Robust Sourcing Under Multi-level Supply Risks: Analysis of Random Yield and Capacity

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We consider the optimal sourcing problem when the available suppliers are subject to ambiguously correlated supply risks. This problem is motivated by the increasing severity of supply risks and difficulty evaluating common sources of vulnerability in upstream supply chains, problems reported by many surveys of goods-producing firms. We propose a distributionally robust model that accommodates *i*) multiple levels of supply disruption, not just full delivery or no delivery, and *ii*) can utilize data-driven estimates of the underlying correlation to develop sourcing strategies in situations where the true correlation structure is ambiguous. Using this framework, we provide analytical results regarding the form of a worst-case supply distribution and show that taking such a worst-case perspective is appealing due to severe consequences associated with supply chain risks. Moreover, we show how our distributionally robust model may be used to offer guidance to firms considering whether to exert additional effort in attempt to better understanding the prevailing correlation structure. Extensive computational experiments further demonstrate the performance of our distributionally robust approach and show how supplier characteristics and the type of supply uncertainty affect the optimal sourcing decision.

Key words: supply uncertainty, distributionally robust, risk management

History:

1. Introduction

Several studies report an increase in the frequency and severity of disturbances affecting today's global supply chains. These disturbances stem from a wide range of causes including natural disasters, labor protests, financial crises, political unrest, and cybersecurity threats. The apparent fragility of today's supply chains is largely a result of companies' efforts to reduce assets and improve financial performance via outsourcing (Tang 2006). Specifically, the increased use of outsourcing has caused supply chains to become long and complex, and companies have struggled to maintain visibility into the operations of their supply chain partners. Interestingly, recent evidence suggests that the source of disruptions in the supply chain is evolving. Specifically, the 2019 Supply Chain Resilience Report published by the Business Continuity Institute, an organization of business continuity and resilience professionals including members from over 100 countries, suggests that firms are getting better at managing the "close-to-home" supply chain risks posed by their tier-1 suppliers. However, disruptions occurring in deeper supply chain tiers, i.e., tiers 2 and above, are increasing. In particular, out of 352 respondents, 24.9% reported experiencing disruptions of a tier-2 supplier and 12.2% reported experiencing disruptions of a tier-3 or higher supplier. Both of these figures represent increases in comparison to the previous year. Moreover, 32.6% of respondents reported being unable to identify the original source of a disruption due to lack of analytical capabilities (BCI 2019).

The difficulty of managing disruptions that occur at deep tiers of the supply chain is exacerbated by data sharing restrictions (Ang et al. 2016). Although historical data allows a firm to estimate the marginal supply distributions for their tier-1 suppliers, contractual obligations or other motives may prevent tier-1 suppliers from sharing information about their suppliers. For example, both Toyota and Boeing have reported that their tier-1 suppliers are reluctant or unwilling to disclose details regarding their suppliers (Grimm 2013). This lack of visibility leaves companies susceptible to concentrated sources of vulnerability deep in the upstream supply chain. A disruption of such a common source can propagate downstream, potentially affecting all immediate suppliers of a buying firm. Such propagation is noted in Chopra & Sodhi (2014), which states that "disruptive risks tend to have a domino effect on the supply chain." Without adequate visibility or time-invariant data, it is all but impossible to account for such correlated risks accurately. Although researchers and organizations recognize that understanding the entire supply network is necessary to build a robust enterprise, attaining such understanding is a nearly insurmountable obstacle in many cases (Tucker & Lynch 2012, Masui & Nishi 2012). Thus, there is a need for methods that allow firms to accommodate interrelated supply uncertainties that arise in deep and complex supply chains, without requiring perfect knowledge of the supply chain and the prevailing correlation structure.

Aside from the challenges associated with deep supply chain risks, most of the OR/MS research on supply chain risk management (SCRM) focuses on the impact of major disruptions. In particular, most studies concentrate on the case where available suppliers are either *up* or *down*. Suppliers in an *up* state can handle all order requests and suppliers in a *down* state cannot fulfill any order requests. Although understanding the described *Bernoulli yield* process offers valuable insights, the ability to accommodate more general supply uncertainties is desirable since many of the day-to-day disruptions affecting companies are not *all-or-nothing*.

Towards addressing the described issues, this research considers the optimal sourcing strategy for a firm subject to general joint supply uncertainties when the immediate suppliers are ambiguously correlated. We propose a novel approach for the described problem that is based on distributionally robust (DR) optimization. Our DR model addresses the potential interrelatedness of suppliers by utilizing a worst-case joint supply distribution that satisfies the available marginal supply distributions and an upper bound on the covariance matrix. Our approach is well-aligned with behavioral tendencies observed in practice, where decision-makers are likely to exhibit risk-aversion (Snyder et al. 2016) when making sourcing decisions due to the negative consequences associated with supply disruptions. We show that the worst-case distribution has a simple and intuitive form that mimics the propagation of adverse effects stemming from an interruption in the supply of a common source of vulnerability in a supply chain. Ultimately, our DR model constitutes a robust approach for supplier selection and order allocation that is suitable for managing risks with limited knowledge of dependencies. Moreover, our DR model yields several insights regarding effective approaches for incorporating partial information regarding supplier correlations and for prioritizing efforts aimed at improving estimates of the true correlation structure.

1.1. Literature Review

The problem of sourcing under supply uncertainty has received substantial attention in the literature under the assumption that the prevailing joint supply distribution is known exactly. A large proportion of this research assumes that suppliers face independent disruption risks. For this case, many, including the seminal work of Anupindi & Akella (1993) and the highly-cited study of Dada et al. (2007), have established that "cost is the order qualifier while reliability is the order winner" when suppliers face independent disruption risk. Others, including Federgruen & Yang (2008, 2009), Tomlin (2009), Feng (2010), Feng & Shi (2012), Li et al. (2013), Hu & Kostamis (2015) and Li et al. (2017) consider additional aspects such as endogenous prices, uncertain demand, and inventory management, along with disruption risks. Although less common, several researchers, e.g., Babich et al. (2007), Federgruen & Yang (2008), Tang & Kouvelis (2011), Li et al. (2013), Sting & Huchzermeier (2014) and Feng et al. (2019), consider the effects of correlated supplier disruptions on the optimal sourcing strategy. However, the cited studies all assume that the true correlation structure is known. In practice, estimating the true correlation structure is difficult, if not impossible. Thus, the assumption of perfect knowledge limits the empirical validity of the associated insights. Our novel DR model constitutes a viable alternative for situations with ambiguous correlations and offers several practical insights such as the potential value of an improved understanding of the prevailing correlation and a method for prioritizing efforts aimed at obtaining such knowledge. Finally, we contribute to the described stream of literature by considering a more comprehensive set of disruption processes that include multiple levels of yield or capacity uncertainty. This latter aspect allows us to observe changes in the optimal sourcing strategy due to the type of supply uncertainty.

Our methodology is based on distributionally robust optimization (DRO), which identifies the optimal solution that hedges against a worst-case distribution. Lu & Shen (2021) recently surveys many applications of robust optimization in operations management. Scarf et al. (1958) provide the foundation for DRO when considering a class of inventory control problems where future demand is uncertain with known mean and standard deviation. DRO under moment uncertainty continues to receive attention in the literature and has been applied to the problems of portfolio optimization, lot-sizing, appointment scheduling, surgery planning, humanitarian logistics, redundancy allocation, targeted display advertising, and sharing economy (Zhang et al. 2016, Jiang et al. 2017, Noyan et al. 2022b, Deng et al. 2019, He et al. 2020, Kong et al. 2020, Wang & Li 2020, Dhara et al. 2021, Shen et al. 2021). Different ambiguity sets and solution approaches for DRO models are presented in Wiesemann et al. (2014), Esfahani & Kuhn (2018), Bertsimas et al. (2018), Ji & Lejeune (2021), Cheramin et al. (2022) and the references therein. Most relevant to this study is the stream of research that considers DRO with known marginal distributions. Agrawal et al. (2010, 2012) consider DRO with a submodular cost function and introduce the concept of the price of correlation. Then, an upper bound is derived by comparing expected cost under different underlying joint distribution. In the sourcing context, we consider submodular profit function and are interested in comparing optimal profit under different underlying joint distribution with the corresponding optimal solution. Mak & Shen (2014) study a DRO model in inventory risk pooling with given marginal moment information. Chen et al. (2022) consider a DRO model with marginal distribution and linear cost over a polytope but the covariance matrix is unknown. In contrast, our DRO model considers second moment information of the joint distribution and offers insights on the relationship between sourcing strategies and correlation structure among suppliers. Gao & Kleywegt (2017) consider a dependence structure whose similarity with the nominal joint distribution is measured by Wasserstein distance with the help of copula theory. Different from their reformulation technique and numerical solution approach, we provide closed-form solutions for a worst-case distribution, which allows us to derive several managerial insights regarding the best approach to gather information regarding the correlation structure. Lu et al. (2015) and Zhao & Freeman (2019) apply DRO in the context of facility location and sourcing, respectively, to manage correlated, *all-or-nothing* disruption risks. Our results generalize the worst-case distribution they present to accommodate multi-level disruption processes that follow discrete or continuous distributions. Therefore, the DRO methodology we describe is suitable for a broader range of practical applications.

In summary, to the best of our knowledge, our research is the first to propose sourcing models for settings where suppliers are subject to multi-level disruption processes that may take the form of both yield and capacity uncertainties, and the prevailing correlation structure is unavailable or ambiguous.

2. Model

This section presents the DR model we consider and provides some preliminary insights into its applicability and performance in a diverse group of settings. Section 2.1 begins by formally defining our problem and presenting the distributionally robust mathematical model. As mentioned previously, we are primarily interested in studying cases where suppliers may experience correlated disruption risks, but the true correlation structure is hard to know with certainty due to issues such as incomplete knowledge of the dependencies beyond tier-1 in the upstream supply chain. In Section 2.2, we first discuss the tractability of our model under very general assumptions regarding the revenue function. We then explore the possibility of deriving analytical solutions for the model under two commonly used revenue functions: 1) a simple quadratic form and 2) a *responsive* quadratic form. Section 2.3 presents the results of two computational experiments that demonstrate the efficacy of our approach using two *key dependence structures* as approximations for the underlying disruption correlation. We present these results to demonstrate the efficacy of our approach and motivate the more technical analyses that follow in subsequent sections.

2.1. Problem Definition

We consider a firm that sources a key component from n suppliers that compose a set $N = \{1, \ldots, n\}$. The procured component is used to produce products that are sold in a single market. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n_+$ represent the vector of orders that the firm places with the available suppliers. The suppliers are exposed to exogenous supply risks captured by random parameters $\xi \in [0,1]^n$. The delivery quantity from supplier $s \in N$, $d_s(x_s, \xi_s)$, is a random variable that depends on the amount ordered from the supplier, x_s, and the supplier's supply, which is captured by ξ_s. We accommodate cases where each supplier is subject to either *capacity* or *yield* uncertainty and use the notation N_C and N_Y to represent sets of suppliers that are subject to capacity and yield uncertainty, respectively, where $N_C \subseteq N$ and $N_Y \subseteq N$. In the case of capacity uncertainty, we assume that each unit of capacity represents the amount needed to deliver a single unit of the component and that the supplier s has a constant capacity of D_s. However, due to the supply uncertainty, the amount of capacity that may be used is $D_s\xi_s \leq D_s$, in which case the quantity delivered is $d_s(x_s, \xi_s) = \min(x_s, D_s\xi_s)$, i.e., the minimum of the order quantity and the available capacity. In the case of yield uncertainty, we assume that the buying firm receives $d_s(x_s, \xi_s) = \xi_s x_s \leq x_s$ units when placing an order of size x_s with supplier s. Because correlated supply risks can arise in both cases (e.g., correlated capacity and yield uncertainty are considered in Li et al. (2013) and Federgruen & Yang (2009), respectively), we let $F(\xi)$ denote the joint *cumulative* distribution function that defines the supply risk among the suppliers. Assuming that the firm spends c_s for each unit *delivered* by supplier s, the firm's problem is:

$$\max_{\mathbf{x}\geq 0} \mathbb{E}_F[V(\mathbf{x}, \xi)] \text{ s.t. } V(\mathbf{x}, \xi) = R(\mathbf{x}, \xi) - \sum_{s \in N} c_s d_s(x_s, \xi_s), \tag{P}$$

where $V(\mathbf{x}, \xi)$ represents the firm's profit function given $\mathbf{x}$ and ξ, $R(\mathbf{x}, \xi)$ represents the firm's revenue function given $\mathbf{x}$ and ξ, and $c_s d_s(x_s, \xi_s)$ represents the amount that the buying firm spends with supplier s.

As noted in Section 1, a common problem faced by buying firms is a lack of visibility into the operations of their upstream suppliers, especially those at higher echelons. Thus, it will often be the case that the joint distribution $F(\xi)$ is unknown. As an approximation of $F(\xi)$, firms may use estimates deduced from historical data regarding the delivery performance of their suppliers. For example, given a set of historical observations $\{\xi^1, \ldots, \xi^M\}$, we may set $F(\xi)$ to the empirical distribution $\hat{F}(\xi) = M^{-1} \sum_{i \in \{1,\ldots,M: \xi^i \leq \xi\}} \delta_{\xi^i}$, where δ_* is the Dirac measure on a single point. This empirical distribution can be used to approximate the sourcing problem via a stochastic programming technique such as sample average approximation (SAA). However, SAA only evaluates the profit function at points derived from observations in the historical data. Thus, the approach will perform best when a large amount of accurate time-invariant statistical information is available

(Delage & Ye 2010, Agrawal et al. 2012). In practice, because disruptions do not occur regularly, the samples available to firms will typically fall short of the time-invariant samples needed for an accurate approximation of $F(\xi)$ (Lim et al. 2013, Snyder et al. 2016). If the amount of available data is limited, then future realizations that substantially differ from the data used in SAA can occur and have significant negative consequences (Kuhn et al. 2019), a situation referred to as *the optimizer's curse* (Smith & Winkler 2006).

We propose a data-driven approach for approximating $F(\xi)$ that is based on i) estimates of the marginal cumulative distribution (probability density) functions $F_s(\xi_s)$ ($f_s(\xi_s)$), for each supplier $s \in N$, and ii) an estimated covariance matrix, Σ, which corresponds to the random vector ξ. For example, with historical data $\{\xi^1, \ldots, \xi^M\}$, we may set $F_s(\xi_s)$ to the empirical marginal distribution $\hat{F}_s(\xi_s) = M^{-1} \sum_{i \in \{1,\ldots,M : \xi_s^i \leq \xi_s\}} \delta_{\xi_s^i}$. Let $\mu = (\mu_s : s \in N)$ and $\sigma = (\sigma_s : s \in N)$ where μ_s and σ_s denote the expected value and standard deviation of the random variable ξ_s, respectively. Using this notation, a firm may account for the supply uncertainty using a joint distribution in the distributional set

$$\mathscr{F} = \left\{ F \,\middle|\, (1a),\ (1b),\ \int_{[0,1]^n} dF(\xi) = 1,\ F(\xi) \geq 0,\ \forall \xi \in [0,1]^n \right\} \neq \emptyset,$$

a subset of all the possible distributions in $[0,1]^n$, where

$$\int_{[0,1]^n} \mathrm{I}(\xi_s \leq \omega_s) dF(\xi) = F_s(\omega_s),\ \forall \omega_s \in [0,1],\ s \in N, \tag{1a}$$

$$\int_{[0,1]^n} (\xi - \mu)(\xi - \mu)^\intercal dF(\xi) \preceq \Sigma. \tag{1b}$$

Constraint set (1a) indicates that the buying firm is confident in their estimates of the marginal distributions, possibly resulting from guarantees stipulated in performance-based procurement contracts (Bernstein & De Véricourt 2008). However, since the suppliers may be linked to a common source of vulnerability further upstream in the supply chain, the firm is not confident in the estimated covariance matrix. In such a case, an upper bound enforced by constraint (1b) with a semidefinite inequality on the second-moment matrix of ξ provides a natural means of quantifying the firm's confidence. We assume $\mathscr{F}$ has a strictly feasible solution. Distinct from the traditional approach that constructs an estimate of $F(\xi)$ and then a risk managing decision in two separated steps, we propose a distributionally robust (DR) model that transforms historical samples directly to a decision with a given choice of Σ as follows,

$$\mathscr{V}(\Sigma) = \max_{\mathbf{x}\geq 0} \min_{F\in\mathscr{F}} \mathbb{E}_F[V(\mathbf{x},\xi)]. \tag{DR}$$

The DR model seeks to maximize the expected profit with respect to the worst-case joint supply distribution that is consistent with available information. If historical data on the supply process is adequate, the decision maker may specify Σ based on the empirical estimate $M^{-1}\sum_{i=1}^{M}(\xi^i - \mu)(\xi^i - \mu)^\intercal$, where $\{\xi^1,\dots,\xi^M\}$ represents M historical data points for the uncertain supply process. If information is available supporting supplier independence, the firm may simply assume Σ is a diagonal matrix, which implies an optimistic viewpoint that the disruption likelihoods of all available suppliers are independent of one another. On the other hand, if data is scarce or difficult to analyze, the firm may ignore constraint (1b) and take a more conservative approach. In the following subsections, we consider each of the described vantage points and show how a reasonably conservative approximation can be employed with good results.

2.2. Model Tractability

As alluded to previously, our DR model may be used effectively when the amount of historical data that is available to the firm is not sufficient to fit an empirical joint supply distribution accurately. Proposition 1 addresses the tractability of the DR model under such a setting, i.e., where finite historical data is available.

Proposition 1. *Let $\Omega = \{\xi^1,\dots,\xi^M\}$ denote the set of historical data regarding the supply process and $\Omega_s = \{\xi_s^1,\dots,\xi_s^M\}$ denote the finite support of the marginal distribution $F_s(\xi_s)$, $\forall s\in N$. If the revenue function, $R(\mathbf{x},\xi)$, is concave with respect to the total delivery amount, i.e., $R(\mathbf{x},\xi)\equiv \mathbf{R}\big(\sum_{s\in N} d_s(x_s,\xi_s)\big)$ and $\mathbf{R}$ is concave, then the DR model is polynomially solvable and can be reformulated as follows:*

$$\max \ \lambda + \sum_{s=1}^{n}\sum_{\xi_s\in\Omega_s}\gamma_{s\xi_s} f_s(\xi_s) + Q\bullet(\mu\mu^\intercal - \Sigma)$$

$$\text{s.t. } \mathbf{R}\left(\sum_{s\in N} t_{s\xi_s}\right) - \sum_{s=1}^{n} c_s t_{s\xi_s} - \lambda - \sum_{s=1}^{n}\gamma_{s\xi_s} + \xi^\intercal Q\xi - 2\xi^\intercal Q\mu \geq 0,\ \forall \xi\in\Omega$$

$$t_{s\xi_s} = x_s\xi_s,\ \forall s\in N_Y,\ \xi_s\in\Omega_s \tag{2}$$

$$t_{s\xi_s} \leq x_s,\ t_{s\xi_s} \leq D_s\xi_s,\ \forall s\in N_C,\ \xi_s\in\Omega_s$$

$$\mathbf{x}\geq 0,\ Q\succeq 0,\ t_{s\xi_s}\geq 0,\ \lambda \textit{ and } \gamma_{s\xi_s} \textit{ are free,}$$

where $\bullet$ *refers to the Frobenius inner product.*

Note that the DR model's reformulation considered in Proposition 1 is able to handle both yield and capacity uncertainties simultaneously. Moreover, since revenue functions are commonly modeled as concave functions, the assumptions required by the proposition are not overly restrictive. For example, a firm may approximate its revenue function using a simple piecewise-linear function $R(\mathbf{x},\xi) = \min_{\ell\in\{1,\dots,L\}}\{\alpha_\ell \sum_{s\in N} d_s(x_s,\xi_s) + \beta_\ell\}$ with some scalars α_ℓ and β_ℓ ($\forall \ell\in\{1,\dots,L\}$). We will also investigate the impact of specific forms for the revenue function.

Although Proposition 1 shows that the DR model is computationally tractable for concave revenue functions, in general, obtaining an analytic solution to the problem is challenging. To better understand how the ambigous correlation affects sourcing strategies, we will consider two specific forms of the revenue functions that are commonly used in the literature: 1) a simple quadratic revenue function $R(\mathbf{x},\xi) = \left\{(a-bq)q \middle| q = \sum_{s\in N} d_s(x_s,\xi_s)\right\}$ and 2) a *responsive* quadratic revenue function $R(\mathbf{x},\xi) = \left\{(a-bq)q \middle| q \leq \sum_{s\in N} d_s(x_s,\xi_s)\right\}$, where $a > c_s$, $\forall s\in N$, denotes the market potential and $b > 0$ represents the sensitivity of the market to the output quantity q. Note that the primary difference between the two revenue function forms is that output q is not a decision in the simple quadratic form whereas the firm can limit output q under the responsive quadratic form. The following proposition demonstrates insights that can be derived regarding the simple quadratic revenue function, which often appears in the sourcing literature (Tang & Kouvelis 2011).

Proposition 2. *If all suppliers are subject to yield uncertainty, then the DR model with the simple quadratic revenue function can be reformulated as the following convex quadratic programming*

$$\max_{\mathbf{x}\geq 0}\left(\sum_{s=1}^{n}(a-c_s)\mu_s x_s - b\mathbf{x}^\intercal\left(\Sigma+\mu\mu^\intercal\right)\mathbf{x}\right).$$

Using the result of Proposition 2, we can show that if the firm sources from all suppliers $s \in S \subseteq N$, the order quantities are

$$\begin{aligned}(x_s : s \in S)^\intercal &= \frac{1}{2b}\left(\Sigma_{S\times S}+\mu_S\mu_S^\intercal\right)^{-1}\left((a-c_s)\mu_s : s\in S\right)^\intercal\\ &= \frac{1}{2b}\left(\Sigma_{S\times S}^{-1}-\frac{\Sigma_{S\times S}^{-1}\mu_S\mu_S^\intercal\Sigma_{S\times S}^{-1}}{1+\mu_S^\intercal\Sigma_{S\times S}^{-1}\mu_S}\right)\left((a-c_s)\mu_s : s\in S\right)^\intercal,\end{aligned}$$

where $\Sigma_{S\times S}$ is the submatrix of Σ with row and column indices in S and $\mu_S = (\mu_s : s \in S)$. Note that prescribed order quantities depend on the precision matrix, i.e., the inverse of the covariance matrix. Thus, in the considered scenario, the partial correlation among suppliers determines the firms ordering strategy. We refer readers to Online Supplement A for a more detailed discussion on the simpler case where all suppliers are subject to Bernoulli disruptions, where $\xi_s \in \{0,1\}$, $\forall s \in N$.

We now turn our attention to the responsive quadratic revenue function, which also has received significant attention in the OR/MS literature. We study this form extensively in Section 4, thus, for now we only discuss its applicability in practice. This responsive quadratic revenue function is also referred to as a *responsive pricing model* since it allows the decision to effectively determine the price, $a - bq$, via the output quantity q. An example of such a responsive pricing situation in practice is the recent microchip shortage, where disrupted supplies have resulted in a surge in vehicle prices. Recently, the CEO of Ford Motor promised investors that the company will run its business with a lower days' supply in the future, resulting in leaner inventories and higher profit (Wayland 2021). This revenue function form, due to its quadratic nature, can also capture non-financial aspects of supply chain disruptions such as humanitarian considerations. In particular, an important optimization criteria in humanitarian aid is the degree to which goods or services provided in a disaster relief context, e.g., medical services, are provided to the affected population (Gutjahr & Nolz 2016, Noyan et al. 2022a). If we let $\mathscr{D}$ represent the demand for such goods and services, and p represent a penalty that is applied to units of demand that is not satisfied by the total delivery, the objective function $\max_q \left\{ -p(\mathscr{D} - q)^2 \middle| q \leq \sum_{s \in N} d_s(x_s, \xi_s) \right\}$ in quadratic penalty form can be applied in such a context. Note that this objective is technically equivalent to the responsive revenue function, $R(\mathbf{x}, \xi)$ with $a = 2p\mathscr{D}$ and $b = p$, minus a constant term $p\mathscr{D}^2$.

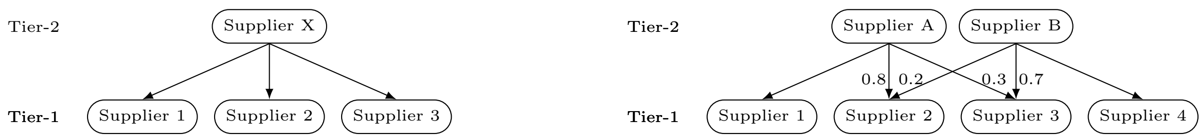
2.3. Computational Experiments on Multiechelon Supply Chain

In this section, we present the results of two computational experiments on multiechelon supply chain. The experimental designs demonstrate the situations that tier-1 suppliers are ambiguously correlated through higher tier suppliers at which the disruptions may occur. To demonstrate the efficacy of our approach in such situations compared to the SAA approach, we assume that the firm can use historical data to construct an *empirical* estimate on Σ in constraint (1b) and two approximations for the joint supply distribution of suppliers. Specifically, the firm may construct the following two approximations:

- an *independent* approximation with Σ being a diagonal matrix, which assumes that the supply disruption process for each supplier is independent from the supply disruption processes of all other suppliers, and
- a *conservative* approximation with constraint (1b) being ignored, which assumes that the supply disruption process for each supplier is dependent on the supply disruption processes of all other suppliers.

