P and rail-only schemes.

Solution	R + P scheme		Rail-only scheme	
	No jump	Jump	No jump	Jump
Optimal concession period (years)	30.39	32.96	37.28	38.84
Optimal number of stations	21	25	13	17
Optimal rail line length (km)	41.5	49.7	24.7	31.7
Optimal headway for private operation (h)	0.23	0.16	0.19	0.15
Optimal fare for private operation (RMB)	0.34	0.25	0.60	0.49
Optimal headway for government operation (h)	0.10	0.09	0.08	0.05
Optimal fare for government operation (RMB)	0.20	0.13	0.42	0.24
Minimum acceptable concession period for private investor (years)	22.82	24.52	26.95	27.51
Maximum acceptable concession period for government (years)	61.21	65.76	71.96	72.48
Resultant city boundary before project investment (km)	53.35	61.27	37.16	43.86
Resultant city boundary during project construction (km)	52.74	60.37	36.89	43.25
Resultant city boundary during private operation (km)	54.18	62.29	38.83	44.56
Resultant city boundary during government operation (km)	55.16	63.25	39.78	45.82
Expected net profit of private investor (billion RMB/year)	78.46	96.54	38.43	47.32
Total expected social welfare (billion RMB/year)	146.37	220.56	68.31	108.53

Columns 2 and 4 of Table 3 also shows that compared to the rail-only scheme without jumps, the R + P scheme without jumps gives a longer city boundary. Specifically, the city boundaries resulting from the R + P scheme before project investment, during project construction, during private operation, and during government operation are 53.35, 52.74, 54.18, and 55.16 km, respectively. The corresponding values for the rail-only scheme are 37.16, 36.89, 38.83, and 39.78 km, respectively, all of which are smaller than those for the R

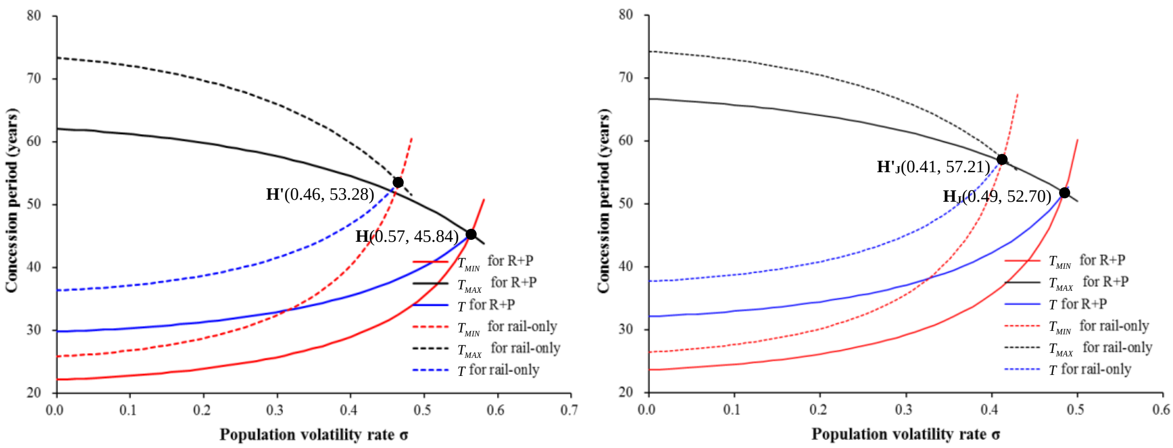
+ P scheme. This means that the R + P scheme can cause urban expansion or sprawl by spreading the property development into the suburb areas, leading to a more decentralized city. In addition, Table 3 shows that the R + P scheme yields the private investor a net profit of RMB78.46 billion, more than double the expected net profit (RMB38.43 billion) generated by the rail-only scheme. From the perspective of the government, the R + P scheme leads to greater expected social welfare (RMB146.37 billion) than the rail-only scheme (RMB68.31billion). Consequently, the R + P scheme can result in a win-win solution for the government and the private investor by significantly improving the expected social welfare and the expected net profit gained by the private investor. Similar results can be observed for the case with jump occurring (see Columns 3 and 5 of Table 3).

We now look at the effects of population jumps on the model solutions. Comparing Columns 2 and 3 (or Columns 4 and 5) of Table 3, it can be seen that the population jump occurring causes a decrease in the headways and fares for both private and government operations, but an increase in the concession period, number of stations and the rail line length, regardless of the R + P scheme or the rail-only scheme. For example, for the R + P scheme (see Columns 2 and 3 of Table 3), after considering the effects of the population jumps, the headway and fare for the private operation, and the headway and fare for the government operation decrease by 0.07 h and RMB0.09, and 0.01h and RMB0.07, whereas the concession period, the number of stations and the rail line length increase by 2.57 years, 4 stations and 8.20 km, respectively. Both the associated expected net profit of the private operator and the associated expected social welfare of the system also increase for either the R + P scheme or the rail-only scheme. Again, we take the R + P scheme as an example. Columns 2 and 3 of Table 3 show that the expected net profit of the private operator and the expected social welfare increase by RMB18.08 billion and RMB74.19 billion, respectively This means that ignoring the effects of population jumps in the BOT contract design leads to an underestimate of the private operator’s expected net profit and the system’s expected social welfare.

5.3. Effects of uncertainty on BOT concession period under R + P and rail-only schemes with and without jumps

Fig. 5(a) and (b) depicts the acceptable minimum concession period curves for the private investor, the acceptable maximum concession period curves for the government, and the optimal concession period curves with and without population jump occurring under the R +

P and rail-only schemes, which can be calculated using Eqs. (57), (58), (C.1), (D.7), (D.8) and (D.9), respectively. Fig. 5(a) and (b) shows that as the population volatility rate increases, the acceptable minimum concession period for the private investor always increases, whereas the acceptable maximum concession period for the government always decreases. These curves intersect at a point (i.e., H, H', H_J or H'_J), which is associated with an allowable maximum value of urban population volatility, denoted as $\sigma_{\max}$. On the left-hand side of this point $\sigma_{\max}$, the acceptable minimum concession period curve (in red) is always below the acceptable maximum concession period curve (in black). As a result of negotiation between the private investor and the government, the optimal concession period curve is in between (in blue). Beyond this point $\sigma_{\max}$, the acceptable minimum concession period exceeds the acceptable maximum concession period, meaning that no bargaining solution exists for the private investor and the government, i.e., their negotiation fails.



(a) (b)

Fig. 5. Effects of population volatility rate and jump occurring on concession period under R + P and rail-only schemes: (a) no jump occurring; (b) jump occurring.

The value of $\sigma_{\max}$ is 0.57 for the R + P scheme without jumps (i.e., point H), 0.46 for the rail-only scheme without jumps (i.e., point H'), 0.49 for the R + P scheme with jumps (i.e., point H_J) and 0.41 for the rail-only scheme with jumps (i.e., point H'_J). These results indicate that for a given population volatility rate σ, the rail-only scheme would overestimate the optimal concession period solution regardless of the jumps or no jumps. The jump occurring can narrow the feasible region of the bargaining solutions between the private investor and the government. Specifically, σ_{max} decreases by 0.08 for the R + P scheme and 0.05 for the rail-only scheme.

