A NEW SECOND-ORDER LOW-REGULARITY INTEGRATOR FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION

JIACHUAN CAO , BUYANG LI , AND YANPING LIN

ABSTRACT. This article is concerned with the question of whether it is possible to construct a time discretization for the one-dimensional cubic nonlinear Schrödinger equation with second-order convergence for initial data with regularity strictly below H^2. We address this question with a positive answer by constructing a new second-order low-regularity integrator for the one-dimensional cubic nonlinear Schrödinger equation. The proposed method can have second-order convergence in L^2 for initial data in $H^{\frac{5}{3}}$, and first-order convergence up to a logarithmic factor for initial data in H^1. This significantly relaxes the regularity requirement for second-order approximations to the cubic nonlinear Schrödinger equation, while retaining the by far best convergence order for initial data in H^1. Numerical experiments are presented to support the theoretical analysis and to illustrate the performance of the proposed method in approximating nonsmooth solutions of the nonlinear Schrödinger equation. The numerical results show that, among the many time discretizations, the proposed method is the only one which has second-order convergence in L^2 for initial data strictly below H^2.

1. Introduction

In this article we consider the numerical approximation of the cubic nonlinear Schrödinger (NLS) equation

$$\begin{cases} i\partial_t u(t,x) + \partial_x^2 u(t,x) = \lambda |u(t,x)|^2 u(t,x) \quad \text{for } x \in \mathbb{T} \text{ and } t \in (0,T], \\ u(0,x) = u^0(x) \quad \text{for } x \in \mathbb{T}, \end{cases} \tag{1.1}$$

on the one-dimensional torus $\mathbb{T} = [-\pi, \pi]$ with initial value $u^0 \in H^r(\mathbb{T})$ for $1 \le r \le 2$, where $\lambda = -1$ or 1 is a given parameter corresponding to the focusing and defocusing cases of the NLS equation, respectively. It is known that problem (1.1) is locally well-posed in $H^r(\mathbb{T})$ for $r \ge 0$, i.e., for $u^0 \in H^r(\mathbb{T})$ problem (1.1) has a unique solution in some subspace of $C([0,T]; H^r(\mathbb{T}))$ satisfying the following relation:

$$u(t,\cdot) = e^{it\partial_x^2} u^0 - i\lambda \int_0^t e^{i(t-s)\partial_x^2} \big[\big| u(s,\cdot) \big|^2 u(s,\cdot) \big] ds \quad \text{for } t \in [0,T],$$

see [6, 27].

The numerical approximation to the solutions of the NLS equation has been studied in many articles with different numerical schemes, including the Lie splitting schemes [18, 37], the Strang splitting [13, 24], the Crank-Nicolson methods [2, 15, 35] and exponential integrators [5, 9, 16]. It is well understood that optimal-order convergence of the numerical approximation can be established for each of these methods if the solution of (1.1) is sufficiently smooth. For instance, the first-order Lie splitting method can be written as

$$\begin{aligned} u^{n+1/2} &= e^{i\tau\partial_x^2} u^n, \\ u^{n+1} &= e^{-i\lambda\tau \left| u^{n+1/2} \right|^2} u^{n+1/2}. \end{aligned} \tag{1.2}$$

where τ denotes the temporal stepsize. If we denote $T(u) = i\partial_x^2 u$ and $V(u) = -i\lambda|u|^2 u$, then the local error of the Lie splitting scheme contains a principle term of type $\tau^2[T,V](u)$, where

2010 *Mathematics Subject Classification.* 65M12, 65M15, 35Q55.

Key words and phrases. cubic nonlinear Schrödinger equation, low regularity, second order, error estimates.

the Lie commutator $[T,V]$ can be expressed as follows (see [24])

$$\frac{1}{2\lambda}[T,V](u) = (\partial_x u\cdot\partial_x\bar{u})u + (\partial_x u\bar{u})\cdot\partial_x u + (u\partial_x\bar{u})\cdot\partial_x u + (u\overline{\partial_x^2 u})u.$$

Due to the presence of $\overline{\partial_x^2 u}$ in the local error, the first-order convergence of the Lie splitting method requires the exact solution of the NLS equation to be in $C([0,T];H^2(\mathbb{T}))$; see the fractional error estimates of splitting schemes in [13]. For the second-order Strang splitting method, the second-order convergence in $L^2(\mathbb{T})$ requires the solution to be in $C([0,T];H^4(\mathbb{T}))$; see the foundational analysis of splitting methods in [24]. The regularity conditions are also required by the classical exponential integrators and finite difference methods; see [16,35,39]. In general, kth-order convergence in $L^2(\mathbb{T})$ of the classical numerical methods, including splitting methods, finite difference methods and exponential integrators, require at least $u\in C([0,T];H^{2k}(\mathbb{T}))$ and therefore requires $u^0\in H^{2k}(\mathbb{T})$.

In various real-world applications, the initial data can be nonsmooth and therefore may not satisfy the smoothness conditions above. For example, in the field of nonlinear optics, the propagation of light in optical fibers can be modeled by the NLS equation. The roughness of the initial data is one of the causes of the formation of rare, large amplitude waves known as rogue waves [1,38]. Therefore, accurate simulation of the NLS equation for rough initial data is crucial for studying the formation and propagation of these phenomena. Since the Schrödinger group $e^{it\partial_x^2}$ does not have the smoothing property as the heat semigroup $e^{t\partial_x^2}$, the convergence order of the classical numerical methods is generally reduced when the initial values are not sufficiently smooth.

To overcome this difficulty, Ostermann & Schratz [28] introduced a low-regularity integrator for the NLS by utilizing the twisted variable $v(t)=e^{-it\partial_x^2}u(t)$ and the equivalent formulation

$$i\partial_t v(t) = \lambda e^{-it\partial_x^2}\Big[\big|e^{it\partial_x^2}v(t)\big|^2 e^{it\partial_x^2}v(t)\Big], \quad v(0)=u^0. \tag{1.3}$$

The solution of this equation can be written as follows by using the variation-of-constants formula:

$$v(t_n+\tau) = v(t_n) - i\lambda\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[\big|e^{i(t_n+s)\partial_x^2}v(t_n+s)\big|^2 e^{i(t_n+s)\partial_x^2}v(t_n+s)\Big]ds, \tag{1.4}$$

where $t_n=n\tau$. By using the Fourier series expansion, Ostermann & Schratz extracted the main oscillation terms from the right-hand side of (1.4), which are computed explicitly such that the remainders only require the boundedness of one additional spatial derivative of the exact solution. Correspondingly, the method can have first-order convergence in $H^r(\mathbb{T})$ for initial value $u^0\in H^{r+1}(\mathbb{T})$ when $r>\frac{1}{2}$. More recently, Wu and Yao [41] proposed a new low-regularity integrator for the one-dimensional NLS equation with different harmonic analysis techniques, with first-order convergence in $H^r(\mathbb{T})$ for initial value $u^0\in H^r(\mathbb{T})$ when $r>\frac{3}{2}$.

The combination of twisted variable and harmonic analysis was also used in the development of low-regularity time discretizations for other nonlinear dispersive equations, such as the KdV equation, the Dirac equation, the Klein-Gordon equation, and the Boussinesq equation. We refer to [17,25,29,33,36,40,42,43] and the references therein.

By introducing and utilizing the discrete Bourgain space, Ostermann, Rousset & Schratz [26,27] proved that some filtered Fourier based integrator methods can have convergence order $\theta(s)$ in $L^2(\mathbb{T})$ for initial value $u^0\in H^s(\mathbb{T}^d)$, where $\theta(s)$ is a piecewise function ranging from 0 to $\frac{7}{8}-\epsilon$ when $s\in(0,1]$. This proves the existence of certain convergence order for some numerical methods under minimal regularity conditions. A fully discrete low-regularity integrator for the NLS equation was constructed in [22] based on harmonic analysis techniques, with almost first-order convergence in $L^2(\mathbb{T})$ for initial value $u^0\in H^1(\mathbb{T})$, which is by far the minimal regularity condition for first-order convergence in $L^2(\mathbb{T})$. Ostermann & Yao [31] proposed a different fully discrete low-regularity integrator with convergence order $\frac{3}{2}s-\frac{1}{2}-\epsilon$ in $L^2(\mathbb{T})$ for initial value $u^0\in H^s(\mathbb{T}^d)$ with $s\in(\frac{1}{2},1]$, relaxing the regularity condition of [22] to $u^0\in H^s(\mathbb{T}^d)$ with $s\in(\frac{1}{2},1)$ without affecting the convergence order in the case $s=1$. For the NLS equation under the Neumann boundary condition, when the Fourier transform based frequency analysis cannot be used, the Littlewood-Paley decomposition technique was used in [3] to construct a low-regularity integrator with first-order convergence for H^1 initial data.

Based on the high-order resonance analysis, several second-order low-regularity integrators for the NLS equation were developed in [8,19,30]. The best result so far is to have second-order convergence in $L^2(\mathbb{T})$ for initial value in $H^2(\mathbb{T})$. A natural question is *whether it is possible to construct low-regularity integrators for the NLS equation with second-order convergence in* $L^2(\mathbb{T})$ *for solutions with regularity strictly below* $H^2(\mathbb{T})$.

The objective of this paper is to address this question by constructing a new time discretization for the NLS equation which can have second-order convergence in $L^2(\mathbb{T})$ for initial value $u^0 \in H^{\frac{5}{3}}(\mathbb{T})$, while keeping to have first-order convergence in $L^2(\mathbb{T})$ under the weaker regularity condition $u^0 \in H^1(\mathbb{T})$ as in [22,31]. Our methodology is based on two main techniques:

(1) Substituting $v(t_n + s) \approx v(t_n) + \Phi_1^n(v(t_n), s)$ into the right-hand side of (1.4), where $v(t_n) + \Phi_1^n(v(t_n), s)$ denotes the first-order low-regularity integrator constructed by Ostermann & Schratz [28].

(2) With the aid of harmonic analysis techniques and a temporal averaging technique developed in [8, 23, 30], we construct the low-regularity integrator by carefully selecting the tractable terms from the exponential integral such that the spatial derivatives are almost uniformly distributed to the product terms in the remainder.

As a result, we manage to prove second-order convergence in $L^2(\mathbb{T})$ for the proposed method under the by far minimal regularity condition, i.e., $u \in C([0,T]; H^{\frac{5}{3}}(\mathbb{T}))$. The numerical experiments in this article indicate that, among the many existing time-stepping methods, the proposed low-regularity integrator is the only method which can have second-order convergence in $L^2(\mathbb{T})$ for initial data below $H^2(\mathbb{T})$. Beyond the numerical analysis, the numerical experiments also indicate that the proposed new time discretization, combined with the Fourier spectral discretization with the number of degrees of freedom $N = O(\tau^{-1})$, has convergence order s for initial data in $H^s(\mathbb{T})$ with $s \in (0,1)$. The rigorous analysis for the stability and convergence of the proposed time discretization in combination with filtered spatial discretizations for initial data in $H^s(\mathbb{T})$ with $s \in (0,1)$ is more challenging and will be investigated in the future.

The rest of this article is organized as follows. In section 2, we present the numerical algorithm and the main theoretical result on the convergence of the algorithm. In section 3, we present the construction of the second-order low-regularity integrator based on the analysis of the consistency errors. The error bounds of the proposed method are proved in section 4. Finally, numerical experiments are presented in section 5 to illustrate the convergence of the proposed method for both smooth and nonsmooth initial data.

2. Notations and main result

In this section we introduce the notations which will be frequently used in this article. Then we present the new low-regularity integrator and the main convergence result.

2.1. Some notations

We denote by $A \lesssim B$ or $A \gtrsim B$ the statement "$A \leq CB$ for some constant $C > 0$ which is independent of τ or n". The inner product and norm on $L^2(\mathbb{T})$ are denoted by

$$(f, g) = \int_{\mathbb{T}} f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\|_{L^2} = \sqrt{(f, f)}, \quad \text{respectively.}$$

The Fourier transform of a function $f \in L^2(\mathbb{T})$ is defined as $\mathcal{F}_k[f] = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} f(x)dx$. For the simplicity of notations, we denote

$$f(x) = \sum_{k\in\mathbb{Z}} e^{ikx}\hat{f}_k \quad \text{with } \hat{f}_k = \mathcal{F}_k[f].$$

It is well known that

$$\|f\|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \qquad \text{(Plancherel identity),}$$

$$\widehat{(fg)}(k) = \sum_{k_1\in\mathbb{Z}} \hat{f}_{k-k_1}\hat{g}_{k_1} \qquad \text{(Fourier transform of a product).}$$

For any function $\varphi : \mathbb{Z} \to \mathbb{C}$ such that $|\varphi(k)| \le C_\varphi(1+|k|)^m$ for some constants C_φ and $m > 0$, we define the operator $\varphi(i^{-1}\partial_x)$ as follows

$$\varphi(i^{-1}\partial_x)f = \sum_{k\in\mathbb{Z}} \varphi(k)\hat{f}_k e^{ikx}.$$

We denote

$$\langle k\rangle = (1+k^2)^{\frac{1}{2}} \quad \text{and} \quad J^s = \langle i^{-1}\partial_x\rangle^s.$$

Then, the equivalent norm on the Sobolev space H^s is denoted by

$$\|f\|_{H^s}^2 := \|J^s f\|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} (1+k^2)^s |\hat{f}_k|^2.$$

By using the Fourier expansion, the free Schrödinger group $e^{it\partial_x^2}$ for $t \in \mathbb{R}$ can be written by

$$e^{it\partial_x^2} f(x) = \sum_{k\in\mathbb{Z}} e^{-itk^2} \hat{f}_k e^{ikx}.$$

Since $|e^{-itk^2}| = 1$, it follows that $e^{it\partial_x^2}$ is a linear isomerty on $H^s(\mathbb{T})$.