Note that the independent approximation is rather *optimistic* in the sense that it assumes an interruption in the supply process for any supplier is isolated. On the other hand, the conservative approximation essentially represents a *worst-case* scenario where an interruption in the supply process for any supplier is highly likely to correspond to an interruption in the supply processes for other suppliers.

We consider two cases of a multi-echelon supply chain network (see Figure 1). Our studied buying firm purchases components from the tier-1 suppliers, which further source materials from the tier-2 suppliers (i.e., Suppliers X, A and B). We consider Suppliers X, A and B are subject to Bernoulli disruptions. In Case 1, we assume that the buying firm adopts a simple quadratic revenue function (see Section 2.2) for its objective, where the parameters $a - c_i$, $\forall i \in \{1, 2, 3\}$, and b are specified in Table 1. In this case, all tier-1 suppliers source from Supplier X, while possibly on different order schedules. Available data allows the buying firm to estimate each supplier's marginal disruption probability, denoted by p_1, p_2 and p_3 for suppliers 1, 2 and 3, respectively.



(a) Case 1 (b) Case 2

Figure 1 Multiechelon Supply Chain

In Case 2, we assume that the buying firm adopts a responsive quadratic revenue function (see Section 2.2). The corresponding general concave profit function $V(\mathbf{x}, \xi)$ is approximated by piecewise linear functions with four segments; that is, $V(\mathbf{x}, \xi) = \min_{\ell \in \{1,\ldots,4\}} \{\alpha_\ell \sum_{s \in N} d_s(x_s, \xi_s) + \beta_\ell\}$, where the parameters $(\alpha_\ell, \beta_\ell)$ for any $\ell \in \{1, \ldots, 4\}$ are specified in Table 1. In this case, four tier-1 suppliers are available and they use different tier-2 supplier bases with potentially different ordering composition. Specifically, Supplier 1 sole sources from Supplier A, Supplier 2 sources 80% of their needs from Supplier A and the remaining 20% from Supplier B, Supplier 3 sources 30% of their needs from Supplier A and the remaining 70% from Supplier B, and Supplier 4 sole sources from Supplier B. For simplicity, we assume that suppliers A and B are subject to Bernoulli disruptions with disruptive probabilities p_A and p_B, respectively. However, from the perspective of our buying firm, suppliers 2 and 3 are subject to multi-level disruptions because they employ dual sourcing strategies. In both Case 1 and Case 2, assuming that the buying firm does not have complete visibility into the tier-2 suppliers, the tier-1 suppliers would be ambiguously correlated due to the lack on information regarding the disruption sources.

We generate random instances for each case depicted in Figure 1. Each instance consists of a time series of 400 data points, and each data point represents a specific disruption scenario for all tier-1 suppliers. For example, an observation in a time series generated for Case 1 is a 3-dimensional binary array (ξ_1, ξ_2, ξ_3), where the values of ξ_1, ξ_2 and ξ_3 represent the delivery status of suppliers 1, 2 and 3, respectively. When the observation $(\xi_1, \xi_2, \xi_3) = (0, 1, 0)$, only supplier 2 delivers components to fulfill the order and the others do not. Such an observation could happen because the tier-1 suppliers source from the common tier-2 supplier, i.e., Supplier X, on a different schedule. Similarly, an observation for Case 2 is represented as a 4-dimensional array $(\xi_1, \xi_2, \xi_3, \xi_4)$. Specifically, we have $\xi_1 = \xi_A$, $\xi_2 = 0.8\xi_A + 0.2\xi_B$, $\xi_3 = 0.3\xi_A + 0.7\xi_B$, and $\xi_4 = \xi_B$, where (ξ_A, ξ_B) is a 2-dimensional binary array indicating the delivery status of the tier-2 suppliers A and B. When $(\xi_A, \xi_B) = (1, 0)$, i.e., Supplier A is available but supplier B is disrupted, we have $(\xi_1, \xi_2, \xi_3, \xi_4) = (1, 0.8, 0.3, 0)$, which denotes the portion of any order amounts that can be fulfilled by each tier-1 supplier. All the instance data used in this section are publicly available in Zhao et al. (2022).

Table 1 Parameters

Used in Case	Parameter	Value(s)
1 and 2	θ	$\{-1, 0, 1\}$
1 and 2	I_ρ	$\{[-0.2, 0], [0, 0.25], [0.25, 0.5]\}$
1	$(a - c_1, a - c_2, a - c_3)$	$(2.1, 1.8, 1.5)$
1	b	0.2
1	(p_1, p_2, p_3)	$(0.3 + 0.05\theta, 0.2 + 0.05\theta, 0.1 + 0.05\theta)$
2	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$	$(2, 0.4, -0.4, -1)$
2	$(\beta_1, \beta_2, \beta_3, \beta_4)$	$(0, 0.5, 1.5, 3)$
2	(p_A, p_B)	$(0.25 + 0.05\theta, 0.15 + 0.05\theta)$

Each instance corresponds to a time series with 400 data points that are governed by a value of θ and a choice of the interval I_ρ (see Table 1). Given θ and I_ρ, we follow Algorithms 1 and 2, which are given in Appendix B of the Online Supplement, to generate instances for Cases 1 and 2, respectively. For both Case 1 and Case 2, we consider three possible values of θ and three choices of the interval I_ρ, leading to 9 different numerical settings. The intuition behind the selected θ and I_ρ settings is as follows. Based on the definition of the marginal probabilities in Table 1, the three values of θ (i.e., -1, 0 and 1) indicate low, moderate and high marginal disruption probabilities among the suppliers, respectively. The pairwise correlation coefficients between any two tier-1 suppliers are controlled by the interval I_ρ. The three possible settings for I_ρ (i.e., $[-0.2, 0], [0, 0.25]$ and $[0.25, 0.5]$) represent negative, weakly positive and strongly positive supplier correlations, respectively. For example, when $\theta = 0$ and $I_\rho = [0, 0.2]$, we have $(p_1, p_2, p_3) = (0.3, 0.2, 0.1)$ for Case 1, $(p_A, p_B) = (0.25, 0.15)$ for Case 2, and the pairwise correlation coefficients between any two tier-1 suppliers are within the interval $[0, 0.2]$. That is, suppliers are subject to moderate marginal disruptive probabilities and weakly positive correlations. Based on this setting, we can generate an instance with 400 data points for each of Case 1 and Case 2.

For each numerical setting of θ and I_ρ, we randomly generate 100 instances. For each instance, we assume that the buying firm observes the first 20 time series observations and uses these observations to estimate the marginal disruption probabilities for all tier-1 suppliers and the supplier correlations. Of course, the estimates for these statistics that are derived using the first 20 observations are likely to deviate from the true values obtained by considering all 400 observations. However, the 20 observations are more likely to give more accurate estimates for the marginal disruption probabilities than the supplier correlations. For each case (Case 1 or Case 2), we consider three scenarios where the firm employs the DR model to optimize the objective function under different choices of Σ using different estimates. (i) The first scenario uses an empirical estimate of the supplier correlations based on the initial 20 time series observations (Empr.). (ii) The second scenario uses an approximation of supplier correlations that assumes all suppliers face independent disruption likelihoods (Indp.). (iii) The third scenario uses an conservative approximation of supplier correlations that assumes suppliers are maximally correlated (Cons.). For each of these three scenarios, the buying firm first determines its optimal sourcing strategy using the initial 20 time series observations, and then simulate the out-of-sample performance of the prescribed strategy using the remaining 380 time series observations. Tables 2 and 3 summarize the results of this experiment for Cases 1 and 2, respectively. Specifically, we compare the above obtained solutions with a benchmark obtained using the SAA approach, and present the corresponding relative gap, which is defined as

$$\frac{\text{(the expected profit of the DR model)} - \text{(the expected profit of the SAA model)}}{\text{(the expected profit of the SAA model)}} \times 100\%.$$

Table 2 Objective Improvement Compared to SAA in Case 1

I_ρ	$[-0.2, 0]$		$[0, 0.25]$		$[0.25, 0.5]$	
θ	Indp.	Cons.	Indp.	Cons.	Indp.	Cons.
-1	1.42%	-6.60%	6.71%	4.35%	-4.90%	7.82%
0	4.90%	-3.63%	10.86%	21.09%	-13.64%	16.48%
1	5.43%	-2.37%	10.93%	21.81%	-22.69%	22.06%

Our computational results offer several interesting insights. First, consider the results for Case 1 (see Table 2). Note that since the simple quadratic revenue function is uniquely determined by the first and second moments of the disruption distributions, the DR model with an empirically estimated correlation structure is equivalent to SAA, hence, there is no column for the empirical estimate scenario in Table 2. We observe that, when compared to SAA, the most extreme positive correlation is better managed by assuming a conservative correlation structure (see column *Cons.* where $I_\rho = [0.25, 0.5]$), and the negative correlation is better managed by assuming an independent correlation structure (see column *Indp.* where $I_\rho = [-0.2, 0]$). Another interesting observation is related to the consequences of incorrect correlation estimates. In particular, note that the firm may construct *i*) an independent approximation when the true disruptions are positively correlated, where the objective improvements compared to SAA are shown in column *Indp.* with $I_\rho = [0.25, 0.5]$; or *ii*) a conservative approximation when the true disruptions are negatively correlated, where the objective improvements compared to SAA are shown in column *Cons.* with $I_\rho = [-0.2, 0]$. By comparing the objective improvement values in the previously mentioned columns, we find that for a given value of θ, the absolute value in column *Indp.* with $I_\rho = [0.25, 0.5]$ is larger than the absolute value in column *Cons.* with $I_\rho = [-0.2, 0]$ (e.g., $|-22.69\%| > |-2.37\%|$ when $\theta = 1$). This observation indicates that incorrectly assuming that disruptions are independent when they are

Table 3 Objective Improvement Compared to SAA in Case 2

I_ρ	$[-0.2, 0]$			$[0, 0.25]$			$[0.25, 0.5]$		
θ	Empr.	Indp.	Cons.	Empr.	Indp.	Cons.	Empr.	Indp.	Cons.
-1	0.38%	-1.37%	0.18%	0.35%	-2.76%	0.08%	0.18%	-2.43%	0.33%
0	0.36%	0.12%	0.23%	1.06%	0.04%	0.81%	0.97%	-0.41%	0.91%
1	0.38%	0.19%	0.12%	0.29%	-0.02%	0.23%	0.79%	-0.84%	0.60%

actually positively correlated is much more detrimental than incorrectly assuming that disruptions are conservative when they are actually negatively correlated.

Next, we consider the results for Case 2 (see Table 3). Before discussing the results it is important to recall that the correlations between any two tier-1 suppliers for this case are dictated by both the correlation among tier-2 suppliers, i.e., Suppliers A and B, and the ordering pattern that the tier-1 suppliers employ, as indicated in Figure 1b. For example, the correlation between suppliers 2 and 3 can be positive even if the correlation between suppliers A and B is negative. This is possible because suppliers 2 and 3 share the same supplier base. Table 3 shows that the DR model with both empirical and conservative correlation estimates outperforms the SAA approach, on average. Moreover, the DR model with an empirical correlation approximation generally outperforms the DR model using a conservative approximation. The only exception is the setting in which suppliers A and B are more reliable and highly correlated (0.33% versus 0.18% when $\theta = -1$ and $I_\rho = [0.25, 0.5]$). In such a setting, the high degree of reliability amongst the tier-2 suppliers may lead to an underestimation of the supplier correlations when using an empirical estimate, while a conservative approximation will perform better.

Our computational experiments have shown that the DR model we propose offers a tractable method that can be used to understand how ambiguity regarding disruption correlations can affect a sourcing strategy. Moreover, the results demonstrate that employing a conservative approximation for the prevailing correlation structure can be an effective strategy for mitigating correlated supply risks. From a practical perspective, obtaining an accurate estimate for Σ requires a time-invariant joint sample for disruptions. There are many cases in which such a sample may not be available in practice, e.g., if the firm has ordered from the available suppliers independently in the past. Hence, a conservative approximation is an attractive alternative when deriving an estimate for Σ is not easy, and as we will show later, it provides an important benchmark that offers several insights into the sourcing process. In Section 3, we study important disruption dependence structures under the assumption of a general revenue function. In particular, we discuss exactly how a conservative approximation of Σ can be derived and how conservatism can be reduced when information is available. Section 4 presents an extensive study for the DR model when a responsive quadratic revenue function is employed. For this case, the specific form of the objective function allows us to demonstrate how the sourcing decision changes as the estimate on Σ and the type of disruption uncertainty, i.e., yield or capacity uncertainty, vary in greater detail.

3. Dependence Structures for Sourcing under Ambiguous Correlation

Regarding the dependence structure emerging from the application of the DR model, in probability theory and statistics, copula theory is used to describe the dependence between random variables. The set of all n-dimensional joint distributions that have the same marginals is referred to as a Fréchet class (Mak & Shen 2014) in copula theory and are comparable according to the supermodular ordering (Shaked & Shanthikumar 1997, 2007). It is well-known that $F^{\mathrm{m}}(\xi) \equiv \min_{s\in N} F_s(\xi_s)$ represents a Fréchet upper bound that specifies a co-monotonic dependence structure. We refer to the distribution $F^{\mathrm{m}}(\xi)$ as the *maximally dependent* distribution because it defines the maximal correlation between suppliers in our problem setting. In Appendix C of the Online Supplement, we discuss the closed-form of the distribution $F^{\mathrm{m}}(\xi)$ further. As shown in Section 2.3, using a conservative approximation for disruption likelihoods in our DR model, such as that given by the worst-case distribution $F^{\mathrm{m}}(\xi)$, is an effective approach for mitigating the negative consequences associated with supply disruptions when the true correlation is ambiguous and difficult to estimate. Thus, we also discuss sourcing strategies that arise when employing the maximally dependent distribution in Appendix C of the Online Supplement. There are other extremes in terms of dependence structure. We introduce the *independent* distribution $F^{\mathrm{i}}(\xi)$ to represent the case where suppliers are completely independent with respect to their likelihood for supply disruptions. We introduce two assumptions as follows:

Assumption 1. *For a given order vector* $\mathbf{x}$*,*

(a) the revenue function $R(\mathbf{x},\xi)$ *is non-decreasing on* ξ*,*

(b) the profit function $V(\mathbf{x},\xi)$ *is submodular on* ξ *and bounded.*

Our assumption of monotonicity of the revenue function holds when *i*) $R(\mathbf{x},\xi) \leq R(\mathbf{x},\xi')$ if $\xi \leq \xi'$, i.e., the firm's revenue increases as supplier reliability improves, and *ii*) all costs to the firm can be captured by c_s ($\forall s \in N$). The assumption of submodularity is justified in practice where increasing the delivered proportion of a fixed set of order quantities offers diminishing marginal returns. Proposition 3 characterizes the retailer's profit under ambiguously correlated supply distributions.

Proposition 3. *We have*

(1) *if* $\Sigma_{ij} = \sigma_i\sigma_j$, $\forall i,j \in N$ *and Assumption 1(b) holds, then* $F^{\mathrm{m}}(\xi)$ *is a worst-case joint distribution for any given order vector* $\mathbf{x}$*.*

(2) $\mathscr{V}(\Sigma)$ *is nonincreasing, i.e.,* $\mathscr{V}(\Sigma') \geq \mathscr{V}(\Sigma'')$ *if* $\Sigma'' \succeq \Sigma'$*. This monotonicity also holds if the order decision* $\mathbf{x}$ *is fixed.*

(3) $\mathscr{V}(\Sigma) \geq \underline{\mathscr{V}} \equiv \max_{\mathbf{x}\geq 0} \mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x},\xi)]$*, and if all correlations between suppliers are positive, then* $\overline{\mathscr{V}} \equiv \max_{\mathbf{x}\geq 0} \mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x},\xi)] \geq \mathscr{V}(\Sigma)$*.*

Because constraint (1b) is relaxed when $\Sigma_{ij} = \sigma_i\sigma_j$ $\forall i,j \in N$, Proposition 3 part (1) shows that $F^{\mathrm{m}}(\xi)$ defines the worst-case, i.e., maximally dependent, distribution that may be derived without utilizing any information regarding supplier correlations. This result is similar to one given in Mak & Shen (2014) and extends similar findings for Bernoulli disruption processes (Lu et al. 2015, Zhao & Freeman 2019), where $\xi \in \{0,1\}^n$, to more general supply uncertainties. Proposition 3 part (2) provides insight regarding a progressive strategy for managing supply uncertainty that can be employed using the DR model. In particular, we conducted computational experiments, see Section 2.3, showing that the DR model is able to manage the spectrum of risks that are possible given historical information on marginal disruption likelihoods effectively. Thus, as long as some data exists, the DR model can be employed using a conservative estimate for Σ to help a buying firm make sourcing decisions. As additional information becomes available over time, the estimate of Σ can be refined, allowing a firm to further improve the outcomes of their sourcing efforts.

In the rest of this section, we discuss several insights that the DR model provides regarding approaches for managing correlated supply uncertainty. A primary concern for a firm sourcing from unreliable suppliers is determining the potential benefit of efforts aiming to improve the current understanding of the supplier dependence structure. If an estimate of such benefits can be obtained, it can be compared to the associated costs to decide whether or not exploration efforts should be taken. Section 3.1 presents a profit decomposition that can be used to determine such an estimate. If this information suggests that exploration costs are not worthwhile or if better information is unattainable, the independent and maximally dependent distributions (F^{i} and F^{m}) offer two options that may be used to determine a sourcing strategy without additional data requirements. In Section 3.2, we compare the sourcing strategies suggested by these two approaches. We show that in terms of regret, which we define as the worst case potential profit loss, an assumption of maximal dependence is advantageous. Even though we demonstrate that assuming maximal dependence among suppliers offers significant benefits with respect to regret, this vantage point can be rather conservative. Moreover, it is often possible to identify obvious factors that would increase or reduce the potential for co-dependence among subsets of supplier. For example, it is reasonable to assume that suppliers located in the same (distant) geographic location will be more (less) susceptible to supply risks stemming from natural disasters, political uncertainty, or diseases. In Section 3.3, we show how such partial information can be incorporated using our independent and maximally dependent distributions. Finally, if it is possible and cost-effective to exert additional effort on better understanding the prevailing correlation structure, the next question to address is how to proceed with such information gathering. In Section 3.4, we study the shadow price associated with constraint (1b) in the DR model. This investigation provides insight regarding the order in which exploration activities should be conducted in order to maximize the cumulative increase in profit as exploration efforts are carried out.

3.1. Profit Decomposition

The efficacy of multi-sourcing as a strategy for mitigating supply risks when the joint supply distributions for available supplier are known is well established in the literature (Tomlin 2009, Wang et al. 2010). However, when the joint supply distribution is unknown, the potential profit gain associated with multi-sourcing can be decomposed into two parts: the value of diversification (i.e., spreading the risk) and the value of understanding supplier correlations (i.e., understanding the best way to spread the risk). Suppose $\mathscr{V}^*$ is the optimal expected profit that may be obtained with exact knowledge of the underlying joint distribution and $\mathscr{V}_s$ is the maximal profit of sole sourcing from supplier $s \in N$. Using these parameters, the potential profit gain due to multi-sourcing can be decomposed as follows

$$\begin{aligned}\text{Value of multi-sourcing} &= \mathscr{V}^* - \max_{s\in N} \mathscr{V}_s = (\mathscr{V}^* - \underline{\mathscr{V}}) + \left(\underline{\mathscr{V}} - \max_{s\in N} \mathscr{V}_s\right) \\ &= \text{Value of correlation information} + \text{Value of diversification}\end{aligned}$$

In all practical cases, the value of correlation information will be positive because of Proposition 3 part (3). Moreover, Corollary 1 shows that the value of diversification is positive.

Corollary 1. $\underline{\mathscr{V}} \geq \max_{s\in N} \mathscr{V}_s$.

Identifying and maximizing the value of correlation information can be very complicated in practice. However, since the value of diversification is fixed, the decomposition provides important insights for decision-makers who want to reach a certain profit goal. For example, selecting a supplier base yielding a large amount of value due to diversification is attractive. However, if the value of diversification is small, a firm that pursues a multi-sourcing strategy will need reliable correlation information to realize substantial gains as a result of the endeavor. Although the value of correlation information cannot be measured accurately, it can be bounded as follows

$$\mathscr{V}(\Sigma_u) - \underline{\mathscr{V}} \leq \text{value of correlation information} \leq \mathscr{V}(\Sigma_\ell) - \underline{\mathscr{V}} \leq \overline{\mathscr{V}} - \underline{\mathscr{V}}, \tag{3}$$

where Σ_ℓ and Σ_u are lower and upper bounds of a suitable confidence region for covariance matrix, respectively. In general, it is hard to know in advance if efforts aimed at improving estimates of supply correlation will yield meaningful insights. However, since the bounds in (3) can be estimated without implementing any ordering decisions, they can be used for budgeting purposes because they provide an estimate of potential benefits *before* allocating resources to such efforts. It is worth noting that Corollary 1 does not suggest that multi-sourcing dominates sole-sourcing because the inequality is not strict. However, if $\underline{\mathscr{V}} > \max_{s\in N} \mathscr{V}_s$, then a multi-sourcing strategy should be pursued, regardless of the true underlying correlation structure.

3.2. The Danger of Optimism

When the task of improving correlation estimates is too costly or impractical, the independent and maximally dependent distributions (F^{i} and F^{m}) offer alternative descriptions of the supply risks that can be used to develop sourcing strategies. Let $\mathbf{x}^{\mathrm{i}} = \arg\max_{\mathbf{x}} \mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x},\xi)]$ and $\mathbf{x}^{\mathrm{m}} = \arg\max_{\mathbf{x}} \mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x},\xi)]$ denote the optimal order quantities under joint distributions F^{i} and F^{m} respectively, where

- $\mathbf{x}^{\mathrm{i}}$ is an optimistic strategy that assumes independent supply risks by setting Σ as the diagonal matrix in the DR model; and
- $\mathbf{x}^{\mathrm{m}}$ is a conservative strategy that assumes maximal dependency among suppliers.

Intuitively, it is in the best interest of a firm to apply the optimistic strategy when they are confident that the degree of correlation among available suppliers is low and the conservative strategy when they are confident that the degree of correlation among available suppliers is high. However, it is not clear which strategy should be selected when the firm is not as confident regarding the degree of supplier correlations. In such a case, the firm should consider basing their strategy decision not on the benefits that can be realized if they correctly estimate the degree of supplier interrelatedness, but instead on the negative consequences that can arise if their estimate is incorrect.

We compare these two strategies by considering the regret that is associated with the firm making an incorrect assumption regarding supplier dependencies. If the firm utilizes the optimistic strategy, $\mathbf{x}^{\mathrm{i}}$, then they experience maximal regret if the supply distribution that is realized is maximally dependent. In this case, the firm's expected profit is $\mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x}^{\mathrm{i}},\xi)]$, whereas it would have been

$\mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x}^{\mathrm{m}},\xi)]$ if the firm implemented the conservative strategy. Following this argument, we define $\mathrm{regret}_{\mathrm{o}} \equiv \mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x}^{\mathrm{m}},\xi)] - \mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x}^{\mathrm{i}},\xi)]$ as the potential regret due to optimism. Now consider the case where the firm employs the conservative strategy $\mathbf{x}^{\mathrm{m}}$. In this case, the firm experiences maximal regret when the true supply distribution is independent. If this occurs, the firm's expected profit is $\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{m}},\xi)]$ but it would have been $\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{i}},\xi)]$ if the optimistic strategy were used instead. Thus, we define $\mathrm{regret}_{\mathrm{c}} \equiv \mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{i}},\xi)] - \mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{m}},\xi)]$ to represent the potential regret due to conservatism. Using these defined expressions, we have

Corollary 2. *Suppose Assumption 1 holds. The potential regret due to conservatism,* $\mathrm{regret}_{\mathrm{c}} \leq \left(n - n\left(1-\frac{1}{n}\right)^{n} - 1\right)\underline{\mathscr{V}}$*, is in* $O(n)\cdot\underline{\mathscr{V}}$*. However, the potential regret due to optimism* $\mathrm{regret}_{\mathrm{o}}$ *can be exponentially large in* $O(2^{n-1})\cdot\underline{\mathscr{V}}$*.*

Corollary 2 suggests that a firm applying an optimistic strategy, i.e., assuming independence among suppliers, should be confident that the correlation among suppliers will stay low. Otherwise, the firm should apply the conservative strategy since the potential regret is lower. However, it is easy to argue that simply applying assumption of maximal dependence to all suppliers is overly conservative. In the following sections, we explore two strategies that a firm may employ to avoid over-conservatism: *i*) utilizing partial information regarding supplier correlations and *ii*) using shadow price information to prioritize efforts aimed at improving understanding of the true correlation structure.