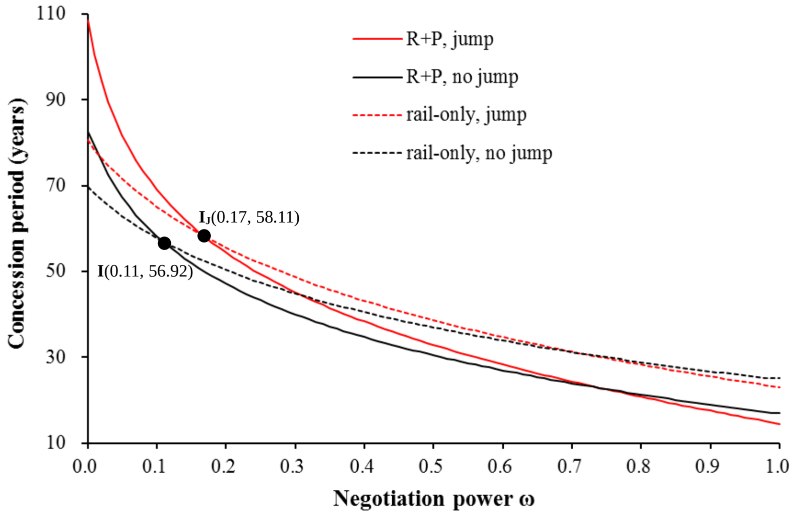


Fig. 6. Effects of negotiation power and jumps on optimal concession period with R + P and rail-only schemes.

5.4. Effects of negotiation power and population jumps on BOT concession period under R + P and rail-only schemes

Fig. 6 displays the changes of optimal concession period with negotiation power under the R + P and rail-only schemes with and without jump occurring. It shows that as the government's negotiation power increases, the optimal concession period always decreases, regardless of which scheme is used or whether the jumps occur. For either jump or no-jump case, there is an indifference negotiation power ω^*, in which the concession period is equal for both R + P and rail-only schemes. However, the rail-only scheme underestimates the optimal concession period for $\omega \leq \omega^*$, and overestimates the optimal concession period for $\omega > \omega^*$, compared to the R + P scheme. Specifically, the indifference negotiation power ω^* is 0.11 for the no jump case (i.e., point I), and 0.17 for the jump case (i.e., point I_J). Accordingly, the jump occurring increases the value of indifference negotiation power ω^*, implying an increase in the range of ω of underestimating the optimal concession period, and a reduction in the range of ω of overestimating the optimal concession period.

5.5. Effects of R + P scheme on urban spatial structure

We now explore the effects of rail line investment on urban spatial structure. We find that the effects with and without population jump occurring are similar. For illustration of the essential findings, we take the case without jumps as an example. Fig. 7(a)-(d) displays the profiles of the equilibrium household residential density, equilibrium housing rental price, equilibrium housing space per household, and equilibrium capital investment intensity for the R + P and rail-only schemes without jumps during the private and government operations when the urban population size is fixed at 13.8 million people (i.e., the trigger population threshold for the private investment under rail-only scheme with jump occurring, see Fig. 4(b′)).

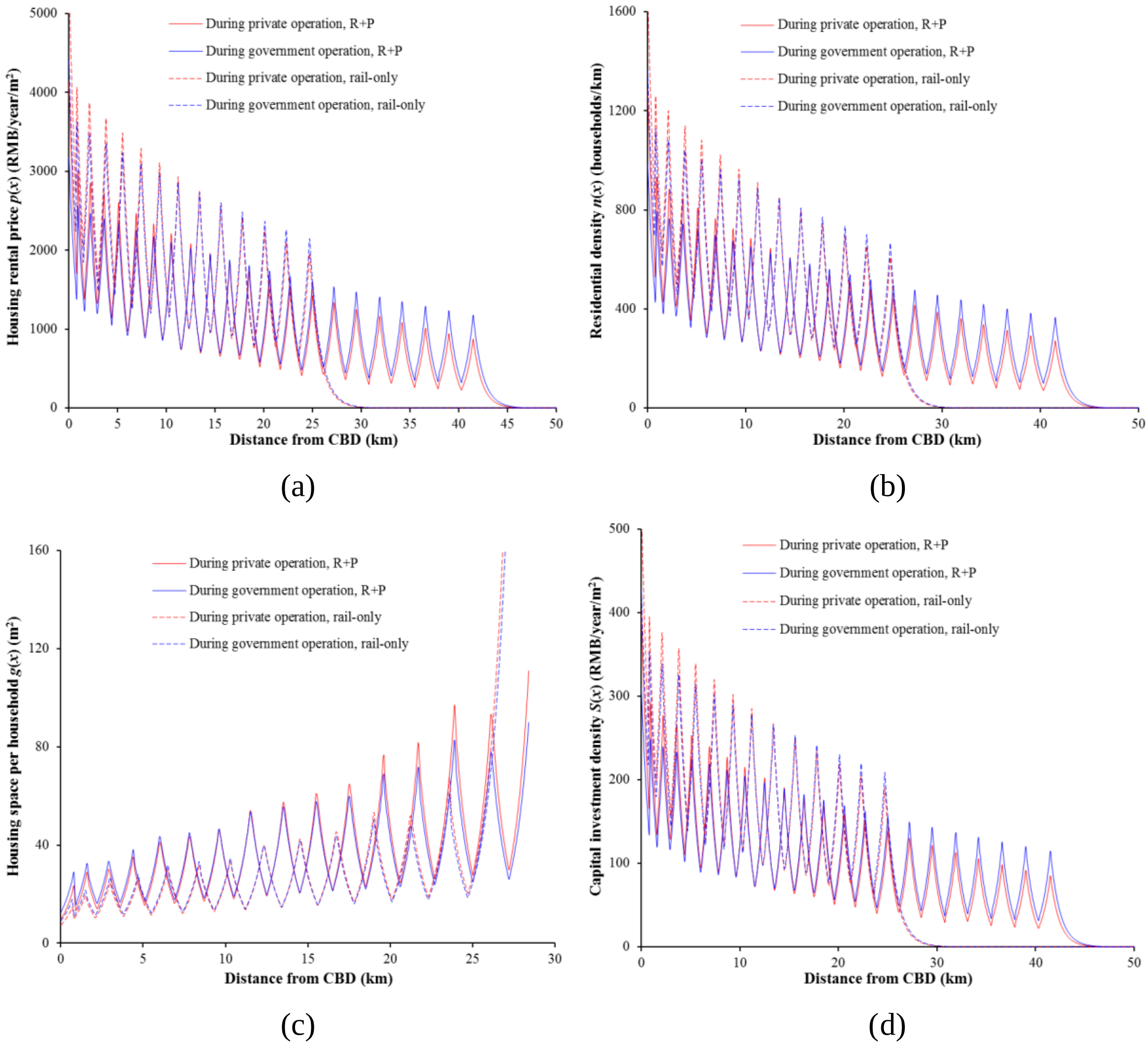


Fig. 7. (a)-(d) represent household residential density, housing rental price, housing space per household, and capital investment density under R + P and rail-only schemes.