Furthermore, for any integer $m \ge 1$ we define a regularization of $\partial_x^{-m} : H^s \to H^{s+m}$ by

$$\mathcal{F}_k[\partial_x^{-m} f] = \begin{cases} (ik)^{-m}\hat{f}_k, & \text{when } k \ne 0, \\ 0, & \text{when } k = 0. \end{cases}$$

The following functions will be used in the definition of the numerical scheme:

$$\varphi_1(z) = \begin{cases} \dfrac{e^z - 1}{z}, & \text{for } z \ne 0, \\ 1, & \text{for } z = 0, \end{cases} \tag{2.1}$$

and

$$\psi_1(z) = \begin{cases} \dfrac{1 - e^z + ze^z}{z^2}, & \text{for } z \ne 0, \\ \dfrac{1}{2}, & \text{for } z = 0, \end{cases} \qquad \psi_2(z) = \begin{cases} \dfrac{e^z - 1 - z}{z^2}, & \text{for } z \ne 0, \\ \dfrac{1}{2}, & \text{for } z = 0. \end{cases} \tag{2.2}$$

It is straightforward to verify that, φ_1, ψ_1 and ψ_2 are bounded on the imaginary axis of the complex plane. Therefore, $\varphi_1(it\partial_x^2)$, $\psi_1(it\partial_x^2)$ and $\psi_2(it\partial_x^2)$ are all bounded operators on $H^s(\mathbb{T})$, $s \ge 0$, uniformly with respect to $t \in \mathbb{R}$.

2.2. The low-regularity integrator and main result

Let $t_n = n\tau$, $n = 0, 1, \ldots, L$, be a uniform partition of the time interval $[0, T]$ with stepsize $\tau = T/L$, where L is any given positive integer. The low-regularity integrator proposed in this paper can be written as

$$u^{n+1} = \widetilde{\Psi}_\tau(u^n), \quad n = 0, 1, \cdots, L-1, \tag{2.3}$$

with

$$\begin{aligned}
\widetilde{\Psi}_\tau(f) :=& \Big\{1 - 2i\lambda\tau\Pi_0(f\partial_x\bar{f})\partial_x^{-1} - 2i\lambda\tau\Pi_0\big(|f|^2\big)\Big\} e^{i\tau\partial_x^2} f \\
&+ \frac{\lambda}{2}\Pi_0\Big\{\bar{f}\cdot e^{-i\tau\partial_x^2}\big(e^{i\tau\partial_x^2}\partial_x^{-1}f\big)^2 - \bar{f}\big(\partial_x^{-1}f\big)^2\Big\} + i\lambda\tau\big|\Pi_0(f)\big|^2\Pi_0(f) \\
&+ \lambda\partial_x^{-1}\Big\{e^{i\tau\partial_x^2}f\cdot\partial_x^{-1}\big(|e^{i\tau\partial_x^2}f|^2\big) - e^{i\tau\partial_x^2}\Big[f\cdot\partial_x^{-1}\big(|f|^2\big)\Big]\Big\} \\
&- \frac{\lambda}{2}\partial_x^{-2}\Big\{e^{-i\tau\partial_x^2}\bar{f}\cdot e^{i\tau\partial_x^2}\big(f\big)^2 - e^{i\tau\partial_x^2}\Big(f\cdot|f|^2\Big)\Big\} \\
&- \frac{\lambda}{2}e^{i\tau\partial_x^2}\partial_x^{-1}\Big\{\partial_x\bar{f}\cdot e^{-i\tau\partial_x^2}\Big(e^{i\tau\partial_x^2}\partial_x^{-1}f\Big)^2 - \partial_x\bar{f}\cdot\big(\partial_x^{-1}f\big)^2\Big\} \\
&- i\lambda\tau e^{i\tau\partial_x^2}\partial_x^{-1}\Big(\partial_x\bar{f}\cdot f^2\Big) + 2i\lambda\tau\Pi_0(f)\cdot e^{i\tau\partial_x^2}\partial_x^{-1}\big(\partial_x\bar{f}\cdot f\big)
\end{aligned}$$

$$
\begin{aligned}
&- i\lambda\tau\big(\Pi_0(f)\big)^2\Pi_{\neq 0}\big(e^{i\tau\partial_x^2}\bar f\big) \qquad (2.4)\\
&+ \frac{i\lambda}{4\tau}\partial_x^{-3}\Big\{e^{-i\tau\partial_x^2}\partial_x^{-1}\bar f\cdot\big(e^{i\tau\partial_x^2}f\big)^2 - e^{-i\tau\partial_x^2}\partial_x^{-1}\bar f\cdot e^{i\tau\partial_x^2}f^2\Big\}\\
&- \frac{i\lambda}{4\tau}e^{i\tau\partial_x^2}\partial_x^{-3}\Big\{\partial_x^{-1}\bar f\cdot e^{-i\tau\partial_x^2}\big(e^{i\tau\partial_x^2}f\big)^2 - \partial_x^{-1}\bar f\cdot f^2\Big\}\\
&- \frac{\lambda}{2}\partial_x^{-2}\Big\{\Pi_{\neq 0}\big(e^{-i\tau\partial_x^2}\bar f\big)\cdot\big(e^{i\tau\partial_x^2}f\big)^2 - \Pi_{\neq 0}\big(e^{-i\tau\partial_x^2}\bar f\big)\cdot e^{i\tau\partial_x^2}f^2\Big\}\\
&- \frac{i\lambda\tau}{2}e^{i\tau\partial_x^2}\partial_x^{-1}\Big\{\partial_x\bar f\cdot e^{-i\tau\partial_x^2}\big(e^{i\tau\partial_x^2}f\big)^2 - \partial_x\bar f\cdot f^2\Big\}\\
&+ \lambda^2\tau^2e^{i\tau\partial_x^2}\Big\{\psi_1(-2i\tau\partial_x^2)\Big(\bar f\cdot|f|^2\Big)\cdot f^2 + \big[\psi_2(-2i\tau\partial_x^2)-\psi_1(-2i\tau\partial_x^2)\big]f\cdot|f|^4\Big\}\\
&- 2\lambda^2\tau^2e^{i\tau\partial_x^2}\Big\{\psi_1(-2i\tau\partial_x^2)\bar f\cdot f^2\cdot|f|^2 + f^2\cdot|f|^2\cdot\big[\psi_2(-2i\tau\partial_x^2)-\frac{1}{2}\big]\bar f\Big\},
\end{aligned}
$$

where

$$
\Pi_0 f = \hat f_0 \quad\text{and}\quad \Pi_{\neq 0}f = \sum_{k\in\mathbb{Z},k\neq 0} e^{ikx}\hat f_k.
$$

Remark 2.1. Although the definition of the algorithm in (2.4) seems complicated, it consists of only $O(1)$ simple terms which can be computed by the fast Fourier transform (FFT) with only $O(N\ln N)$ computational cost at every time level, where N is the degrees of freedom in a spatial discretization; see [11,22,31] for more details of the spatial discretization by the Fourier spectral method. For the simplicity of presentation, we focus on semi-discretization in time for the construction of the time-stepping method in this paper.

The main theoretical result in this paper is the following theorem.

Theorem 2.2. *Let u^n be the numerical solution of the NLS equation given by (2.3)–(2.4). Then, under the regularity condition $u^0\in H^{\frac{5}{3}}(\mathbb{T})$, there exist positive constants τ_0 and C_0 such that for any $\tau\in(0,\tau_0]$ there holds*

$$
\max_{1\le n\le L}\|u(t_n,\cdot)-u^n\|_{L^2}\le C_0\tau^2, \qquad (2.5)
$$

where C_0 and τ_0 depend only on the final time T and $\|u^0\|_{H^{\frac{5}{3}}}$.

In addition, under the weaker regularity condition $u^0\in H^1(\mathbb{T})$, the following error estimate holds:

$$
\max_{1\le n\le L}\|u(t_n,\cdot)-u^n\|_{L^2}\le C_1\tau\sqrt{\ln\tau^{-1}}, \qquad (2.6)
$$

for any $\tau\in(0,\tau_1]$, where C_1 and τ_1 depend only on T and $\|u^0\|_{H^1}$.

The construction of the algorithm in (2.3)–(2.4) and the proof of Theorem 2.2 are presented in the rest of this paper.

Remark 2.3. The proposed method presented in the next section has the potential to be extended to seek a new scheme similar to (2.3) in higher space dimensions. Compared to the second-order schemes in [8, 19, 30], which require $u_0\in H^{\frac{d}{2}+2}(\mathbb{T}^d)$ for $d=2,3$, the proposed method has a lower regularity requirement for the initial data for the numerical solutions to have second-order convergence. If the spatial derivatives are almost uniformly distributed in the consistency error of the new scheme, then the regularity requirement of the initial data might be improved to $H^{\frac{d}{3}+\frac{4}{3}}(\mathbb{T}^d)$ for second-order convergence in $L^2(\mathbb{T}^d)$. However, it should be noted that one of the main difficulties in extending the proposed method to higher dimensions is the problem of resonance, which becomes more complicated in higher dimensions, as discussed in [19]. Therefore, further research is needed to address this issue and develop efficient and accurate low-regularity integrators for higher-dimensional problems.

Remark 2.4. The spatial discretization of (3.1) can be handled using trigonometric interpolation, and a fully discrete error estimate can be established by combining the current proof for semi-discretization in time and the approach of [22] for full discretizations, albeit with some additional technical details. Specifically, assuming that the initial data u^0 satisfies the regularity condition $u^0 \in H^{\frac{5}{3}}(\mathbb{T})$, there exist positive constants τ_0, N_0, and C_0 such that for any time step size $\tau \in (0, \tau_0]$ and any number of spatial grid points $N \geq N_0$, the error between the numerical solution u_N^n obtained by the fully discrete scheme and the exact solution $u(t_n, \cdot)$ at each time step $t_n = n\tau$ satisfies

$$\max_{1\leq n\leq L} \|u(t_n, \cdot) - u_N^n\|_{L^2} \leq C_0\big(\tau^2 + N^{-\frac{5}{3}}\big),$$

where τ_0, N_0, and C_0 depend only on the final time T and $\|u^0\|_{H^{\frac{5}{3}}}$.

3. The construction of the method

For simplicity, we only consider the case $\lambda = 1$; the case $\lambda = -1$ can be treated in the same way. We will frequently use the following version of the Kato–Ponce inequalities, which were originally proved in [20] and further improved in [7, 21].

Lemma 3.1. (*The Kato–Ponce inequalities*).

(i) If $\gamma > \frac{1}{2}$ and $f, g \in H^\gamma(\mathbb{T})$ then

$$\|fg\|_{H^\gamma} \lesssim \|f\|_{H^\gamma}\|g\|_{H^\gamma}.$$

(ii) If $\gamma \geq 0$, $\gamma_1 > \frac{1}{2}$, $f \in H^{\gamma+\gamma_1}(\mathbb{T})$ and $g \in H^\gamma(\mathbb{T})$, then

$$\|fg\|_{H^\gamma} \lesssim \|f\|_{H^{\gamma+\gamma_1}}\|g\|_{H^\gamma}.$$

From the previous articles (for example, see [22, 28, 31, 41]) it is known that the twisted function $v(t) = e^{-it\partial_x^2}u(t)$, with $u(t)$ being the solution of the NLS equation (1.1), can be expressed as follows:

$$v(t_n + \tau) = v(t_n) - i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[\big|e^{i(t_n+s)\partial_x^2}v(t_n + s)\big|^2 e^{i(t_n+s)\partial_x^2}v(t_n + s)\Big]ds. \tag{3.1}$$

The objective of this section is to construct a fully explicit approximation of (3.1) with local error $O(\tau^3)$. To this end, we first reformulate $v(t_n + s)$ as

$$v(t_n + s) = v(t_n) + \Phi_1^n(v(t_n), s) + \mathcal{R}^s(t_n), \tag{3.2}$$

where

$$v(t_n) + \Phi_1^n(v(t_n), s) = v(t_n) - ise^{-it_n\partial_x^2}\Big[\varphi_1(-2is\partial_x^2)e^{-it_n\partial_x^2}\bar{v}(t_n)\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big], \tag{3.3}$$

is the first-order low-regularity integrator constructed in [28]. For the remainder term $\mathcal{R}^s(t_n)$, Ostermann & Schratz [28] have proved the following estimate for any $r > \frac{3}{2}$:

$$\|\mathcal{R}^s(t_n)\|_{H^{r-1}(\mathbb{T})} \leq s^2 \cdot C\big(\|v\|_{C([0,T];H^r)}\big), \tag{3.4}$$

where $C\big(\|v\|_{C([0,T];H^r)}\big)$ denotes a constant which depends only on $\|v\|_{C([0,T];H^r)}$.

By inserting (3.2) into the right-hand side of (3.1) we derive that

$$\begin{aligned} v(t_n + \tau) = v(t_n) - i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big\{&\overline{e^{i(t_n+s)\partial_x^2}\big(v(t_n) + \Phi_1^n(v(t_n), s) + \mathcal{R}^s(t_n)\big)} \\ &\cdot\Big[e^{i(t_n+s)\partial_x^2}\big(v(t_n) + \Phi_1^n(v(t_n), s) + \mathcal{R}^s(t_n)\big)\Big]^2\Big\}ds. \end{aligned} \tag{3.5}$$

Then direct calculation yields the following expression:

$$\begin{aligned} v(t_n + \tau) &= v(t_n) - i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[e^{-i(t_n+s)\partial_x^2}\bar{v}(t_n)\big(e^{i(t_n+s)\partial_x^2}v(t_n)\big)^2\Big]ds \\ &\quad -i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[e^{-i(t_n+s)\partial_x^2}\overline{\Phi_1^n(v(t_n), s)}\big(e^{i(t_n+s)\partial_x^2}v(t_n)\big)^2\Big]ds \\ &\quad -2i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[\Big|e^{i(t_n+s)\partial_x^2}v(t_n)\Big|^2\big(e^{i(t_n+s)\partial_x^2}\Phi_1^n(v(t_n), s)\big)\Big]ds \\ &\quad +\mathcal{R}_1^\tau(t_n) \\ &=: v(t_n) + I^\tau(t_n) + J_1^\tau(t_n) + J_2^\tau(t_n) + \mathcal{R}_1^\tau(t_n). \end{aligned} \tag{3.6}$$

The remainder $R_1^\tau(t_n)$ is estimated below.

Lemma 3.2. *Let $r > \frac{3}{2}$ and $\tau \in (0,1]$. Then the following estimate holds:*

$$\|\mathcal{R}_1^\tau(t_n)\|_{H^{r-1}} \le C\tau^3, \tag{3.7}$$

where C is some constant which only depends on $\|v\|_{C([0,T];H^r)}$.