3.3. A Partially Dependent Distribution

We have shown that the maximally dependent distribution offers high-quality solutions in comparison to the independent distribution when the underlying joint distribution is unknown (Lu et al. 2015, Zhao & Freeman 2019). However, it is a rather conservative approximation of the prevailing supply distribution due to the fact that it utilizes no information regarding supplier correlations. In many cases, the firm may have information suggesting that certain groups of suppliers face mutually independent supply risks. For example, suppliers in different geographic locations can be considered independent with respect to disturbances resulting from extreme weather conditions. We can use such "partial" information to reduce the conservatism associated with maximally dependent distribution.

THEOREM 1. *Suppose the suppliers can be partitioned into r groups, where two suppliers that belong to different groups are independent with respect to supply uncertainty. Let $N_1, \dots, N_r$ denote such a partition of the suppliers comprising the set N. If Assumption 1 holds, then we have*

(1) *the partially dependent distribution* $F^{\mathrm{p}}(\xi) = \prod_{i=1}^{r} F^{\mathrm{m}}(\xi_{N_i})$ *as a worst-case distribution, where* $\xi_{N_i} = (\xi_j : j \in N_i)$ *for any* $i \in \{1, 2, \dots, r\}$, *and*

(2) $\overline{\mathscr{V}} \le \vartheta(\overline{n}) \max_{\mathbf{x} \ge 0} \mathbb{E}_{F^{\mathrm{p}}}[V(\mathbf{x}, \xi)]$, *where* $\vartheta(k) = k\left(1 - \left(1 - \frac{1}{k}\right)^k\right)$ *and* $\overline{n} = \max_{i \in \{1,\dots,r\}} |N_i|$.

Theorem 1 shows that the independent and maximally dependent distributions provide fundamental building blocks that allow a buying firm to incorporate "partial information" regarding supply correlations. If the estimated covariance matrix Σ_0 demonstrates weak correlation among groups of suppliers, Theorem 1 provides a closed-form worst-case distribution F^{p} that utilizes such information. This insight provides a useful scenario analysis tool, especially for firms in the initial phases of supplier selection. In particular, the firm can construct partitioning scenarios based on reasonable beliefs regarding supplier independence and apply the insights of Theorem 1 to identify the potential impact on profitability for each partition. As was true for the value of correlation information, these projections can be made without implementing any new strategy since F^{p} does not rely on ordering decisions.

If evaluating the potential profit for the previously described partitioning scenarios is difficult, then the bound in Theorem 1 part (2) admits a simple heuristic approach for constructing a supplier base that is composed of independent supplier groups with an equal number of suppliers in each group. As an example, consider a firm that currently transacts with 3 suppliers. Assume that the only available information regarding supplier correlations is that supplier 1's supply distribution is independent of the other two suppliers, 2 and 3. Thus, we have two groups, the first composed of supplier 1 and the second composed of suppliers 2 and 3. Suppose that the firm is considering adding one new supplier from a set of two candidates, a and b, and knows that supplier a faces supply risks that are independent of those faced by supplier 1, and supplier b faces supply risks that are independent of those faced by suppliers 2 and 3. With only this information, the firm should select supplier b because having $\overline{n}=2$ lowers the value of $\vartheta(\overline{n})$ in Theorem 1 part (2).

3.4. Shadow Price of Correlation Constraints

In practice, it is unreasonable to expect that accurate information regarding all supplier correlations will be available at the time when a sourcing decision needs to be made. Thus, a buying firm may need to further explore dependencies among suppliers based on current information. In the process of deciding where to focus investigation efforts aimed at improving the firm's understanding of prevailing correlations, the shadow price of constraint (1b) offers important insights. Let Q be the dual variable for constraint (1b), where Q_{ij} corresponds to the shadow price of correlation between suppliers i and j for any $i,j\in N$ and $i\neq j$. We characterize the structural property of the shadow price as follows.

THEOREM 2. *Given any order quantities* $\mathbf{x}$, *let* $F^*(\xi;\mathbf{x})$ *with support supp*(F^*) *be an optimizer of* $\min_{F\in\mathcal{F}}\mathbb{E}[V(\mathbf{x},\xi)]$ *and* Q^* *be the value of the dual optimizer. For any* $i,j\in N$, $i\neq j$, *if there exist* $\xi\in$ *supp*(F^*), *scalar* δ_i, *and scalar* δ_j *such that* $\{\xi+\delta_i e_i,\xi+\delta_j e_j,\xi+\delta_i e_i+\delta_j e_j\}\subseteq\operatorname{supp}(F^*)$, *where* e_i *and* e_j *are unit vectors, then we have*

$$Q^*_{ij}=-\frac{\big(R(\mathbf{x},\xi+\delta_j e_j+\delta_i e_i)-R(\mathbf{x},\xi+\delta_j e_j)\big)-\big(R(\mathbf{x},\xi+\delta e_i)-R(\mathbf{x},\xi)\big)}{2\delta_i\delta_j}. \tag{4}$$

Moreover, if $R(\mathbf{x},\xi)$ *is twice-differentiable on* ξ *and* $d(x_s,\xi_s)$ *is differentiable on* ξ_s, $\forall s\in N$, *for any* $\xi\in$ *supp*(F^*), *then we have*

$$Q^*_{ij}=-\int_{supp(F^*)}\frac{\partial R(\mathbf{x},\xi)}{\partial\xi_i\partial\xi_j}dF^*(\xi;\mathbf{x}),\ \forall i,j\in N, i\neq j.$$

REMARK 1. Note that $F^*(\xi;\mathbf{x})$ is an optimizer of $\min_{F\in\mathcal{F}}\mathbb{E}[V(\mathbf{x},\xi)]$. For any choice of ξ, δ_i, and δ_j such that $\{\xi,\xi+\delta_i e_i,\xi+\delta_j e_j,\xi+\delta_i e_i+\delta_j e_j\}\subseteq\operatorname{supp}(F^*)$, the constraints (EC.23) with respect to $\{\xi,\xi+\delta_i e_i,\xi+\delta_j e_j,\xi+\delta_i e_i+\delta_j e_j\}$ in the dual formulation hold at equality for the optimal dual solution because of the complementarity conditions. The resulting four equations allow us to obtain Q^*_{ij} $(\forall i, j \in N, i \neq j)$ as in (4), and this result applies when F^* has discrete support (see Example EC.3 in Appendix D of the Online Supplement).

Theorem 2 shows that the shadow price of the correlation constraints depends on revenue function, optimal order quantities, and the worst-case distribution. However, the shadow price does not depend on the supplier's unit costs, i.e., expenditure. This is reminiscent of the analysis presented in Simchi-Levi et al. (2015) that finds Ford's expenditure with a supplier is uncorrelated to the profit loss they would incur if the supplier were suddenly interrupted. Theorem 2 offers a possible explanation to their findings by demonstrating that a firm's expenditure with suppliers does not affect the shadow price associated with the supply correlation structure. Moreover, Theorem 2 suggests that suppliers' unit costs should not factor into decisions on how to prioritize efforts towards exploring the prevailing correlation structure. However, we must point out that Theorem 2 does not suggest the unit costs are irrelevant to the correlation structure because the shadow prices depend on the ordering quantities. For example, if a supplier changes its unit cost, then it may affect the firm's optimal order quantities. As the shadow price relates to the order quantities, the change of unit cost may have impact on the firms' priority on exploring correlations. Therefore, the priority ordering related to discovering suppliers' correlation is determined by the shadow prices, not unit costs, *after* the firm has the optimal sourcing decision given by the DR model.

4. Sourcing Strategy with Quadratic Revenue Functions

Distinct from the existing literature, we focus on robust sourcing under multi-level disruptive processes where the prevailing correlation structure is unavailable or ambiguous. Based on the responsive quadratic revenue function, a DR equivalent is

$$\max_{\mathbf{x}\geq 0} \min_{F\in\mathscr{F}} \mathbb{E}_F \left[\max_q \left\{ (a-bq)q \;\middle|\; q \leq \sum_{s=1}^n d_s(x_s,\xi_s),\ \forall \xi \in [0,1]^n \right\} - \sum_{s=1}^n c_s d_s(x_s,\xi_s) \right]. \tag{5}$$

Considering the DR model with a specific revenue function allows us to investigate how the optimal supplier base is affected by the correlation structure and to demonstrate how we may use the insights of Section 3.4 to define a best sequence for exploring supplier correlations. We also present the results of a comprehensive numerical study that allows us to investigate the performance of the DR model.

4.1. Yield Uncertainty

Although the profit function is concave under yield uncertainty, developing a full characterization of the optimal solution for the setting is cumbersome even for all-or-nothing disruptions (Hu & Kostamis 2015). In order to develop analytical solutions, we define a supplier s as being stable if $\sigma_s = 0$. For example, a reliable supplier is obviously stable. Because of its zero variance, the unit cost of a stable supplier can be used as a price "anchor" to measure other suppliers' performance and derive order quantities in closed form. Proposition 4 considers a distributional set $\mathscr{F}$ where Σ is determined by a single parameter $0 \le \rho < 1$ and the case with $\rho = 1$ is discussed in Remark EC.2.

PROPOSITION 4. *Consider yield uncertainty. Suppose supplier* 1 *is stable, i.e.,* $\sigma_1 = 0$, *and* $\Sigma_{ij} = \rho\sigma_i\sigma_j \ \forall i \neq j \in N$. *We have two cases*[1]*:*

i) The firm orders from supplier 1. Let the suppliers in $N \setminus \{1\}$ *be ordered to satisfy* $(c_1 - c_2)\mu_2/\sigma_2 \ge \cdots \ge (c_1 - c_n)\mu_n/\sigma_n$ *and*

$$S_\rho = \max\left\{ k \left| \frac{(c_1 - c_k)\mu_k}{2b\sigma_k} \ge \frac{\rho}{\rho(k-2)+1} \sum_{i=2}^{k} \frac{(c_1 - c_i)\mu_i}{2b\sigma_i} \right. \right\}.$$

If the firm outputs all available supply, in particular, if $(c_1 + a(\mu_1 - 1))(1-\rho) \ge \sum_{s=2}^{n}(c_1 - c_s)\mu_s(1 - \mu_s)/\sigma_s^2$, *then the optimal order quantities are* $x_s = 0$, $\forall s = S_\rho + 1, \ldots, n$,

$$x_1 = \frac{a - c_1}{2b\mu_1} - \sum_{i=2}^{S_\rho} \frac{(c_1 - c_i)\mu_i^2}{2b(1-\rho)\sigma_i^2\mu_1} + \frac{\rho}{\rho(S_\rho - 2)+1} \sum_{i=2}^{S_\rho} \frac{(c_1 - c_i)\mu_i}{2b(1-\rho)\sigma_i\mu_1} \left(\sum_{j=2}^{S_\rho} \frac{\mu_j}{\sigma_j} \right), \quad \textit{and} \tag{6}$$

$$x_s = \frac{1}{(1-\rho)\sigma_s} \left(\frac{(c_1 - c_s)\mu_s}{2b\sigma_s} - \frac{\rho}{\rho(S_\rho - 2)+1} \sum_{i=2}^{S_\rho} \frac{(c_1 - c_i)\mu_i}{2b\sigma_i} \right), \ \forall s = 2, \ldots, S_\rho.$$

[1] The conditions that the buyer sources (or not) from the reliable supplier and outputs all available supply can be stated explicitly by considering $x_1 > 0$ (or $x_1 \le 0$) in (6) and $\sum_{s=1}^{n} x_s \le a/2b$, respectively.

ii) The firm does not order from supplier 1. Suppose suppliers in the set $\mathcal{J} \subseteq N \setminus \{1\}$ receive orders. If the firm outputs all available supply, in particular, if $a(1-\rho) \geq \sum_{s=2}^{n}(c_1 - c_s)\mu_s/\sigma_s^2$, then there exists λ_1 such that the optimal order quantities are

$$x_s = \frac{1}{(1-\rho)\sigma_s}\left(\frac{(c_1-c_s)\mu_1\mu_s - \lambda_1\mu_s}{2b\mu_1\sigma_s} - \frac{\rho}{\rho(|\mathcal{J}|-1)+1}\sum_{i\in\mathcal{J}}\frac{(c_1-c_i)\mu_1\mu_i - \lambda_1\mu_i}{2b\mu_1\sigma_i}\right), \forall s \in \mathcal{J},$$

and $x_s = 0$, $\forall s \notin \mathcal{J}$. If $\rho = 0$ and suppliers are ordered to satisfy $c_2 \leq \cdots \leq c_n$, then $\mathcal{J} = \{2, \ldots, j^*\}$ and $\lambda_1 = \alpha_{j^*}$ where $j^* = \max\{j | \mu_1(c_1 - c_j) \geq \alpha_j\}$ and

$$\alpha_j = \mu_1\left(c_1 - a + \sum_{s=2}^{j}\frac{(c_1-c_s)\mu_s^2}{\sigma_s^2}\right) \Bigg/ \left(1 + \sum_{s=2}^{j}\frac{\mu_s^2}{\sigma_s^2}\right).$$

Proposition 4 provides several interesting insights that relate to observations made in the literature. First, when suppliers are independent, i.e., $\rho = 0$, Proposition 4 supports the conventional wisdom that "cost is an order qualifier and reliability is an order winner," and similar to the findings presented in Li et al. (2013), it also suggests that procurement cost is not a sufficient qualifier when suppliers are correlated. Moreover, the sequence $(c_1 - c_2)\mu_1/\sigma_1 \geq \cdots \geq (c_1 - c_n)\mu_n/\sigma_n$ provides an explicit ratio to select suppliers under certain conditions. Second, when the buying firm sources from the stable supplier, the size of the order placed with the stable supplier is coupled with the market size whereas the amounts ordered from other suppliers are not. This finding, first noted by Hu & Kostamis (2015) when considering the problem of sourcing from a set of reliable and unreliable suppliers where the likelihood of simultaneous disruption of more than one supplier is negligible, suggests that a cost-effective and reliable supplier provides an effective counter against disruptions as the market size varies. Our result generalizes the insight of Hu & Kostamis (2015) to a setting with generally correlated supply risks.

A more interesting insight of Proposition 4 is that the size of the optimal supplier base decreases in a nested fashion as the degree of correlation among the suppliers increases.

Corollary 3. *Let $\mathbb{S}(\rho) = \{1, \ldots, S_\rho\}$ be the optimal supplier base in the case of Proposition 4 part i). We have $\mathbb{S}(\rho)$ is nested decreasing with ρ.*

Consider the case where the firm orders from the stable supplier. When $\rho = 0$, the firm sources from all suppliers $s \in N$ with unit cost $c_s < c_1$. However, as the correlation coefficient increases, the set of suppliers that receive an order shrinks, i.e., S_ρ decreases. Thus, the optimal degree of supplier diversification decreases as suppliers become increasingly interrelated. Moreover, if the firm were able to determine a maximal threshold for ρ, then the optimal supplier base associated with this threshold will represent a set of "core" suppliers with whom it is advantageous for the firm to pursue strategic alliances, a valuable insight regardless of the true correlation structure.

4.2. Capacity Uncertainty

The model (5) becomes non-concave under capacity uncertainty and the model does not admit a closed-form solution, even if the supply distribution is given (Li et al. 2013). However, it is interesting to note that structurally the optimal supplier base under capacity uncertainty is different from what we observed for the case of yield uncertainty in Corollary 3.

THEOREM 3. *Consider a case where no supplier experiences a complete disruption, i.e., $F_s(0) = 0, \forall s \in N$. Let $\mathbb{S}(\Sigma)$ denote the optimal supplier base given upper bound Σ in constraint* (1b)*. Then, $\mathbb{S}(\Sigma') \subseteq \mathbb{S}(\Sigma'')$ if $\Sigma' \preceq \Sigma''$, i.e., $\mathbb{S}(\Sigma)$ is nested increasing in Σ.*

The difference between the structural properties of the optimal supplier base is a result of the risk preferences implied by the associated supply distributions and uncertainty types. Regardless of the estimated correlation level between suppliers, the fact that yield is uncertain allows the buyer to increase the supply of goods by inflating order quantities, i.e., increasing the amount ordered from members of the supplier base to increase the expected delivery amount. Such *shortage gaming* is not practical when capacity is random. In particular, when capacity is random, increasing the amount ordered from a fixed set of suppliers is not beneficial when a disruption occurs since the total capacity of the selected suppliers limits the amount delivered. Thus, in the case of yield uncertainty, weak supplier dependencies make it attractive to spread the total order amount among a relatively large number of suppliers. As supplier dependencies increase, it becomes attractive to concentrate the order among a smaller set of more reliable suppliers. On the other hand, under capacity uncertainty, increasing supplier dependencies requires the firm to prepare for potential supply risks by increasing the total capacity that is available in the supply base through increased diversification.

4.3. Sequence for Exploring Supplier Correlations

It can be computationally challenging to find the shadow prices through the equations given in Theorem 2. However, considering the model (5) under yield uncertainty gives a simple expression for the shadow prices.

COROLLARY 4. *When yield is uncertain, if the firm outputs all available supply, then* $Q_{ij} = bx_i^* x_j^*$, $\forall i, j \in N, i \neq j$, *where* $\mathbf{x}^*$ *is the optimal order quantities.*

Following conventional wisdom, it would seem that we should first identify the correlation among "bad" suppliers who are more expensive and less reliable. Such suppliers typically send up *red flags*, even though they may typically receive smaller orders from the firm. In contrast, Corollary 4 suggests that the firm should prioritize investigating the correlation between "good" suppliers, i.e., suppliers that are less expensive and/or more reliable, because they are more likely to receive larger orders. Since the DR model is rather conservative and may assume a high degree correlation among suppliers based on given information in the distribution set $\mathscr{F}$, the possible correlations between those "bad" suppliers are already considered in the model via the order quantities under a worst-case distribution. In order to improve profitability, the firm needs to reduce unnecessary conservatism by uncovering the true correlation between "good" suppliers. Therefore, Corollary 4 is consistent with our DR approach and offers an effective strategy for prioritizing efforts on exploring correlation information in a supply chain.

4.4. Computational Experiments

This section presents two sets of computational experiments that demonstrate the performance of our proposed models when applied to the model (5) with $a = 20$ and $b = 1$. Throughout our experiments, we assume that there are three suppliers, each experiencing four possible disruption levels, and we consider both cases of yield and capacity uncertainty. In particular, when yield is uncertain, we assume that each supplier may experience disruption levels at 0, 0.2, 0.4, and 1, representing cases where the supplier delivers 0%, 20%, 40%, and 100% of the requested quantity, respectively. When capacity is uncertain, we assume that each supplier may experience disruption levels at 0, 0.1, 0.3, and 1, representing cases where the suppliers capacity is 0, 1, 3, or 10 units, respectively (i.e., $D_s = 10$). We use a parameter θ to vary the mean likelihood of disruption levels that are associated with reduced yield or capacity. For each supplier $s \in N$, we have $F_s(1) = 1$ by its definition. Recall that we have four possible disruption levels for both cases. Thus, we generate three random numbers uniformly in range $[10\% + 5\%\theta, 40\% + 5\%\theta]$ and use them to specify the cumulative probability for each disruption level $F_s(\xi_s)$, $\xi_s \in [0, 1)$, in nondecreasing order. For the upper bound of constraint (1b), we generate a sequence of random matrices Σ_κ with $\kappa \in \{0.00, 0.05, \ldots, 0.95, 1.00\}$ such that all non-diagonal entries of Σ_κ are random numbers in $[\kappa - 0.05, \kappa]$ if $\kappa > 0$ and 0 otherwise. Table 4 summarizes the parameter settings we use throughout the numerical study. For each combination of θ and κ values that are included in Table 4, we

Table 4 Experimental Parameters

Parameter	Values
n	3
c_s	uniform$[1, 3]$
ξ	$\{0, 0.2, 0.4, 1\}^n$ under yield uncertainty, and
	$\{0, 0.1, 0.3, 1\}^n$ under capacity uncertainty, with $D_s = 10$, $\forall s \in N$
θ	$-1, 0, 1$
$F_s(\xi_s)$	selected from a uniform$[10\% + 5\%\theta, 40\% + 5\%\theta]$ if $\xi_s < 1$
Σ_κ	where $\kappa \in \{0.00, 0.05, \ldots, 0.95, 1.00\}$

randomly generate 100 instances. For the sake of space, all graphical presentations for the results of our numerical experiments are based on the case where $\theta = 0$. Although this excludes some settings of our experimental design, we did verify that all of the reported insights are consistent across the excluded θ levels. All the instance data used in this section are publicly available in Zhao et al. (2022).

Table 5 **Summary of the experiment with independent and maximally dependent distributions**

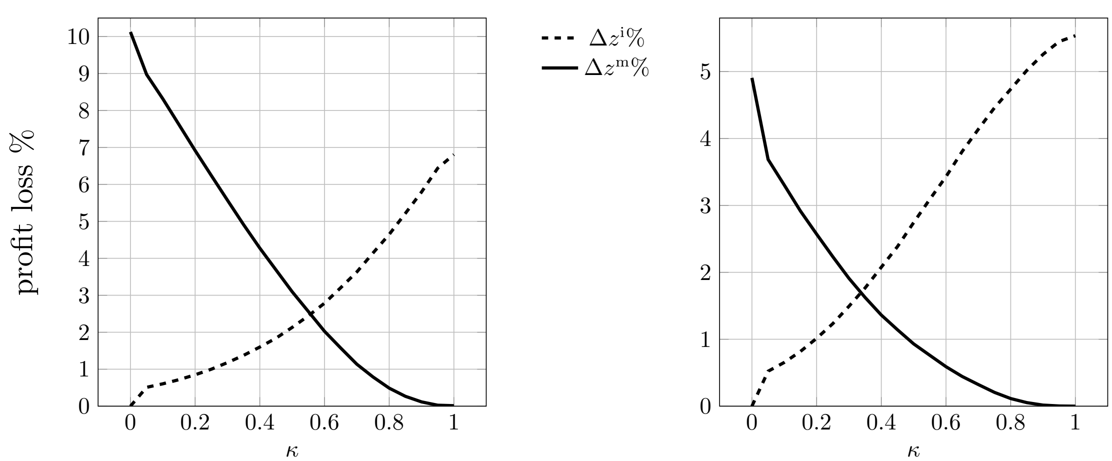
Uncertainty Type	θ	$\Delta z^{\mathrm{i}}\%$	$\Delta z^{\mathrm{m}}\%$	$\mathbf{x}^{\mathrm{i}}/\mathbf{x}$			$\mathbf{x}^{\mathrm{m}}/\mathbf{x}$			$\frac{\mathrm{VoM}}{\Delta\mathscr{V}}\%$	$\frac{\Delta\mathscr{V}}{\overline{\mathscr{V}}}\%$
				s_0	s_1	s_2	s_0	s_1	s_2		
Yield	-1	1.95	2.85	3.04	2.16	1.68	0.41	0.57	0.61	1.85	8.98
	0	2.71	3.75	3.14	2.64	2.04	0.35	0.51	0.69	1.38	11.91
	1	3.62	5.26	2.97	2.43	2.25	0.30	0.70	0.61	0.85	16.18
Capacity	-1	1.78	0.86	3.25	2.47	1.80	0.74	0.84	0.89	52.37	8.08
	0	2.84	1.39	3.20	2.50	2.15	0.68	0.81	0.82	31.96	11.12
	1	3.83	1.66	2.71	2.80	1.90	0.78	0.78	0.90	28.61	15.18

4.4.1. Fully Correlated Case In the first experiment, we assume all suppliers are positively correlated to some degree, i.e., that $\Sigma_\kappa > 0$ for all $\kappa > 0$. We define $\Delta\mathscr{V} = \overline{\mathscr{V}} - \underline{\mathscr{V}}$, an upper bound for the value of correlation information (see (3)). Table 5 summarizes the experimental results.

We discuss three insights based on the results given in Table 5. First, the column $\frac{\Delta\mathscr{V}}{\overline{\mathscr{V}}}\%$ suggests that the value of information regarding the underlying correlation structure increases as the likelihood of supplier disruptions increases, which is captured by increasing value of θ. This trend is consistent for both yield and capacity uncertainties and results in corresponding decreases in the ratio of the value of multisourcing (VoM) to $\Delta\mathscr{V}$, which is given in column $\frac{\mathrm{VoM}}{\Delta\mathscr{V}}\%$. Ultimately, this suggests that a firm experiencing an increase in the frequency of supply disturbances is better served by investigating and understanding the relationship among these risks than by simply increasing redundancy. This observation makes sense in the considered setting because any newly acquired sources may share commonalities with the existing supplier base, making the marginal benefit of the increased diversification small.