Fig. 7(a) indicates the household residential density along the linear corridor under the R + P scheme (solid lines) and the rail-only scheme (dotted lines), respectively. It shows that all of the residential density curves change in a zigzag pattern with respect to the distance from the CBD. Household residential density near the station achieves its local maximum due to the highest level of station accessibility. On both sides of the station, household residential density descends with an increase in the distance from the station, and vice versa. Implementing the R + P scheme will reduce residential density in the urban central area and increase it in the city's outskirts, making the city more decentralized, particularly after the government has taken over the rail BOT project.

Fig. 7(b) shows that the housing rental price curves under the R + P and rail-only schemes decrease wavily with the distance from the CBD. Specifically, the housing rental price at each station achieves its local maximum due to the highest level of station accessibility. On both sides of each station, the station accessibility decreases and thus the housing rental price decreases with the distance from the station, and vice versa. Similar to household residential density (shown in Fig. 7(a)), compared with the rail-only scheme, the housing rental price with the R + P scheme is lower for the urban central area but higher for the suburbs. This implies that the R + P scheme helps to promote the healthy development of the real estate market in both the urban central area and the suburbs.

Fig. 7(c) plots the variation in housing space per household under the R + P and rail-only schemes. It shows that the housing space curves change in a zigzag pattern with the distance from the CBD, and that the average housing space per household near a station is lower than that per household far away from the same station for each of the two schemes. This is due to the high household residential density near the station. Fig. 7(d) shows the change in the capital investment density under the R + P and rail-only schemes. As expected, the trend of change in the capital investment density is consistent with that in household residential density, as shown in Fig. 7(a). That is, under the R + P scheme, compared with the rail-only scheme, property developers wish to provide more houses in the suburbs by reducing housing supply in the urban central area. When the government takes over the rail BOT project, more houses should be built in suburban areas, because more residents wish to live in the suburbs due to improved public transit services.

5.6. Comparison of the results with uneven and even station spacings

In order to look at the effects of the station configuration along the rail line on the BOT contract design, we consider two types of station deployment schemes, namely even (or average) and uneven station spacings. The even (or average) station spacing, meaning that $D_s - D_{s+1}$ is a constant for any $s = 1, 2, ..., M$, is often used as an estimate or approximation of the station deployments in the BOT contract design. This is because it can provide useful information for the rail operator on the planning and design of rail transit line, particularly at an early stage. Again, we take the case without population jump occurring as an example due to similar observations found for the case with jump occurring. Fig. 8 shows the changes of the negotiation value (i.e., Eq. (59)) with the number of stations for the even and uneven station deployments without jump occurring. It can be seen that as the number of stations in the rail corridor increases, the negotiation value first increases and then decreases regardless of the R + P or rail-only scheme. Given an investment scheme (R + P or rail-only), the negotiation value curve with the uneven station spacing would be above that with the even station spacing. This means that the even station deployment is inferior to the uneven one for a given number of stations, in terms of the negotiation value. Moreover, the optimal even station configuration requires more stations to be built than the optimal uneven one. Specifically, for the R + P scheme, the optimal numbers of stations with the even and uneven station spacings are 23 and 21, associated with negotiation values of RMB91.92 billion and RMB95.32 billion, respectively. For the rail-only scheme, the optimal numbers of stations are 14 and 13 for the even and uneven station deployments, respectively.

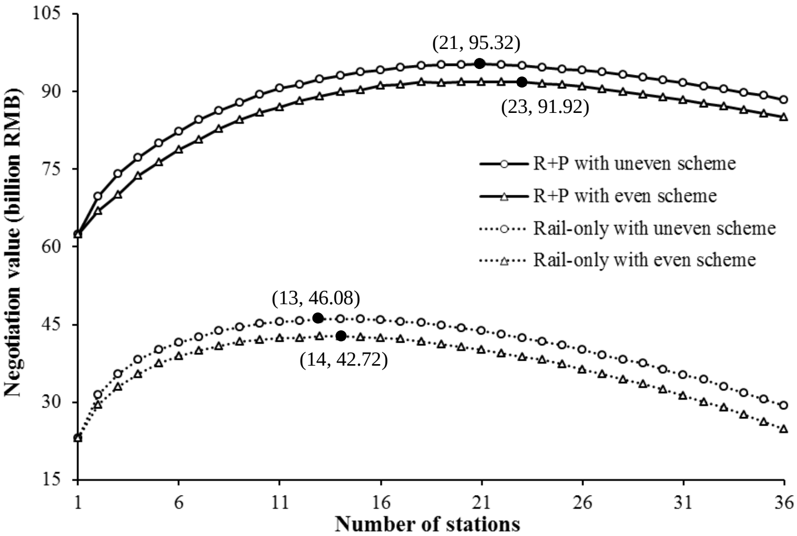


Fig. 8. Changes of negotiation value with the number of stations for even and uneven station deployments without jump occurring.

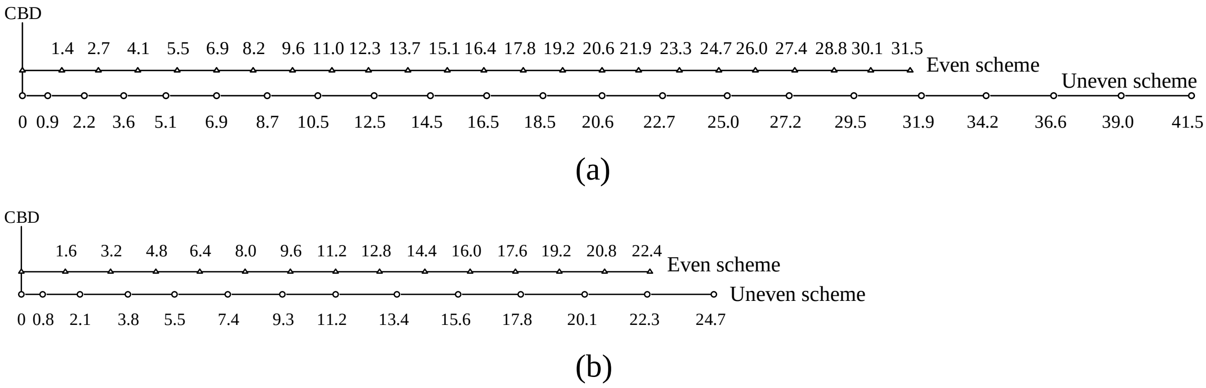


Fig. 9. Optimal station deployments under even and uneven schemes without jump occurring: (a) R + P scheme; (b) rail-only scheme.

Table 4 The solutions with even and uneven station deployments for the R + P scheme.