Proof. By comparing (3.5) and (3.6) we see that the remainder $\mathcal{R}_1^\tau(t_n)$ is a sum of following terms:

$$-i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[e^{-i(t_n+s)\partial_x^2}\overline{\mathcal{W}}_j(t_n)\cdot e^{i(t_n+s)\partial_x^2}\mathcal{W}_k(t_n)\cdot e^{i(t_n+s)\partial_x^2}\mathcal{W}_l(t_n)\Big]ds \tag{3.8}$$

for $j+k+l \ge 2$, where

$$\mathcal{W}_0(t_n) = v(t_n), \quad \mathcal{W}_1(t_n) = \Phi_1^n(v(t_n), s), \quad \mathcal{W}_2(t_n) = \mathcal{R}^s(t_n). \tag{3.9}$$

Note that φ_1 is bounded on the imaginary axis, by applying (i) of Lemma 3.1 we get

$$\|\mathcal{W}_1(t_n)\|_{H^{r-1}} \le \|\mathcal{W}_1(t_n)\|_{H^r} = \|\Phi_1^n(v(t_n), s)\|_{H^r} \lesssim s\cdot\|v\|^3_{C([0,T];H^r)}, \tag{3.10}$$

where we have used the equality $\|e^{it\partial_x^2}f\|_{H^r} = \|f\|_{H^r}$ for all $t\in\mathbb{R}$ and $r\ge 0$. Then (3.7) is an easy consequence of (3.4), (3.8), (3.10) and the Kato–Ponce inequality. □

The main objective is to find some computable second-order approximations of $I^\tau(t_n)$, $J_1^\tau(t_n)$ and $J_2^\tau(t_n)$ in the expression (3.6).

3.1. Approximation of $I^\tau(t_n)$

We begin with decomposing $I^\tau(t_n)$ into two parts based on its Fourier series expansion, i.e.,

$$\begin{aligned}
I^\tau(t_n) &= -i\int_0^\tau e^{-i(t_n+s)\partial_x^2}\Big[e^{-i(t_n+s)\partial_x^2}\bar v(t_n)\Big(e^{i(t_n+s)\partial_x^2}v(t_n)\Big)^2\Big]ds \\
&= -i\sum_{k\in\mathbb{Z}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=k_1+k_2+k_3}}\int_0^\tau e^{i(t_n+s)[k^2+k_1^2-k_2^2-k_3^2]}ds\,\hat{\bar v}_{k_1}(t_n)\hat v_{k_2}(t_n)\hat v_{k_3}(t_n)e^{ikx} \\
&= -i\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ 0=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar v}_{k_1}(t_n)\hat v_{k_2}(t_n)\hat v_{k_3}(t_n)\cdot\int_0^\tau e^{is\phi_3}ds \\
&\quad -i\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar v}_{k_1}(t_n)\hat v_{k_2}(t_n)\hat v_{k_3}(t_n)e^{ikx}\cdot\int_0^\tau e^{is\phi_3}ds \\
&:= I_1^\tau(t_n) + I_2^\tau(t_n),
\end{aligned} \tag{3.11}$$

where $\phi_3 = \phi_3(k;k_1,k_2,k_3)$ denotes the following expression:

$$\phi_3(k;k_1,k_2,k_3) = k^2 + k_1^2 - k_2^2 - k_3^2. \tag{3.12}$$

Since $k = k_1+k_2+k_3$, it follows that $\phi_3(k;k_1,k_2,k_3) = 2(kk_1 + k_2k_3)$. In the case $k=0$ and $k_2k_3\neq 0$ we have

$$\int_0^\tau e^{is\phi_3}ds = \frac{1}{2ik_2k_3}\big(e^{2i\tau k_2k_3}-1\big).$$

By using the expression above, $I_1^\tau(t_n)$ can be decomposed into three parts corresponding to the following four cases (the second and third cases can be combined): $k_2\neq 0$ and $k_3\neq 0$, $k_2\neq 0$ and $k_3 = 0$, $k_2 = 0$ and $k_3\neq 0$, and $k_2 = k_3 = 0$, i.e.,

$$\begin{aligned}
I_1^\tau(t_n) = -i&\sum_{\substack{0=k_1+k_2+k_3\\ k_2\neq 0,k_3\neq 0}} e^{it_n[k_1^2-k_2^2-k_3^2]}\hat{\bar v}_{k_1}(t_n)\hat v_{k_2}(t_n)\hat v_{k_3}(t_n)\Big(\frac{1}{2ik_2k_3}\big(e^{2i\tau k_2k_3}-1\big)\Big) \\
&-2i\tau\sum_{k_1+k_2=0} e^{it_n(k_1^2-k_2^2)}\hat{\bar v}_{k_1}(t_n)\hat v_{k_2}(t_n)\hat v_0(t_n) + i\tau\hat{\bar v}_0(t_n)\hat v_0(t_n)\hat v_0(t_n).
\end{aligned}$$

By using the relation $2k_2k_3 = (k_2+k_3)^2 - k_2^2 - k_3^2$, we note that $I_1^\tau(t_n)$ can be expressed in the physical space as

$$
\begin{aligned}
I_1^\tau(t_n) = &\frac{1}{2}\Pi_0\Big(e^{-it_n\partial_x^2}\bar{v}(t_n)\cdot e^{-i\tau\partial_x^2}\big(e^{it_{n+1}\partial_x^2}\partial_x^{-1}v(t_n)\big)^2\Big)\\
&-\frac{1}{2}\Pi_0\Big(e^{-it_n\partial_x^2}\bar{v}(t_n)\cdot\big(e^{it_n\partial_x^2}\partial_x^{-1}v(t_n)\big)^2\Big)\\
&-2i\tau\Pi_0\big(|v(t_n)|^2\big)\cdot\Pi_0(v(t_n)) + i\tau\big|\Pi_0\big(v(t_n)\big)\big|^2\Pi_0\big(v(t_n)\big).
\end{aligned}
\tag{3.13}
$$

In order to approximate the integral $I_2^\tau(t_n)$, we divide $e^{is\phi_3}$ into the following three parts:

$$
\frac{(k_1+k_2)}{k}e^{is\phi_3} + \frac{(k_1+k_3)}{k}e^{is\phi_3} - \frac{k_1}{k}e^{is\phi_3}. \tag{3.14}
$$

By the symmetry between k_2 and k_3, we can collect the first two parts and obtain the following expression:

$$
\begin{aligned}
I_2^\tau(t_n) = &-2i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^\tau\frac{(k_1+k_2)}{k}e^{is\phi_3}ds\\
&+i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^\tau\frac{k_1}{k}e^{is\phi_3}ds\\
:= &\, I_{2,1}^\tau(t_n) + I_{2,2}^\tau(t_n).
\end{aligned}
\tag{3.15}
$$

From Section 4 in [22], we see that $I_{2,1}^\tau(t_n)$ can be calculated explicitly as follows

$$
\begin{aligned}
I_{2,1}^\tau(t_n) = &\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k=k_1+k_2+k_3\\k_1+k_3\neq 0}}\frac{e^{it_{n+1}\phi_3}-e^{it_n\phi_3}}{ik\cdot i(k_1+k_3)}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\\
&-2i\tau\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{k_1+k_3=0}\Big(\frac{k_1}{k}+1\Big)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_k(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\\
= &\, e^{-it_{n+1}\partial_x^2}\partial_x^{-1}\Big[\Big(e^{it_{n+1}\partial_x^2}v(t_n)\Big)\cdot\partial_x^{-1}\Big(\Big|e^{it_{n+1}\partial_x^2}v(t_n)\Big|^2\Big)\Big]\\
&-e^{-it_n\partial_x^2}\partial_x^{-1}\Big[\Big(e^{it_n\partial_x^2}v(t_n)\Big)\cdot\partial_x^{-1}\Big(\Big|e^{it_n\partial_x^2}v(t_n)\Big|^2\Big)\Big]\\
&-2i\tau\Pi_0\big(v(t_n)\partial_x\bar{v}(t_n)\big)\partial_x^{-1}v(t_n)\\
&-2i\tau\Pi_0\big(|v(t_n)|^2\big)v(t_n) + 2i\tau\Pi_0\big(|v(t_n)|^2\big)\Pi_0\big(v(t_n)\big).
\end{aligned}
\tag{3.16}
$$

However, $I_{2,2}^\tau(t_n)$ can not be integrated in the physical space explicitly. We need to find a "sufficiently good" approximation of $I_{2,2}^\tau(t_n)$ with desired consistency error.

Firstly, since $\phi_3 = 2kk_1 + 2k_2k_3$, the integrand $\frac{k_1}{k}e^{is\phi_3}$ can be decomposed into the following three parts:

$$
\frac{k_1}{k}e^{2iskk_1} + \frac{k_1}{k}\big(e^{2isk_2k_3}-1\big) + \frac{k_1}{k}\big(e^{2iskk_1}-1\big)\big(e^{2isk_2k_3}-1\big). \tag{3.17}
$$

Next, inspired by Ostermann, Yao & Wu [30], we furthermore split the third term of (3.17) into two parts, i.e.,

$$
\frac{k_1}{k}\big(e^{2iskk_1}-1\big)\big(e^{2isk_2k_3}-1\big) = 2i\frac{k_1k_2k_3}{k}e^{2isk_2k_3}\mathcal{M}_\tau(kk_1) + R^s(k;k_1,k_2,k_3),
$$

where

$$
\mathcal{M}_\tau(\alpha) = \frac{1}{\tau}\int_0^\tau\sigma\big(e^{2i\sigma\alpha}-1\big)d\sigma \tag{3.18}
$$

is a temporal average of $s(e^{2is\alpha}-1)$, and

$$
R^s(k;k_1,k_2,k_3) = \frac{k_1}{k}\big(e^{2iskk_1}-1\big)\big(e^{2isk_2k_3}-1\big) - 2i\frac{k_1k_2k_3}{k}e^{2isk_2k_3}\mathcal{M}_\tau(kk_1). \tag{3.19}
$$

Therefore, $I^{\tau}_{2,2}(t_n)$ can be decomposed into the following four terms:

$$
\begin{aligned}
&I^{\tau}_{2,2}(t_n)\\
&= i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^{\tau}\frac{k_1}{k}e^{2iskk_1}ds\\
&\quad +i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^{\tau}\frac{k_1}{k}\big(e^{2isk_2k_3}-1\big)ds\\
&\quad +i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^{\tau}2i\frac{k_1k_2k_3}{k}e^{2isk_2k_3}\mathcal{M}_{\tau}(kk_1)ds\\
&\quad +\mathcal{R}^{\tau}_2(t_n)\\
&:= I^{\tau}_{2,2,1}(t_n)+I^{\tau}_{2,2,2}(t_n)+I^{\tau}_{2,2,3}(t_n)+\mathcal{R}^{\tau}_2(t_n),
\end{aligned}
\tag{3.20}
$$

where

$$
\mathcal{R}^{\tau}_2(t_n) = i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^{\tau}R^s(k;k_1,k_2,k_3)ds. \tag{3.21}
$$

We note that $I^{\tau}_{2,2,1}(t_n)$, $I^{\tau}_{2,2,2}(t_n)$ and $I^{\tau}_{2,2,3}(t_n)$ are tractable. For the remainder term $\mathcal{R}^{\tau}_2(t_n)$, we shall prove the following estimates in subsection 3.3,:

$$
\|\mathcal{R}^{\tau}_2(t_n)\|_{L^2}\le \tau^3 C(\|v\|_{C([0,T];H^r)}) \quad\text{and}\quad \|\mathcal{R}^{\tau}_2(t_n)\|_{H^r}\le \tau^2 C(\|v\|_{C([0,T];H^r)}) \ \text{ for } r\ge \tfrac{5}{3}. \tag{3.22}
$$

We can evaluate the integral in $I^{\tau}_{2,2,1}(t_n)$ by using the relation $2kk_1 = k^2+k_1^2-(k_2+k_3)^2$, i.e.,

$$
\int_0^{\tau}\frac{k_1}{k}e^{2iskk_1}ds = \frac{1}{2ik^2}\Big(e^{i\tau[k^2+k_1^2-(k_2+k_3)^2]}-1\Big). \tag{3.23}
$$

Then, by using this expression, we can write $I^{\tau}_{2,2,1}(t_n)$ in the physical space as follows:

$$
\begin{aligned}
I^{\tau}_{2,2,1}(t_n) = &-\frac{1}{2}e^{-it_{n+1}\partial_x^2}\partial_x^{-2}\Big[\Big(e^{-it_{n+1}\partial_x^2}\bar{v}(t_n)\Big)\cdot e^{i\tau\partial_x^2}\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big]\\
&+\frac{1}{2}e^{-it_n\partial_x^2}\partial_x^{-2}\Big[\Big|e^{it_n\partial_x^2}v(t_n)\Big|^2 e^{it_n\partial_x^2}v(t_n)\Big].
\end{aligned}
\tag{3.24}
$$

As for $I^{\tau}_{2,2,2}(t_n)$, we note that its integrand satisfies $e^{2isk_2k_3}-1=0$ when $k_2=0$ or $k_3=0$. Therefore, $I^{\tau}_{2,2,2}(t_n)$ can be written as

$$
\begin{aligned}
I^{\tau}_{2,2,2}(t_n) &= i\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k=k_1+k_2+k_3\\k_2\neq 0,k_3\neq 0}} e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\cdot\int_0^{\tau}\frac{k_1}{k}\big(e^{2isk_2k_3}-1\big)ds\\
&= -\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{\substack{k=k_1+k_2+k_3\\k_2\neq 0,k_3\neq 0}} \frac{ik_1}{2ik\cdot ik_2\cdot ik_3}e^{it_n\phi_3}\big(e^{2i\tau k_2k_3}-1\big)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\\
&\quad -i\tau\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{k=k_1+k_2+k_3}\frac{ik_1}{ik}e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}\\
&\quad +2i\tau\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\sum_{k=k_1+k_2}\frac{ik_1}{ik}e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_0(t_n)e^{ikx} - i\tau\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}e^{it_n\phi_3}\hat{\bar{v}}_k(t_n)\big(\hat{v}_0(t_n)\big)^2e^{ikx}.
\end{aligned}
$$