Second, we note that there are substantial differences in the ratio of the value of multisourcing to $\Delta\mathscr{V}$ under the two types of supply uncertainties in Table 5. In particular, we observe that the values of multisourcing, relative to $\Delta\mathscr{V}$, are much lower under yield uncertainty than under capacity uncertainty, even though the value of multisourcing decreases for both as the mean disruption likelihood increases. The difference is related to our observations in Section 4. In particular, when yield is uncertain and the available suppliers are maximally correlated, the optimal sourcing strategy will use a more reliable supplier as an anchor supplier that receives the lion's share of the total order allotment, with the remaining suppliers receiving orders based on their relative competitiveness with respect to both cost and reliability. Thus, the benefit of multi-sourcing is dampened because any additional suppliers included in the supply base may only receive smaller orders, which limits their ability to impact profitability. In such a setting, a better understanding of the correlation structure offers the most substantial improvements in the expected profit. When capacity is uncertain, it may not be possible to simply increase the amount ordered from a more reliable supplier due to capacity constraints and diversification takes on a more important role in the buying firm's risk mitigation strategy. This more prominent role of multisourcing is reflected in the higher values of $\frac{\text{VoM}}{\Delta \mathscr{V}}\%$ that we observe in Table 5 when capacity is uncertain.

Finally, Table 5 compares the optimal solutions $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{m}}$ under both independent and maximally dependent distributions, respectively. The column "$\mathbf{x}^{\mathrm{i}}(\mathbf{x}^{\mathrm{m}})/\mathbf{x}$" captures $\mathbf{x}^{\mathrm{i}}(\mathbf{x}^{\mathrm{m}})$ divided by $\mathbf{x}$, the average order quantity across all values of κ for each supplier. Following intuition, we observe that the firm tends to order more than the average under the assumption of independent supply risks and less than the average under the assumption of maximally dependent supply risks. The column "$\Delta z^{\mathrm{i}}\%$" ("$\Delta z^{\mathrm{m}}\%$") provides the expected profit loss, expressed as a percentage, that occurs if the solution $\mathbf{x}^{\mathrm{i}}$ ($\mathbf{x}^{\mathrm{m}}$) is employed regardless of the underlying correlation estimate defined by Σ_κ, where the loss percentages are averaged over all values of κ. Figure 2 compares the profit loss percentages for each value of κ when $\theta = 0$. We observe in Figure 2 that the performance

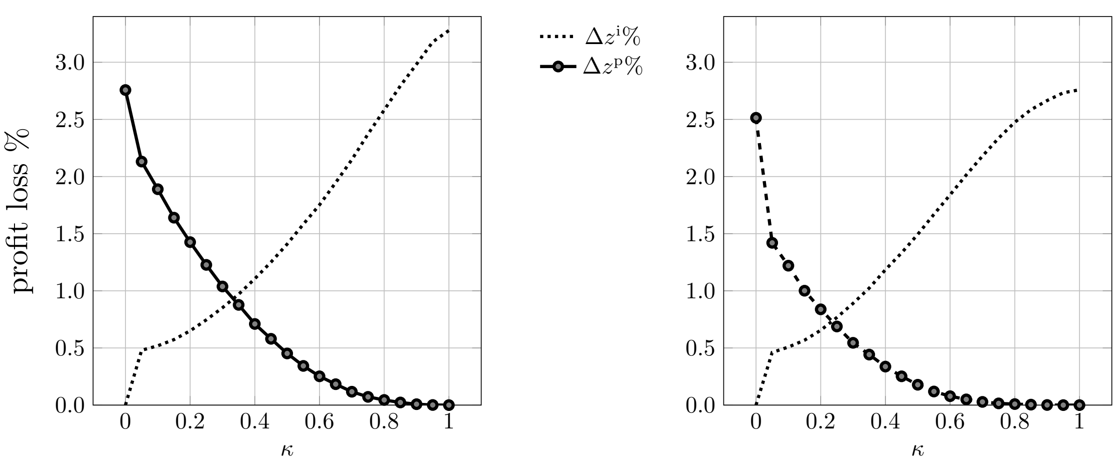


(a) Yield Uncertainty (b) Capacity Uncertainty

Figure 2 Profit loss by applying $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{m}}$

of solution $\mathbf{x}^{\mathrm{m}}$ ($\mathbf{x}^{\mathrm{i}}$) improves (deteriorates) as the correlation increases. The plots also provide compelling evidence to suggest that regardless of the type of uncertainty and characteristics of the available suppliers, at least one of the solutions is able to offer a sourcing strategy with a relatively low degree of profit loss, i.e., less than 3%. Thus, before exerting any effort on investigating the prevailing correlation structure, our DR models allow the firm to determine *i*) budget limits related to potential investigations into supplier correlations; *ii*) a maximum expected profit loss if the measure of correlation is incorrect or missing; and *iii*) bounds on the supplier order quantities given by the solutions $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{m}}$. The latter can be particulary useful for firms considering suppliers that require quantity commitment contracts.

4.4.2. Partially Correlated Case In our second experiment, we partition the available suppliers into two groups where supplier 1 composes the first group and suppliers 2 and 3 compose the second group. We assume that the suppliers in each group are mutually independent from those in the other group with respect to supply risks. Let $\mathbf{x}^{\mathrm{p}}$ denote the optimal solution based on the partially dependent distribution given in Theorem 1. Figure 3 plots the profit loss associated with implementing solutions $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{p}}$, respectively, regardless of the underlying correlation estimate defined by Σ_{κ}. Figure 3 shows that the solution $\mathbf{x}^{\mathrm{i}}$ may result in a profit loss exceeding 1.5% if



(a) Yield Uncertainty (b) Capacity Uncertainty

Figure 3 Compare profit loss between uncorrelated and DR models

supplier 2 and 3 are highly correlated (i.e., $\kappa \geq 0.5$). However, the solution $\mathbf{x}^{\mathrm{p}}$ results in profit loss values below 1.0% if suppliers 2 and 3 are moderately or highly correlated, i.e., $\kappa \geq 0.3$. Ultimately, these results show that our partial information can substantially improve the profitability of a firm facing correlated supply risks when they are able to use available information or judgment to partition suppliers into mutually independent groups.

5. Conclusion

Although the existing OR/MS literature on SCRM offers useful insights, an overwhelming majority of this research is based on the assumption of perfect knowledge regarding supply risk correlations that may limit empirical validity and practical utility. The overarching goal of this research is to propose and explore viable sourcing techniques for the less stylized settings that may be encountered in practice. First, our DR model constitutes a risk-averse approach for accommodating the interrelated risks that arise due to the depth and complexity of today's global supply chains. Second, instead of limiting our study to one uncertainty type, we consider both the cases of yield and capacity uncertainties. Third, our model accommodates multiple levels of supply risks, allowing us to determine sourcing strategies that account for less severe, day-to-day issues that arise in a supply chain in addition to major, all-or-nothing disruptions. Finally, when applied to problems in the area of sourcing, our model suggests an efficient approach for prioritizing efforts aimed at better understanding the correlation among suppliers.

Our numerical experiments offer many additional insights. First, we show that our DR models can outperform models based on erroneous estimates of the prevailing correlation. This insight suggests that firms' SCRM efforts can be simplified and may require no effort be expended to better understand the true correlation that prevails. Such a decision can be made by using the solutions to the independent and maximally dependent distributions to estimate the maximal return on such efforts, via the value of correlation information, and comparing this maximal return to expected costs of data collection. Second, we identify several fundamental differences between the optimal sourcing strategies under conditions of random yield and random capacity. In general, the setting of random capacity is more difficult to analyze and the resulting solutions are starkly different to those for the case of random yield since increasing the amount ordered from more reliable suppliers is not sufficient to offset the detrimental effects of disruptions.

Research on supply chain risk management when information regarding the prevailing correlation is unavailable or ambiguous is in a nascent stage. Thus, there are several promising avenues for future research for this application area. For example, we can consider multiple buyers that compete and utilize a common set of ambiguously correlated suppliers. We may also consider the case where supplier ordering costs are endogenous and suppliers compete on pricing. In this study, we consider cases where *i*) the correlation structure is specific and the objective function is general and *ii*) the correlation structure is general and the objective function is specific. Given this, future research is needed to better understand the cases where both the correlation structure and objective are general. Finally, in Section 2, we discuss how quadratic objective functions can be used to model a variety of problems that fall outside of those traditionally covered in the sourcing literature. For example, quadratic objectives can be used to model problems such as the allocation of supplies during a humanitarian emergency or the allocation of emergency medical resources. Thus, future research could investigate how the proposed DR framework can be applied to these new contexts.

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Online Supplement for "Robust Sourcing Under Multi-level Supply Risks: Analysis of Random Yield and Capacity"

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Appendix A: Quadratic Revenue Function under Bernoulli Disruptions

In this appendix, we assume that all suppliers experience Bernoulli disruptions with $\xi_s \in \{0,1\}$ $\forall s \in N$, and the buying firm adopts the simple quadratic revenue function $R(\mathbf{x},\xi) = \{(a-bq)q | q = \sum_{s\in N} d_s(x_s,\xi_s)\}$. Hence, the DR model is a quadratic programming as follows:

$$\max_{\mathbf{x}\geq 0} \left(\sum_{s=1}^{n} (a-c_s)\mu_s x_s - b\mathbf{x}^\intercal (\Sigma + \mu\mu^\intercal)\mathbf{x} \right). \tag{EC.1}$$

Although the model is computationally tractable, it is cumbersome to derive analytical solutions (see Section 2.2). In order to understand how the correlation structure affects sourcing strategies, we will study the optimal sourcing strategies under several key correlation structures. We assume that $1 > \mu_1 \geq \cdots \geq \mu_n > 0$.

Lemma EC.1. *For given order quantities x_i $\forall i \in N$, the firm achieves*

- *the highest profit if the correlation ρ_{ij} between firm i and j with $i \neq j$ attains its lower bound $\underline{\rho}_{ij} \equiv -\sqrt{[(1-\mu_i)(1-\mu_j)]/\mu_i\mu_j}$. This optimistic correlation structure reflects a situation where the likelihood that multiple suppliers experience simultaneous disruptions is negligible.*

- *the lowest profit if the correlation ρ_{ij} between firm i and j with $i \neq j$ attains its upper bound $\overline{\rho}_{ij} \equiv \sqrt{[\mu_i(1-\mu_j)]/[\mu_j(1-\mu_i)]}$. This conservative correlation structure reflects a situation where supply disruptions exhibit a domino effect. In other words, when a supplier experiences a disruption, all suppliers with higher marginal disruptive probabilities also experience disruptions.*

For notational simplicity, for any $i \in N$, we let $r_i \equiv (a-c_i)\mu_i/2b > 0$ denote the expected profit margin, scaled by $2b$, if the firm sources from supplier i.

Proposition EC.1. *Optimal solutions under different correlation structures are characterized as follows:*

- *under the overly optimistic correlation structure with $\rho_{ij} = \underline{\rho}_{ij}$, the firm sources from all firms and the order quantities $x_i^{\mathrm{O}} = r_i/(1-\mu_i)$, $\forall i \in N$, where $r_i/(1-\mu_i)$ also represents the return–risk ratio of supplier i.*
- *under the independent correlation structure with $\rho_{ij} = 0$, the firm sources only from suppliers whose return–risk ratios are greater than a scalar[2] λ and the order quantities $x_i^{\mathrm{I}} = \max(r_i/(1-\mu_i) - \lambda, 0)/\mu_i$, $\forall i \in N$.*
- *under the conservative correlation structure with $\rho_{ij} = \overline{\rho}_{ij}$, the firm sources only with suppliers that are on an efficiency frontier defined by return r_i and risk $1-\mu_i$, $\forall i \in N$. Suppose all suppliers are on the efficiency frontier and $\mu_i \neq \mu_j$ $\forall i \neq j \in N$, then the order quantities*

$$x_i^{\mathrm{C}} = \frac{r_i - r_{i-1}}{\mu_{i-1} - \mu_i} - \frac{r_{i+1} - r_i}{\mu_i - \mu_{i+1}}, \ \forall i \in N \setminus \{n\}, \ \textit{and } x_n^{\mathrm{C}} = \frac{r_n - r_{n-1}}{\mu_{n-1} - \mu_n},$$

where $r_0 = 1 - \mu_0 = 0$ for notational convenience.

Proposition EC.1 demonstrates the sourcing strategies under different correlation structures. Under the overly optimistic correlation structure, the firm employs a high degree of diversification, sourcing from all suppliers since it assumes a negligible probability of simultaneous disruptions. Under the independent correlation structure, which also implies an optimistic view of the supply risk, the firm sources from suppliers based on a return-risk ratio threshold, λ. Under the conservative correlation structure, the firm compares the return-risk ratio between suppliers and sources from suppliers lying on the described efficient frontier.

A.1. Technical Proofs for Appendix A

Proof of Lemma EC.1 With a given marginal distribution that $\Pr(\xi_i = 1) = \mu_i \in (0,1)$, $\forall i \in N$, we can attain the lower and upper bounds of the correlation coefficient ρ_{ij} for any $i, j \in N$ and $i \neq j$ as follows (note that $\rho_{ii} = 1$ for any $i \in N$). First, we have

$$\rho_{ij} = \frac{\mathbb{E}[\xi_i \cdot \xi_j] - \mathbb{E}[\xi_i]\mathbb{E}[\xi_j]}{\sqrt{(\mathbb{E}[\xi_i^2] - \mathbb{E}[\xi_i]^2)(\mathbb{E}[\xi_j^2] - \mathbb{E}[\xi_j]^2)}}$$

[2] The definition of λ is presented in the proof.

$$
\begin{aligned}
&= \frac{\Pr(\xi_i=1,\xi_j=1)-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} \\
&= \frac{(1-\Pr(\xi_i=0,\xi_j=0)-\Pr(\xi_i=0,\xi_j=1)-\Pr(\xi_i=1,\xi_j=0))-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} \\
&\geq \frac{(1-\Pr(\xi_i=0)-\Pr(\xi_j=0))-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} \\
&= \frac{\mu_i-(1-\mu_j)-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} = -\sqrt{\frac{(1-\mu_i)(1-\mu_j)}{\mu_i\mu_j}} = \underline{\rho}_{ij},
\end{aligned}
$$

where the first equality follows from the definition of the correlation coefficient, the second and third equalities hold because ξ_i follows a Bernoulli distribution for any $i \in N$, the first inequality holds because $\Pr(\xi_i = 0, \xi_j = 0) + \Pr(\xi_i = 0, \xi_j = 1) = \Pr(\xi_i = 0)$ and $\Pr(\xi_i = 1, \xi_j = 0) \leq \Pr(\xi_j = 0)$.

Second, we have

$$
\begin{aligned}
\rho_{ij} &= \frac{\Pr(\xi_i=1,\xi_j=1)-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} \leq \frac{\Pr(\xi_i=1)-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} \\
&= \frac{\mu_i-\mu_i\mu_j}{\sqrt{\mu_i\mu_j(1-\mu_i)(1-\mu_j)}} = \sqrt{\frac{\mu_i(1-\mu_j)}{\mu_j(1-\mu_i)}} = \overline{\rho}_{ij},
\end{aligned}
$$

where the inequality holds because $\Pr(\xi_i = 1, \xi_j = 1) \leq \Pr(\xi_i = 1)$. When the order quantities x_i $\forall i \in N$ are given, the optimal profit in (EC.1) is a linear function of Σ and accordingly a linear function of ρ_{ij} for any $i, j \in N$ and $i \neq j$. Hence, the firm obtains its maximal (resp. minimal) profit if ρ_{ij} attains its lower (rep. upper) bound. □

Proof of Proposition EC.1 Suppose $\rho_{ij} = \underline{\rho}_{ij}$ for any $i, j \in N$ and $i \neq j$. By Lemma EC.1, Problem (EC.1) becomes

$$
\begin{aligned}
&\max_{\mathbf{x}\geq 0} \left(\sum_{i=1}^{n} \left[(a-c_i)\mu_i x_i - b(1-\mu_i)x_i^2 \right] - 2b \sum_{i,j=1; i<j}^{n} x_i x_j \left(\rho_{ij}\sqrt{\mu_i(1-\mu_i)\mu_j(1-\mu_j)} + (1-\mu_i)(1-\mu_j) \right) \right) \\
&= \max_{\mathbf{x}\geq 0} \sum_{i=1}^{n} \left[(a-c_i)\mu_i x_i - b(1-\mu_i)x_i^2 \right].
\end{aligned}
$$

The first order condition gives the optimal solution

$$
x_i = \frac{(a-c_i)\mu_i}{2b(1-\mu_i)} \equiv \frac{r_i}{1-\mu_i} \geq 0, \ \forall i \in N.
$$

Now, suppose $\rho_{ij} = \overline{\rho}_{ij}$ for any $i, j \in N$ and $i \neq j$. By Lemma EC.1, Problem (EC.1) becomes

$$
\max_{\mathbf{x}\geq 0} \left(\sum_{i=1}^{n} (a-c_i)\mu_i x_i - 2b \sum_{i,j=1; i\leq j}^{n} (1-\mu_i) x_i x_j \right).
$$

The first order conditions are

$$r_i \equiv \frac{(a-c_i)\mu_i}{2b} = \sum_{j=1}^{i-1}(1-\mu_j)x_j + (1-\mu_i)\sum_{j=i}^{n} x_j,\ \forall i \in N.$$

Then, we have

$$\frac{r_{i+1}-r_i}{\mu_i-\mu_{i+1}} = \sum_{j=i+1}^{n} x_j,\ \forall i \in N\setminus\{n\}.$$

Hence,

$$x_i = \frac{r_i - r_{i-1}}{\mu_{i-1}-\mu_i} - \frac{r_{i+1}-r_i}{\mu_i-\mu_{i+1}},\ \forall i \in N\setminus\{n\},\ \text{and}\ x_n = \frac{r_n - r_{n-1}}{\mu_{n-1}-\mu_n},$$

where $r_0 = 1-\mu_0 = 0$. Then, $x_i \geq 0\ \forall i \in N\setminus\{n\}$ if and only if

$$\frac{r_i - r_{i-1}}{\mu_{i-1}-\mu_i} - \frac{r_{i+1}-r_i}{\mu_i-\mu_{i+1}} \geq 0,\ \forall i \in N\setminus\{n\}.$$

The condition implies that connecting all the points $\{(0,0),(1-\mu_i,r_i): i\in N\}$ on a plane forms a convex efficiency frontier.

Last, suppose $\rho_{ij}=0$ for any $i,j\in N$ and $i\neq j$. By Lemma EC.1, Problem (EC.1) becomes

$$\max_{\mathbf{x}\geq 0}\left(\sum_{i=1}^{n}\left[(a-c_i)\mu_i x_i - b(1-\mu_i)x_i^2\right] - 2b\sum_{i,j=1;i<j}^{n}(1-\mu_i)(1-\mu_j)x_i x_j\right),$$

and the first order condition gives

$$\frac{r_i}{1-\mu_i} = \mu_i x_i + \sum_{j=1}^{n}(1-\mu_j)x_j,\ \forall i \in N.$$

Let $\mathcal{J}$ be the set of suppliers with positive order quantities. The first order condition gives

$$\sum_{i\in\mathcal{J}}\frac{r_i}{\mu_i} = \sum_{i\in\mathcal{J}}(1-\mu_i)x_i + \sum_{i\in\mathcal{J}}\frac{1-\mu_i}{\mu_i}\left(\sum_{i\in\mathcal{J}}(1-\mu_i)x_i\right)$$
$$\Rightarrow \sum_{i\in\mathcal{J}}(1-\mu_i)x_i = \frac{\sum_{i\in\mathcal{J}} r_i/\mu_i}{1+\sum_{i\in\mathcal{J}}(1-\mu_i)/\mu_i}.$$

Hence, the optimal solution is

$$x_i = \frac{1}{\mu_i}\left(\frac{r_i}{1-\mu_i} - \lambda\right),\ \forall i\in\mathcal{J},\ \text{and}\ x_i = 0,\ \forall i \in N\setminus\mathcal{J},$$

where

$$\lambda = \frac{\sum_{i\in\mathcal{J}} r_i/\mu_i}{1+\sum_{i\in\mathcal{J}}(1-\mu_i)/\mu_i}.$$

We reorder the indices in N such that $r_1/(1-\mu_1)\geq\cdots\geq r_n/(1-\mu_n)$. Hence, $\mathcal{J}=\{1,\ldots,J\}$ where

$$J = \arg\max\left\{j: \frac{r_j}{(1-\mu_j)} \geq \frac{\sum_{i=1}^{j} r_i/\mu_i}{1+\sum_{i=1}^{j}(1-\mu_i)/\mu_i}\right\}.$$

The proposition is proved. □

Appendix B: Instance Generation Procedures

For Case 1, given a 3-dimensional array $(p_1, p_2, p_3) = (0.3 + 0.05\theta, 0.2 + 0.05\theta, 0.1 + 0.05\theta)$ and an interval I_ρ, let Σ' be a 3×3 correlation matrix where all the off-diagonal elements are -0.3, 0.2, or 0.7 if I_ρ is $[-0.2, 0]$, $[0, 0.25]$, or $[0.25, 0.5]$, respectively. We then follow Algorithm 1 to generate an instance.

Algorithm 1 Generate an instance for Case 1, given (p_1, p_2, p_3), I_ρ and Σ'

1: Generate a sample $\{(\xi'_{1,i}, \xi'_{2,i}, \xi'_{3,i}) : i = 1, \ldots, 400\}$ that follows a multivariate normal $(0, \Sigma')$.

2: Let r_s be the $p_s \times 100$th percentile of the data set $\{\xi'_{s,i} : i = 1, \ldots, 400\}$, $\forall s \in \{1, 2, 3\}$.

3: Get a sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}) : i = 1, \ldots, 400\}$, such that $\xi_{s,i} = 1$, if $\xi'_{s,i} \geq r_s$; $\xi_{s,i} = 0$, otherwise, $\forall s = 1, 2, 3$. Note that $\sum_{i=1}^{400} \xi_{s,i}/400 = (1 - p_s)$, $\forall s = 1, 2, 3$. Since $\xi_s = 0$ indicates supplier s is disrupted and $\xi_s = 1$ represents no disruption, each supplier s has marginal disruptive probability p_s, $\forall s = 1, 2, 3$.

4: Calculate the pairwise correlation coefficients ρ_{st}, $\forall s \neq t \in \{1, 2, 3\}$, of the sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}) : i = 1, \ldots, 400\}$.

5: Return sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}) : i = 1, \ldots, 400\}$ if $\rho_{st} \in I_\rho$, $\forall s \neq t \in \{1, 2, 3\}$, otherwise go to Step 1.

For Case 2, given a 2-dimensional array $(p_A, p_B) = (0.25 + 0.05\theta, 0.15 + 0.05\theta)$ and an interval I_ρ, let Σ' be a 2×2 correlation matrix where all the off-diagonal elements are 0.05, 0.25, or 0.5 if I_ρ is $[-0.2, 0]$, $[0, 0.25]$, or $[0.25, 0.5]$, respectively. We then follow Algorithm 2 to generate an instance.

Algorithm 2 Generate an instance for Case 2, given Σ', (p_A, p_B) and I_ρ

1: Generate a sample $\{(\xi'_{A,i}, \xi'_{B,i}) : i = 1, \ldots, 400\}$ that follows a multivariate normal $(0, \Sigma')$.

2: Let r_s be the $p_s \times 100$th percentile of the data set $\{\xi'_{s,i} : i = 1, \ldots, 400\}$, $\forall s \in \{A, B\}$.

3: Get a sample $\{(\xi_{A,i}, \xi_{B,i}) : i = 1, \ldots, 400\}$, such that $\xi_{s,i} = 1$, if $\xi'_{s,i} \geq r_s$; $\xi_{s,i} = 0$, otherwise, $\forall s = A, B$. Note that $\sum_{i=1}^{400} \xi_{s,i}/400 = (1 - p_s)$, $\forall s = A, B$. Since $\xi_s = 0$ indicates the supplier s is disrupted and $\xi_s = 1$ represents no disruption, each supplier s has marginal disruptive probability p_s, $\forall s = A, B$.

4: Derive a sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}, \xi_{4,i}) = (\xi_{A,i}, .8\xi_{A,i} + .2\xi_{B,i}, .3\xi_{A,i} + .7\xi_{B,i}, \xi_{B,i}) : i = 1, \ldots, 400\}$ for the tier-1 suppliers.

5: Calculate the pairwise correlation coefficients ρ_{st}, $\forall s \neq t \in \{1, 2, 3, 4\}$ of the sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}, \xi_{4,i}) : i = 1, \ldots, 400\}$.

6: Return sample $\{(\xi_{1,i}, \xi_{2,i}, \xi_{3,i}, \xi_{4,i}) : i = 1, \ldots, 400\}$, if $\rho_{st} \in I_\rho$ $\forall s \neq t \in \{1, 2, 3, 4\}$, otherwise go to Step 1.

Appendix C: Distribution $F^{\mathrm{m}}(\xi)$ and Associated Sourcing Strategies

We use $f^{\mathrm{m}}(\xi)$ to denote the probability distribution function of the maximally dependent distribution $F^{\mathrm{m}}(\xi)$, which denotes a cumulative distribution function. In this appendix, we first demonstrate the support of the distribution $f^{\mathrm{m}}(\xi)$ as shown in Corollary EC.1. Then, we study the sourcing strategies under the supply distribution $F^{\mathrm{m}}(\xi)$. As shown in Section 2.3, the sourcing strategies associated with the worst-case distribution $F^{\mathrm{m}}(\xi)$ can effectively mitigate supply disruptions when the correlation is ambiguous and difficult to estimate.