Solutions	Uneven scheme	Even scheme
Optimal concession period (years)	30.39	33.47
Optimal number of stations	21	23
Optimal rail line length (km)	41.5	31.5
Optimal headway for private operation (h)	0.23	0.20
Optimal fare for private operation (RMB)	0.34	0.40
Optimal headway for government operation (h)	0.10	0.09
Optimal fare for government operation (RMB)	0.20	0.28
Minimum acceptable concession period for private investor (years)	22.82	24.56
Maximum acceptable concession period for government (years)	61.21	57.64
Resultant city boundary before project investment (km)	53.35	43.72
Resultant city boundary during project construction (km)	52.74	43.45
Resultant city boundary during private operation (km)	54.18	44.69
Resultant city boundary during government operation (km)	55.16	45.78
Expected net profit of private investor (billion RMB/year)	78.46	70.34
Total expected social welfare (billion RMB/year)	146.37	132.57

Fig. 9(a) and (b) shows the detailed locations of the stations along the rail line with the optimal station deployment solutions for the R + P and rail-only schemes, respectively. It can be seen that the rail line length with the even station spacing is shorter than that with the uneven station spacing, although the even one requires more stations. As a result, the average station spacing with the even station deployment is shorter than that with the uneven station deployment. Specifically, the rail line length with the even station spacing is 10 km shorter than that with the uneven station spacing for the R + P scheme. This number is 2.3 km for the rail-only scheme.

Table 4 further summarizes the solutions with even and uneven station deployments for the R + P scheme. One can see that compared to the uneven station deployment scheme, the even scheme causes an increase in the concession period by about 3 years. It also causes a decrease in the city boundary or length (thus a more compact city) and in the headways during the private and government operations, but an increase in the fares during both operations. As a result, both the expected net profit of the private operator and the expected social welfare decrease by RMB8.12 billion and RMB13.8 billion, respectively. Thereby, the uneven station deployment scheme is a better option in practice due to the “win-win” situation achieved in terms of the expected net profit and the social welfare.

6. Conclusion and further studies

This paper addresses the issues of designing a BOT contract for an integrated R + P development scheme under future urban population uncertainty. A real option-based two-stage model is proposed to determine the optimal concession period (including investment timing) and rail line parameters. The first stage of the model determines the concession period, rail line length, and number and locations of stations through a bilateral Nash bargaining game between a private investor and the government. The second stage optimizes the headway and fare during the private concession period of the BOT project, and the headway and fare after transferring the BOT project to the government. The proposed model is also extended to consider the effects of random jumps in urban population size and rail station deployments (even and uneven station spacings) on the BOT contract design. The results from the R + P and rail-only schemes with considerations of population jump occurring and even/uneven station spacings are compared, together with their effects on the urban spatial structure.

The results generated by the proposed model offer some new and important insights on the issues concerned. First, we show that compared with the rail-only scheme, the R + P scheme leads to urban sprawl and early investment. It also causes an increase in rail line length, number of stations, and headway, but a reduction in concession period and rail fare. In addition, the R + P scheme can significantly affect urban residential spatial distribution and the housing market, contributing to the sustainable development of the urban housing market by reducing the gap in housing rental prices in both the urban central area and suburban areas, particularly after the government has taken over the rail BOT project. Implementing the R + P scheme can generate a “win-win” situation for a profit-driven private investor and a welfare-driven government. For a given development model, the government wishes to invest earlier than the private investor. Second, a higher rate of population volatility results in a higher option value and a higher trigger population threshold, thus implying later investment. Third, population jump occurring can lead to a decrease in the headways and fares for both private and government operations, but an increase in the concession period, number of stations and the rail line length. It can also reduce the feasible region of the bargaining solutions between the private investor and the government, and increase the critical value of the indifference negotiation power between the R + P and rail-only schemes. Ignoring the effects of population jumps in the BOT contract design can lead to an underestimate of project value, early investment and a decreased duration of concession period. Fourth, in the BOT contract design, using an average (or even) station spacing as an estimate or approximation of the actual station deployment can cause a large bias of the parameter values designed in the contract and an underestimate of the project values in terms of the expected net profit and the expected social welfare. Therefore, an optimal uneven station deployment scheme should be adopted in practice to design a more professional, meticulous, and efficient BOT contract.

Although the proposed model provides useful insights for BOT contract design for integrated R + P development project under future population uncertainty, some important extensions can be made for further studies.

(1) The paper focuses on rail BOT project investment. The public-private-partnership (PPP) pattern has become a hot topic in the field of mass public transit investment. The proposed model should thus be extended to explore the problem of designing PPP contracts for mass public transit investment projects, which involves revenue and risk sharing issues.
(2) Only one single investment project is considered in this paper. In reality, multiple projects may require investment over a planning horizon. The effects of correlation between these projects (e.g., project investment sequence) should be taken into consideration.
(3) The source of uncertainty in this paper is fluctuation in urban population size (i.e., the demand side). In reality, various other sources of uncertainty exist, such as variation in investment cost or interest rate. Therefore, it is necessary to consider hybrid sources of

uncertainty from the supply and demand sides when designing contracts for rail investment projects.

(4) The housing developed near stations may be more attractive to residents than the general type of housing due to high accessibility to railway stations. Such attraction can be considered in the model by using an amenity function, which is left for a further study. In addition, the radius of the property development over stations, as an important decision variable, can also be optimized in a future study.

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Appendix A: Derivations of Eqs. (22), (23) and (26) and proof of Proposition 1

(i) Substituting Eq. (19) into Eq. (20) yields

$$\int_0^{B_i} n_i(x)dx = \int_0^{L_{M,i}} n_i(x)dx + \sum_{s=1}^{M}\left(\int_{L_{s,i}}^{D_s} n_i(x)dx + \int_{D_s}^{L_{s-1,i}} n_i(x)dx\right) = N\,, \tag{A.1}$$

where $L_{s,i}$ is given by Eq. (5) and $L_{0,i} = B_i$. The first term on the right-hand side of Eq. (A.1) denotes the number of households between the CBD and the watershed line $L_{M,i}$. Note that the residents between the CBD and $L_{M,i}$ walk to the CBD. We then have

$$\int_0^{L_{M,i}} n_i(x)dx = \frac{\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_0^{L_{M,i}} \exp\left(-\frac{\varphi_i(x)}{\alpha}\right)^{\frac{1}{1-\theta}} dx$$

$$= \frac{(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{2\rho\tau_a}\left(1-\exp\left(-\frac{2\rho\tau_a L_{M,i}}{\alpha(1-\theta)\bar{V}}\right)\right). \tag{A.2}$$