By using the identity $2k_2k_3 = (k_2+k_3)^2 - k_2^2 - k_3^2$, we have

$$
\begin{aligned}
I^\tau_{2,2,2}(t_n) = & -\frac{1}{2}e^{-it_n\partial_x^2}\partial_x^{-1}\Big[\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)\Big)\cdot e^{-i\tau\partial_x^2}\Big(e^{it_{n+1}\partial_x^2}\partial_x^{-1}v(t_n)\Big)^2\Big] \\
& +\frac{1}{2}e^{-it_n\partial_x^2}\partial_x^{-1}\Big[\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)\Big)\cdot\Big(e^{it_n\partial_x^2}\partial_x^{-1}v(t_n)\Big)^2\Big] \\
& -i\tau e^{-it_n\partial_x^2}\partial_x^{-1}\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)(e^{it_n\partial_x^2}v(t_n))^2\Big) \\
& +2i\tau\Pi_0 v(t_n)e^{-it_n\partial_x^2}\partial_x^{-1}\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)e^{it_n\partial_x^2}v(t_n)\Big) \\
& -i\tau(\Pi_0 v(t_n))^2 e^{-it_n\partial_x^2}\Pi_{\neq 0}\big(e^{-it_n\partial_x^2}\bar{v}(t_n)\big).
\end{aligned}
\tag{3.25}
$$

As for $I^\tau_{2,2,3}(t_n)$, we know from (3.18) that $\mathcal{M}_\tau(kk_1) = 0$ when $k_1 = 0$. This implies that

$$
I^\tau_{2,2,3}(t_n) = i\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\ \sum_{\substack{k=k_1+k_2+k_3\\ k_1\neq 0}} \frac{k_1}{k}\mathcal{M}_\tau(kk_1)\Big(e^{2i\tau k_2k_3}-1\Big)e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}.
$$

From the definition of $\mathcal{M}_\tau(\alpha)$ in (3.18) we can calculate that

$$
\frac{k_1}{k}\mathcal{M}_\tau(kk_1) = \frac{k_1}{k\tau}\int_0^\tau \sigma\big(e^{2i\sigma kk_1}-1\big)d\sigma = \frac{e^{2i\tau kk_1}-1}{4\tau k^3k_1} - \frac{ie^{2i\tau kk_1}}{2k^2} - \frac{k_1\tau}{2k}.
$$

Then, $I^\tau_{2,2,3}(t_n)$ can be decomposed into the following three terms

$$
i\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\ \sum_{\substack{k=k_1+k_2+k_3\\ k_1\neq 0}} \frac{e^{2i\tau kk_1}-1}{4\tau k^3k_1}\Big(e^{2i\tau k_2k_3}-1\Big)e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx} \tag{3.26a}
$$

$$
+\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\ \sum_{\substack{k=k_1+k_2+k_3\\ k_1\neq 0}} \frac{e^{2i\tau kk_1}}{2k^2}\Big(e^{2i\tau k_2k_3}-1\Big)e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx} \tag{3.26b}
$$

$$
-i\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\ \sum_{\substack{k=k_1+k_2+k_3\\ k_1\neq 0}} \frac{k_1\tau}{2k}\Big(e^{2i\tau k_2k_3}-1\Big)e^{it_n\phi_3}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)e^{ikx}. \tag{3.26c}
$$

By using the two relations $2kk_1 = k^2 + k_1^2 - (k_2+k_3)^2$ and $2k_2k_3 = (k_2+k_3)^2 - k_2^2 - k_3^2$, we can find the following explicit expression of $I^\tau_{2,2,3}(t_n)$ in the physical space:

$$
\begin{aligned}
I^\tau_{2,2,3}(t_n) = & \left.\begin{aligned}
& \frac{i}{4\tau}e^{-it_{n+1}\partial_x^2}\partial_x^{-3}\Big[\Big(e^{-it_{n+1}\partial_x^2}\partial_x^{-1}\bar{v}(t_n)\Big)\cdot\Big(e^{it_{n+1}\partial_x^2}v(t_n)\Big)^2\Big] \\
& -\frac{i}{4\tau}e^{-it_{n+1}\partial_x^2}\partial_x^{-3}\Big[\Big(e^{-it_{n+1}\partial_x^2}\partial_x^{-1}\bar{v}(t_n)\Big)\cdot e^{i\tau\partial_x^2}\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big] \\
& -\frac{i}{4\tau}e^{-it_n\partial_x^2}\partial_x^{-3}\Big[\Big(e^{-it_n\partial_x^2}\partial_x^{-1}\bar{v}(t_n)\Big)\cdot e^{-i\tau\partial_x^2}\Big(e^{it_{n+1}\partial_x^2}v(t_n)\Big)^2\Big] \\
& +\frac{i}{4\tau}e^{-it_n\partial_x^2}\partial_x^{-3}\Big[\Big(e^{-it_n\partial_x^2}\partial_x^{-1}\bar{v}(t_n)\Big)\cdot\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big]
\end{aligned}\right\}(3.26a) \\
& \left.\begin{aligned}
& -\frac{1}{2}e^{-it_{n+1}\partial_x^2}\partial_x^{-2}\Big[\Pi_{\neq 0}\Big(e^{-it_{n+1}\partial_x^2}\bar{v}(t_n)\Big)\cdot\Big(e^{it_{n+1}\partial_x^2}v(t_n)\Big)^2\Big] \\
& +\frac{1}{2}e^{-it_{n+1}\partial_x^2}\partial_x^{-2}\Big[\Pi_{\neq 0}\Big(e^{-it_{n+1}\partial_x^2}\bar{v}(t_n)\Big)\cdot e^{i\tau\partial_x^2}\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big]
\end{aligned}\right\}(3.26b) \\
& \left.\begin{aligned}
& -\frac{i\tau}{2}e^{-it_n\partial_x^2}\partial_x^{-1}\Big[\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)\Big)\cdot e^{-i\tau\partial_x^2}\Big(e^{it_{n+1}\partial_x^2}v(t_n)\Big)^2\Big] \\
& +\frac{i\tau}{2}e^{-it_n\partial_x^2}\partial_x^{-1}\Big[\Big(e^{-it_n\partial_x^2}\partial_x\bar{v}(t_n)\Big)\cdot\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big].
\end{aligned}\right\}(3.26c)
\end{aligned}
$$

We see that the explicit expressions of $I^\tau_{2,2,1}(t_n)$, $I^\tau_{2,2,2}(t_n)$ and $I^\tau_{2,2,3}(t_n)$ in the physical space can be found. Then, by combining (3.11) and (3.20), we obtain the following result:

$$I^\tau(t_n) = \Psi^n_1(v(t_n), \tau) + \mathcal{R}^\tau_2(t_n), \tag{3.27}$$

where

$$\Psi^n_1(v(t_n), \tau) = I^\tau_1(t_n) + I^\tau_{2,1}(t_n) + I^\tau_{2,2,1}(t_n) + I^\tau_{2,2,2}(t_n) + I^\tau_{2,2,3}(t_n)$$

has an explicit expression in the physical space and therefore can be effectively computed.

3.2. Approximations of $J^\tau_1(t_n)$ and $J^\tau_2(t_n)$

It remains to find computable approximations of $J^\tau_1(t_n)$ and $J^\tau_2(t_n)$ defined in (3.6), which have the following expressions:

$$\begin{aligned} J^\tau_1(t_n) &= -i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \Big[e^{-i(t_n+s)\partial_x^2} \overline{\Phi^n_1(v(t_n), s)} \cdot \big(e^{i(t_n+s)\partial_x^2} v(t_n)\big)^2 \Big] ds, \\ J^\tau_2(t_n) &= -2i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \Big[\Big| e^{i(t_n+s)\partial_x^2} v(t_n) \Big|^2 \cdot \big(e^{i(t_n+s)\partial_x^2} \Phi^n_1(v(t_n), s)\big) \Big] ds, \end{aligned} \tag{3.28}$$

where

$$\Phi^n_1(v(t_n), s) = -is e^{-it_n\partial_x^2} \Big[\varphi_1(-2is\partial_x^2) e^{-it_n\partial_x^2} \bar{v}(t_n) \cdot \Big(e^{it_n\partial_x^2} v(t_n) \Big)^2 \Big], \tag{3.29}$$

and

$$\overline{\Phi^n_1(v(t_n), s)} = is e^{it_n\partial_x^2} \Big[\varphi_1(2is\partial_x^2) e^{it_n\partial_x^2} v(t_n) \cdot \Big(e^{-it_n\partial_x^2} \bar{v}(t_n) \Big)^2 \Big]. \tag{3.30}$$

By substituting (3.30) into (3.28) we obtain the following updated expression of $J^\tau_1(t_n)$:

$$\begin{aligned} J^\tau_1(t_n) = \int_0^\tau s \cdot e^{-i(t_n+s)\partial_x^2} \Big\{ & e^{-is\partial_x^2} \Big[\varphi_1(2is\partial_x^2) e^{it_n\partial_x^2} v(t_n) \cdot \Big(e^{-it_n\partial_x^2} \bar{v}(t_n) \Big)^2 \Big] \\ & \cdot \big(e^{i(t_n+s)\partial_x^2} v(t_n) \big)^2 \Big\} ds. \end{aligned} \tag{3.31}$$

The Fourier series expansion of (3.31) can be written as

$$J^\tau_1(t_n) = \sum_{k\in\mathbb{Z}} \sum_{k=k_1+\cdots+k_5} K^\tau_1 \cdot e^{it_n\phi_5} \hat{v}_{k_1}(t_n) \hat{\bar{v}}_{k_2}(t_n) \hat{\bar{v}}_{k_3}(t_n) \hat{v}_{k_4}(t_n) \hat{v}_{k_5}(t_n) e^{ikx}, \tag{3.32}$$

where

$$\phi_5 = k^2 - k_1^2 + k_2^2 + k_3^2 - k_4^2 - k_5^2, \tag{3.33}$$

$$K^\tau_1 = \int_0^\tau s \cdot e^{is[k^2 + (k_1+k_2+k_3)^2 - k_4^2 - k_5^2]} \cdot \frac{1 - e^{-2isk_1^2}}{2isk_1^2} ds. \tag{3.34}$$

Note that $J^\tau_1(t_n)$ can not be explicitly integrated in the physical space. Hence to select the dominant terms from (3.34), we use the relation

$$k^2 + (k_1 + k_2 + k_3)^2 - k_4^2 - k_5^2 = 2(k_1 + k_2 + k_3)^2 + 2\beta_1, \tag{3.35}$$

where $\beta_1 = (k_1 + k_2 + k_3)(k_4 + k_5) + k_4 k_5$ is a summation of cross terms. Then we decompose the integrand of (3.34) into

$$\begin{aligned} & s \cdot e^{2is[(k_1+k_2+k_3)^2+\beta_1]} \cdot \frac{1 - e^{-2isk_1^2}}{2isk_1^2} \\ &= s \cdot e^{2is[(k_1+k_2+k_3)^2+\beta_1]} + s \cdot e^{2is[(k_1+k_2+k_3)^2+\beta_1]} \cdot \Big(\frac{1 - e^{-2isk_1^2}}{2isk_1^2} - 1 \Big) \\ &= s \cdot e^{2is(k_1+k_2+k_3)^2} + s \cdot e^{2is(k_1+k_2+k_3)^2} \big(e^{2is\beta_1} - 1 \big) \\ &\quad + s \cdot e^{2isk_1^2} \cdot \Big(\frac{1 - e^{-2isk_1^2}}{2isk_1^2} - 1 \Big) \\ &\quad + s \cdot e^{2isk_1^2} \cdot \Big(e^{2is[(k_1+k_2+k_3)^2 - k_1^2 + \beta_1]} - 1 \Big) \cdot \Big(\frac{1 - e^{-2isk_1^2}}{2isk_1^2} - 1 \Big), \end{aligned} \tag{3.36}$$

Then K_1^τ can be split into the following three terms

$$K_1^\tau = \int_0^\tau s\cdot e^{2is(k_1+k_2+k_3)^2}ds + \int_0^\tau s\cdot e^{2isk_1^2}\cdot\Big(\frac{1-e^{-2isk_1^2}}{2isk_1^2}-1\Big)ds + K_{1,R}^\tau, \tag{3.37}$$

where

$$\begin{aligned} K_{1,R}^\tau =& \int_0^\tau s\cdot e^{2is(k_1+k_2+k_3)^2}\big(e^{2is\beta_1}-1\big)ds \\ &+\int_0^\tau s\cdot e^{2isk_1^2}\cdot\Big(e^{2is[(k_1+k_2+k_3)^2-k_1^2+\beta_1]}-1\Big)\cdot\Big(\frac{1-e^{-2isk_1^2}}{2isk_1^2}-1\Big)ds. \end{aligned} \tag{3.38}$$

From the definitions of ψ_1 and ψ_2 in (2.2), we can calculate that

$$\psi_1(z) = \int_0^1 s\cdot e^{sz}ds, \quad \text{and} \quad \psi_2(z)=\int_0^1 \frac{e^{sz}-1}{z}ds, \tag{3.39}$$

and therefore (3.37) implies

$$K_1^\tau = \tau^2\psi_1\big(2i\tau(k_1+k_2+k_3)^2\big) + \tau^2\Big(\psi_2(2i\tau k_1^2)-\psi_1(2i\tau k_1^2)\Big) + K_{1,R}^\tau. \tag{3.40}$$