Recall that $F^{\mathrm{m}}(\xi) \equiv \min_{s\in N} F_s(\xi_s)$ for any $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in [0,1]^n$. We define $\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t)) \in [0,1]^n$ with $t \in [0,1]$ where $\zeta_s(t) = \min\{\xi_s \in [0,1] : F_s(\xi_s) \geq t\}$, $\forall s \in N$. For an arbitrarily given $\xi \in [0,1]^n$, we claim that

$$F^{\mathrm{m}}\left(\zeta(\tau)\right) = \min_{s\in N} F_s(\zeta_s(\tau)) = F^{\mathrm{m}}(\xi), \tag{EC.2}$$

where $\tau = \min_{i\in N} F_i(\xi_i)$. Let $i^* \in \arg\min_{i\in N} F_i(\xi_i)$. First, we have

$$\min_{s\in N} F_s(\zeta_s(\tau)) = \min_{s\in N} F_s(\zeta_s(F_{i^*}(\xi_{i^*}))) \leq F_{i^*}(\zeta_{i^*}(F_{i^*}(\xi_{i^*}))) = F_{i^*}(\xi_{i^*}) = \min_{i\in N} F_i(\xi_i) = F^{\mathrm{m}}(\xi), \tag{EC.3}$$

where the second equality holds because $\zeta_{i^*}(F_{i^*}(\xi_{i^*})) = \xi_{i*}$. Second, we have

$$\min_{s\in N} F_s(\zeta_s(\tau)) \geq \tau = \min_{i\in N} F_i(\xi_i) = F^{\mathrm{m}}(\xi), \tag{EC.4}$$

where the inequality holds because of the definition of $\zeta_s(t)$ for any $s \in N$. Combining (EC.3) and (EC.4) gives equation (EC.2). The equation (EC.2) suggests that we can focus on the domain $\Gamma = \{\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t)) : t \in [0,1]\}$ for the distribution $F^{\mathrm{m}}(\xi)$.

Corollary EC.1. *If the available marginal disruption distributions are discrete, then f^{m} is defined on a finite support $\Gamma = \{\zeta^0 = \mathbf{0}, \cdots, \zeta^L = \mathbf{1}\}$ for an integer L, where $\zeta^{i+1} \in [0,1]^n$ is different from $\zeta^i \in [0,1]^n$ by only one element and $\zeta^{i+1} \geq \zeta^i$ for any $i \in \{0, 1, \ldots, L-1\}$.*

The support of $f^{\mathrm{m}}(\xi)$ described in Corollary EC.1 mimics a domino effect where increasingly higher degrees of supply disruption propagate through a supply chain. To illustrate this more concretely, we provide Example EC.1 as follows.

Example EC.1. Consider a set of three suppliers, i.e., $N = \{1, 2, 3\}$, which are subject to supply risks captured by $\xi \in \{0, .3, 1\}^3$. Assume that the marginal cumulative supply distributions for the suppliers are

Table EC.1 Supplier Information

Supplier (s)	$F_s(0)$	$F_s(.3)$	c_s
1	0.25	0.70	20
2	0.45	0.60	21
3	0.30	0.40	22

Table EC.2 Supply Uncertainty Scenarios

i	Scenario	Supplier 1	Supplier 2	Supplier 3	f^{m}
0	ζ^0	0	0	0	0.25
1	ζ^1	.3	0	0	0.05
2	ζ^2	.3	0	.3	0.10
3	ζ^3	.3	0	1	0.05
4	ζ^4	.3	.3	1	0.15
5	ζ^5	.3	1	1	0.10
6	ζ^6	1	1	1	0.30

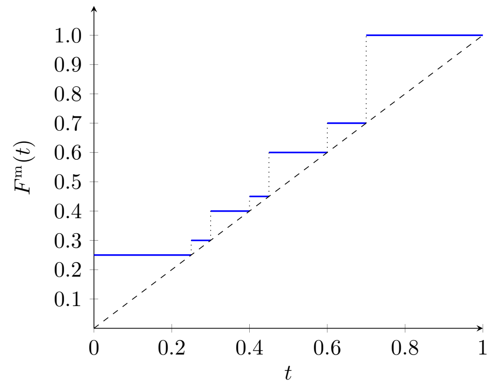
as given in Table EC.1, along with the unit costs in column c_s. The support of f^{m} includes 7 scenarios, i.e., $\zeta^0, \ldots, \zeta^6$. Scenario ζ^6 corresponds to a case where all suppliers deliver everything that is ordered and ζ^0 corresponds to a scenario where none of the suppliers delivers. The maximally dependent distribution attributes the supply risk expressed by the marginals to a common source that affects all suppliers progressively. To see this, note that the probability of supplier 1 experiencing a supply disturbance is the largest in Table EC.1 with $F_1(0.3) = 0.7$. Thus, the maximally dependent distribution implicitly assumes that supplier 1 is most sensitive to disruptions of the common source with a 70% likelihood of delivering less than the ordered amount. As a result, we determine the likelihood that supplier 1 experiences no supply disturbance by considering the probability of scenario ζ^6 and the scenario that only supplier 1 experiences supply disruption as scenario ζ^5. The resulting probability is $1 - F_1(0.3) = 0.3$. Referring back to Table EC.1, we observe that $F_2(0.3) = 0.6$ is the second largest cumulative disruption probability in the table. Under the assumption of maximal dependence, supplier 2 experiences a supply disturbance at level 0.3 only if supplier 1 is also at level 0.3. This situation is represented by scenario ζ^4 and the probability of scenario ζ^5 is given by $F_1(0.3) - F_2(0.3) = 0.1$. Continuing this logic, we arrive at the supply distribution shown in Table EC.2.

The definition of F^{m} implies that the support of $f^{\mathrm{m}}(\xi)$ is $\{\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t)) : t \in [0,1]\}$ where $\zeta_s(t) = \min\{\xi_s \in [0,1] : F_s(\xi_s) \geq t\}$ $\forall s \in N$. Since ξ follows a discrete distribution, $\zeta(t)$ has the same value when t is within a defined interval. For example, when $t \in (.25, .30]$, $\zeta(t) = (0.3, 0, 0)$, which is ζ^1 defined in Table EC.2, because $F_1(0) = 0.25$, $0 = \arg\min\{\xi_2 : F_2(\xi_2) \geq 0.25\}$ and $0 = \arg\min\{\xi_3 : F_3(\xi_3) \geq 0.25\}$. The cumulative probability $F^{\mathrm{m}}(t)$ in Table EC.3 can be calculated from column f^{m} in Table EC.2. That is, $F^{\mathrm{m}}(t) = f^{\mathrm{m}}(\zeta^0) + f^{\mathrm{m}}(\zeta^1) = 0.30$ for $t \in (.25, .30]$. Figure EC.1 plots $F^{\mathrm{m}}(t)$ over the domain $t \in [0,1]$ and shows that $F^{\mathrm{m}}(t)$ is a step function sitting on the dashed line that corresponds to the cumulative probability function for the uniform distribution on the same support. □

Table EC.3 Supply Uncertainty Scenarios

t	Scenario	$\eta_1(t)$	$\eta_2(t)$	$\eta_3(t)$	$F^{\mathrm{m}}(t)$
$[0.00, 0.25]$	ζ^0	0	0	0	0.25
$(0.25, 0.30]$	ζ^1	.3	0	0	0.30
$(0.30, 0.40]$	ζ^2	.3	0	.3	0.40
$(0.40, 0.45]$	ζ^3	.3	0	1	0.45
$(0.45, 0.60]$	ζ^4	.3	.3	1	0.60
$(0.60, 0.70]$	ζ^5	.3	1	1	0.70
$(0.70, 1.00]$	ζ^6	1	1	1	1.00

Figure EC.1 $F^{\mathrm{m}}(t)$ **on its support**



The above example shows that if the available marginal distributions are continuous, then f^{m} is a uniform distribution defined on its support. It is true because equation (EC.2) implies $(F^{\mathrm{m}} \circ \zeta)(\tau) = F^{\mathrm{m}}(\zeta(\tau)) = F^{\mathrm{m}}(\xi) = \min_{i \in N} F_i(\xi_i) = \tau$. That is, F^{m} is equivalent to a uniform distribution on domain Γ.

In the rest of this appendix, we assume that the available marginal disruption distributions are discrete. This assumption is not overly limiting since continuous marginal distributions can be accurately approximated by discrete distributions. In order to derive clear insights on sourcing strategies, we consider the responsive quadratic revenue function $R(\mathbf{x}, \xi) = \max_q \left\{(a - bq)q \middle| q \leq \sum_{s \in N} d_s(x_s, \xi_s)\right\}$. Using the notations and the worst-case distribution given in Corollary EC.1, the DR model with the maximally dependent supply distribution is

$$\max \quad \sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k) \left[(a - bq_k)q_k - \sum_{s \in N_Y} c_s \zeta_s^k x_s - \sum_{s \in N_C} c_s \min\left(D_s \zeta_s^k, x_s\right)\right] \tag{EC.5}$$

$$\text{s.t.} \quad q_k \leq \sum_{s \in N_Y} \zeta_s^k x_s + \sum_{s \in N_C} \min\left(D_s \zeta_s^k, x_s\right), \ \forall k \in \{0, 1, \ldots, L\}; \text{ and } x_s \geq 0, \ \forall s \in N.$$

Let $\mathbf{x} = (x_i, \forall i \in N)$ and $\mathbf{q} = (q_k, \forall k \in \{0, 1, \ldots, L\})$ represent the vectors of the optimal order and output quantities of Problem (EC.5), respectively. We describe an important property in the following lemma.

Lemma EC.2. *For the optimal solution of* (EC.5)*, we have*

$$q_k = \min\left\{a/2b, \sum_{s \in N_Y} \zeta_s^k x_s + \sum_{s \in N_C} \min\left(D_s \zeta_s^k, x_s\right)\right\}, \ \forall k \in \{0, 1, \ldots, L\}, \tag{EC.6}$$

and $0 = q_0 \leq q_1 \leq \cdots \leq q_L$.

Although it is easy for commercial solvers to obtain a numeric solution for (EC.5), it is cumbersome to derive an analytic solution, even for the case of two suppliers, because of the disjunctive output decisions

q_k ($\forall k \in \{0,1,\dots,L\}$) (e.g. see details in Hu & Kostamis 2015). To provide some insights on the relationship between supplier reliability and supplier selection, we consider random yields and capacities in Sections C.1 and C.2, respectively. The technical proofs are presented in Section C.3.

C.1. Random Yields

In this section, we consider that all suppliers are subject to yield uncertainties that $N_Y = N$. Suppose a supplier s' is *less reliable* than another supplier s. Specifically, we use the description *less reliable* to indicate that a particular supplier s' is more susceptible to disruptions at all levels when compared to another supplier s, i.e., where $F_s(t) \le F_{s'}(t)$, $\forall t \in [0,1]$. In the optimal solution, the output level in any disruption scenario will be less than or equal to the revenue maximizing quantity $a/2b$ because of Lemma EC.2. However, the total order quantities may exceed $a/2b$ to account for the possibility of supplier disruptions. Recall that the support of f^{m} contains $L+1$ scenarios $\zeta^0,\dots,\zeta^L$. Let $k^* \in \{0,1,\dots,L\}$ denote the superscript (i.e, index) of a scenario in Γ such that the optimal solution $\mathbf{q}$ satisfies $q_{k^*} < a/2b \le q_{k^*+1}$, where $k^* = L$ if $q_k < a/2b$ $\forall k \in \{0,1,\dots,L\}$. Using this definition, we may derive the relationship between supplier reliability and supplier selection in the following proposition.

Proposition EC.2. *Suppose a supplier s' is less reliable than another supplier s and all the suppliers are ambiguously correlated. If the firm orders from supplier s', then $c_{s'} < c_s/(1+\eta)$ where*

$$\eta = \left(\sum_{k=0}^{\arg\max\left\{k:\zeta_{s'}^k=0\right\}} \zeta_s^k f^{\mathrm{m}}(\zeta^k) \right) \Bigg/ \sum_{k=0}^{k^*} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k).$$

If the total order quantity increases in the optimal solution (e.g., when the market size increases), then k^* decreases and η increases because of its denominator. Proposition EC.2 shows that $c_{s'} < c_s/(1+\eta)$ is a necessary condition on ordering from supplier s'. When η increases, the buying firm expects lower unit cost from the less reliable supplier s'. Thus, the supplier s' becomes less attractive as the total order quantity increases. The finding, the reliability trumps cost in the worst-case distribution, is the opposite of the conventional wisdom developed under the uncorrelated assumption, which indicates that reliability weighs in more as correlation increases.

With respect to the terms composing η, the numerator captures the expected proportion delivered by supplier s when supplier s' is down because $\zeta_{s'}^k = 0$ for any $k \in \{0,1,\dots,\arg\max\{j : \zeta_{s'}^j = 0\}\}$. And the denominator captures the expected proportion that supplier s' delivers in the scenarios where all procured goods are output to the market because $q_0 \leq \cdots \leq q_{k^*} < a/2b$ in the optimal solution. Thus, the ratio η essentially provides a measure of the incremental reliability of supplier s in comparison to supplier s' and the inequality $(c_{s'} < c_s/(1+\eta))$ translates this measure into a cost reduction required for supplier s' to be attractive. From a managerial perspective, Proposition EC.2 provides a means to eliminate unreliable suppliers on the basis of η. Moreover, it provides a concrete reduction in cost that a less reliable supplier must offer in order to be competitive, a potentially useful piece of information for negotiation purposes.

C.2. Random Capacities

In this section, we consider that all suppliers are subject to capacity uncertainties that $N_C = N$. Although many researchers have shown that the notion that "cost is an order qualifier" does not hold in general when suppliers face correlated disruption risks, the supplier with the lowest ordering cost plays a major role in defining the optimal order quantities for the case of random capacity.

Proposition EC.3. *Let $\underline{s}$ represent the supplier with lowest ordering cost. Under the maximally dependent distribution, when $\underline{s}$ receives no order, the quantity ordered from any other supplier s, with $c_s > c_{\underline{s}}$, will be less than or equal to $D_s\zeta_s^k$, where $k = \arg\max\left\{k' : \zeta_{\underline{s}}^{k'} = 0\right\}$.*

Proposition EC.3 describes an interesting relationship between the order quantities and the scenarios composing the worst-case distribution for the case of uncertain capacity. In particular, when no order is placed with the cheapest supplier, the amount ordered from any other supplier will be bounded by its capacity level in the scenario where the cheapest supplier first becomes completely unavailable. Thus, the cheapest supplier is increasingly likely to receive an order as its probability of being fully down, i.e., $f_{\underline{s}}(0)$, decreases. Because supplier $\underline{s}$ is most attractive from the perspective of cost, this insight can be described by the following corollary.

Corollary EC.2. *The firm always orders from the supplier $\underline{s}$ with the lowest ordering cost if $f_{\underline{s}}(0) = \min_{s\in N} f_s(0)$.*

C.3. Technical Proofs for Appendix C

Proof of Corollary EC.1 Suppose the marginal distributions $f_s(\xi_s)$, $\forall s \in N$, are discrete on support Ω_s and $\{0,1\} \in \Omega_s$ for any $s \in N$. Because the marginal distributions are discrete, $\Gamma = \{\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t)) : t \in \{F_s(\xi_s), \forall s \in N, \xi_s \in \Omega_s\}\}$. Note that Γ depends only on the different values in the set $\{F_s(\xi_s), \forall s \in$

$N, \xi_s \in \Omega_s\}$. We introduce a set $\mathcal{L} = \{0, 1, \ldots, L\}$ such that $\{(s_i, k_i) : \forall i \in \mathcal{L}\} = \{(s, \xi_s) : \forall s \in N, \xi_s \in \Omega_s\}$ and $F_{s_i}(k_i) < F_{s_{i+1}}(k_{i+1})$ for any $i \in \mathcal{L} \setminus \{L\}$. Basically, we reorder the set $\{(s, \xi_s) : \forall s \in N, \xi_s \in \Omega_s\}$ based on their marginal distributions. Hence, $\Gamma = \{\zeta(F_{s_i}(k_i)) : \forall i \in \mathcal{L}\}$, which can be simply denoted by $\{\zeta^0, \zeta^1, \ldots, \zeta^L\}$. Because $F_{s_0}(k_0)$ is the smallest marginal probability, we have $F_s(\xi_s) \geq F_{s_0}(k_0)$, $\forall s \in N,\ \xi_s \in \Omega_s$. Hence, $\zeta_s^0 = \zeta_s(F_{s_0}(k_0)) = \min\{\xi_s \in [0,1] : F_s(\xi_s) \geq F_{s_0}(k_0)\} = 0\ \forall s \in N$, i.e., $\zeta^0 = \mathbf{0}$. For any $i \in \mathcal{L}$, the definition of $\zeta(t)$ implies $F_s(\zeta_s(F_{s_i}(k_i))) \geq F_{s_i}(k_i)\ \forall s \in N$. Because $\{(s_i, k_i) : i \in \mathcal{L}\}$ includes all different values of the marginal probabilities $F_{s_i}(k_i)$, we have $F_s(\zeta_s(F_{s_i}(k_i))) > F_{s_i}(k_i)\ \forall s \in N \setminus \{s_i\}$. Hence, $F_s(\zeta_s(F_{s_i}(k_i))) \geq F_{s_{i+1}}(k_{i+1})$, $\forall s \in N \setminus \{s_i\}$. Then, following the definition of $\zeta(t)$, $\zeta_s^{i+1} = \zeta_s(F_{s_i}(k_i)) = \zeta_s^i$, $\forall s \in N \setminus \{s_i\}, i \in \mathcal{L} \setminus \{L\}$, and $\zeta_{s_i}^{i+1} = \zeta_{s_i}(F_{s_{i+1}}(k_{i+1})) = \min\{\xi_{s_i} \in \Omega_{s_i} : \xi_{s_i} > k_i\}$. That is, ζ^{i+1} is different from ζ^i by only one element and $\zeta^{i+1} \geq \zeta^i$ for any $i \in \mathcal{L} \setminus \{L\}$. Because $F_{s_L}(k_L) = \max\{F_s(\xi_s), \forall s \in N, \xi_s \in \Omega_s\} = 1$, we have $\zeta^L = \zeta(F_{s_L}(k_L)) = \zeta(1) = \mathbf{1}$. $\square$

Proof of Lemma EC.2 For any given order quantity x_i ($\forall i \in N$), equation (EC.6) holds for the optimal output quantity q_k ($\forall k \in \{0, 1, \ldots, L\}$). Hence, equation (EC.6) is necessary for the optimal solution. Because $\zeta_s^k \leq \zeta_s^{k+1}$ for any $k \in \{0, 1, \ldots, L-1\}$ and $s \in N$ by its definition, it follows that $q_0 \leq q_1 \leq \cdots \leq q_L$ due to equation (EC.6). $\square$

Proof of Proposition EC.2 The Lagrangian function of Problem (EC.5) is

$$\mathcal{L}(x, q, \lambda, \delta) = \sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)(a - bq_k)q_k - \sum_{i=1}^{n} c_i x_i + \sum_{k=0}^{L} \delta_k \left(\sum_{i=1}^{n} \zeta_i^k x_i - q_k \right) + \sum_{i=1}^{n} \lambda_i x_i,$$

where $\delta_k \geq 0$, $\forall k \in \{0, 1, \ldots, L\}$, and $\lambda_i \geq 0$, $\forall i \in N$, are Lagrangian multipliers corresponding to the upper bound and non-negativity constraints, respectively. The associated KKT conditions, which guarantee the optimality and characterize an optimal solution, are:

$$\frac{\partial \mathcal{L}}{\partial x_i} = -c_i + \lambda_i + \sum_{k=0}^{L} \zeta_i^k \delta_k = 0, \qquad \forall i \in N; \tag{EC.7a}$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = f^{\mathrm{m}}(\zeta^k)(a - 2bq_k) - \delta_k = 0, \qquad \forall k \in \{0, 1, \ldots, L\}. \tag{EC.7b}$$

We have $q_k = \min(a/2b, \sum_{i=1}^{n} \zeta_i^k x_i)$ for any $k \in \{0, 1, \ldots, L\}$ and $q_0 \leq q_1 \leq \cdots \leq q_L$ by Lemma EC.2. Thus, (EC.7b) implies $\delta_0 \geq \delta_1 \geq \cdots \geq \delta_L \geq 0$ because $f^{\mathrm{m}}(\zeta^k) \geq 0$ for any $k \in \{0, 1, \ldots, L\}$. Let $K = L$ if $\delta_L > 0$, otherwise there exists a $K \in \{0, 1, \ldots, L\}$ such that $\delta_K > 0$ and $\delta_{K+1} = 0$. We denote $k' = \arg\max\{k : \zeta_{s'}^k = 0\}$, i.e., $\zeta_{s'}^0 = \cdots = \zeta_{s'}^{k'} = 0$ but $\zeta_{s'}^{k'+1} = 1$. Then, the output level q_k ($\forall k \leq k'$) does not depend on the order quantity $x_{s'}$, because supplier s' cannot deliver under scenarios $\zeta_{s'}^0, \ldots, \zeta_{s'}^{k'}$. We show that $K \geq k'$ as follows.

Suppose, on the contrary, that $K+1 \le k'$. Then the output level q_k $(\forall k > k')$ does not depend on the order quantity $x_{s'}$, because $\delta_{K+1} = 0 \Rightarrow q_{K+1} = a/2b \Rightarrow q_k = a/2b \; \forall k > k'$. That is, the order quantity $x_{s'}$ has no effect on the output level. It follows that $x_{s'} = 0$ would be the optimal solution to minimize the ordering cost. This contradicts our assumption that the buying firm orders from supplier s'. Therefore, we conclude that $K \ge k'$.

Next, we first prove the following claim and then apply it to finish proving Proposition EC.2.