In the following, we determine the terms in the brackets of Eq. (A.1). For any $s = 1, 2, ..., M$,

$$\int_{L_{s,i}}^{D_s} n_i(x)dx = \frac{\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{L_{s,i}}^{D_s} \exp\left(-\frac{\varphi_i(x)}{\alpha}\right)^{\frac{1}{1-\theta}} dx$$

$$= \frac{\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{L_{s,i}}^{D_s} \exp\left(-\frac{2\rho}{\alpha}\left(\frac{\tau_a}{\bar{V}}(D_s - x) + C_{s,i}(D_s)\right)\right)^{\frac{1}{1-\theta}} dx$$

$$= \frac{(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{2\rho\tau_a}\left(\exp\left(-\frac{2\rho}{\alpha(1-\theta)}C_{s,i}(D_s)\right) - \exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(D_s - L_{s,i}) + C_{s,i}(D_s)\right)\right)\right). \tag{A.3}$$

$$\int_{D_s}^{L_{s-1,i}} n_i(x)dx = \frac{\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{D_s}^{L_{s-1,i}} \exp\left(-\frac{\varphi_i(x)}{\alpha}\right)^{\frac{1}{1-\theta}} dx$$

$$= \frac{\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{D_s}^{L_{s-1,i}} \exp\left(-\frac{2\rho}{\alpha}\left(\frac{\tau_a}{\bar{V}}(x - D_s) + C_{s,i}(D_s)\right)\right)^{\frac{1}{1-\theta}} dx$$

$$= \frac{(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{2\rho\tau_a}\left(\exp\left(-\frac{2\rho}{\alpha(1-\theta)}C_{s,i}(D_s)\right) - \exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(L_{s-1,i} - D_s) + C_{s,i}(D_s)\right)\right)\right). \tag{A.4}$$

Substituting Eqs. (A.2)-(A.4) into Eq. (A.1) gives

$$N=\frac{(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{2\rho\tau_a}\left\{\begin{array}{l}\left(1-\exp\left(-\frac{2\rho\tau_a L_{M,i}}{\alpha(1-\theta)\bar{V}}\right)\right)+\sum_{s=1}^{M}\left(2\exp\left(-\frac{2\rho}{\alpha(1-\theta)}C_{s,i}(D_s)\right)-\exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(D_s-L_{s,i})+C_{s,i}(D_s)\right)\right)\right)\\ -\sum_{s=2}^{M}\exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(L_{s-1,i}-D_s)+C_{s,i}(D_s)\right)\right)-\exp\left(-\frac{2\rho}{\alpha(1-\theta)}\left(\frac{\tau_a}{\bar{V}}(B_i-D_s)+C_{s,i}(D_s)\right)\right)\end{array}\right\}. \quad \text{(A.5)}$$

From the boundary condition $r(B_i)=r_A$ and Eq. (18), we have

$$r_A=r(B_i)=(1-\theta)\left(a\theta^{\theta}k^{-\theta}p_i(0)\exp\left(-\frac{2\rho C_{s,i}(B_i)}{\alpha}\right)\right)^{\frac{1}{1-\theta}}$$

$$=(1-\theta)\left(a\theta^{\theta}k^{-\theta}p_i(0)\exp\left(-\frac{2\rho}{\alpha}\left(\frac{\tau_a}{\bar{V}}(B_i-D_1)+C_{s,i}(D_1)\right)\right)\right)^{\frac{1}{1-\theta}}. \quad \text{(A.6)}$$

Substituting Eq. (A.6) into (A.5) yields Eq. (22). In addition, we can easily derive Eq. (23) from Eq. (A.6).

(ii) The daily number of passengers boarding trains at station s is given by

$$Q_{s,i}=\int_{L_{s,i}}^{L_{s-1,i}}q_{s,i}(x)dx=\int_{L_{s,i}}^{D_s}q_{s,i}(x)dx+\int_{D_s}^{L_{s-1,i}}q_{s,i}(x)dx$$

$$=\frac{\eta\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{L_{s,i}}^{D_s}\exp\left(-\frac{2\rho C_{s,i}(x)}{\alpha}\right)^{\frac{1}{1-\theta}}\exp\left(-\pi C_{s,i}(x)\right)dx$$

$$+\frac{\eta\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\alpha}\int_{D_s}^{L_{s-1,i}}\exp\left(-\frac{2\rho C_{s,i}(x)}{\alpha}\right)^{\frac{1}{1-\theta}}\exp\left(-\pi C_{s,i}(x)\right)dx$$

$$=\frac{\eta(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{2\rho\tau_a+\alpha\pi(1-\theta)\tau_a}\left(\begin{array}{l}2\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)C_{s,i}(D_s)\right)-\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)\left(\frac{\tau_a}{\bar{V}}(D_s-L_{s,i})+C_{s,i}(D_s)\right)\right)\\ -\exp\left(-\left(\frac{2\rho}{\alpha(1-\theta)}+\pi\right)\left(\frac{\tau_a}{\bar{V}}(L_{s-1,i}-D_s)+C_{s,i}(D_s)\right)\right)\end{array}\right). \quad \text{(A.7)}$$

(iii) The partial derivatives of $p_i(0)$, B_i and $Q_{s,i}$ with regard to N and r_A are as follows.

$$\frac{\partial p_i(0)}{\partial N}=\frac{2\rho\tau_a}{\Omega_i\bar{V}}a^{-1}\left(k\theta^{-1}\right)^{\theta}\left(\frac{r_A}{1-\theta}\right)^{-\theta}\left(\frac{1}{\Omega_i}\left(\frac{2\rho\tau_a N}{r_A\bar{V}}+1\right)\right)^{-\theta}, \quad \text{(A.8)}$$

$$\frac{\partial p_i(0)}{\partial r_A}=a^{-1}\left(k\theta^{-1}\right)^{\theta}\left(\frac{r_A}{1-\theta}\right)^{-\theta}\left(\frac{1}{\Omega_i}\right)^{1-\theta}\left(\frac{2\rho\tau_a N}{r_A\bar{V}}+1\right)^{-\theta}, \quad \text{(A.9)}$$

$$\frac{\partial B_i}{\partial N}=\frac{\alpha\bar{V}}{2\rho\tau_a N+r_A\bar{V}}, \quad \text{(A.10)}$$

$$\frac{\partial B_i}{\partial r_A} = -\frac{\alpha \bar{V} N}{r_A(2\rho\tau_a N + r_A \bar{V})}, \tag{A.11}$$

$$\frac{\partial Q_{s,i}}{\partial N} = \frac{\eta(1-\theta)\bar{V}\left(\frac{1}{1-\theta}\right)\left(a\theta^{\theta}k^{-\theta}\left(p_i(0)\right)^{\theta}\right)^{\frac{1}{1-\theta}}\Phi_{s,i}}{2\rho\tau_a + \alpha\pi(1-\theta)\tau_a}\frac{\partial p_i(0)}{\partial N}, \tag{A.12}$$

$$\frac{\partial Q_{s,i}}{\partial r_A} = \frac{\eta(1-\theta)\bar{V}\left(\frac{1}{1-\theta}\right)\left(a\theta^{\theta}k^{-\theta}\left(p_i(0)\right)^{\theta}\right)^{\frac{1}{1-\theta}}\Phi_{s,i}}{2\rho\tau_a + \alpha\pi(1-\theta)\tau_a}\frac{\partial p_i(0)}{\partial r_A}. \tag{A.13}$$

As $0 < \theta < 1$ holds, we can immediately obtain

$$\frac{\partial p_i(0)}{\partial N} > 0,\ \frac{\partial p_i(0)}{\partial r_A} > 0,\ \frac{\partial B_i}{\partial N} > 0,\ \frac{\partial B_i}{\partial r_A} < 0,\ \frac{\partial Q_{s,i}}{\partial N} > 0, \text{ and } \frac{\partial Q_{s,i}}{\partial r_A} > 0. \tag{A.14}$$