Substituting (3.40) into (3.32), we derive that

$$\begin{aligned} J_1^\tau(t_n) =& \tau^2\sum_{k\in\mathbb{Z}}\sum_{k=k_1+\cdots+k_5} e^{it_n\phi_5}\hat{v}_{k_1}(t_n)\hat{\bar{v}}_{k_2}(t_n)\hat{\bar{v}}_{k_3}(t_n)\hat{v}_{k_4}(t_n)\hat{v}_{k_5}(t_n)e^{ikx} \\ &\qquad \cdot\psi_1\big(2i\tau(k_1+k_2+k_3)^2\big) \\ &+\tau^2\sum_{k\in\mathbb{Z}}\sum_{k=k_1+\cdots+k_5} e^{it_n\phi_5}\hat{v}_{k_1}(t_n)\hat{\bar{v}}_{k_2}(t_n)\hat{\bar{v}}_{k_3}(t_n)\hat{v}_{k_4}(t_n)\hat{v}_{k_5}(t_n)e^{ikx} \\ &\qquad \cdot\Big(\psi_2(2i\tau k_1^2)-\psi_1(2i\tau k_1^2)\Big) \\ &+\mathcal{R}_3^\tau(t_n) \\ =& \tau^2 e^{-it_n\partial_x^2}\Big\{\psi_1(-2i\tau\partial_x^2)\Big[e^{it_n\partial_x^2}v(t_n)\cdot\Big(e^{-it_n\partial_x^2}\bar{v}(t_n)\Big)^2\Big]\cdot\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\Big\} \\ &+\tau^2 e^{-it_n\partial_x^2}\Big\{\Big(\psi_2(-2i\tau\partial_x^2)-\psi_1(-2i\tau\partial_x^2)\Big)e^{it_n\partial_x^2}v(t_n)\cdot\Big|e^{-it_n\partial_x^2}v(t_n)\Big|^4\Big\} \\ &+\mathcal{R}_3^\tau(t_n) \\ :=& \Psi_2^n(v(t_n),\tau)+\mathcal{R}_3^\tau(t_n), \end{aligned}$$

where

$$\mathcal{R}_3^\tau(t_n) = \sum_{k\in\mathbb{Z}}\sum_{k=k_1+\cdots+k_5} K_{1,R}^\tau\cdot e^{it_n\phi_5}\hat{v}_{k_1}(t_n)\hat{\bar{v}}_{k_2}(t_n)\hat{\bar{v}}_{k_3}(t_n)\hat{v}_{k_4}(t_n)\hat{v}_{k_5}(t_n)e^{ikx}. \tag{3.41}$$

In subsection 3.3 we shall prove the following estimate of the remainder $\mathcal{R}_3^\tau(t_n)$:

$$\|\mathcal{R}_3^\tau(t_n)\|_{L^2} \lesssim \tau^3\|v\|^5_{C([0,T];H^r)}, \quad \text{for any} \quad r\geq\frac{3}{2}. \tag{3.42}$$

Meanwhile, it follows from Lemma 3.1 (i) and the boundedness of ψ_1 and ψ_2 on the imaginary axis that

$$\|\Psi_2^n(v(t_n),\tau)\|_{H^r} \lesssim \tau^2\|v\|^5_{C([0,T];H^r)}, \quad \text{for any} \quad r>\frac{1}{2}. \tag{3.43}$$

As for $J_2^\tau(t_n)$, we substitute (3.29) into (3.28). This yields

$$\begin{aligned} J_2^\tau(t_n) =& -2\int_0^\tau s\cdot e^{-i(t_n+s)\partial_x^2}\Big\{\Big|e^{i(t_n+s)\partial_x^2}v(t_n)\Big|^2 \\ &\qquad \cdot e^{is\partial_x^2}\Big[\Big(e^{it_n\partial_x^2}v(t_n)\Big)^2\cdot\varphi_1(-2is\partial_x^2)e^{-it_n\partial_x^2}\bar{v}(t_n)\Big]\Big\}ds. \end{aligned} \tag{3.44}$$

The Fourier series expansion of $J_2^\tau(t_n)$ is

$$J_2^\tau(t_n) = -2\sum_{k\in\mathbb{Z}}\sum_{k=k_1+\cdots+k_5} K_2^\tau\cdot e^{it_n\tilde{\phi}_5}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\hat{v}_{k_4}(t_n)\hat{\bar{v}}_{k_5}(t_n)e^{ikx}, \tag{3.45}$$

where

$$\tilde{\phi}_5 = k^2 + k_1^2 - k_2^2 - k_3^2 - k_4^2 + k_5^2, \tag{3.46}$$

$$K_2^\tau = \int_0^\tau s \cdot e^{is[k^2+k_1^2-k_2^2-(k_3+k_4+k_5)^2]} \cdot \frac{e^{2isk_5^2}-1}{2isk_5^2} ds. \tag{3.47}$$

By using the equality $k = k_1 + \cdots + k_5$, we get

$$\begin{aligned} & k^2 + k_1^2 - k_2^2 - (k_3+k_4+k_5)^2 \\ & = 2k_1^2 + 2k_1(k_2+k_3+k_4+k_5) + 2k_2(k_3+k_4+k_5) \\ & =: 2k_1^2 + 2\beta_2. \end{aligned} \tag{3.48}$$

As before, we decompose the integrand of (3.47) by

$$\begin{aligned} s \cdot e^{2is(k_1^2+\beta_2)} \cdot \frac{e^{2isk_5^2}-1}{2isk_5^2} &= s \cdot e^{2isk_1^2} + s \cdot e^{2isk_1^2} \cdot \big(e^{2is\beta_2} - 1\big) + s \cdot \Big(\frac{e^{2isk_5^2}-1}{2isk_5^2} - 1\Big) \\ & + s \cdot \big(e^{2is(k_1^2+\beta_2)} - 1\big) \cdot \Big(\frac{e^{2isk_5^2}-1}{2isk_5^2} - 1\Big). \end{aligned} \tag{3.49}$$

Hence

$$K_2^\tau = \tau^2 \psi_1\big(2i\tau k_1^2\big) + \tau^2 \Big(\psi_2(2i\tau k_5^2) - \frac{1}{2}\Big) + K_{2,R}^\tau. \tag{3.50}$$

where

$$K_{2,R}^\tau = \int_0^\tau s \cdot e^{2isk_1^2} \cdot \big(e^{2is\beta_2} - 1\big) ds + \int_0^\tau s \cdot \big(e^{2is(k_1^2+\beta_2)} - 1\big) \cdot \Big(\frac{e^{2isk_5^2}-1}{2isk_5^2} - 1\Big) ds. \tag{3.51}$$

We get

$$\begin{aligned} J_2^\tau(t_n) = & -2\tau^2 e^{-it_n\partial_x^2} \Big\{ \psi_1(-2i\tau\partial_x^2) e^{-it_n\partial_x^2} \bar{v}(t_n) \cdot \Big(e^{it_n\partial_x^2} v(t_n)\Big)^2 \cdot \Big| e^{it_n\partial_x^2} v(t_n) \Big|^2 \Big\} \\ & -2\tau^2 e^{-it_n\partial_x^2} \Big\{ \Big(e^{it_n\partial_x^2} v(t_n)\Big)^2 \cdot \Big| e^{it_n\partial_x^2} v(t_n) \Big|^2 \\ & \qquad \cdot \Big(\psi_2(-2i\tau\partial_x^2) - \frac{1}{2}\Big) e^{-it_n\partial_x^2} \bar{v}(t_n) \Big\} \\ & + \mathcal{R}_4^\tau(t_n) \\ := & \Psi_3^n(v(t_n), \tau) + \mathcal{R}_4^\tau(t_n), \end{aligned} \tag{3.52}$$

where

$$\mathcal{R}_4^\tau(t_n) = -2 \sum_{k\in\mathbb{Z}} \sum_{k=k_1+\cdots+k_5} K_{2,R}^\tau \cdot e^{it_n\tilde{\phi}_5} \hat{\bar{v}}_{k_1}(t_n) \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \hat{v}_{k_4}(t_n) \hat{\bar{v}}_{k_5}(t_n) e^{ikx}. \tag{3.53}$$

In subsection 3.3 we shall prove the following result:

$$\|\mathcal{R}_4^\tau(t_n)\|_{L^2} \lesssim \tau^3 \|v\|_{C([0,T];H^r)}^5 \quad \text{for any} \quad r \geq \frac{3}{2}. \tag{3.54}$$

Therefore, by the similar argument as that for (3.43), we have

$$\|\Psi_3^n(v(t_n), \tau)\|_{H^r} \lesssim \tau^2 \|v\|_{C([0,T];H^r)}^5, \quad \text{for any} \quad r > \frac{1}{2}. \tag{3.55}$$

Overall, by collecting all the cases together, we obtain

$$v(t_{n+1}) = \Psi^n(v(t_n), \tau) + \mathcal{R}_1^\tau(t_n) + \mathcal{R}_2^\tau(t_n) + \mathcal{R}_3^\tau(t_n) + \mathcal{R}_4^\tau(t_n), \tag{3.56}$$

where

$$\Psi^n(v(t_n), \tau) = v(t_n) + \Psi_1^n(v(t_n), \tau) + \Psi_2^n(v(t_n), \tau) + \Psi_3^n(v(t_n), \tau), \tag{3.57}$$

and

$$\|\mathcal{R}_1^\tau(t_n)\|_{L^2} \le \|\mathcal{R}_1^\tau\|_{H^{r-1}} \le \tau^3 \cdot C\Big(\|v\|_{C([0,T];H^r)}\Big), \quad \text{for any} \quad r > \frac{3}{2},$$

$$\|\mathcal{R}_2^\tau(t_n)\|_{L^2} \lesssim \tau^3 \|v\|^3_{C([0,T];H^r)}, \quad \|\mathcal{R}_2^\tau\|_{H^r} \lesssim \tau^2 \|v\|^3_{C([0,T];H^r)}, \quad \text{for any} \quad r \ge \frac{5}{3}, \tag{3.58}$$

$$\|\mathcal{R}_3^\tau(t_n)\|_{L^2} + \|\mathcal{R}_4^\tau(t_n)\|_{L^2} \lesssim \tau^3 \|v\|^5_{C([0,T];H^r)}, \quad \text{for any} \quad r \ge \frac{3}{2},$$

$$\|\Psi_2^n(v(t_n),\tau)\|_{H^r} + \|\Psi_3^n(v(t_n),\tau)\|_{H^r} \lesssim \tau^2 \|v\|^5_{C([0,T];H^r)}, \quad \text{for any} \quad r > \frac{1}{2}. \tag{3.59}$$

The numerical scheme can be defined by dropping the remainders $\mathcal{R}_1^\tau(t_n)$, $\mathcal{R}_2^\tau(t_n)$, $\mathcal{R}_3^\tau(t_n)$ and $\mathcal{R}_4^\tau(t_n)$, i.e.,

$$v^{n+1} = \Psi^n(v^n, \tau), \quad n \ge 0; \quad \text{for } v^0 = u^0. \tag{3.60}$$

Then, by substituting $v^n = e^{-it_n\partial_x^2} u^n$ into (3.60), we obtain

$$u^{n+1} = e^{it_{n+1}\partial_x^2} \Psi^n(e^{-it_n} u^n, \tau) =: \widetilde{\Psi}_\tau(u^n),$$

where the expression of $\widetilde{\Psi}_\tau(u^n)$ is given in (2.4). This yields the method in (2.3)–(2.4).

4. Proof of Theorem 2.2

In this section, we prove the error estimates in (2.5) and (2.6) for $H^{\frac{5}{3}}$ and H^1 initial data, respectively.

4.1. Proof of (2.5): Error estimates in the case $v \in C([0,T]; H^{\frac{5}{3}}(\mathbb{T}))$

The consistency errors of the proposed method consists of $\mathcal{R}_1^\tau(t_n)$, $\mathcal{R}_2^\tau(t_n)$, $\mathcal{R}_3^\tau(t_n)$ and $\mathcal{R}_4^\tau(t_n)$, where the estimate of $\mathcal{R}_1^\tau(t_n)$ has already been proved in Lemma 3.2. In this subsection, we prove the estimates in (3.22), (3.42) and (3.54) for the remainders $\mathcal{R}_2^\tau(t_n)$, $\mathcal{R}_3^\tau(t_n)$ and $\mathcal{R}_4^\tau(t_n)$, respectively. The following lemma will be frequently used.

Lemma 4.1. *For any $\alpha \in \mathbb{R}$ and $\gamma \in [0,1]$, there hold*

$$\big|e^{i\alpha} - 1\big| \lesssim |\alpha|^\gamma, \quad \text{and} \quad \big|e^{i\alpha} - 1 - i\alpha\big| \lesssim |\alpha|^{1+\gamma}. \tag{4.1}$$

Proof. Note that

$$|e^{i\alpha} - 1| \le 2, \quad |e^{i\alpha} - 1| = \Big|\int_0^\alpha ie^{is} ds\Big| \le |\alpha|. \tag{4.2}$$

Hence, we have $|e^{i\alpha} - 1| \le 2^{1-\gamma}|\alpha|^\gamma$ for $\gamma \in [0,1]$. Moreover, we have

$$|e^{i\alpha} - 1 - i\alpha| \le 2|\alpha|, \quad |e^{i\alpha} - 1 - i\alpha| = \Big|\int_0^\alpha \int_0^s ie^{i\sigma} d\sigma ds\Big| \le \frac{1}{2}|\alpha|^2. \tag{4.3}$$

It follows that $|e^{i\alpha} - 1 - i\alpha| \le 2^{1-2\gamma}|\alpha|^{1+\gamma}$ for $\gamma \in [0,1]$. □

The estimate of $\mathcal{R}_2^\tau(t_n)$ is presented in the following lemma.