CLAIM EC.1. *We have*

$$\sum_{k=0}^{K} \zeta_s^k f^{\mathrm{m}}(\zeta^k) \ge (1+\eta) \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k), \text{ and} \tag{EC.8}$$

$$\sum_{i=1}^{n} \left(\sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \zeta_s^k f^{\mathrm{m}}(\zeta^k) \right) x_i > (1+\eta) \sum_{i=1}^{n} \left(\sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) \right) x_i. \tag{EC.9}$$

Proof of Claim EC.1 Note that supplier s stochastically dominates s' with $F_s(t) \le F_{s'}(t), \forall t \in [0,1]$. Hence, $\zeta_s^k \ge \zeta_{s'}^k$, $\forall k \in \{0,1,\ldots,L\}$. It follows that

$$\begin{aligned}
\sum_{k=0}^{K} \zeta_s^k f^{\mathrm{m}}(\zeta^k) &= \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) + \sum_{k=0}^{K} (\zeta_s^k - \zeta_{s'}^k) f^{\mathrm{m}}(\zeta^k) \ge \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) + \sum_{k=0}^{k'} (\zeta_s^k - \zeta_{s'}^k) f^{\mathrm{m}}(\zeta^k) \\
&= \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) + \sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k) \\
&= \left(1 + \frac{\sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k)}{\sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)} \right) \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) = (1+\eta) \sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k),
\end{aligned}$$

where the inequality holds because $K \ge k'$ and $\zeta_s^k \ge \zeta_{s'}^k$, $\forall k \in \{0,1,\ldots,L\}$. Therefore, (EC.8) holds. Before showing (EC.9), we show that for any $i \in N$, we have

$$\begin{aligned}
\frac{1}{\eta} \sum_{k=0}^{k'} (\zeta_i^K - \zeta_i^k) \zeta_s^k f^{\mathrm{m}}(\zeta^k) &= \left(\frac{\sum_{k=0}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)}{\sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k)} \right) \sum_{k=0}^{k'} (\zeta_i^K - \zeta_i^k) \zeta_s^k f^{\mathrm{m}}(\zeta^k) \\
&> \left(\frac{\sum_{k=k'}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)}{\sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k)} \right) \sum_{k=0}^{k'} (\zeta_i^K - \zeta_i^{k'}) \zeta_s^k f^{\mathrm{m}}(\zeta^k) \qquad \text{(EC.10)} \\
&= \left(\frac{\sum_{k=k'}^{K} \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)}{\sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k)} \right) (\zeta_i^K - \zeta_i^{k'}) \sum_{k=0}^{k'} \zeta_s^k f^{\mathrm{m}}(\zeta^k) \\
&= \sum_{k=k'}^{K} (\zeta_i^K - \zeta_i^{k'}) \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) \ge \sum_{k=k'}^{K} (\zeta_i^K - \zeta_i^k) \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k), \qquad \text{(EC.11)}
\end{aligned}$$

where the strictly inequality (EC.10) holds because $\zeta_i^{k'} > \cdots > \zeta_i^0$. Thus, we have

$$\begin{aligned}
\sum_{i=1}^{n} \left(\sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \zeta_s^k f^{\mathrm{m}}(\zeta^k) \right) x_i &= \sum_{i=1}^{n} \left(\sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) + \sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \left(\zeta_s^k - \zeta_{s'}^k\right) f^{\mathrm{m}}(\zeta^k) \right) x_i \\
&\ge \sum_{i=1}^{n} \left(\sum_{k=0}^{K} \left(\zeta_i^K - \zeta_i^k\right) \zeta_{s'}^k f^{\mathrm{m}}(\zeta^k) + \sum_{k=0}^{k'} \left(\zeta_i^K - \zeta_i^k\right) \left(\zeta_s^k - \zeta_{s'}^k\right) f^{\mathrm{m}}(\zeta^k) \right) x_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)+\sum_{k=0}^{k'}\left(\zeta_i^K-\zeta_i^k\right)\zeta_s^k f^{\mathrm{m}}(\zeta^k)\right)x_i\\
&> \sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)+\eta\sum_{k=k'}^{K}(\zeta_i^K-\zeta_i^k)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)\right)x_i\\
&= \sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)+\eta\sum_{k=0}^{K}(\zeta_i^K-\zeta_i^k)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)\right)x_i\\
&= (1+\eta)\sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)\right)x_i,
\end{aligned}
$$

where the first inequality holds because $K\geq k'$ and $(\zeta_i^K-\zeta_i^k)(\zeta_s^k-\zeta_{s'}^k)f^{\mathrm{m}}(\zeta^k)\geq 0$, the second equality holds because $\zeta_{s'}^k=0$ for $k=0,\ldots,k'$, and the second inequality (i.e., the strict inequality) is due to (EC.11). Therefore, (EC.9) holds. $\square$

Using (EC.7a) for supplier s, along with (EC.7b), we get

$$
\sum_{i=1}^{n}\left(\sum_{k=0}^{K}\zeta_i^k\zeta_s^k f^{\mathrm{m}}(\zeta^k)\right)x_i=\frac{1}{2b}\left(a\sum_{k=0}^{K}\zeta_s^k f^{\mathrm{m}}(\zeta^k)-c_s+\lambda_s\right). \tag{EC.12}
$$

Similarly, using (EC.7a) for supplier s', along with (EC.7b), we get

$$
\sum_{i=1}^{n}\left(\sum_{k=0}^{K}\zeta_i^k\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)\right)x_i=\frac{1}{2b}\left(a\sum_{k=0}^{K}\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)-c_{s'}+\lambda_{s'}\right). \tag{EC.13}
$$

The inequality $\delta_K>0$ implies that there exists $\epsilon>0$ such that $\sum_{i=1}^{n}\zeta_i^K x_i+\epsilon=a/2b$. Then, (EC.12) and (EC.13) can be rewritten as:

$$
\sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_s^k f^{\mathrm{m}}(\zeta^k)\right)x_i+\epsilon\sum_{k=0}^{K}\zeta_s^k f^{\mathrm{m}}(\zeta^k)=\frac{1}{2b}(c_s-\lambda_s),\ \text{and} \tag{EC.14}
$$

$$
\sum_{i=1}^{n}\left(\sum_{k=0}^{K}\left(\zeta_i^K-\zeta_i^k\right)\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)\right)x_i+\epsilon\sum_{k=0}^{K}\zeta_{s'}^k f^{\mathrm{m}}(\zeta^k)=\frac{1}{2b}(c_{s'}-\lambda_{s'}). \tag{EC.15}
$$

Because of Claim EC.1, we have $(c_s-\lambda_s)>(1+\eta)(c_{s'}-\lambda_{s'})$. When $c_s\leq(1+\eta)c_{s'}$, $\lambda_{s'}>\lambda_s/(1+\eta)\geq 0$ implies $x_{s'}=0$. It contradicts the assumption that the firm orders from supplier s'. Hence, we must have $c_s>(1+\eta)c_{s'}$. The proposition is proved by showing a contradiction. $\square$

LEMMA EC.3. *Suppose $x_s>0$ in the optimal solution $\mathbf{x}$ for a supplier $s\in N$. Let $\hat{k}$ be an index in $\{1,\ldots,L\}$ such that $D_s\zeta_s^{\hat{k}}\geq x_s>D_s\zeta_s^{\hat{k}-1}$. We have $a-2bq_k\geq c_s$, $\forall k\in\{0,1,\ldots,\hat{k}-1\}$.*

Proof Note that $\hat{k}$ exists because $\zeta_s^0=0$ and $\zeta_s^L=1$. Because $q_k=\min(a/2b,\sum_{i=1}^{n}\min(D_i\zeta_i^k,x_i))$ and $q_1\leq\cdots\leq q_L$ by Lemma EC.6, we have $a-2bq_1\geq\cdots\geq a-2bq_L$. Thus, the lemma is proved if we can show $a-2bq_k\geq c_s$ for some $k\geq\hat{k}$. We will prove the lemma by introducing a contradiction. Suppose $a-2bq_k<$

$c_s, \forall k \geq \hat{k}$. We construct (i) $\mathbf{x}' = (x'_i, \forall i \in N)$ such that $x'_s = x_s - \epsilon$ for some arbitrarily small scalar $\epsilon > 0$ and $x'_i = x_i$ for any $i \in N \setminus \{s\}$, where $\epsilon \leq \min\{\sum_{i=0}^n \min(D_i\zeta_i^k, x_i) - a/2b : \forall k \geq \hat{k}, \sum_{i=0}^n \min(D_i\zeta_i^k, x_i) > a/2b\}$ and $\epsilon \leq q_k$ for any $k \geq \hat{k}$, and (ii) $\mathbf{q}' = (q'_k, \forall k \in \{0, 1, \ldots, L\})$ such that $q'_k = \min(a/2b, \sum_{i=1}^n \min(D_i\zeta_i^k, x'_i))$, $\forall k \in \{0, 1, \ldots, L\}$. It is easy to check that the constructed solution $(\mathbf{x}', \mathbf{q}')$ is feasible to Problem (EC.5). Then, for all $k < \hat{k}$, $\min(D_s\zeta_s^k, x'_s) = \min(D_s\zeta_s^k, x_s - \epsilon) = D_s\zeta_s^k = \min(D_s\zeta_s^k, x_s)$, hence $q'_k = q_k$. And, for all $k \geq \hat{k}$, $\min(D_s\zeta_s^k, x'_s) = \min(D_s\zeta_s^k, x_s - \epsilon) = x_s - \epsilon = \min(D_s\zeta_s^k, x_s) - \epsilon$. The profit difference with respect to the two solutions (i.e., $(\mathbf{x}, \mathbf{q})$ and $(\mathbf{x}', \mathbf{q}')$) can be expressed as follows:

$$\begin{aligned}
&\sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)\left[(a - bq_k)q_k - \sum_{i=1}^n c_i \min(D_i\zeta_i^k, x_i)\right] - \sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)\left[(a - bq'_k)q'_k - \sum_{i=1}^n c_i \min(D_i\zeta_i^k, x'_i)\right] \\
&= \sum_{k \geq \hat{k}} f^{\mathrm{m}}(\zeta^k)\left(\left[(a - bq_k)q_k - c_s \min(D_s\zeta_s^k, x_s)\right] - \left[(a - bq'_k)q'_k - c_s \min(D_s\zeta_s^k, x'_s)\right]\right) \\
&= \sum_{k \geq \hat{k}} f^{\mathrm{m}}(\zeta^k)\left((a - bq_k)q_k - (a - bq'_k)q'_k - \epsilon c_s\right). \qquad \text{(EC.16)}
\end{aligned}$$

For any $k \geq \hat{k}$, if $\sum_{i=0}^n \min(D_i\zeta_i^k, x_i) > a/2b$, then we have $q'_k = q_k$ because $a/2b \leq \sum_{i=0}^n \min(D_i\zeta_i^k, x'_i) = \sum_{i=0}^n \min(D_i\zeta_i^k, x_i) - \epsilon$; otherwise, $\sum_{i=0}^n \min(D_i\zeta_i^k, x_i) \leq a/2b$, we have $q'_k = q_k - \epsilon$. When $q'_k = q_k$ for some $k \geq \hat{k}$, the kth term in (EC.16) follows $f^{\mathrm{m}}(\zeta^k)\left((a - bq_k)q_k - (a - bq'_k)q'_k - \epsilon c_s\right) = -\epsilon c_s < 0$; when $q'_k = q_k - \epsilon$ for some $k \geq \hat{k}$, the kth term in (EC.16) follows

$$\begin{aligned}
&(a - bq_k)q_k - (a - bq'_k)q'_k - \epsilon c_s = (a - bq_k)q_k - (a - bq_k + \epsilon b)(q_k - \epsilon) - \epsilon c_s \\
&= \epsilon(a - 2bq_k) + \epsilon^2 b - \epsilon c_s = \epsilon(a - bq_k - c_s) - \epsilon b(q_k - \epsilon) < 0.
\end{aligned}$$

The last inequality holds because of our assumption that $a - bq_k < c_s, \forall k \geq \hat{k}$, and that ϵ is a very small scalar. Moreover, the inequality implies that (EC.16) is strictly less than 0 and that $\mathbf{x}'$ gives a higher profit than $\mathbf{x}$. This implication contradicts our assumption that $\mathbf{x}$ is optimal. Thus, the lemma is proved. □

Proof of Proposition EC.3 We prove this proposition by introducing a contradiction. Consider suppliers $\underline{s}$ and s defined in the proposition. Suppose $x_{\underline{s}} = 0$ and $x_s > D_s\zeta_s^{k'}$ with $k' = \arg\max\{k : \zeta_{\underline{s}}^k = 0\}$. Then, there exists a $\hat{k} > k'$ such that $D_s\zeta_s^{\hat{k}} \geq x_s > D_s\zeta_s^{\hat{k}-1}$. We construct another solution $\mathbf{x}' = (x'_i, \forall i \in N)$ and $\mathbf{q}' = (q'_k, \forall k \in \{0, 1, \ldots, L\})$ as follows. First, we let $x'_s = x_s - \epsilon$ and $x'_{\underline{s}} = x_{\underline{s}} + \epsilon = \epsilon$ for some arbitrarily small scalar $\epsilon > 0$ and $x'_i = x_i$ for any $i \in N \setminus \{s, \underline{s}\}$, where ϵ is chosen to satisfy $x'_s \geq D_s\zeta_s^{\hat{k}-1}$. Also, note that

$$\sum_{i=1}^n \min(D_i\zeta_i^k, x'_i) = \begin{cases} \displaystyle\sum_{i=1}^n \min(D_i\zeta_i^k, x_i) + \epsilon, & \text{when } k = k' + 1, \ldots, \hat{k} - 1; \\ \displaystyle\sum_{i=1}^n \min(D_i\zeta_i^k, x_i), & \text{otherwise.} \end{cases}$$

Second, we let $q'_k = q_k + \epsilon \; \forall k \in \{k'+1, \ldots, \hat{k}-1\}$ and $q'_k = q_k$ otherwise. It is easy to check that the constructed solution $(\mathbf{x}', \mathbf{q}')$ is feasible to Problem (EC.5). The profit difference with respect to the two solutions (i.e., $(\mathbf{x}, \mathbf{q})$ and $(\mathbf{x}', \mathbf{q}')$) can be expressed as follows,

$$
\begin{aligned}
&\sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)\left[(a-bq_k)q_k - \sum_{i=1}^{n} c_i \min(D_i\zeta_i^k, x_i)\right] - \sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)\left[(a-bq'_k)q'_k - \sum_{i=1}^{n} c_i \min(D_i\zeta_i^k, x'_i)\right] \\
=& \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k)\left[(a-bq_k)q_k - (a-bq'_k)q'_k\right] \\
&\quad - \sum_{k=0}^{L} f^{\mathrm{m}}(\zeta^k)\left[c_s \min(D_s\zeta_s^k, x_s) - c_{\underline{s}} \min(D_{\underline{s}}\zeta_{\underline{s}}^k, x'_{\underline{s}}) - c_s \min(D_s\zeta_s^k, x'_s)\right] \\
=& \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k)\left[(a-bq_k)q_k - (a-bq_k-\epsilon b)(q_k+\epsilon)\right] + \epsilon c_{\underline{s}} \sum_{k=k'+1}^{L} f^{\mathrm{m}}(\zeta^k) - \epsilon c_s \sum_{k=\hat{k}}^{L} f^{\mathrm{m}}(\zeta^k) \\
=& \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k)\left[-\epsilon(a-2bq_k) + \epsilon^2 b\right] + \epsilon c_{\underline{s}} \sum_{k=k'+1}^{L} f^{\mathrm{m}}(\zeta^k) - \epsilon c_s \sum_{k=\hat{k}}^{L} f^{\mathrm{m}}(\zeta^k) \\
\leq& -\epsilon c_s \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k) + \epsilon^2 b \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k) + \epsilon c_{\underline{s}} \sum_{k=k'+1}^{L} f^{\mathrm{m}}(\zeta^k) - \epsilon c_s \sum_{k=\hat{k}}^{L} f^{\mathrm{m}}(\zeta^k) \qquad \text{(because of Lemma EC.3)} \\
=& \, \epsilon(c_{\underline{s}} - c_s) \sum_{k=\hat{k}+1}^{L} f^{\mathrm{m}}(\zeta^k) + \epsilon^2 b \sum_{k=k'+1}^{\hat{k}-1} f^{\mathrm{m}}(\zeta^k) < 0.
\end{aligned}
$$

The last inequality holds because $c_{\underline{s}} - c_s < 0$ and ϵ is very small. Thus, the solution $\mathbf{x}'$ gives a higher profit than $\mathbf{x}$, which contradicts the optimality of $\mathbf{x}$. □

Proof of Corollary EC.2 Following the proof of Corollary EC.1, we have $s_0 = \underline{s}$ and hence $\zeta_{\underline{s}}^1 > 0$. Since $\zeta^{i+1} \geq \zeta^i \; \forall i \in \mathcal{L} \setminus \{L\}$, $0 = \arg\max\{k' : \zeta_{\underline{s}}^{k'} = 0\}$. Since $\zeta_s^0 = 0$ for all $s \in N$, Proposition EC.3 indicates that if supplier $\underline{s}$ receives no order, then no supplier receives order and the profit is 0. However, the firm can have nonzero profit by ordering a small amount from supplier $\underline{s}$ becasue $a > c_{\underline{s}}$. Hence, the corollary is proved by showing the contradition and the firm always orders from the supplier $\underline{s}$. □

Appendix D: Technical Proofs

This appendix contains all technical proofs for the results in the main body of the paper.

Definition EC.1. A function $f:[0,1]^n \to \Re$ is submodular if for all $\xi^1 \le \xi^2 \in [0,1]^n$, $f(\xi^1)+f(\xi^2) \ge f(\xi^1 \vee \xi^2) + f(\xi^1 \wedge \xi^2)$, where $\wedge$ and $\vee$ are the coordinate-wise minimum and maximum operations, respectively.

The inner optimization problem of the DR model is a conic linear problem:

$$
\begin{aligned}
\min_{F\in\mathscr{F}} \mathbb{E}_F[V(\mathbf{x},\xi)] = \min_F \ & \int_{[0,1]^n} V(\mathbf{x},\xi)dF(\xi) \\
\text{s.t.} \ & \int_{[0,1]^n} \mathrm{I}(\xi_i \le \omega_i)dF(\xi) = F_i(\omega_i),\ \forall \omega_i \in [0,1],\ i \in N \\
& \int_{[0,1]^n} (\xi-\mu)(\xi-\mu)^\intercal dF(\xi) \preceq \Sigma \\
& \int_{[0,1]^n} dF(\xi) = 1,\ F(\xi) \ge 0,\ \forall \xi \in [0,1]^n.
\end{aligned}
$$

It follows that the corresponding Lagrangian function can be described as (see also (5.10) in Shapiro 2001):

$$
\begin{aligned}
\mathcal{L}(F,\lambda,\gamma,Q) = & \int_{[0,1]^n} V(\mathbf{x},\xi)dF(\xi) + \lambda\left(1 - \int_{[0,1]^n} dF(\xi)\right) + Q \bullet \left(\int_{[0,1]^n} (\xi-\mu)(\xi-\mu)^\intercal dF(\xi) - \Sigma\right) \\
& + \sum_{i=1}^n \int_0^1 \left(\int_{[0,1]^n} I(\xi_i \le \omega_i)dF(\xi) - F_i(\omega_i)\right) d\gamma_i(\omega_i) \\
= & \ \lambda - \sum_{i=1}^n \int_0^1 F_i(\omega_i)d\gamma_i(\omega_i) + Q\bullet(\mu\mu^\intercal - \Sigma) \\
& + \int_{[0,1]^n} \left(V(\mathbf{x},\xi) - \lambda + \sum_{i=1}^n \int_0^1 \mathrm{I}(\xi_i \le \omega_i)d\gamma_i(\omega_i) + \xi^\intercal Q \xi - 2\xi^\intercal Q\mu\right) dF(\xi) \\
= & \ \lambda - \sum_{i=1}^n \int_0^1 F_i(\omega_i)d\gamma_i(\omega_i) + Q\bullet(\mu\mu^\intercal - \Sigma) \\
& + \int_{[0,1]^n} \left(V(\mathbf{x},\xi) - \lambda + \sum_{i=1}^n \gamma_i(\xi_i) + \xi^\intercal Q \xi - 2\xi^\intercal Q\mu\right) dF(\xi),
\end{aligned} \tag{EC.17}
$$

where $\bullet$ refers to the Frobenius inner product and $\gamma_s \in C[0,1]^*$ is the set of right continuous functions with bounded variation on $[0,1]$ normalized by $\gamma_s(1)=0$, and the corresponding integrals are Lebesgue-Stieltjes integrals. By making use of the duality theorem (see Section 5 in Shapiro 2001, Delage & Ye 2010, and Chen et al. 2022), we have the dual formulation for the inner optimization as follows.

$$
\begin{aligned}
\max \ & \lambda + \sum_{s=1}^n \int_0^1 \gamma_s(\omega_s)dF_s(\omega_s) + Q\bullet(\mu\mu^\intercal - \Sigma) \\
\text{s.t.} \ & V(\mathbf{x},\xi) - \lambda - \sum_{s=1}^n \gamma_s(\xi_s) + \xi^\intercal Q\xi - 2\xi^\intercal Q\mu \ge 0,\ \forall \xi \in [0,1]^n \\
& Q \succeq 0,\ \gamma_s \in C[0,1]^*,\ \forall s \in N.
\end{aligned} \tag{EC.18}
$$

The strong duality holds because $\mathscr{F}$ is assumed to having a strictly feasible solution and the Slater condition holds (see Proposition 5.2 in Shapiro 2001). Therefore, the DR model is equivalent to

$$\begin{aligned}\max\ & \lambda + \sum_{s=1}^{n} \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s) + Q \bullet (\mu\mu^\intercal - \Sigma) \\ \text{s.t. } & V(\mathbf{x}, \xi) - \lambda - \sum_{s=1}^{n} \gamma_s(\xi_s) + \xi^\intercal Q \xi - 2\xi^\intercal Q \mu \geq 0,\ \forall \xi \in [0,1]^n \\ & \mathbf{x} \geq 0,\ Q \succeq 0,\ \gamma_s \in C[0,1]^*,\ \forall s \in N.\end{aligned} \tag{DR-D}$$

Proof of Proposition 1 Because F_s are discrete with finite supports $\Omega_s = \{\xi_s^1, \ldots, \xi_s^M\}$, $\forall s \in N$, the dual variable $\gamma_s(\xi_s)$ defined in infinite dimension can be redefined as a set of finite dual variables $\gamma_{s\xi_s^1}, \ldots, \gamma_{s\xi_s^M}$. Hence, the DR-D model becomes a convex program as follows:

$$\begin{aligned}\max\ & \lambda + \sum_{s=1}^{n} \sum_{\xi_s \in \Omega_s} \gamma_{s\xi_s} f_s(\xi_s) + Q \bullet (\mu\mu^\intercal - \Sigma) \\ \text{s.t. } & \mathbf{R}\left(\sum_{s\in N} t_{s\xi_s}\right) - \sum_{s=1}^{n} c_s t_{s\xi_s} - \lambda - \sum_{s=1}^{n} \gamma_{s\xi_s} + \xi^\intercal Q \xi - 2\xi^\intercal Q \mu \geq 0,\ \forall \xi \in \Omega \\ & t_{s\xi_s} = x_s \xi_s,\ \forall s \in N_Y,\ \xi_s \in \Omega_s \\ & t_{s\xi_s} \leq x_s, t_{s\xi_s} \leq D_s \xi_s,\ \forall s \in N_C,\ \xi_s \in \Omega_s \\ & \mathbf{x} \geq 0,\ Q \succeq 0,\ t_{s\xi_s} \geq 0,\ \lambda \text{ and } \gamma_{s\xi_s} \text{ are free},\ \forall s \in N,\end{aligned} \tag{EC.19}$$

where $\bullet$ refers to the Frobenius inner product, suppliers in set N_Y are subject to yield uncertainty, and suppliers in set N_C are subject to capacity uncertainty. □

Proof of Proposition 2 Consider the simple quadratic revenue function and yield uncertainties. We have $R(\mathbf{x}, \xi) = \{(a - bq)q | q = \sum_{s\in N} d_s(x_s, \xi_s)\} = (a - b\sum_{s\in N} \xi_s x_s) \sum_{s\in N} \xi_s x_s$. Hence,

$$\begin{aligned}\min_{F\in\mathscr{F}} \mathbb{E}_F[V(\mathbf{x}, \xi)] &= \min_{F\in\mathscr{F}} \mathbb{E}_F \left[(a - b\sum_{s\in N} \xi_s x_s) \sum_{s\in N} \xi_s x_s - \sum_{s\in N} c_s \xi_s x_s\right] \\ &= \sum_{s\in N} (a - c_s)\mu_s x_s - \min_{F\in\mathscr{F}} b\mathbf{x}^\intercal \mathbb{E}_F\left(\xi\xi^\intercal\right) \mathbf{x} \\ &= \sum_{s\in N} (a - c_s)\mu_s x_s - b\mathbf{x}^\intercal \left(\Sigma + \mu\mu^\intercal\right) \mathbf{x}\end{aligned}$$

The DR model by setting Σ as its empirical estimation is equivalent to the SAA approach. □

Proof of Proposition 3 First, we prove part (1). Note that the constraint (1b) is redundant if $\Sigma_{ij} = \sigma_i \sigma_j$. So, we can set $Q = \mathbf{0}$. The dual of the inner problem of model (DR) with given $\mathbf{x}$ becomes

$$\begin{aligned} \max\ & \lambda + \sum_{s=1}^{n} \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s) \\ \text{s.t. } & V(\mathbf{x}, \xi) - \lambda - \sum_{s=1}^{n} \gamma_s(\xi_s) \geq 0,\ \forall \xi \in [0,1]^n \\ & \gamma_s \in C[0,1]^*,\ \forall s \in N. \end{aligned}$$

We partition the interval $[0,1]$ into n mutually disjoint Borel sets $\mathscr{S}_s$, $\forall s \in N$, such that

$$\mathscr{S}_s = \{t \in [0,1] : F_j(\zeta_j(t)) > F_s(\zeta_s(t))\ \forall j < s \text{ and } F_j(\zeta_j(t)) \geq F_s(\zeta_s(t))\ \forall j > s\}.$$

By the definition of F^{m}, it is clear that $F^{\mathrm{m}}(\zeta(t)) = F_i(\zeta_i(t))$ if $t \in \mathscr{S}_i$ for some $i \in N$. Let $\lambda = V(\mathbf{x}, \zeta(1))$ and

$$\gamma_s(\omega_s) = -\int_{\mathscr{S}_s} \mathrm{I}(\zeta_s(t) \geq \omega_s) dV(\mathbf{x}, \zeta(t)),\ \forall s \in N.$$

The solution is feasible because

$$\begin{aligned} \lambda + \sum_{s=1}^{n} \gamma_s(\xi_s) &= V(\mathbf{x}, \zeta(1)) - \sum_{s=1}^{n} \int_{\mathscr{S}_s} \mathrm{I}(\zeta_s(t) \geq \xi_s) dV(\mathbf{x}, \zeta(t)) \\ &= V(\mathbf{x}, \zeta(1)) - \sum_{s=1}^{n} \int_{\mathscr{S}_s} dV(\mathbf{x}, \zeta(t) \vee \xi_s e_s) \\ &\leq V(\mathbf{x}, \zeta(1)) - \sum_{s=1}^{n} \int_{\mathscr{S}_s} dV(\mathbf{x}, \zeta(t) \vee \xi) \\ &= V(\mathbf{x}, \zeta(1)) - \int_0^1 dV(\mathbf{x}, \zeta(t) \vee \xi) \\ &= V(\mathbf{x}, \zeta(1)) - V(\mathbf{x}, \zeta(1)) + V(\mathbf{x}, \xi) = V(\mathbf{x}, \xi). \end{aligned}$$

The inequality holds because $V(\mathbf{x}, \xi)$ is submodular. The solution is also optimal because the duality gap is zero, as demonstrated below:

$$\begin{aligned} \int_{[0,1]^n} V(\mathbf{x}, \xi) dF(\xi) &= \int_0^1 V(\mathbf{x}, \zeta(t)) dF^{\mathrm{m}}(\zeta(t)) \\ &= V(\mathbf{x}, \zeta(1)) - \int_0^1 F^{\mathrm{m}}(\zeta(t)) dV(\mathbf{x}, \zeta(t)) \\ &= V(\mathbf{x}, \zeta(1)) - \sum_{s=1}^{n} \int_{\mathscr{S}_s} F^{\mathrm{m}}(\zeta(t)) dV(\mathbf{x}, \zeta(t)) \\ &= V(\mathbf{x}, \zeta(1)) - \sum_{s=1}^{n} \int_{\mathscr{S}_s} F_s(\zeta_s(t)) dV(\mathbf{x}, \zeta(F_s(\zeta_s(t)))) \\ &= \lambda - \sum_{s=1}^{n} \int_{\mathscr{S}_s} \int_0^1 \mathrm{I}(\zeta_s(t) \geq \omega_s) dF_s(\omega_s) dV(\mathbf{x}, \zeta(t)) \end{aligned}$$

$$
\begin{aligned}
&= \lambda - \sum_{s=1}^{n} \int_0^1 \int_{\mathscr{S}_s} \mathrm{I}(\zeta_s(t) \geq \omega_s) dV(\mathbf{x}, \zeta(t)) dF_s(\omega_s) \\
&= \lambda + \sum_{s=1}^{n} \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s).
\end{aligned}
$$

Because, given the primal solution F^{m}, we had found a feasible dual solution with zero duality gap. Hence, F^{m} is the primal optimal, i.e., $F^* = F^{\mathrm{m}}$.