Appendix B: Proof of Proposition 2

(i) The annual fare revenue for the private investor is

$$FR_i = 2\rho\sum_{s=1}^{M}\left(f_f + f_v D_s\right)Q_{s,i}$$

$$= 2\rho\sum_{s=1}^{M}\left(f_f + f_v D_s\right)\frac{\eta(1-\theta)\bar{V}\left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}}{\tau_a\left(2\rho+\alpha\pi(1-\theta)\right)}\Phi_{s,i}$$

$$= \left(a\theta^{\theta}k^{-\theta}p_i(0)\right)^{\frac{1}{1-\theta}}\frac{2\rho\eta(1-\theta)\bar{V}}{\tau_a\left(2\rho+\alpha\pi(1-\theta)\right)}\sum_{s=1}^{M}\left(f_f + f_v D_s\right)\Phi_{s,i}$$

$$= \frac{2\rho\eta(2\rho\tau_a N + r_A\bar{V})}{\tau_a\Omega_i(2\rho+(1-\theta)\alpha\pi)}\sum_{s=1}^{M}\left(f_f + f_v D_s\right)\Phi_{s,i}, \tag{B.1}$$

where the daily passenger demand $Q_{s,i}$ at station s is defined by Eq. (26).

The annual property revenue is

$$PR_i = \sum_{s=1}^{M+1}\int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0}\left(p_i(x)h\left(S_i(x)\right)-r_i(x)\right)dx$$

$$= \sum_{s=1}^{M+1}\int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0}\left(k\left(k^{-1}a\theta p_i(0)\exp\left(-\frac{\varphi_i(x)}{\alpha}\right)\right)^{\frac{1}{1-\theta}}\right)dx$$

$$= \frac{\alpha\theta(2\rho\tau_a N + r_A\bar{V})}{2\rho\tau_a\Omega_i}\sum_{s=1}^{M+1}2\left(\exp\left(-\frac{2\rho C_{s,i}(D_s)}{\alpha(1-\theta)}\right)-\exp\left(-\frac{2\rho\left(\frac{\tau_a}{\bar{V}}\Delta_s^0 + C_{s,i}(D_s)\right)}{\alpha(1-\theta)}\right)\right). \tag{B.2}$$

When $r_A\bar{V} << 2\rho\tau_a N$ holds, Eqs. (B.1) and (B.2) can be expressed, respectively, as

$$FR_i = \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,i}D_s\right)\Phi_{s,i}N}{(2\rho+(1-\theta)\alpha\pi)\Omega_i}, \text{ and} \tag{B.3}$$

$$PR_i = \frac{\alpha\theta\Psi_i N}{\Omega_i}, \tag{B.4}$$

where

$$\Psi_i = \sum_{s=1}^{M+1} 2\left(\exp\left(-\frac{2\rho C_{s,i}(D_s)}{\alpha(1-\theta)} \right) - \exp\left(-\frac{2\rho\left(\frac{\tau_a}{\bar{V}}\Delta_s^0 + C_{s,i}(D_s) \right)}{\alpha(1-\theta)} \right) \right). \quad \text{(B.5)}$$

We now derive $U_1 - U_0$, $U_2 - U_1$, and $U_3 - U_1$. By Eqs. (14) and (22), we have

$$U_1 - U_0 = Y - \alpha + \alpha\log\left(\frac{\alpha}{p_1(0)} \right) - \left(Y - \alpha + \alpha\log\left(\frac{\alpha}{p_0(0)} \right) \right)$$

$$= \alpha(1-\theta)\log\left(\frac{\Omega_1}{\Omega_0} \right). \quad \text{(B.6)}$$

Similarly, we can obtain

$$U_2 - U_1 = Y - \alpha + \alpha\log\left(\frac{\alpha}{p_2(0)} \right) - \left(Y - \alpha + \alpha\log\left(\frac{\alpha}{p_1(0)} \right) \right)$$

$$= \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1} \right). \quad \text{(B.7)}$$

$$U_3 - U_1 = Y - \alpha + \alpha\log\left(\frac{\alpha}{p_3(0)} \right) - \left(Y - \alpha + \alpha\log\left(\frac{\alpha}{p_1(0)} \right) \right)$$

$$= \alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1} \right). \quad \text{(B.8)}$$

Substituting Eqs. (B.3) and (B.4) into Eq. (35) gives

$$\Lambda_p(N) = E_N\left[\int_\Delta^T \left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,i}D_s \right)\Phi_{s,2}N_t}{(2\rho + (1-\theta)\alpha\pi)\Omega_2} \right) e^{-kt}dt + \int_\Delta^T \left(\frac{\alpha\theta\Psi_2 N_t}{\Omega_2} \right) e^{-kt}dt \right.$$

$$\left. - \int_\Delta^T C_2^O e^{-kt}dt - \int_0^\Delta (C^L + C^S) e^{-kt}dt \right]. \quad \text{(B.9)}$$

Similarly, substituting Eqs. (B.3)-(B.8) into Eq. (43) yields

$$W_g(N) = E_N\left[\int_0^{+\infty} N_t\left(\alpha(1-\theta)\log\left(\frac{\Omega_1}{\Omega_0} \right) \right) e^{-kt}dt + \int_\Delta^T N_t\left(\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1} \right) \right) e^{-kt}dt \right.$$

$$+\int_{\Delta}^{T}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,i}D_s\right)\Phi_{s,2}N_t}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}\right)e^{-kt}dt+\int_{\Delta}^{T}\left(\frac{\alpha\theta\Psi_2N_t}{\Omega_2}\right)e^{-kt}dt-\int_{\Delta}^{T}C_2^Oe^{-kt}dt-\int_{0}^{\Delta}(C^L+C^S)e^{-kt}dt$$

$$+\int_{T}^{+\infty}N_t\left(\alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1}\right)\right)e^{-kt}dt+\int_{T}^{+\infty}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,i}D_s\right)\Phi_{s,3}N_t}{(2\rho+(1-\theta)\alpha\pi)\Omega_3}\right)e^{-kt}dt+\int_{T}^{+\infty}\left(\frac{\alpha\theta\Psi_3N_t}{\Omega_3}\right)e^{-kt}dt-\int_{T}^{+\infty}C_3^Oe^{-kt}dt\Bigg]. \quad \text{(B.10)}$$

As N_t follows a GBM, the expectation of N_t is

$$E\left[N_t\right]=Ne^{\mu t}. \quad \text{(B.11)}$$

Substituting Eq. (B.11) into Eqs. (B.9) and (B.10), one can obtain Eqs. (52) and (53).