Lemma 4.2. *Let $r \ge \frac{5}{3}$ and $v \in C([0,T]; H^r)$. Then the remainder $\mathcal{R}_2^\tau(t_n)$ defined in (3.21) has the following upper bounds:*

$$\|\mathcal{R}_2^\tau(t_n)\|_{L^2} \lesssim \tau^3 \|v\|^3_{C([0,T];H^r)}, \tag{4.4}$$

$$\|\mathcal{R}_2^\tau(t_n)\|_{H^r} \lesssim \tau^2 \|v\|^3_{C([0,T];H^r)}. \tag{4.5}$$

Proof. We first recall the following expression of $R^s(k; k_1, k_2, k_3)$:

$$R^s(k; k_1, k_2, k_3) = \frac{k_1}{k}\big(e^{2iskk_1} - 1\big)\big(e^{2isk_2k_3} - 1\big) - 2i\frac{k_1k_2k_3}{k} e^{2isk_2k_3} \mathcal{M}_\tau(kk_1), \tag{4.6}$$

which can be divided into the following three terms

$$R^s(k; k_1, k_2, k_3) = \frac{k_1}{k}\big(e^{2iskk_1} - 1\big)\big(e^{2isk_2k_3} - 1\big) - 2i\frac{k_1k_2k_3}{k} e^{2isk_2k_3} \mathcal{M}_\tau(kk_1)$$

$$= \frac{k_1}{k}\big(e^{2iskk_1}-1\big)\big(e^{2isk_2k_3}-1-2isk_2k_3e^{2isk_2k_3}\big) \tag{4.7a}$$

$$+2i\frac{sk_1k_2k_3}{k}\big(e^{2iskk_1}-1\big)\big(e^{2isk_2k_3}-1\big) \tag{4.7b}$$

$$+2i\frac{sk_1k_2k_3}{k}\big(e^{2iskk_1}-1\big)-2i\frac{k_1k_2k_3}{k}e^{2isk_2k_3}\mathcal{M}_\tau(kk_1) \tag{4.7c}$$

$$=: R_1^s(k;k_1,k_2,k_3)+R_2^s(k;k_1,k_2,k_3)+R_3^s(k;k_1,k_2,k_3).$$

The second factor in the expression of $R_1^s(k;k_1,k_2,k_3)$ can be written as

$$e^{2isk_2k_3}-1-2isk_2k_3e^{2isk_2k_3} = e^{2isk_2k_3}-1-2isk_2k_3+2isk_2k_3\big(1-e^{2isk_2k_3}\big).$$

This and Lemma 4.1 imply that

$$\big|R_1^s(k;k_1,k_2,k_3)\big| \lesssim \frac{|k_1|}{|k|}\cdot|skk_1|^\gamma\cdot|sk_2k_3|^{2-\gamma} = s^2|k|^{\gamma-1}|k_1|^{1+\gamma}|k_2k_3|^{2-\gamma}, \tag{4.8}$$

for $\gamma\in[0,1]$. Analogously,

$$\big|R_2^s(k;k_1,k_2,k_3)\big| \lesssim \frac{|sk_1k_2k_3|}{|k|}\cdot|skk_1|^\gamma\cdot|sk_2k_3|^{1-\gamma} = s^2|k|^{\gamma-1}|k_1|^{1+\gamma}|k_2k_3|^{2-\gamma}, \tag{4.9}$$

To summarize, we have

$$\Big|\int_0^\tau R_1^s(k;k_1,k_2,k_3)ds\Big| + \Big|\int_0^\tau R_2^s(k;k_1,k_2,k_3)ds\Big| \lesssim \tau^3|k|^{\gamma-1}|k_1|^{1+\gamma}|k_2k_3|^{2-\gamma}. \tag{4.10}$$

Direct calculation of the integral of $R_3^s(k;k_1,k_2,k_3)$ yields the following result:

$$\int_0^\tau R_3^s(k;k_1,k_2,k_3)ds = 2i\frac{k_1k_2k_3}{k}\int_0^\tau s(e^{2iskk_1}-1)ds - \frac{k_1}{k}\mathcal{M}_\tau(kk_1)\int_0^\tau 2ik_2k_3e^{2isk_2k_3}ds$$

$$= \frac{k_1}{k}\Big(2i\tau k_2k_3 - e^{2i\tau k_2k_3}+1\Big)\mathcal{M}_\tau(kk_1).$$

Meanwhile, by the definition of $\mathcal{M}_\tau(kk_1)$ in (3.18) and Lemma 4.1, we have

$$\big|\mathcal{M}_\tau(kk_1)\big| = \Big|\frac{1}{\tau}\int_0^\tau s(e^{2ikk_1s}-1)ds\Big| \lesssim \frac{1}{\tau}\int_0^\tau s^{1+\gamma}|kk_1|^\gamma ds \lesssim \tau^{1+\gamma}|kk_1|^\gamma,$$

for $\gamma\in[0,1]$. This implies that

$$\Big|\int_0^\tau R_3^s(k;k_1,k_2,k_3)ds\Big| \lesssim \frac{|k_1|}{|k|}\cdot|\tau k_2k_3|^{2-\gamma}\cdot\tau^{1+\gamma}|kk_1|^\gamma = \tau^3|k|^{\gamma-1}|k_1|^{1+\gamma}|k_2k_3|^{2-\gamma}. \tag{4.11}$$

By substituting the inequalities (4.10) and (4.11) into (3.21), we see that $\hat{\mathcal{R}}^\tau_{2,0}=0$ and

$$\big|\hat{\mathcal{R}}^\tau_{2,k}(t_n)\big| \lesssim \tau^3\cdot\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}}\langle k\rangle^{\gamma-1}|k_1|^{1+\gamma}|k_2k_3|^{2-\gamma}|\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)| \quad \text{for } k\neq 0, \tag{4.12}$$

where we have used the inequality $|k|^{\gamma-1}\lesssim\langle k\rangle^{\gamma-1}$, for $k\in\mathbb{Z}$ such that $k\neq0$.

Now, we take $\gamma=\frac12+\varepsilon$ for $0<\varepsilon\le\frac16$, and denote

$$\widetilde{\mathcal{R}}_2(t_n) = \sum_{k\in\mathbb{Z}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\k=k_1+k_2+k_3}}\langle k\rangle^{-(\frac12-\varepsilon)}|k_1|^{\frac32+\varepsilon}\cdot|k_2k_3|^{\frac32-\varepsilon}\cdot|\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)|e^{ikx}.$$

Since the Sobolev embedding $W^m_p(\mathbb{T})\hookrightarrow L^2(\mathbb{T})$ holds for $p=(1-\varepsilon)^{-1}$ and $m=\frac12-\varepsilon$ (see [4,14]), where $W^m_p(\mathbb{T})$ denotes the Sobolev space endowed with the norm

$$\|f\|_{W^m_p(\mathbb{T})} = \Big\|\sum_{k\in\mathbb{Z}}\langle k\rangle^m\hat{f}_k\cdot e^{ikx}\Big\|_{L^p(\mathbb{T})},$$

it follows from (4.12) and the Sobolev embedding theorem that

$$\|\mathcal{R}_2^\tau(t_n)\|_{L^2} \lesssim \tau^3\|\widetilde{\mathcal{R}}_2(t_n)\|_{L^2} \lesssim \tau^3\|J^{(\frac12-\varepsilon)}\widetilde{\mathcal{R}}_2(t_n)\|_{L^{\frac{1}{1-\varepsilon}}}. \tag{4.13}$$

If we denote

$$V_1(t_n)=\sum_{k\in\mathbb{Z}}|k|^{\frac{3}{2}+\varepsilon}|\hat{\bar{v}}_k(t_n)|e^{ikx} \quad \text{and} \quad V_2(t_n)=\sum_{k\in\mathbb{Z}}|k|^{\frac{3}{2}-\varepsilon}|\hat{v}_k(t_n)|e^{ikx}, \tag{4.14}$$

then the following equality holds:

$$\begin{aligned} J^{(\frac{1}{2}-\varepsilon)}\widetilde{\mathcal{R}}_2(t_n) &= \sum_{k\in\mathbb{Z}}\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=k_1+k_2+k_3}}|k_1|^{\frac{3}{2}+\varepsilon}\cdot|k_2k_3|^{\frac{3}{2}-\varepsilon}|\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)|e^{ikx} \\ &= V_1(t_n)\big(V_2(t_n)\big)^2. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.13) and employing the Hölder inequality, we get

$$\|\mathcal{R}_2^\tau(t_n)\|_{L^2}\lesssim\tau^3\big\|V_1(t_n)\big(V_2(t_n)\big)^2\big\|_{L^{\frac{1}{1-\varepsilon}}}\lesssim\tau^3\|V_1(t_n)\|_{L^{q_1}}\cdot\big(\|V_2(t_n)\|_{L^{q_2}}\big)^2, \tag{4.16}$$

where $q_1=(1/3+\varepsilon)^{-1}$ and $q_2=(1/3-\varepsilon)^{-1}$. Then the Sobolev embedding theorem shows that $H^{\frac{1}{6}-\varepsilon}\hookrightarrow L^{q_1}$ and $H^{\frac{1}{6}+\varepsilon}\hookrightarrow L^{q_2}$, consequently, by (4.14) and (4.16), one has

$$\|\mathcal{R}_2^\tau(t_n)\|_{L^2}\lesssim\tau^3\|V_1(t_n)\|_{H^{\frac{1}{6}-\varepsilon}}\cdot\big(\|V_2(t_n)\|_{H^{\frac{1}{6}+\varepsilon}}\big)^2\lesssim\tau^3\|v\|^3_{C([0,T];H^{\frac{5}{3}})}. \tag{4.17}$$

Therefore, we finish the proof of (4.4).

Next, observing the expression of $R^s(k;k_1,k_2,k_3)$ and the equalities

$$\Big|e^{2iskk_1}-1\Big|\lesssim 1,\quad \Big|e^{2isk_2k_3}-1\Big|\lesssim s|k_2k_3|,\quad \Big|e^{2isk_2k_3}\Big|\lesssim 1,\quad \Big|\mathcal{M}_\tau(kk_1)\Big|\lesssim\tau,$$

we have

$$\big|\hat{\mathcal{R}}^\tau_{2,k}(t_n)\big|\lesssim\tau^2\cdot\sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=k_1+k_2+k_3}}\langle k\rangle^{-1}|k_1k_2k_3||\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)|. \tag{4.18}$$

Hence, by using the notation

$$V(t_n)=\sum_{k\in\mathbb{Z}}|\hat{v}_k(t_n)|e^{ikx}, \tag{4.19}$$

we obtain from Lemma 3.1 (i) that

$$\|\mathcal{R}_2^\tau(t_n)\|_{H^r}\lesssim\tau^2\Big\||\partial_x|V(t_n)\cdot\big(|\partial_x|V(t_n)\big)^2\Big\|_{H^{r-1}}\lesssim\tau^2\|v\|^3_{C([0,T];H^r)}\quad\text{for } r>\frac{3}{2}. \tag{4.20}$$

This completes the proof of Lemma 4.2. □

We now present estimates for $\mathcal{R}_3^\tau(t_n)$ and $\mathcal{R}_4^\tau(t_n)$.

Lemma 4.3. *Let $r\geq\frac{3}{2}$ and $v\in C([0,T];H^r(\mathbb{T}))$. Then the remainders $\mathcal{R}_3^\tau(t_n)$ and $\mathcal{R}_4^\tau(t_n)$ defined in* (3.41) *and* (3.53)*, respectively, have the following upper bound:*

$$\|\mathcal{R}_3^\tau(t_n)\|_{L^2}+\|\mathcal{R}_4^\tau(t_n)\|_{L^2}\lesssim\tau^3\|v\|^5_{C([0,T];H^r)}, \tag{4.21}$$

Proof. We recall the expression of $\mathcal{R}_3^\tau(t_n)$ in (3.41), which implies that

$$\big|\hat{\mathcal{R}}^\tau_{3,k}(t_n)\big|\leq\sum_{k=k_1+\cdots+k_5}\big|K^\tau_{1,R}\big|\cdot|\hat{v}_{k_1}(t_n)|\cdot|\hat{\bar{v}}_{k_2}(t_n)|\cdot|\hat{\bar{v}}_{k_3}(t_n)|\cdot|\hat{v}_{k_4}(t_n)|\cdot|\hat{v}_{k_5}(t_n)|. \tag{4.22}$$

In order to estimate the upper bound of $\|\mathcal{R}_3^\tau(t_n)\|_{L^2}$, we observe from (3.38) that

$$\big|K^\tau_{1,R}\big|\leq\tau^2\max_{s\in[0,\tau]}\bigg\{\Big|e^{2is(k_1+k_2+k_3)^2}\big(e^{2is\beta_1}-1\big)\Big|\bigg\} \tag{4.23a}$$

$$+\tau^2\max_{s\in[0,\tau]}\bigg\{\Big|e^{2isk_1^2}\cdot\Big(e^{2is[(k_1+k_2+k_3)^2-k_1^2+\beta_1]}-1\Big)\cdot\Big(\frac{1-e^{-2isk_1^2}}{2isk_1^2}-1\Big)\Big|\bigg\}. \tag{4.23b}$$

We now estimate (4.23a) and (4.23b) respectively. For the term (4.23a), we note that β_1 is a summation of cross terms. Thus, it follows from Lemma 4.1 that

$$\Big|e^{2is(k_1+k_2+k_3)^2}\big(e^{2is\beta_1}-1\big)\Big|\lesssim s|\beta_1|\lesssim s\cdot\sum_{j\neq l}|k_jk_l|. \tag{4.24}$$

For the term (4.23b), we recall the expression of β_1 in (3.35) and consider the following identity:

$$\begin{aligned}(k_1+k_2+k_3)^2-k_1^2+\beta_1 &= 2k_1(k_2+k_3)+(k_2+k_3)^2+\beta_1\\ &= 2k_1(k_2+k_3)+(k_2+k_3)^2+(k_1+k_2+k_3)(k_4+k_5)+k_4k_5\\ &= \big[(k_2+k_3)(k_2+k_3+k_4+k_5)+k_4k_5\big]+k_1(2k_2+2k_3+k_4+k_5),\end{aligned} \tag{4.25}$$

which implies

$$\big|(k_1+k_2+k_3)^2-k_1^2+\beta_1\big| \le \Big(\sum_{j\neq 1}|k_j|\Big)^2+2|k_1|\sum_{j\neq 1}|k_j|. \tag{4.26}$$

By using Lemma 4.1, we get

$$\begin{aligned}&\Big|e^{2is[(k_1+k_2+k_3)^2-k_1^2+\beta_1]}-1\Big| \lesssim s^{\frac12}\big|(k_1+k_2+k_3)^2-k_1^2+\beta_1\big|^{\frac12}\\ &\Big|\frac{1-e^{-2isk_1^2}}{2isk_1^2}-1\Big| = \frac{\big|1-e^{-2isk_1^2}-2isk_1^2\big|}{\big|2sk_1^2\big|} \lesssim \frac{s^{\frac32}|k_1|^3}{s|k_1|^2} = s^{\frac12}|k_1|.\end{aligned} \tag{4.27}$$

Hence

$$\begin{aligned}&\Big|e^{2isk_1^2}\cdot\Big(e^{2is[(k_1+k_2+k_3)^2-k_1^2+\beta_1]}-1\Big)\cdot\Big(\frac{1-e^{-2isk_1^2}}{2isk_1^2}-1\Big)\Big|\\ &\lesssim s\Big\{\Big(\sum_{j\neq 1}|k_j|\Big)^2+|k_1|\sum_{j\neq 1}|k_j|\Big\}^{\frac12}\cdot|k_1|\\ &\lesssim s\Big(\sum_{j\neq 1}|k_j|+|k_1|^{\frac12}\sum_{j\neq 1}|k_j|^{\frac12}\Big)\cdot|k_1|.\end{aligned} \tag{4.28}$$