Next, we prove Part (2). Let $(\mathbf{x}^*, \lambda^*, \gamma_s^*, Q^*)$ be the optimal solution of the DR-D model when $\Sigma = \Sigma''$. The solution is certainly feasible for the DR-D model with any other choices of Σ. We have

$$
\begin{aligned}
&\mathscr{V}(\Sigma') - \mathscr{V}(\Sigma'') \\
\geq &\left(\lambda^* + \sum_{s=1}^{n} \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s) + Q^* \bullet (\mu\mu^{\intercal} - \Sigma')\right) - \left(\lambda^* + \sum_{s=1}^{n} \int_0^1 \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s) + Q^* \bullet (\mu\mu^{\intercal} - \Sigma'')\right) \\
= &Q^* \bullet (\Sigma'' - \Sigma') \geq 0.
\end{aligned}
$$

The last term is positive because $Q^* \succeq 0$ and $\Sigma'' \succeq \Sigma'$. Since the ordering decision $\mathbf{x}$ is not involved, the monotonicity of the profit also holds when $\mathbf{x}$ is given and fixed.

Finally, we prove Part (3). The covariance between suppliers i and j is $\mathrm{Cov}(\xi_i, \xi_j) = \mathbb{E}[\xi_i \xi_j]$ where $-\xi_i \xi_j$ is submodular. So, the previous proof shows that $\mathrm{Cov}_{F^{\mathrm{m}}}(\xi_i, \xi_j)$ gives the maximal covariance with supply distribution F^{m}, and hence $\mathrm{Corr}_{F^{\mathrm{m}}}(\xi_i, \xi_j)$ is the maximal correlation. Therefore, Part (3) holds because of the monotonicity. □

Proof of Corollary 1 When the firm sources from a single supplier s, the profit is

$$
\mathscr{V}_s = \max_{\mathbf{x} \geq 0} \int V(x_s, \mathbf{0}, \xi_s, \mathbf{0}) dF_s(\xi_s).
$$

For any Σ, we have

$$
\max_{\mathbf{x} \geq 0} \int V(x_s, \mathbf{0}, \xi_s, \mathbf{0}) dF_s(\xi_s) = \max_{\mathbf{x} \geq 0} \int V(x_s, \mathbf{0}, \xi) dF^{\mathrm{m}}(\xi) \leq \max_{\mathbf{x} \geq 0} \int V(\mathbf{x}, \xi) dF^{\mathrm{m}}(\xi) \leq \mathscr{V}(\Sigma).
$$

Therefore, the DR model always gives a higher profit than sole sourcing does. □

Proof of Corollary 2 We have $\overline{\mathscr{V}} \leq n\big(1 - (1 - 1/n)^n\big)\underline{\mathscr{V}}$ by Theorem 1 part (2) with $r = 1$. Thus,

$$
\mathrm{regret_c} \equiv \mathbb{E}_{F^{\mathrm{i}}}\left[V(\mathbf{x}^{\mathrm{i}}, \xi)\right] - \mathbb{E}_{F^{\mathrm{i}}}\left[V(\mathbf{x}^{\mathrm{m}}, \xi)\right] \leq \overline{\mathscr{V}} - \underline{\mathscr{V}} \leq \left(n - 1 - \frac{n}{e}\right) \underline{\mathscr{V}}.
$$

Next, we demonstrate that the potential regret due to optimism can be exponentially large. The exponential gap is also demonstrated in Lemma 4 in Agrawal et al. (2012) with submodular cost function. Consider $\xi \in \{0,1\}^n$ with $f_s(1) = 1/2$ and $c_s = 0\ \forall s \in N$. Let the profit function be

$$V(\mathbf{x},\xi) = 4\sum_{s\in N} x_s\xi_s - 2^n \max\left(0, \sum_{s\in N} x_s\xi_s - (n-1)\right)$$

with $\mathbf{x} \in [0,1]^n$, where the firm has a fixed selling price of 4 and a salvage cost of 2^n per unit if the total delivery is larger than the demand $n-1$. It is easy to check that the profit function is submodular. The joint distribution $f^{\mathrm{m}}(\xi)$ is defined with non-zero values on only two vectors $\mathbf{0}$ and $\mathbf{1}$, where $f^{\mathrm{m}}(\mathbf{1}) = 1/2$ and $f^{\mathrm{m}}(\mathbf{0}) = 1/2$. For any joint distribution $F \in \mathscr{F}$, we have

$$\mathbb{E}_F\left[V(\mathbf{x},\xi)\right] = 2\sum_{s\in N} x_s - 2^n \mathbb{E}_F\left[\max\left(0, \sum_{s\in N} x_s\xi_s - (n-1)\right)\right].$$

Under the independent distribution F^{i}, the salvage cost occurs with probability $1/2^n$. So, $\mathbf{x}^{\mathrm{i}} = (1,\dots,1)$ and $\mathbb{E}_{F^{\mathrm{i}}}\left[V(\mathbf{x}^{\mathrm{i}},\xi)\right] = 2n-1$. However, if the underlying joint distribution is F^{m}, then employing strategy $\mathbf{x}^{\mathrm{i}}$ may result in a large loss of $\mathbb{E}_{F^{\mathrm{m}}}\left[V(\mathbf{x}^{\mathrm{i}},\xi)\right] = 2n - 2^{n-1}$. On the other hand, when supply uncertainty follows the maximally dependent distribution F^{m}, we have $\mathbf{x}^{\mathrm{m}} = (1,\dots,1,0)$ in order to avoid the high salvage cost and $\mathbb{E}_{F^{\mathrm{m}}}\left[V(\mathbf{x}^{\mathrm{m}},\xi)\right] = 2n-2 = \mathbb{E}_{F^{\mathrm{i}}}\left[V(\mathbf{x}^{\mathrm{m}},\xi)\right]$. Therefore, the potential regret due to conservatism is only 1, but the potential regret due to optimism is $2^{n-1} - 1$. □

Proof of Theorem 1 If Theorem 1 holds when $r = 2$, then the procedure can be applied repetitively in order to prove the case with general r. So, we only prove the theorem for the case with $r = 2$. Since ξ_{s_1} and ξ_{s_2} are independent for any $s_1 \in N_1$ and $s_2 \in N_2$, the joint distribution can be denoted as $G_1(\xi_{N_1}) \times G_2(\xi_{N_2})$ for some joint supply distributions G_1 and G_2 where $\xi_{N_i} = (\xi_s : s \in N_i)$ for $i = 1,2$. For any $i \neq j \in \{1,2\}$, given G_j, the inner problem of the DR model is

$$\min_{G_i\in\mathscr{F}_i} \int_{[0,1]^{|N_i|}} \left(\int_{[0,1]^{|N_j|}} V(\mathbf{x},\xi_{N_i},\xi_{N_j}) dG_j(\xi_{N_j})\right) dG_i(\xi_{N_i}) \equiv \min_{G_i\in\mathscr{F}_i} \int_{[0,1]^{|N_i|}} V'(\mathbf{x},\xi_{N_i}) dG_i(\xi_{N_i}),$$

where $\mathscr{F}_i$ is the distributional set $\mathscr{F}$ projected to random parameter ξ_{N_i}. Since $V'(\mathbf{x},\xi_{N_i})$ is submodular following the submodularity of $V(\mathbf{x},\xi)$, we have

$$G_i(\xi_{N_i}) = \min_{s\in N_i} F_s(\xi_s) = F^{\mathrm{m}}(\xi_{N_i}),\ \forall i = 1,2,$$

because of Proposition 3 part (1).

Suppose $r=1$. We first show that, if $\xi \in \{0,1\}^n$, then Theorem 1 part (2) holds for some simplified instances which are called "nice" instances in Agrawal et al. (2012). The definition of "nice" instances in Agrawal et al. (2012) is: (a) $f_s(1)=1/K$, $\forall s \in N$, for some finite integer $K>0$, and (b) the worst-case distribution f^{m} is a "K-partition-type" distribution. That is, f^{m} has support on K disjoint sets $\{A_1,\ldots,A_K\}$ that form a partition of N and each A_k occurs with probability $1/K$. So, the expected revenue on the F^{m} is given by $(1/K)\sum_{i=1}^K R(\mathbf{x}^{\mathrm{i}},\mathbf{1}_{A_i},\mathbf{0})$ where $\xi=(\mathbf{1}_{A_i},\mathbf{0})$ with $\xi_{A_i}=\mathbf{1}$ and $\xi_{N\setminus A_i}=\mathbf{0}$. Because of submodularity and $R(\mathbf{x}^{\mathrm{i}},\mathbf{0})=0$, if $K=2$, then we have

$$R(\mathbf{x}^{\mathrm{i}},\mathbf{1})=R(\mathbf{x}^{\mathrm{i}},\mathbf{1})+R(\mathbf{x}^{\mathrm{i}},\mathbf{0})=R(\mathbf{x}^{\mathrm{i}},(\mathbf{1}_{A_1},\mathbf{0})\vee(\mathbf{1}_{A_2},\mathbf{0}))+R(\mathbf{x}^{\mathrm{i}},(\mathbf{1}_{A_1},\mathbf{0})\wedge(\mathbf{1}_{A_2},\mathbf{0}))\le\sum_{i=1}^{2}R(\mathbf{x}^{\mathrm{i}},\mathbf{1}_{A_i},\mathbf{0}).$$

Continuing the process by increasing the value of K, we get

$$R(\mathbf{x}^{\mathrm{i}},\mathbf{1})\le\sum_{i=1}^{K}R(\mathbf{x}^{\mathrm{i}},\mathbf{1}_{A_i},\mathbf{0})$$

for any integer $K\ge 2$. Thus, for "nice" instances, we have

$$\begin{aligned}\mathbb{E}_{F^{\mathrm{i}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]&\le\mathbb{E}_{F^{\mathrm{i}}}[R(\mathbf{x}^{\mathrm{i}},\xi)|\xi=\mathbf{0}]+\mathbb{E}_{F^{\mathrm{i}}}[R(\mathbf{x}^{\mathrm{i}},\xi)|\xi>\mathbf{0}]\\&\le 0+\left(1-\left(1-\frac{1}{K}\right)^{K}\right)R(\mathbf{x}^{\mathrm{i}},\mathbf{1})\\&\le K\left(1-\left(1-\frac{1}{K}\right)^{K}\right)\frac{1}{K}\sum_{i=1}^{K}R(\mathbf{x}^{\mathrm{i}},\mathbf{1}_{A_i},\mathbf{0})\\&\le n\left(1-\left(1-\frac{1}{n}\right)^{n}\right)\mathbb{E}_{F^{\mathrm{m}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]=\vartheta(n)\mathbb{E}_{F^{\mathrm{m}}}[R(\mathbf{x}^{\mathrm{i}},\xi)].\end{aligned}$$

Since $R(\mathbf{x},\xi)$ is monotone and submodular (inherited from the submodularity of $V(\mathbf{x},\xi)$), the splitting procedure in Agrawal et al. (2012) does not change $\mathbb{E}_{F^{\mathrm{m}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]$ and can only decrease $\mathbb{E}_{F^{\mathrm{i}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]$. Thus, following the same proofs in Section 2.3, Appendix C and D in Agrawal et al. (2012), the inequality derived in the following (EC.20) holds for general $\xi\in[0,1]^n$. That is,

$$\begin{aligned}\overline{\mathscr{V}}&=\mathbb{E}_{F^{\mathrm{i}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]-\sum_{s=1}^{n}c_sE_{F_s}[d_s(x_s,\xi_s)]\\&\le\vartheta(n)\mathbb{E}_{F^{\mathrm{m}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]-\sum_{s=1}^{n}c_sE_{F_s}[d_s(x_s,\xi_s)]\\&\le\vartheta(n)\left(\mathbb{E}_{F^{\mathrm{m}}}[R(\mathbf{x}^{\mathrm{i}},\xi)]-\sum_{s=1}^{n}c_sE_{F_s}[d_s(x_s,\xi_s)]\right)\le\vartheta(n)\underline{\mathscr{V}},\end{aligned}\tag{EC.20}$$

where the second inequality holds because $\vartheta(n)\ge 1$ for any integer $n\ge 1$.

Now, we consider $r > 1$. By adding reliable suppliers with ∞ unit cost, we can assume $|N_1| = \cdots = |N_r| \equiv \overline{n}$. Because of the minimization operators in the definition of F^{p}, these reliable suppliers will not change the value of F^{p}. Let $\mathcal{J} = \{J_1, \ldots, J_{\overline{n}}\}$ be a partition of N such that $|J_i \cap N_j| = 1$, $\forall i = 1, \ldots, \overline{n}, j = 1, \ldots, r$. Let

$$F'(\xi) = \min_{\{j_1, \ldots, j_{\overline{n}}\} \in \mathcal{J}} (F_{j_1}(\xi_{j_1}) \times \cdots \times F_{j_{\overline{n}}}(\xi_{j_{\overline{n}}})).$$

Following (EC.20), we get

$$\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{i}}, \xi)] = \mathbb{E}_{\prod_{(j_1, \ldots, j_{\overline{n}}) \in \mathcal{J}} (F_{j_1} \times \cdots \times F_{j_m})}[V(\mathbf{x}^{\mathrm{i}}, \xi)] \leq \vartheta(\overline{n}) \mathbb{E}_{F'}[V(\mathbf{x}^{\mathrm{i}}, \xi)].$$

Based on the definition of $\mathcal{J}$,

$$F'(\xi) = \min_{\{j_1, \ldots, j_{\overline{n}}\} \in \mathcal{J}} (F_{j_1}(\xi_{j_1}) \times \cdots \times F_{j_{\overline{n}}}(\xi_{j_{\overline{n}}})) \geq \prod_{i=1}^{r} \min_{s \in N_i} F_s(\xi_s) = F^{\mathrm{p}}(\xi).$$

Thus, F^{p} first-order stochastically dominates $F'(\xi)$, which indicates that suppliers under distribution $F'(\xi)$ are less reliable than under F^{p}, and

$$\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{i}}, \xi)] \leq \vartheta(\overline{n}) \mathbb{E}_{F'}[V(\mathbf{x}^{\mathrm{i}}, \xi)] \leq \vartheta(\overline{n}) \mathbb{E}_{F^{\mathrm{p}}}[V(\mathbf{x}^{\mathrm{i}}, \xi)] \leq \vartheta(\overline{n}) \max_{\mathbf{x} \geq 0} \mathbb{E}_{F^{\mathrm{p}}}[V(\mathbf{x}, \xi)].$$

Therefore, Theorem 1 part (2) is proved. □

Before continuing, we make a remark regarding the relationship between Theorem 1 part (2) and a seemingly related result given in Agrawal et al. (2012).

Remark EC.1. When $r = 1$, Theorem 1 part (2) gives

$$\overline{\mathscr{V}} \leq n \left(1 - \left(1 - \frac{1}{n}\right)^n\right) \underline{\mathscr{V}}, \tag{EC.21}$$

which is quite different from the upper bound of the price of correlation derived in Agrawal et al. (2012). This difference occurs because they consider a submodular cost function. In particular, applying Theorem 4 in Agrawal et al. (2012) gives

$$\frac{\sup_F \mathbb{E}_F[V(\mathbf{x}^{\mathrm{i}}, \xi)]}{\sup_F \mathbb{E}_F[V(\mathbf{x}^{\mathrm{m}}, \xi)]} \leq \frac{e}{e-1}.$$

In contrast, part (2) of our Proposition 3 yields

$$\frac{\sup_F \mathbb{E}_F[V(\mathbf{x}^{\mathrm{i}}, \xi)]}{\sup_F \mathbb{E}_F[V(\mathbf{x}^{\mathrm{m}}, \xi)]} = \frac{\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{i}}, \xi)]}{\mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x}^{\mathrm{m}}, \xi)]} \leq \frac{\overline{\mathscr{V}}}{\underline{\mathscr{V}}}.$$

It is easy to see that the upper bound derived in Agrawal et al. (2012) cannot describe the relationship between $\overline{\mathscr{V}}$ and $\underline{\mathscr{V}}$. Moreover, Example EC.2 indicates that the bound in (EC.21) is tight.

Example EC.2. Consider a setting with $\xi \in \{0,1\}^n$ and $f_s(1) = 1/n$, $\forall s \in N$. Define the corresponding profit function as

$$V(\mathbf{x},\xi) = \begin{cases} 0, & \text{if } \sum_{s\in N} x_s \xi_s = 0; \\ n, & \text{otherwise.} \end{cases}$$

It is easy to check that the profit function is submodular. The maximally dependent distribution f^{m} is defined on two vectors, $\mathbf{0} = (0,\ldots,0)$ and $\mathbf{1} = (1,\ldots,1)$, where $f^{\mathrm{m}}(\mathbf{1}) = 1/n$ and $f^{\mathrm{m}}(\mathbf{0}) = 1 - 1/n$. Obviously, the optimal order quantity $\mathbf{x}^* \neq \mathbf{0}$. Thus, we have

$$\underline{\mathscr{V}} = \max_{\mathbf{x}\geq 0} \mathbb{E}_{F^{\mathrm{m}}}[V(\mathbf{x},\xi)] = n f^{\mathrm{m}}(\mathbf{1}) = 1, \text{ and}$$
$$\overline{\mathscr{V}} = \max_{\mathbf{x}\geq 0} \mathbb{E}_{F^{\mathrm{i}}}[V(\mathbf{x},\xi)] = n\left(1 - F^{\mathrm{i}}(\mathbf{0})\right) = n\left(1 - \left(1 - \frac{1}{n}\right)^n\right)\underline{\mathscr{V}}.$$

Proof of Theorem 2 Given any order quantities $\mathbf{x}$, the inner optimization problem of the DR model is $\min_{F\in\mathcal{F}} \mathbb{E}_F[V(\mathbf{x},\xi)]$. Following the duality theorem in Shapiro (2001), we have derived its dual formulation in (EC.18) and rewritten it here:

$$\max\ \lambda + \sum_{s=1}^n \int_0^1 \gamma_s(\omega_s) dF_s(\omega_s) + Q \bullet (\mu\mu^\intercal - \Sigma) \tag{EC.22}$$

$$\text{s.t. } V(\mathbf{x},\xi) - \lambda - \sum_{s=1}^n \gamma_s(\xi_s) + \xi^\intercal Q \xi - 2\xi^\intercal Q\mu \geq 0,\ \forall \xi \in [0,1]^n, \tag{EC.23}$$

$$Q \succeq 0,\ \gamma_s \in C[0,1]^*,\ \forall s \in N.$$

Note that $F^*(\xi;\mathbf{x})$ denotes an optimizer of $\min_{F\in\mathcal{F}} \mathbb{E}_F[V(\mathbf{x},\xi)]$ and $\operatorname{supp}(F^*)$ denotes the support of F^*. Since $\mathscr{F}$ is assumed to have a strictly feasible solution (see Section 2.1) and $V(\xi,\mathbf{x})$ is bounded (see Assumption 1 in Section 3), the primal problem is feasible and bounded. Hence, the above dual problem has optimal solution (see Proposition 5.2 in Shapiro 2001). We use λ^*, Q^*, and γ_s^* to denote an optimal solution of the above dual model. Recall that we use the last term of the Lagrangian function (EC.17) to derive constraints (EC.23) of the above dual formulation, which is shown as follows:

$$\int_{[0,1]^n} \left(V(\mathbf{x},\xi) - \lambda + \sum_{s=1}^n \gamma_s(\xi_s) + \xi^\intercal Q\xi - 2\xi^\intercal Q\mu \right) dF(\xi). \tag{EC.24}$$

Because of the complementarity conditions of the infinite-dimensional optimization duality (see Equations (5.18) and (5.21) in Section 5 of Shapiro 2001), the integrand in (EC.24) is 0 for any $\xi \in \operatorname{supp}(F^*)$. In other words, for all $\xi \in \operatorname{supp}(F^*)$, constraints (EC.23) hold at equality, i.e.,

$$\Omega(\xi) \equiv V(\mathbf{x},\xi) - \lambda^* + \sum_{s=1}^n \gamma_s^*(\xi_s) + \xi^\intercal Q^*\xi - 2\xi^\intercal Q^*\mu = 0. \tag{EC.25}$$

If there exist $\xi \in \text{supp}(F^*)$, scalar δ_i, and scalar δ_j such that $\{\xi, \xi+\delta_i e_i, \xi+\delta_j e_j, \xi+\delta_i e_i+\delta_j e_j\} \subseteq \text{supp}(F^*)$, where e_i and e_j are unit vectors, then we have

$$\begin{aligned}
0 = &\ (\Omega(\xi+\delta_j e_j+\delta_i e_i) - \Omega(\xi+\delta_j e_j)) - (\Omega(\xi+\delta_i e_i) - \Omega(\xi)) \\
= &\ \big(V(\mathbf{x}, \xi+\delta_j e_j+\delta_i e_i) - V(\mathbf{x}, \xi+\delta_j e_j) + \gamma_i^*(\xi_i+\delta_i) - \gamma_i^*(\xi_i) + \delta_i^2 e_i^\intercal Q^* e_i + 2\delta_i(\xi+\delta_j e_j)^\intercal Q^* e_i - 2\delta_i e_i^\intercal Q^* \mu\big) \\
&\qquad - \big(V(\mathbf{x}, \xi+\delta e_i) - V(\mathbf{x}, \xi) + \gamma_i^*(\xi_i+\delta_i) - \gamma_i^*(\xi_i) + \delta_i^2 e_i^\intercal Q^* e_i + 2\delta_i \xi^\intercal Q^* e_i - 2\delta_i e_i^\intercal Q^* \mu\big) \\
= &\ \big(V(\mathbf{x}, \xi+\delta_j e_j+\delta_i e_i) - V(\mathbf{x}, \xi+\delta_j e_j)\big) - \big(V(\mathbf{x}, \xi+\delta e_i) - V(\mathbf{x}, \xi)\big) + 2\delta_i\delta_j Q_{ij}^* \\
= &\ \big(R(\mathbf{x}, \xi+\delta_j e_j+\delta_i e_i) - R(\mathbf{x}, \xi+\delta_j e_j) - c_i d_i(x_i, \xi_i+\delta_i) + c_i d_i(x_i, \xi_i)\big) \\
&\qquad - \big(R(\mathbf{x}, \xi+\delta e_i) - R(\mathbf{x}, \xi) - c_i d_i(x_i, \xi_i+\delta_i) + c_i d_i(x_i, \xi_i)\big) + 2\delta_i\delta_j Q_{ij}^* \\
= &\ \big(R(\mathbf{x}, \xi+\delta_j e_j+\delta_i e_i) - R(\mathbf{x}, \xi+\delta_j e_j)\big) - \big(R(\mathbf{x}, \xi+\delta e_i) - R(\mathbf{x}, \xi)\big) + 2\delta_i\delta_j Q_{ij}^*.
\end{aligned}$$

Therefore, we get

$$Q_{ij}^* = -\frac{\big(R(\mathbf{x}, \xi+\delta_j e_j+\delta_i e_i) - R(\mathbf{x}, \xi+\delta_j e_j)\big) - \big(R(\mathbf{x}, \xi+\delta e_i) - R(\mathbf{x}, \xi)\big)}{2\delta_i\delta_j}.$$

Moreover, if $R(\mathbf{x}, \xi)$ is twice-differentiable on ξ and $d(x_s, \xi_s)$ is differentiable on ξ_s, $\forall s \in N$, for any $\xi \in \text{supp}(F^*)$, then, for any $\xi \in \text{supp}(F^*)$, taking the second derivative of both the left- and right-hand sides of (EC.25) yields

$$0 = \frac{\partial V(\mathbf{x}, \xi)}{\partial \xi_i \partial \xi_j} + Q_{ij}^* = \frac{\partial R(\mathbf{x}, \xi)}{\partial \xi_i \partial \xi_j} - \frac{\partial\left(\sum_{s\in N} c_s d_s(x_s, \xi_s)\right)}{\partial \xi_i \partial \xi_j} + Q_{ij}^* = \frac{\partial R(\mathbf{x}, \xi)}{\partial \xi_i \partial \xi_j} + Q_{ij}^*,\ \forall i, j \in N,\ i \neq j. \quad \text{(EC.26)}$$

The third equality holds because $d(x_s, \xi_s)$ only depends on a single component ξ_s for each $s \in N$, thus, $\partial\left(\sum_{s\in N} c_s d_s(x_s, \xi_s)\right)/\partial\xi_i\partial\xi_j = 0$. Since one may not obtain a specific value of $\xi \in \text{supp}(F^*)$, we resort to derive a better presentation for obtaining the value of Q_{ij}^*. We further take the integral of both sides of (EC.26) with respect to the distribution $F^*(\xi; \mathbf{x})$ and obtain

$$Q_{ij}^* = -\int_{\text{supp}(F^*)} \frac{\partial R(\mathbf{x}, \xi)}{\partial \xi_i \partial \xi_j} dF^*(\xi; \mathbf{x}).$$

This completes the proof. □

We introduce two examples to demonstrate Theorem 2 as follows.