(ii) From the value-matching condition in Eq. (48) and Eqs. (50), (52), and (53), we obtain

$$A_pN^{*\beta}=\frac{e^{-(k-\mu)\Delta}-e^{-(k-\mu)T}}{k-\mu}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}+\frac{\alpha\theta\Psi_2}{\Omega_2}\right)N^*$$

$$-\frac{1}{k}\left(C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})\right). \quad \text{(B.12)}$$

$$A_gN^{*\beta}=\frac{\alpha(1-\theta)\log\left(\frac{\Omega_1}{\Omega_0}\right)N^*}{k-\mu}+\frac{e^{-(k-\mu)\Delta}-e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)+\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}+\frac{\alpha\theta\Psi_2}{\Omega_2}\right)N^*$$

$$+\frac{e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1}\right)+\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,3}D_s\right)\Phi_{s,3}}{(2\rho+(1-\theta)\alpha\pi)\Omega_3}+\frac{\alpha\theta\Psi_3}{\Omega_3}\right)N^*$$

$$-\frac{1}{k}\left(C_2^O(e^{-k\Delta}-e^{-kT})+C_3^Oe^{-kT}+(C^L+C^S)(1-e^{-k\Delta})\right). \quad \text{(B.13)}$$

Using the smooth-pasting condition in Eq. (49), we have

$$\beta A_pN^{*\beta-1}=\frac{e^{-(k-\mu)\Delta}-e^{-(k-\mu)T}}{k-\mu}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}+\frac{\alpha\theta\Psi_2}{\Omega_2}\right), \text{ and} \quad \text{(B.14)}$$

$$\beta A_g N^{*\beta-1} = \frac{\alpha(1-\theta)\log\left(\frac{\Omega_1}{\Omega_0}\right)}{k-\mu} + \frac{e^{-(k-\mu)\Delta} - e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2} + \frac{\alpha\theta\Psi_2}{\Omega_2}\right)$$

$$+\frac{e^{-(k-\mu)T}}{k-\mu}\left(\alpha(1-\theta)\log\left(\frac{\Omega_3}{\Omega_1}\right) + \frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f + f_{v,3}D_s\right)\Phi_{s,3}}{(2\rho+(1-\theta)\alpha\pi)\Omega_3} + \frac{\alpha\theta\Psi_3}{\Omega_3}\right). \qquad \text{(B.15)}$$

Solving the system of Eqs. (B.12)-(B.15), one obtains the expressions for N_p^* and N_g^* (i.e., Eqs. (54) and (55)), and for A_p and A_g, as shown in Eq. (51).

(iii) For comparison purpose, the population threshold of the traditional NPV model is derived as follows. Under the NPV approach, a private investor (or the government) would like to invest in a transit project if the resultant expected net profit (or the expected welfare gain) is larger than zero, i.e.,

$$\Lambda_p\left(N^*_{p(NPV)}\right) \geq 0 \text{ for the private investor; } W_g\left(N^*_{g(NPV)}\right) \geq 0 \text{ for the government.} \qquad \text{(B.16)}$$

From Eq. (B.16), the population thresholds for the private investor and the government under the NPV approach are, respectively, given by

$$N^*_{p(NPV)} = \frac{k-\mu}{k}\frac{C_2^O(e^{-k\Delta} - e^{-kT}) + (C^L + C^S)(1-e^{-k\Delta})}{(e^{-(k-\mu)\Delta} - e^{-(k-\mu)T})\left(\xi_2 - \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)}, \qquad \text{(B.17)}$$

$$N^*_{g(NPV)} = \frac{k-\mu}{k}\frac{C_2^O(e^{-k\Delta} - e^{-kT}) + C_3^O e^{-kT} + (C^L + C^S)(1-e^{-k\Delta})}{\xi_1 + (e^{-(k-\mu)\Delta} - e^{-(k-\mu)T})\xi_2 + e^{-(k-\mu)T}\xi_3}. \qquad \text{(B.18)}$$

From Eqs. (54), (55), (B.17), and (B.18), one obtains Eq. (56).

Appendix C: The solutions for concession period, headway and fare

Setting $\partial W(T)/\partial T = 0$, one obtains the optimal concession period solution satisfying

$$\omega\left(e^{-(k-\mu)T}(\xi_2-\xi_3)N - \frac{\beta e^{-kT}(C_2^O - C_3^O)}{\beta-1} \right)\left(\Lambda_p(T) - \Lambda_p(T_{N_p^*}) \right)$$

$$+(1-\omega)\left(e^{-(k-\mu)T}\left(\xi_2 - \alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)N - \frac{\beta C_2^O e^{-kT}}{\beta-1} \right)\left(W_g(T) - W_g(T_{N_g^*}) \right) = 0\,. \tag{C.1}$$

From $\partial\Lambda_p/\partial H_2 = 0$ and $\partial\Lambda_p/\partial f_{v,2} = 0$, one obtains

$$H_2 = \sqrt{\frac{-C_v^O\left(\zeta t_c + 2\left(\frac{D_1}{V_2} + t_0 M\right)\right)\frac{e^{-k\Delta} - e^{-kT}}{k}}{E_N\left[\int_\Delta^T 2\rho \sum_{s=1}^{M}\left((f_f + f_{v,2}D_s)\frac{\partial Q_{s,2}}{\partial H_2}\right) e^{-kt}dt + \int_\Delta^T \left(\sum_{s=1}^{M+1} k \int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0} \frac{\partial S_2(x)}{\partial H_2}dx \right) e^{-kt}dt \right]}}\,, \tag{C.2}$$

$$f_{v,2} = \frac{E_N\left[\int_\Delta^T 2\rho \sum_{s=1}^{M}\left(D_s Q_{s,2} + f_f\frac{\partial Q_{s,2}}{\partial f_{v,2}}\right) e^{-kt}dt + \int_\Delta^T \left(\sum_{s=1}^{M+1} k \int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0} \frac{\partial S_2(x)}{\partial f_{v,2}}dx \right) e^{-kt}dt \right]}{E_N\left[\int_\Delta^T 2\rho \sum_{s=1}^{M}\left(D_s\left(-\frac{\partial Q_{s,2}}{\partial f_{v,2}}\right)\right) e^{-kt}dt \right]}\,. \tag{C.3}$$

Setting $\partial W_g/\partial H_3 = 0$ and $\partial W_g/\partial f_{v,3} = 0$ yields

$$H_3 = \sqrt{\frac{-C_v^O\left(\zeta t_c + 2\left(\frac{D_1}{V_3} + t_0 M\right)\right)\frac{e^{-kT}}{k}}{E_N\left[\int_T^{+\infty} N_t\alpha(1-\theta)\frac{\partial\log(\Omega_3/\Omega_1)}{\partial H_3}e^{-kt}dt + \int_T^{+\infty} 2\rho\sum_{s=1}^{M}(f_f + f_{v,3}D_s)\frac{\partial Q_{s,3}}{\partial H_3}e^{-kt}dt + \int_T^{+\infty}\left(\sum_{s=1}^{M+1} k \int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0} \frac{\partial S_3(x)}{\partial H_3}dx \right) e^{-kt}dt \right]}}\,, \tag{C.4}$$

$$f_{v,3} = \frac{E_N\left[\int_T^{+\infty} N_t\alpha(1-\theta)\frac{\partial\log(\Omega_3/\Omega_1)}{\partial f_{v,3}}e^{-kt}dt + \int_T^{+\infty} 2\rho\sum_{s=1}^{M}\left(D_s Q_{s,3} + f_f\frac{\partial Q_{s,3}}{\partial f_{v,3}}\right) e^{-kt}dt + \int_T^{+\infty}\left(\sum_{s=1}^{M+1} k \int_{D_s-\Delta_s^0}^{D_s+\Delta_s^0} \frac{\partial S_3(x)}{\partial f_{v,3}}dx \right) dt \right]}{E_N\left[\int_T^{+\infty} 2\rho \sum_{s=1}^{M}\left(D_s\left(-\frac{\partial Q_{s,3}}{\partial f_{v,3}}\right)\right) e^{-kt}dt \right]}\,. \tag{C.5}$$