Then substituting (4.24) and (4.28) into (4.23), we get

$$|K_{1,R}^\tau| \lesssim \tau^3\Big(\sum_{j\neq 1}|k_1|^{\frac32}|k_j|^{\frac12}+\sum_{j\neq l}|k_jk_l|\Big). \tag{4.29}$$

Again, by denoting $V(t_n)=\sum_{k\in\mathbb{Z}}|\hat{v}_k(t_n)|e^{ikx}$ as in (4.19), inequalities (4.22), (4.29) and Plancherel's identity imply that

$$\begin{aligned}\|\mathcal{R}_3^\tau(t_n)\|_{L^2} &\lesssim \tau^3\Big(\big\||\partial_x|^{\frac32}V\cdot|\partial_x|^{\frac12}V\cdot V^3\big\|_{L^2}+\big\|\big(|\partial_x|V\big)^2\cdot V^3\big\|_{L^2}\Big)\\ &\le \tau^3\Big(\big\||\partial_x|^{\frac32}V\big\|_{L^2}\cdot\big\||\partial_x|^{\frac12}V\big\|_{L^\infty}\cdot\big\|V\big\|_{L^\infty}^3+\big\||\partial_x|V\big\|_{L^4}\cdot\big\||\partial_x|V\big\|_{L^4}\cdot\big\|V\big\|_{L^\infty}^3\Big)\\ &\le \tau^3\|v\|_{C([0,T];H^r)}^5 \quad \text{for } r\ge\frac32.\end{aligned} \tag{4.30}$$

For the remainder $\mathcal{R}_4^\tau(t_n)$ defined in (3.53), it follows from the inequalities in Lemma 4.1 that

$$\begin{aligned}&\Big|e^{2isk_1^2}\big(e^{2is\beta_2}-1\big)\Big| \le \Big|\big(e^{2is\beta_2}-1\big)\Big| \lesssim s|\beta_2|,\\ &\Big|\big(e^{2is(k_1^2+\beta_2)}-1\big)\Big(\frac{e^{2isk_5^2}-1}{2isk_5^2}-1\Big)\Big| = \Big|\big(e^{2is(k_1^2+\beta_2)}-1\big)\Big(\frac{e^{2isk_5^2}-1-2isk_5^2}{2isk_5^2}\Big)\Big|\\ &\qquad\lesssim s^{\frac12}|k_1^2+\beta_2|^{\frac12}\cdot\frac{s^{\frac32}|k_5|^3}{s|k_5|^2} = s|k_1^2+\beta_2|^{\frac12}\cdot|k_5|.\end{aligned} \tag{4.31}$$

Analogously, we have

$$|K_{2,R}^\tau| \lesssim \tau^3\Big(\sum_{j\neq 5}|k_5|^{\frac32}|k_j|^{\frac12}+\sum_{j\neq l}|k_jk_l|\Big). \tag{4.32}$$

Moreover there holds

$$\|\mathcal{R}_4^\tau(t_n)\|_{L^2} \lesssim \tau^3\|v\|_{C([0,T];H^r)}^5, \tag{4.33}$$

for any $r\ge\frac32$. This estimate together with (4.30) give the desired result in (4.21). □

The difference between (3.56) and (3.58) gives the following error equation:

$$v(t_{n+1})-v^{n+1} = \Psi^n(v(t_n),\tau)-\Psi^n(v^n,\tau)+\mathcal{R}_1^\tau(t_n)+\mathcal{R}_2^\tau(t_n)+\mathcal{R}_3^\tau(t_n)+\mathcal{R}_4^\tau(t_n), \tag{4.34}$$

where the consistency errors $\mathcal{R}_j^\tau(t_n)$, $j = 1,2,3,4$, have been estimated in Lemma 3.2 and Lemmas 4.2–4.3, i.e.,

$$\|\mathcal{R}_1^\tau(t_n)\|_{L^2} + \|\mathcal{R}_2^\tau(t_n)\|_{L^2} + \|\mathcal{R}_3^\tau(t_n)\|_{L^2} + \|\mathcal{R}_4^\tau(t_n)\|_{L^2} \lesssim \tau^3 \quad \text{for } v \in C([0,T]; H^{\frac{5}{3}}). \tag{4.35}$$

The following stability estimate can be proved in the same way as [22,31] (therefore the detailed proof is omitted):

$$\|\Psi^n(v(t_n),\tau) - \Psi^n(v^n,\tau)\|_{L^2} \le (1 + C\tau)\|v(t_n) - v^n\|_{L^2}. \tag{4.36}$$

By combining the consistency estimates and the stability estimates, we obtain

$$\|v(t_{n+1}) - v^{n+1}\|_{L^2} \le (1 + C\tau)\|v(t_n) - v^n\|_{L^2} + C\tau^3, \tag{4.37}$$

which implies (2.5). This completes the error estimates in the case $v \in C([0,T]; H^{\frac{5}{3}}(\mathbb{T}))$.

4.2. Proof of (2.6): Error estimates in the case $v \in C([0,T]; H^1(\mathbb{T}))$

In this subsection, we sketch the error estimates for initial data in $H^1(\mathbb{T})$, i.e., the solution is in $C([0,T]; H^1(\mathbb{T}))$. In this case, we split the consistency error into the following three parts:

$$\begin{aligned} v(t_{n+1}) - \Psi^n(v(t_n),\tau) =& \big(v(t_{n+1}) - v(t_n) - I^\tau(t_n)\big) + \big(I^\tau(t_n) - \Psi_1^n(v(t_n),\tau)\big) \\ &+ \Psi_2^n(v(t_n),\tau) + \Psi_3^n(v(t_n),\tau). \end{aligned} \tag{4.38}$$

Firstly, by substituting the variation-of-constants formula (1.4) into the first term on the right-hand side of (4.38) and applying the Kato–Ponce inequality, we can obtain the following result (the details are similar as [22, 28, 41] and therefore omitted)

$$\big\|v(t_{n+1}) - v(t_n) - I^\tau(t_n)\big\|_{H^1} \lesssim \tau^2 \|v\|^5_{C([0,T];H^1)}. \tag{4.39}$$

Secondly, for the term $\mathcal{R}_2^\tau(t_n) = I^\tau(t_n) - \Psi_1^n(v(t_n),\tau)$, which is related to $R^s(k, k_1, k_2, k_3)$ through (3.21), we consider the following triangle inequality:

$$\begin{aligned} \Big|\int_0^\tau R^s(k;k_1,k_2,k_3)ds\Big| \le& \int_0^\tau \Big|\frac{k_1}{k}\big(e^{2iskk_1} - 1\big)\big(e^{2isk_2k_3} - 1\big)\Big|ds \\ &+ \Big|\frac{k_1}{k}\big(e^{2i\tau k_2k_3} - 1\big)\mathcal{M}_\tau(kk_1)\Big|. \end{aligned} \tag{4.40}$$

In order to estimate (4.40), we consider the following two cases

$$|k| \ge \min\{|k_2|,|k_3|\} \quad \text{and} \quad |k| < \min\{|k_2|,|k_3|\}.$$

For the moment we assume that $\min\{|k_2|,|k_3|\} = k_3$ and consider the two cases $|k| \ge |k_3| = \min\{|k_2|,|k_3|\}$ and $|k| < |k_3| = \min\{|k_2|,|k_3|\}$, respectively.

Case 1: If $|k| \ge |k_3| = \min\{|k_2|,|k_3|\}$, then we apply the following estimates

$$\big|e^{2iskk_1} - 1\big| \le 2, \qquad \big|e^{2isk_2k_3} - 1\big| \le 2s|k_2k_3|,$$

$$\big|\mathcal{M}_\tau(kk_1)\big| = \Big|\frac{1}{\tau}\int_0^\tau \sigma\big(e^{2i\sigma kk_1} - 1\big)d\sigma\Big| \le \Big|\frac{1}{\tau}\int_0^\tau 2\sigma d\sigma\Big| = \tau,$$

which imply that

$$\Big|\int_0^\tau R^s(k;k_1,k_2,k_3)ds\Big| \le 4\tau^2|k|^{-1}|k_1||k_2||k_3| \lesssim \tau^2|k|^{-\alpha}|k_1||k_2||k_3|^\alpha \quad \text{for } \alpha \in [0,1]. \tag{4.41}$$

Case 2. If $|k| \le |k_3| = \min\{|k_2|,|k_3|\}$, then the relation $k = k_1 + k_2 + k_3$ implies that

$$|k_1| \le |k| + |k_2| + |k_3| \lesssim |k_2|.$$

We use the inequality above and the following inequalities in estimating the right-hand side of (4.40):

$$\begin{aligned} &\big|e^{2iskk_1} - 1\big| \le 2s|kk_1|, \quad \big|e^{2isk_2k_3} - 1\big| \le 2, \\ &\big|\mathcal{M}_\tau(kk_1)\big| \le \Big|\frac{1}{\tau}\int_0^\tau 2s|kk_1|\sigma d\sigma\Big| = \tau|kk_1|, \end{aligned} \tag{4.42}$$

which imply the following result:

$$\Big|\int_0^\tau R^s(k;k_1,k_2,k_3)ds\Big| \le 4\tau^2|k_1|^2 \lesssim \tau^2|k_1||k_2| \lesssim \tau^2|k_1||k_2|\Big(\frac{|k_3|}{|k|}\Big)^\alpha \quad \text{for } \alpha\in[0,1]. \tag{4.43}$$

The assumption $\min\{|k_2|,|k_3|\}=k_3$ above actually does not lose generality when we use the symmetry between k_2 and k_3 in the expression of (3.21). In fact, by using the symmetry between k_2 and k_3, substituting (4.41) and (4.43) into (3.21) yields the following estimate:

$$\begin{aligned} |\hat{\mathcal{R}}^\tau_{2,k}(t_n)| \lesssim{}& \tau^2 \sum_{\substack{k_1+k_2+k_3=k\\ |k_2|\ge|k_3|,|k|\ge|k_3|}} |k|^{-1}|k_1||k_2||k_3||\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)| \\ &+\tau^2 \sum_{\substack{k_1+k_2+k_3=k\\ |k_2|\ge|k_3|,|k|<|k_3|}} |k_1||k_2||\hat{\bar{v}}_{k_1}(t_n)|\cdot|\hat{v}_{k_2}(t_n)|\cdot|\hat{v}_{k_3}(t_n)|. \end{aligned} \tag{4.44}$$

Thirdly, the following results can be proved by following the lines in the proof of Lemma 4 in [22] (the details are omitted):

$$\|\mathcal{R}^\tau_2(t_n)\|_{L^2} \lesssim \tau^2\sqrt{\ln\tau^{-1}}\|v\|^3_{C([0,T];H^1)}, \tag{4.45}$$

$$\|\mathcal{R}^\tau_2(t_n)\|_{H^s} \lesssim \tau^{\frac32}\|v\|^3_{C([0,T];H^1)} \quad \text{for} \quad s\in\Big(\frac12,1\Big). \tag{4.46}$$

Finally, substituting (3.59), (4.39) and (4.45)–(4.46) into (4.38) yields

$$\begin{aligned} &\|v(t_{n+1})-\Psi^n(v(t_n),\tau)\|_{L^2} \lesssim \tau^2\sqrt{\ln\tau^{-1}}\cdot C\big(\|v\|_{C([0,T];H^1)}\big),\\ &\|v(t_{n+1})-\Psi^n(v(t_n),\tau)\|_{H^s} \lesssim \tau^{\frac32}\cdot C\big(\|v\|_{C([0,T];H^1)}\big) \quad \text{for} \quad s\in\Big(\frac12,1\Big). \end{aligned} \tag{4.47}$$

This proves the desired estimates for the consistency errors in the case $v\in C([0,T];H^1(\mathbb{T}))$. The stability estimate (4.36) can be proved in the same way as [22, 31] and therefore omitted here. The error estimate in (2.6) can be obtained by combining the consistency estimate in (4.47) with the stability estimate (4.36).

The proof of Theorem 2.2 is complete. □

5. Numerical experiments

In this section, we present numerical experiments to test the convergence of the proposed method for both smooth and nonsmooth initial data. We consider the NLS equation (1.1) with $\lambda=1$ and the following initial value:

$$u^0(x)=\frac{1}{10}\sum_{0\ne k\in\mathbb{Z}}|k|^{-0.51-r}e^{ikx}, \tag{5.1}$$

which satisfies that $u^0\in H^r(\mathbb{T})$ but $u^0\notin H^{r+0.01}(\mathbb{T})$. We compare the numerical solution given by the proposed method (2.3) and several existing time-stepping methods, including the Strang splitting method, i.e.,

$$\begin{aligned} u_-^{n+1/2} &= e^{i\frac{\tau}{2}\partial_x^2}u^n,\\ u_+^{n+1/2} &= e^{-i\tau\left|u_-^{n+1/2}\right|^2}u_-^{n+1/2},\\ u^{n+1} &= e^{i\frac{\tau}{2}\partial_x^2}u_+^{n+1/2}, \end{aligned} \tag{5.2}$$

a mass- and energy-conservative Crank-Nicolson method (which conserves the mass and energy of the NLS equation; see [10, 12]), i.e.,

$$i\frac{u^{n+1}-u^n}{\tau}+\frac12\Big(\partial_x^2u^{n+1}+\partial_x^2u^n\Big)=\frac14\Big(|u^{n+1}|^2+|u^n|^2\Big)\cdot\big(u^{n+1}+u^n\big), \tag{5.3}$$

the first-order low-regularity integrator in [22] (which we refer to as `First-order LRI`), and two second-order low-regularity integrators in [19] and [8, 30] (referred to as `Second-order LRI_1` and `Second-order LRI_2`, respectively). The reference solution is computed by the proposed method (2.3) with a sufficiently small stepsize $\tau_{\text{ref}}=2^{-12}$.

The L^2 errors of these methods at $T = 1$ are shown in Figure 1 for H^r initial data with $r = 1, \frac{5}{3}, 2$ and 3, where a sufficiently large degrees of freedom $N = 2^{12}$ is chosen in the spatial discretization (by the Fourier spectral method with FFT). The numerical results in Figure 1 show that the proposed method in (2.3) can have second-order convergence when $u^0 \in H^{\frac{5}{3}}(\mathbb{T})$ and first-order convergence when $u^0 \in H^1(\mathbb{T})$. This is consistent with the theoretical results in Theorem 2.2. Moreover, the proposed method is the only one which has second-order convergence for $H^{\frac{5}{3}}$ initial data.