Example EC.3. Consider two suppliers (i.e., $N = \{1, 2\}$) with Bernoulli yield process such that $\xi_1, \xi_2 \in \{0, 1\}$. We consider $F_1(0) = F_2(0) = 0.5$, $V(\mathbf{x}, \xi) = R(\mathbf{x}, \xi) = \min\{0.1(\xi_1 x_1 + \xi_2 x_2) + 0.2, -0.1(\xi_1 x_1 + \xi_2 x_2) + 1.5\}$, and $c_i = 0,\ \forall i \in N$. Let

$$\Sigma = \begin{bmatrix} 0.25 & -0.05 \\ -0.05 & 0.25 \end{bmatrix}.$$

By solving the DR model with the above setting, we obtain the primal and dual solutions as follows: (i) the optimal order quantities are $x^* = (x_1^*, x_2^*) = (6.5, 6.5)$ and the cumulative distribution function F^* has $F^*(\{0,0\}) = 0.2$, $F^*(\{1,0\}) = 0.5$, $F^*(\{0,1\}) = 0.5$, and $F^*(\{1,1\}) = 1$, indicating that the probability mass function f^* has $f^*(\{0,0\}) = 0.2$, $f^*(\{1,0\}) = 0.3$, $f^*(\{0,1\}) = 0.3$, and $f^*(\{1,1\}) = 0.2$; (ii) the dual solution $Q_{12}^* = Q_{21}^* = 0.65$.

In addition, because $\{\xi, \xi + \delta_1 e_1, \xi + \delta_2 e_2, \xi + \delta_1 e_1 + \delta_2 e_2\} \subseteq \text{supp}(F^*)$ with $\xi = \{0, 0\}$ and $\delta_1 = \delta_2 = 1$, Theorem 2 shows that

$$\begin{aligned} Q_{12}^* = Q_{21}^* &= -\frac{\big(R(\{1,1\}, x^*) - R(\{1,0\}, x^*)\big) - \big(R(\{0,1\}, x^*) - R(\{0,0\}, x^*)\big)}{2\delta_1\delta_2} \\ &= -\frac{(0.2 - 0.85) - (0.85 - 0.2)}{2} = 0.65, \end{aligned}$$

which matches the obtained dual solution.

EXAMPLE EC.4. Consider $V(\mathbf{x}, \xi) = R(\mathbf{x}, \xi) = -b\left(\sum_{i=1}^n \xi_i x_i\right)^2$ with $b > 0$ and $c_i = 0$, $\forall i \in N$. Given the order quantities $x_i \geq 0$, $\forall i \in N$, the objective value of the inner (primal) problem of the DR model is

$$\min_{F \in \mathscr{F}} \mathbb{E}_F\left[-b\left(\sum_{i=1}^n \xi_i x_i\right)^2\right] = -b\sum_{i=1}^n\sum_{j=1}^n \max_{F\in\mathscr{F}} \mathbb{E}_F\left[\xi_i\xi_j\right] x_i x_j = -b\sum_{i=1}^n\sum_{j=1}^n \left(\Sigma_{ij} + \mu_i\mu_j\right) x_i x_j. \tag{EC.27}$$

The last equality holds because the optimal value is obtained when the constraint (1b) holds at equality. Following Proposition 1, the DR model is tractable based on the above setting. Following Theorem 2, the dual variable Q of the constraint (1b) has an optimal solution as follows:

$$\begin{aligned} Q_{ij}^* &= -\int_{\text{supp}(F^*)} \frac{\partial R(\mathbf{x}, \xi)}{\partial \xi_i \partial \xi_j} dF^*(\xi; \mathbf{x}) = -\int_{\text{supp}(F^*)} \frac{\partial(-b\sum_{s=1}^n\sum_{t=1}^n x_s x_t \xi_s \xi_t)}{\partial \xi_i \partial \xi_j} dF^*(\xi; \mathbf{x}) \\ &= -\int_{\text{supp}(F^*)} (-b x_i x_j) dF^*(\xi; \mathbf{x}) = b x_i x_j \int_{\text{supp}(F^*)} dF^*(\xi; \mathbf{x}) = b x_i x_j, \ \forall i, j \in N. \end{aligned} \tag{EC.28}$$

We now demonstrate that the above Q_{ij}^*, $\forall i, j \in N$, together with specific values of λ and γ, is indeed dual optimal. Specifically, we consider the dual formulation (EC.22) and construct a solution (λ, Q, γ) by letting $\lambda = -2b\sum_{s=1}^n x_s \sum_{t=1}^n x_t \mu_t$, $Q_{ij} = Q_{ij}^* = b x_i x_j$, $\forall i, j \in N$, and $\gamma_s(\xi_s) = 2b x_s (1 - \xi_s) \sum_{t=1}^n x_t \mu_t$, $\forall s \in N$. This solution is feasible for the formulation (EC.22) because (i) $Q = b\mathbf{x}\mathbf{x}^\intercal \succeq 0$. (ii) $\gamma_s(\xi_s)$ is continuous on ξ_s with bounded variation, i.e., $\gamma_s \in C[0,1]^*$, and $\gamma_s(1) = 0$, $\forall s \in N$. (iii) We have

$$\begin{aligned} & V(\mathbf{x}, \xi) - \lambda - \sum_{s=1}^n \gamma_s(\xi_s) + \xi^\intercal Q \xi - 2\xi^\intercal Q \mu \\ = \; & -b\sum_{s=1}^n\sum_{t=1}^n x_s x_t \xi_s \xi_t + 2b\sum_{s=1}^n\sum_{t=1}^n x_s \xi_s x_t \mu_t + b\sum_{s=1}^n\sum_{t=1}^n x_s x_t \xi_s \xi_t - 2b\sum_{s=1}^n\sum_{t=1}^n x_s \xi_s x_t \mu_t = 0. \end{aligned}$$

That is, the constructed solution (λ, Q, γ) satisfies all the constraints in (EC.22). The objective value of (EC.23) corresponding to this solution is

$$\begin{aligned}
&\lambda + \sum_{s=1}^{n} \int_{0}^{1} \gamma_s(\omega_s) dF_s(\omega_s) + Q \bullet (\mu\mu^\intercal - \Sigma) \\
= &-2b \sum_{s=1}^{n} x_s \left(\sum_{t=1}^{n} x_t \mu_t \int_{0}^{1} \xi_s dF_s(\omega_s) \right) + b \sum_{s=1}^{n} \sum_{t=1}^{n} x_s x_t \left(\mu_s \mu_t - \Sigma_{st} \right) \\
= &-2b \sum_{s=1}^{n} \sum_{t=1}^{n} x_s x_t \mu_t \mu_s + b \sum_{s=1}^{n} \sum_{t=1}^{n} x_s x_t \left(\mu_s \mu_t - \Sigma_{st} \right) \\
= &-b \sum_{s=1}^{n} \sum_{t=1}^{n} \left(\Sigma_{st} + \mu_s \mu_t \right) x_s x_t.
\end{aligned} \tag{EC.29}$$

Because the primal objective (EC.27) and the dual objective (EC.29) are the same, the constructed solution (λ, Q, γ) with $Q = Q^*$ in (EC.28) is dual optimal.

Proof of Proposition 4 We assume that the firm always outputs all available supply, i.e., $\sum_{s=1}^{n} x_s \le a/2b$. The required conditions for this assumption can be derived after we obtain the optimal solution. The firm's problem is

$$\begin{aligned}
&\max_{\mathbf{x} \ge 0} \min_{F \in \mathscr{F}} \mathbb{E}_F \left[\left(a - b \left(\sum_{s=1}^{n} \xi_s x_s \right) \right) \left(\sum_{s=1}^{n} \xi_s x_s \right) - \sum_{s=1}^{n} c_s \xi_s x_s \right] \\
= &\max_{\mathbf{x} \ge 0} \min_{\rho_{st} \le \rho} \left(\sum_{s=1}^{n} (a - c_s) \mu_s x_s - b \sum_{s=1}^{n} \sum_{t=1 \ne s}^{n} (\rho_{st} \sigma_s \sigma_t + \mu_s \mu_t) x_s x_t - b \sum_{s=1}^{n} (\sigma_s^2 + \mu_s^2) x_s^2 \right) \\
= &\max_{\mathbf{x} \ge 0} \left(\sum_{s=1}^{n} (a - c_s) \mu_s x_s - b \sum_{s=1}^{n} \sum_{t=1 \ne s}^{n} (\rho \sigma_t \sigma_s + \mu_t \mu_s) x_s x_t - b \sum_{s=1}^{n} (\sigma_s^2 + \mu_s^2) x_s^2 \right).
\end{aligned}$$

The objective function is jointly concave on x_s, $\forall s \in N$, and the KKT conditions are

$$\lambda_s x_s = 0, \ \forall s \in N, \ (a - c_1)\mu_1 + \lambda_1 = 2b\mu_1 \sum_{i=1}^{n} \mu_i x_i,$$
$$(a - c_s)\mu_s + \lambda_s = 2b \left(\sum_{i=1}^{n} (\rho \sigma_i \sigma_s + \mu_i \mu_s) x_i + (1 - \rho) \sigma_s^2 x_s \right), \ \forall s \in \{2, \ldots, n\}.$$

Then, we have

$$\frac{(a - c_1)\mu_1 + \lambda_1}{2b\mu_1} = \sum_{i=1}^{n} \mu_i x_i \quad \text{and} \quad \frac{(c_1 - c_s)\mu_1 \mu_s + \lambda_s \mu_1 - \lambda_1 \mu_s}{2b\mu_1 \sigma_s} = \sum_{i=2}^{n} \rho \sigma_i x_i + (1 - \rho) \sigma_s x_s, \tag{EC.30}$$

$\forall s \in \{2, \ldots, n\}$. Suppose the buyer sources from the stable supplier. We demonstrate the primal and dual feasibility of the solution given in (6). It is clear that the primal feasibility is ensured by the definition of S_ρ, which implies that the complementarity constraints give $\lambda_s = 0$, $\forall s = 1, \ldots, S_\rho$. From (EC.30), we have

$$\sum_{s=2}^{S_\rho} \frac{(c_1 - c_s)\mu_s}{2b\sigma_s} = \rho(S_\rho - 1) \sum_{s=2}^{S_\rho} \sigma_s x_s + (1 - \rho) \sum_{s=2}^{S_\rho} \sigma_s x_s$$

$$\Rightarrow \frac{1}{\rho(S_\rho-2)+1}\sum_{s=2}^{S_\rho}\frac{(c_1-c_s)\mu_s}{2b\sigma_s}=\sum_{s=2}^{S_\rho}\sigma_s x_s=\sum_{s=2}^{n}\sigma_s x_s \tag{EC.31}$$

$$\Rightarrow \lambda_i=\frac{\rho}{\rho(S_\rho-2)+1}\sum_{s=2}^{S_\rho}\frac{(c_1-c_s)\mu_s}{2b\sigma_s}-\frac{(c_1-c_i)\mu_i}{2b\sigma_i}\geq 0,\ \forall i=S_\rho+1,\dots,n.$$

So, the dual feasibility, $\lambda_i\geq 0$, $\forall i=S_\rho+1,\dots,n$, follows the definition of S_ρ. Note that x_s, $\forall s=1,\dots,S_\rho$, is derived from (EC.30) and (EC.31). Then, we have

$$\sum_{s=1}^{n}x_s=\frac{a-c_1}{2b\mu_1}+\sum_{i=2}^{S_\rho}\frac{(c_1-c_i)\mu_i(1-\mu_i/\mu_1)}{2b(1-\rho)\sigma_i^2}-\frac{\rho}{\rho(S_\rho-2)+1}\sum_{i=2}^{S_\rho}\frac{(c_1-c_i)\mu_i}{2b(1-\rho)\sigma_i}\sum_{j=2}^{S}\frac{1-\mu_j/\mu_1}{\sigma_j}.$$

The condition $\sum_{s=1}^{n}x_s\leq a/2b$ holds if

$$\sum_{i=2}^{S}\frac{(c_1-c_i)\mu_i(1-\mu_i)}{(1-\rho)\sigma_i^2}\leq c_1+a(\mu_1-1)\Rightarrow\frac{a-c_1}{2b\mu_1}+\sum_{i=2}^{S}\frac{(c_1-c_i)\mu_i(1-\mu_i/\mu_1)}{2b(1-\rho)\sigma_i^2}\leq\frac{a}{2b}\Rightarrow\sum_{s=1}^{n}x_s\leq a/2b.$$

When the buyer does not source from the stable supplier, the complementarity constraints give $\lambda_s=0$, $\forall s\in\mathcal{J}$. From (EC.30), we have

$$\sum_{s\in\mathcal{J}}\frac{(c_1-c_s)\mu_1\mu_s-\lambda_1\mu_s}{2b\mu_1\sigma_s}=\rho|\mathcal{J}|\sum_{s\in\mathcal{J}}\sigma_s x_s+(1-\rho)\sum_{s\in\mathcal{J}}\sigma_s x_s$$

$$\Rightarrow \frac{1}{\rho(|\mathcal{J}|-1)+1}\sum_{s\in\mathcal{J}}\frac{(c_1-c_s)\mu_1\mu_s-\lambda_1\mu_s}{2b\mu_1\sigma_s}=\sum_{s\in\mathcal{J}}\sigma_s x_s=\sum_{s=1}^{n}\sigma_s x_s$$

$$\Rightarrow x_i=\frac{1}{(1-\rho)\sigma_i}\left(\frac{(c_1-c_i)\mu_1\mu_i-\lambda_1\mu_i}{2b\mu_1\sigma_i}-\frac{\rho}{\rho(|\mathcal{J}|-1)+1}\sum_{s\in\mathcal{J}}\frac{(c_1-c_s)\mu_1\mu_s-\lambda_1\mu_s}{2b\mu_1\sigma_s}\right)\quad\forall i\in\mathcal{J}.$$

Then λ_1 satisfies

$$\frac{(a-c_1)\mu_1+\lambda_1}{2b\mu_1}=\sum_{i=1}^{n}\mu_i x_i=\sum_{i\in\mathcal{J}}\mu_i x_i$$

$$\Rightarrow \lambda_1\left[1+\frac{1}{(1-\rho)}\left(\sum_{s\in\mathcal{J}}\frac{\mu_s^2}{\sigma_s^2}-\frac{\rho}{\rho(|\mathcal{J}|-1)+1}\left(\sum_{s\in\mathcal{J}}\frac{\mu_s}{\sigma_s}\right)^2\right)\right]=$$

$$(c_1-a)\,\mu_1+\frac{\mu_1}{(1-\rho)}\left[\sum_{s\in\mathcal{J}}\frac{(c_1-c_s)\mu_s^2}{\sigma_s^2}-\frac{\rho}{\rho(|\mathcal{J}|-1)+1}\sum_{s\in\mathcal{J}}\frac{(c_1-c_s)\mu_s}{\sigma_s}\left(\sum_{s\in\mathcal{J}}\frac{\mu_s}{\sigma_s}\right)\right].$$

The condition that the buyer outputs all available supply can be stated explicitly by considering $\sum_{s=1}^{n}x_s\leq a/2b$ which holds if

$$\sum_{s=2}^{n}\frac{(c_1-c_s)\mu_s}{2b(1-\rho)\sigma_s^2}\leq\frac{a}{2b}\Leftarrow\sum_{s=2}^{n}\frac{(c_1-c_s)\mu_s}{(1-\rho)\sigma_s^2}\leq a.$$

When $\rho=0$, it is clear that $x_s>0$ if and only if $\mu_1(c_1-c_s)\geq\alpha_s$. Then $\mathcal{J}$ can be ordered by suppliers' costs and $\lambda_1=\alpha_{j^*}$. □

Remark EC.2. When $\rho = 1$, (EC.30) becomes

$$\frac{(a-c_1)\mu_1+\lambda_1}{2b\mu_1}=\sum_{i=1}^{n}\mu_i x_i \quad \text{and} \quad \frac{(c_1-c_s)\mu_1\mu_s+\lambda_s\mu_1-\lambda_1\mu_s}{2b\mu_1\sigma_s}=\sum_{i=2}^{n}\sigma_i x_i, \tag{EC.32}$$

$\forall s\in\{2,\ldots,n\}$. First, suppose the firm orders from supplier 1. Hence, $\lambda_1=0$. If the firm sources from both suppliers j and j' with $j\neq j'\in N\setminus\{1\}$, then $\lambda_j=\lambda_{j'}=0$ and

$$\frac{(c_1-c_j)\mu_1\mu_j}{2b\mu_1\sigma_j}=\sum_{i=2}^{n}\sigma_i x_i=\frac{(c_1-c_{j'})\mu_1\mu_{j'}}{2b\mu_1\sigma_{j'}}. \tag{EC.33}$$

The equation implies that the firm will only source from suppliers in the set $\arg\max_s\{(c_1-c_s)\mu_s/\sigma_s : s\in N\}$. Let $j\in\arg\max_s\{(c_1-c_s)\mu_s/\sigma_s : s\in N\}$. The optimal solution is

$$x_1=\frac{(a-c_1)}{2b}-\frac{(c_1-c_j)\mu_j^2}{2b\sigma_j^2},\ x_j=\frac{(c_1-c_j)\mu_j}{2b\sigma_j^2}.$$

Now, suppose the firm does not order from supplier 1. Combining two equations in (EC.32), we have

$$\frac{(a-c_s)\mu_s}{2b\sigma_s}=\sum_{i=1}^{n}(\mu_i\mu_s+\sigma_i\sigma_s)x_i,\ \forall s\in\{2,\ldots,n\}. \tag{EC.34}$$

If the firm sources from all suppliers in $S\subseteq N\setminus\{1\}$, then the optimal solution is

$$(x_s : s\in S)=\frac{1}{2b}\left(\mu_S\mu_S^{\intercal}+\sigma_S\sigma_S^{\intercal}\right)^{-1}\left(\frac{(a-c_s)\mu_s}{\sigma_s} : s\in S\right)$$

where $\mu_S=(\mu_s : s\in S)$ and $\sigma_S=(\sigma_s : s\in S)$.

Proof of Theorem 3 Suppose $c_1\leq\cdots\leq c_n$. We will show that our model with a given joint supply distribution F is equivalent to the model in Li et al. (2013). Thus, our theorem can be proved by applying Theorem 3 and Theorem 4 in Li et al. (2013). Similar to Li et al. (2013), we introduce salvage price $\gamma<c_s$ $\forall s\in N$, costs $c_s=c_s-\gamma$, the cost of lost goodwill c_u and a demand function $D(p)=(a-\gamma)/b-p/b\equiv a'-b'p$ with price p. We denote

$$Q(x,\xi)=\sum_{s=1}^{n}Q_s(x_s,\xi_s).$$

With given order quantity $\mathbf{x}$, our profit function is

$$\begin{aligned}
&\max_q\{(a-bq)q \mid q\leq Q(x,\xi)\}-\sum_{s=1}^{n}c_sQ_s(x_s,\xi_s)\\
=\ &\max_q\{(a-bq)q-\gamma\left(Q(x,\xi)-q\right) \mid q\leq Q(x,\xi)\}-\sum_{s=1}^{n}(c_s-\gamma)Q_s(x_s,\xi_s)\\
=\ &\max_q\{(a-bq)q-\gamma\left(Q(x,\xi)-q\right) \mid q\leq Q(x,\xi)\}-\sum_{s=1}^{n}c_s'Q_s(x_s,\xi_s).
\end{aligned}$$

Now, the revenue function is

$$
\begin{aligned}
&\max_{q}\left\{(a-\gamma-bq)q-\gamma\left(Q(x,\xi)-q\right)\ \mid q\le Q(x,\xi)\right\}\\
=\ &\max_{q,p\ge 0}\left\{(a-\gamma-bD(p))q-\gamma\left(Q(x,\xi)-D(p)\right)^{+}-c_u\left(D(p)-Q(x,\xi)\right)^{+}\ \middle|\ q\le S(x,\xi),q=D(p)\right\}\\
=\ &\max_{q,p\ge 0}\left\{pq+\gamma\left(Q(x,\xi)-D(p)\right)^{+}-c_u\left(D(p)-S(x,\xi)\right)^{+}\middle|\ q\le Q(x,\xi),q\le D(p)\right\} \qquad \text{(EC.35)}\\
=\ &\max_{q,p\ge 0}\left\{p\min\left\{D(p),Q(x,\xi)\right\}+\gamma\left(Q(x,\xi)-D(p)\right)^{+}-c_u\left(D(p)-Q(x,\xi)\right)^{+}\right\},
\end{aligned}
$$

where the last equality holds obviously because the maximal revenue is obtained when $q=\min\{D(p),Q(x,\xi)\}$. The equation (EC.35) holds because the maximization problem in (EC.35) satisfies $q=D(p)$ in the optimal solution. If $D(p)\le Q(\mathbf{x},\xi)$ and $q<D(p)$, then the revenue is $pq+\gamma(Q(x,\xi)-D(p))$ and increasing q such that $q=D(p)$ results in higher revenue. On the other hand, if $D(p)>Q(\mathbf{x},\xi)$ and $q<D(p)$, then the revenue is $pq-c_u(D(p)-Q(x,\xi))$ and increasing price p obtains a better revenue. Since we have the same form of the revenue and cost functions, the model (P) is equivalent to the one in Li et al. (2013). Note that Li et al. (2013) assume the marginal probability density function is defined on $\xi_s>0$ for each supplier $s\in N$. The assumption requires the probability of shutdown is zero for any supplier.

Following (12) in Li et al. (2013), the effective cost of supplier base $\mathbb{S}$ under joint supply distribution F:

$$
C_{\mathbb{S}}(F)=\begin{cases}\dfrac{\mathbb{E}_F\left[a'-2Q_{\mathbb{S}}\right]}{b'} & \text{if } \sum_{s\in\mathbb{S}} x_s^*\le (a'-\gamma b')/2\\[2ex] \dfrac{\mathbb{E}_F\left[\max\{a'-2Q_{\mathbb{S}},\gamma\}\right]}{b'} & \text{otherwise,}\end{cases}
$$

where $\sum_{s\in\mathbb{S}} x_s^*$ and $Q_{\mathbb{S}}$ are the total order quantity and total delivery quantity, respectively, after the firm makes the optimal sourcing decisions by only considering suppliers in set $\mathbb{S}$. Since $Q_{\mathbb{S}}$ is supermodular on ξ given $\mathbf{x}$, $C_{\mathbb{S}}(F)$ is submodular on ξ given $\mathbf{x}$. Let F' (resp. F'') and $\mathbb{S}'$ (resp. $\mathbb{S}''$) be the optimal supply distribution and supplier base of the DR model with Σ' (resp. Σ''). Theorem 4.(ii) in Li et al. (2013) suggests that $C_{\mathbb{S}''}(F'')\le c_i'\ \forall i\in N\setminus\mathbb{S}'$ since $\mathbb{S}''$ is the optimal supplier base with supply distribution F''. If we only consider the suppliers in $\mathbb{S}''$, Theorem 3 in Li et al. (2013) indicates $C_{\mathbb{S}''}(F')\le C_{\mathbb{S}''}(F'')$ because the correlation between each pair of suppliers decreases when the supply distribution changes from F'' to F'. Then, Theorem 4.(i) in Li et al. (2013) implies that the optimal supplier base with supply distribution F' is a subset of $\mathbb{S}''$. That is, $\mathbb{S}'\subseteq\mathbb{S}''$. □

Proof of Corollary 4 In the case of the model (5) with yield uncertainty and the firm outputs all supply, we have

$$R(\mathbf{x},\xi)=\left(a-b\left(\sum_{s=1}^{n}\xi_s x_s\right)\right)\left(\sum_{s=1}^{n}\xi_s x_s\right)\Rightarrow\frac{\partial R(\mathbf{x},\xi)}{\partial\xi_i\partial\xi_j}=-bx_ix_j,\ \forall i,j\in N,i\neq j.$$

It follows that $Q_{ij}=bx_ix_j$ for any $i,j\in N$ and $i\neq j$. Since it is a quadratic program and $b>0$, the optimal solution $\mathbf{x}$ exists and is unique. □