Appendix D: The results with population random jumps

As $N_{t(J)}$ follows a mixed geometric Brownian motion with jumps, its expectation is

$$E\left[N_{t(J)}\right]=N_J e^{(\mu+\lambda\phi)t}. \tag{D.1}$$

The expected net profit $\Lambda_{p(J)}(N_J)$ and the expected welfare gain $W_{g(J)}\left(N_J\right)$ with jumps can then be given as

$$\Lambda_{p(J)}(N_J)=\frac{e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}}{k-(\mu+\lambda\phi)}\left(\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}+\frac{\alpha\theta\Psi_2}{\Omega_2}\right)N_J-\frac{1}{k}\left(C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})\right), \tag{D.2}$$

$$\begin{aligned}W_{g(J)}\left(N_J\right)=&\frac{\alpha(1-\theta)\log\left(\Omega_1/\Omega_0\right)N_J}{k-(\mu+\lambda\phi)}+\frac{e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}}{k-(\mu+\lambda\phi)}\left(\alpha(1-\theta)\log\left(\Omega_2/\Omega_1\right)+\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,2}D_s\right)\Phi_{s,2}}{(2\rho+(1-\theta)\alpha\pi)\Omega_2}+\frac{\alpha\theta\Psi_2}{\Omega_2}\right)N_J\\&+\frac{e^{-(k-(\mu+\lambda\phi))T}}{k-(\mu+\lambda\phi)}\left(\alpha(1-\theta)\log\left(\Omega_3/\Omega_1\right)+\frac{4\rho^2\eta\sum_{s=1}^{M}\left(f_f+f_{v,3}D_s\right)\Phi_{s,3}}{(2\rho+(1-\theta)\alpha\pi)\Omega_3}+\frac{\alpha\theta\Psi_3}{\Omega_3}\right)N_J-\frac{1}{k}\left(C_2^O(e^{-k\Delta}-e^{-kT})+C_3^Oe^{-kT}+(C^L+C^S)(1-e^{-k\Delta})\right).\end{aligned} \tag{D.3}$$

According to the value-matching condition, $A_{j(J)}$ can be given as

$$\begin{cases}A_{p(J)}=\left(\frac{e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}}{\beta_J(k-(\mu+\lambda\phi))}\left(\xi_2-\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)\right)^{\beta_J}\left(\frac{k(\beta_J-1)}{C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})}\right)^{\beta_J-1},\\ A_{g(J)}=\left(\frac{\xi_1+(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T})\xi_2+e^{-(k-(\mu+\lambda\phi))T}\xi_3}{\beta_J(k-(\mu+\lambda\phi))}\right)^{\beta_J}\left(\frac{k(\beta_J-1)}{C_2^O(e^{-k\Delta}-e^{-kT})+C_3^Oe^{-kT}+(C^L+C^S)(1-e^{-k\Delta})}\right)^{\beta_J-1}.\end{cases} \tag{D.4}$$

For a given concession period T_J, the population thresholds $N^*_{p(J)}$ and $N^*_{g(J)}$ for the private investor and the government are given, respectively, by

$$N^*_{p(J)}=\frac{\beta_J(k-(\mu+\lambda\phi))}{k(\beta_J-1)}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})}{\left(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}\right)\left(\xi_2-\alpha(1-\theta)\log\left(\Omega_2/\Omega_1\right)\right)}, \text{ and} \tag{D.5}$$

$$N^*_{g(J)}=\frac{\beta_J(k-(\mu+\lambda\phi))}{k(\beta_J-1)}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+C_3^Oe^{-kT}+(C^L+C^S)(1-e^{-k\Delta})}{\xi_1+\left(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}\right)\xi_2+e^{-(k-(\mu+\lambda\phi))T}\xi_3}. \tag{D.6}$$

The minimum acceptable concession period $T_{MIN(J)}$ for the private investor and the maximum acceptable concession period $T_{MAX(J)}$ for the government satisfy

$$\frac{\beta_J(k-(\mu+\lambda\phi))}{k(\beta_J-1)}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})}{(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T})\left(\xi_2-\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)}-N_J=0 \text{, and} \tag{D.7}$$

$$\frac{\beta_J(k-(\mu+\lambda\phi))}{k(\beta_J-1)}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+C_3^O e^{-kT}+(C^L+C^S)(1-e^{-k\Delta})}{\xi_1+(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T})\xi_2+e^{-(k-(\mu+\lambda\phi))T}\xi_3}-N_J=0\,. \tag{D.8}$$

The optimal concession period T_J^* can be calculated by solving the following equation

$$\omega\left(e^{-(k-(\mu+\lambda\phi))T}(\xi_2-\xi_3)N_J-\frac{\beta_J e^{-kT}(C_2^O-C_3^O)}{\beta_J-1}\right)\left(\Lambda_p(T_J)-\Lambda_p(T_{N_p^*(J)})\right)+(1-\omega)\left(e^{-(k-(\mu+\lambda\phi))T}\left(\xi_2-\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)N_J-\frac{\beta_J C_2^O e^{-kT}}{\beta_J-1}\right)\left(W_g(T_J)-W_g(T_{N_g^*(J)})\right)=0\,. \tag{D.9}$$

where $W_g(T_J)-W_g(T_{N_g^*(J)})$ and $\Lambda_p(T_J)-\Lambda_p(T_{N_p^*(J)})$ are the relative expected welfare of the government and the relative expected net profit of the private investor, given as

$$W_g(T_J)-W_g(T_{N_g^*(J)})=\frac{\left(\xi_1+(e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T})\xi_2+e^{-(k-(\mu+\lambda\phi))T}\xi_3\right)N_J}{k-(\mu+\lambda\phi)}-\frac{\beta_J}{\beta_J-1}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+C_3^O e^{-kT}+(C^L+C^S)(1-e^{-k\Delta})}{k}\,, \tag{D.10}$$

$$\Lambda_p(T_J)-\Lambda_p(T_{N_p^*(J)})=\frac{e^{-(k-(\mu+\lambda\phi))\Delta}-e^{-(k-(\mu+\lambda\phi))T}}{k-(\mu+\lambda\phi)}\left(\xi_2-\alpha(1-\theta)\log\left(\frac{\Omega_2}{\Omega_1}\right)\right)N_J-\frac{\beta_J}{\beta_J-1}\frac{C_2^O(e^{-k\Delta}-e^{-kT})+(C^L+C^S)(1-e^{-k\Delta})}{k}\,. \tag{D.11}$$