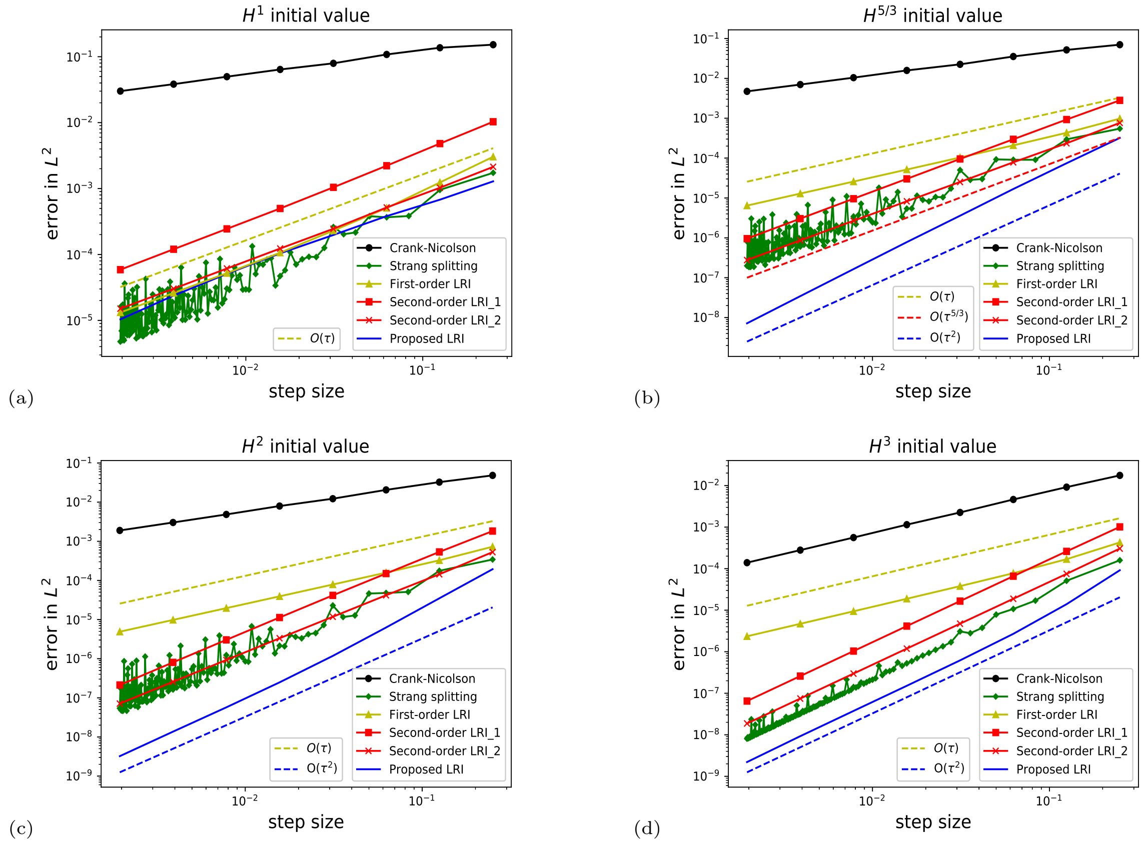

FIGURE 1. L^2 error of several time-stepping methods for H^s initial data, with $s \geq 1$.

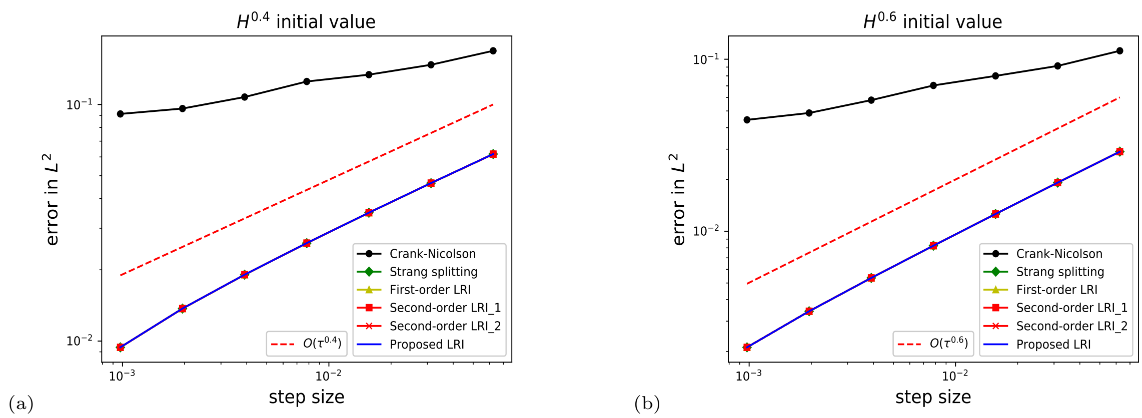

FIGURE 2. L^2 error of several time-stepping methods for $H^{0.4}$ and $H^{0.6}$ initial data.

For lower-regular H^r solutions with $r = 0.4$ and 0.6, we present the L^2 errors of the several numerical methods at $T = 1$ in Figure 2, where the numerical solutions are computed with the CFL condition $N = \tau^{-1}$, which improves the stability of the numerical solutions for extremely nonsmooth solutions like filters; see [32]. The numerical results show that all methods perform similarly for such extremely nonsmooth initial data.

Overall, the numerical results show that the proposed method improves the convergence order for nonsmooth initial data in $H^r(\mathbb{T})$ with $1 \le r < 2$, and is equally accurate as the other low-regularity integrators for extremely nonsmooth initial data in $H^r(\mathbb{T})$ with $0 < r < 1$.

Acknowledgment

This work was partially supported by the CAS AMSS-PolyU Joint Laboratory of Applied Mathematics, a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (GRF Project No. PolyU15302120), and an internal grant at The Hong Kong Polytechnic University (Project ID: P0038843, Work Programme: ZVX7).

References

[1] Dmitry S. Agafontsev, and Vladimir Zakharov: Integrable turbulence and formation of rogue waves. *Nonlinearity* **28** (2015), pp. 2791–2821.

[2] Georgios Akrivis: Finite difference discretization of the cubic Schrödinger equation. *IMA J. Numer. Anal.* **13** (1993), pp. 115–124.

[3] Genming Bai, Buyang Li, and Yifei Wu: A constructive low-regularity integrator for the 1d cubic nonlinear Schrödinger equation under the Neumann boundary condition. *IMA J. Numer. Anal.* (Accepted) DOI: 10.1093/imanum/drac075

[4] Árpád Bényi, Tadahiro Oh, The Sobolev inequality on the torus revisited. *Publ. Math. Debrecen* **83**(3)(2013), pp. 359–374.

[5] Havard Berland, Alvaro L. Islas, and Constance M. Schober, Conservation of phase space properties using exponential integrators on the cubic Schrödinger equation. *J. Comput. Phys.* **225**(2007a), pp. 284–299.

[6] Jean Bourgain: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* **3**(1993), pp. 107–156.

[7] Jean Bourgain, and Dong Li: On an endpoint Kato–Ponce inequality. *Differential Integral Equations* **27**(2014), pp. 1037–1072.

[8] Yvain Bruned and Katharina Schratz: Resonance based schemes for dispersive equations via decorated trees. *Forum Math. Pi* **10**(2022), pp. 1–76.

[9] Elena Celledoni, David Cohen, and Brynjulf Owren, Symmetric exponential integrators with an application to the cubic Schrödinger equation. *Found. Comp. Math.* **8**(2008), pp. 303–317.

[10] Qianshun Chang, Erhui Jia, and Weiwei Sun: Difference schemes for solving the generalized nonlinear Schrödinger equation. *J. Comput. Phys.* **148**(1999), pp. 397–415.

[11] Eleanor Chu: *Discrete and continuous Fourier transforms analysis, applications and fast algorithms.* CRC Press, New York, 2008.

[12] Michel C. Delfour, Michael Fortin, and G. Payr: Finite-difference solutions of a non-linear Schrödinger equation. *J. Comput. Phys.* **44** (1981), pp. 277–288.

[13] Johannes Eilinghoff, Roland Schnaubelt, and Katharina Schratz: Fractional error estimates of splitting schemes for the nonlinear Schrödinger equation. *J. Math. Anal. Appl.* **442**(2016), pp. 740–760.

[14] Loukas Grafakos, *Modern Fourier Analysis, second ed.*, Grad. Texts in Math., vol. 250, Springer, New York, 2009.

[15] Patrick Henning and Daniel Peterseim: Crank–Nicolson Galerkin approximations to nonlinear Schrödinger equations with rough potentials. *Math. Models Methods Appl. Sci.*, **27**(2017), pp. 2147–2184.

[16] Marlis Hochbruck and Alexander Ostermann: Exponential integrators. *Acta Numerica* **19**(2010), pp. 209–286.

[17] Martina Hofmanová and Katharina Schratz: An exponential-type integrator for the KdV equation. *Numer. Math.* **136**(2017), pp. 1117–1137.

[18] Liviu I. Ignat: A splitting method for the nonlinear Schrödinger equation. *J. Differential Equations* **250**(2011), pp. 3022–3046.

[19] Marvin Knöller, Alexander Ostermann, and Katharina Schratz: A Fourier integrator for the cubic nonlinear Schrödinger equation with rough initial data. *SIAM J. Numer. Anal.* **57**(2019), pp. 1967–1986.

[20] Tosio Kato and Gustavo Ponce: Commutator estimates and the Euler and Navier-Stokes equations. *Commun. Pure Appl. Math.* **41**(1988), pp. 891–907.

[21] Dong Li: On Kato–Ponce and fractional Leibniz. *Rev. Mat. Iberoam.* **35**(2019), pp. 23–100.

[22] Buyang Li and Yifei Wu, A full discrete low-regularity integrator for the 1D period cubic nonlinear Schrödinger equation, *Numer. Math.* **149**(2021), pp. 151–183.

[23] Buyang Li and Yifei Wu, An unfiltered low-regularity integrator for the KdV equation with solutions below H^1. Preprint, `https://arxiv.org/abs/2206.09320`

[24] Christian Lubich: On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations. *Math. Comp.* **77**(2008), pp. 2141–2153.

[25] Cui Ning, Yifei Wu, and Xiaofei Zhao: An embedded exponential-type low-regularity integrator for mKdV equation. *SIAM J. Numer. Anal.*, **60**(2022), pp. 999–1025.

[26] Alexander Ostermann, Frédéric Rousset, and Katharina Schratz: Error estimates of a Fourier integrator for the cubic Schrödinger equation at low regularity. *Found. Comput. Math.* **21**(2021), pp. 725–765.

[27] Alexander Ostermann, Frédéric Rousset, and Katharina Schratz: Fourier integrator for periodic NLS: low regularity estimates via discrete Bourgain spaces. *J. Eur. Math. Soc.* (2022), DOI: 10.4171/JEMS/1275.

[28] Alexander Ostermann and Katharina Schratz: Low regularity exponential-type integrators for semilinear Schrödinger equations. *Found. Comput. Math.* **18**(2018), pp. 731–755.

[29] Alexander Ostermann and Chunmei Su: A Lawson-type exponential integrator for the Korteweg-de Vries equation. *IMA J. Numer. Anal.* **40**(2020), pp. 2399–2414.

[30] Alexander Ostermann, Yifei Wu, and Fangyan Yao: A second order low-regularity integrator for the nonlinear Schrödinger equation. *Advances in Continuous and Discrete Models* (2022), DOI: 10.1186/s13662-022-03695-8

[31] Alexander Ostermann and Fangyan Yao: A fully discrete low-regularity integrator for the nonlinear Schrödinger equation. *J. Sci. Comput.* (2022), DOI:10.1007/s10915-022-01786-y

[32] Alexander Ostermann, Frédéric Rousset and Katharina Schratz: Error estimates at low regularity of splitting schemes for NLS. *Math. Comp.* **91**(2022), pp. 169–182.

[33] Alexander Ostermann, Chunmei Su: Two exponential-type integrator for the "good" Boussinesq equation. *Numer. Math.*, **143**(2019), pp. 683–712.

[34] Frédéric Rousset and Katharina Schratz: A general framework of low regularity integrators. *SIAM J. Numer. Anal.* **59**(2021), pp. 1735–1768.

[35] J. M. Sanz-Serna: Methods for the numerical solution of the nonlinear Schrödinger equation. *Math. Comp.* **43**(1984), pp. 21–27.

[36] Katharina Schratz, Yan Wang, and Xiaofei Zhao: Low-regularity integrators for nonlinear Dirac equations. *Math. Comp.* **90**(2021), pp. 189–214.

[37] Chunmei Su and Xiaofei Zhao: On time-splitting methods for nonlinear Schrödinger equation with highly oscillatory potential. *ESAIM Math. Model. Numer. Anal.* **54**(2020), pp. 1491–1508.

[38] Pierre Walczak, Stéphane Randoux, and Pierre Suret: Optical rogue waves in integrable turbulence. *Phys. Rev. Lett.* **114**, 143903(2015).

[39] Jilu Wang: A new error analysis of Crank–Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation. *J. Sci. Comput.* **60**(2014), pp. 390–407.

[40] Yan Wang and Xiaofei Zhao: A symmetric low-regularity integrator for nonlinear Klein–Gordon equation. *Math. Comp.*, **91**(2022), pp. 2215–2245.

[41] Yifei Wu and Fangyan Yao: A first-order Fourier integrator for the nonlinear Schrödinger equation on $\mathbb{T}$ without loss of regularity. *Math. Comp.* **91**(2022), pp. 1213–1235.

[42] Yifei Wu and Xiaofei Zhao: Optimal convergence of a first order low-regularity integrator for the KdV equation. *IMA J. Numer. Anal.* **42**(2022), pp. 3499–3528.

[43] Yifei Wu and Xiaofei Zhao: Embedded exponential-type low-regularity integrators for KdV equation under rough data. *BIT* **62**(2022), pp. 1049–1090.

Jiachuan Cao: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong. E-mail address: caojiachuan16@mails.ucas.edu.cn

Buyang Li: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong. E-mail address: buyang.li@polyu.edu.hk

Yanping Lin: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong. E-mail address: malin@polyu.edu.hk