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Wave propagation and stabilization in the Boussinesq–Burgers system

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ABSTRACT

This paper considers the existence and stability of traveling wave solutions of the Boussinesq–Burgers system describing the propagation of bores. Assuming the fluid is weakly dispersive, we establish the existence of three different wave profiles by the geometric singular perturbation theory alongside phase plane analysis. We further employ the method of weighted energy estimates to prove the nonlinear asymptotic stability of the traveling wave solutions against small perturbations. The technique of taking antiderivative is utilized to integrate perturbation functions because of the conservative structure of the Boussinesq–Burgers system. Using a change of variable to deal with the dispersion term, we perform numerical simulations for the Boussinesq–Burgers system to showcase the generation and propagation of various wave profiles in both weak and strong dispersions. The numerical simulations not only confirm our analytical results, but also illustrate that the Boussinesq–Burgers system can generate numerous propagating wave profiles depending on the profiles of initial data and the intensity of fluid dispersion, where in particular the propagation of bores can be generated from the system in the case of strong dispersion.

1. Introduction

A bore is a sudden increase in water depth in a free-surface flow (cf. [1,2]) due to dam-break, initial mound of water, and solitary-wave breaking and so on. A typical example is the tidal bore — sudden elevation of the water surface that travels upstream an estuary with the incoming flood tide. The first study in a laboratory was made in [3] where a barrier separating different levels of water is suddenly removed and the surface motion of water flows is described by bores. In general two classes of wave motions can be observed depending on the levels of water on the two sides of the barrier, named as strong (or turbulent) bores and weak (or undular) bores. Strong bores represent the motion of the sudden violent change of water level and the weak bores describe the gently sloping or oscillatory transition between the different levels (see more detailed description in [4]). Strong bores are often more difficult to study mathematically than the weak bores.

There are various physical models for the description of weak bores and many theories have been developed (cf. [5–11]). Among them is the Boussinesq system [7,9]:

$$\begin{cases} \rho_t + u_x + (u\rho)_x = 0, \\ u_t + \rho_x + uu_x - \epsilon u_{xxt} = 0, \end{cases} \tag{1.1}$$

where $\rho(x,t)$ and $u(x,t)$ represent the height and velocity of the free surface of the fluid above the bottom, respectively, and $\epsilon > 0$ is a parameter accounting for the intensity of fluid dispersion. The Boussinesq system includes the nonlinear and dispersive effects but not fluid viscosity. Nevertheless it was pointed out in [12–14] that dissipative effects must be considered to accurately predict wave propagation, at least at the laboratory scale. Appending diffusion and viscosity to the Boussinesq system, one arrives at the following so-called Boussinesq–Burgers system (cf. [15]):

$$\begin{cases} \rho_t + u_x + (u\rho)_x = \mu\rho_{xx}, \\ u_t + \rho_x + uu_x - \epsilon u_{xxt} = \mu u_{xx}. \end{cases} \tag{1.2}$$

For convenience, one can make a change of variable $w(x,t) = 1 + \rho(x,t)$ and transform (1.2) into the following one:

$$\begin{cases} w_t + (wu)_x = \mu w_{xx}, \\ u_t + (w + \frac{u^2}{2})_x = \epsilon u_{xxt} + \mu u_{xx}. \end{cases} \tag{1.3}$$

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When we ignore the diffusion, viscosity and dispersion terms, i.e. $\epsilon = \mu = 0$, the system is reduced to a shallow water wave equation (cf. [16]).

There are a few mathematical results developed for the Boussinesq–Burgers system. In paper [17], the global existence and asymptotic behavior of solutions of (1.3) in a bounded interval with Neumann–Dirichlet boundary conditions were established. Furthermore when the boundary conditions are time-dependent (i.e., dynamical boundary conditions), the global existence and asymptotic profile of solutions were studied in [18]. The large-time behavior of solutions to the Cauchy problem of a generalized Boussinesq–Burgers system was investigated in [19] followed by a work [20] giving the decay rates of solutions.

By making the following change of variables:

$$t \to t/\mu, \quad x \to x/\mu, \quad \epsilon \to \epsilon/\mu^2,$$

we can rewrite the Boussinesq–Burgers system (1.3) by reducing one parameter as

$$\begin{cases} w_t + (wu)_x = w_{xx}, \\ u_t + (w + \frac{u^2}{2})_x = \epsilon u_{xxt} + u_{xx}. \end{cases} \tag{1.4}$$

Hence without loss of generality we may assume $\mu = 1$ in (1.3). The goal of this paper is to study the existence and stability of traveling wave solutions of (1.4) with one-dimensional space variable and supplemented with the initial data

$$(w, u)(x, 0) = (w_0, u_0)(x) \to \begin{cases} (w_-, u_-) & \text{as } x \to -\infty, \\ (w_+, u_+) & \text{as } x \to +\infty. \end{cases} \tag{1.5}$$

A traveling wave solution of (1.4) in $(x, t) \in \mathbb{R} \times [0, \infty)$ is a non-constant solution in the form

$$(w, u)(x, t) = (W, U)(z), \quad z = x - ct \tag{1.6}$$

with $W, U \in C^\infty(\mathbb{R})$ satisfying the boundary conditions (i.e. end states)

$$W(\pm\infty) = w_\pm, \quad U(\pm\infty) = u_\pm, \tag{1.7}$$

where c is the wave speed assumed to be non-negative without loss of generality, z is called the wave variable. The constants $w_\pm$ and $u_\pm$ are called end states of w and u, respectively, describing the asymptotic behavior of traveling wave solutions as $z \to \pm\infty$.

Substituting the wave ansatz (1.6) into (1.4) yields

$$\begin{cases} -cW_z + (WU)_z = W_{zz}, & z \in \mathbb{R}, \\ -cU_z + (W + \frac{U^2}{2})_z = U_{zz} - c\epsilon U_{zzz}, & z \in \mathbb{R}, \\ (W, U)(\pm\infty) = (w_\pm, u_\pm). \end{cases} \tag{1.8}$$

By assuming $w_\pm = 0$ and $u_+ = 0$, under the condition $4\epsilon c^2 \le 1$, it was shown in [15, Theorem 2.4] that the Boussinesq–Burgers system (1.3) admits a traveling wave solution (W, U) in $\mathbb{R}$ connecting $(0, 2c)$ to $(0, 0)$, where $U_z < 0$ (see [15, Theorem 2.6]). This was proved by phase plane analysis performed to the system (1.8) in a three-dimensional phase space. Note that the work [15] only considered the existence of traveling wave solutions for a special case $w_\pm = 0, u_+ = 0$. However, whether there are possible traveling wave solutions connecting other possible end states $w_\pm$ and $u_\pm$ remains unknown as mentioned in [15]. The goal of this paper is twofold. First, we prove that the Boussinesq–Burgers system (1.4) admits traveling wave solutions connecting any two critical points between $O(0, 0)$, $A(0, 2c)$ and $B(\frac{c^2}{2}, c)$ if $\epsilon > 0$ is small. Our proof strategy is to first prove the existence of traveling wave solutions for the dispersion-free system (1.8) (i.e. $\epsilon = 0$) and then prove the results hold true for small $\epsilon > 0$ by means of the geometric singular perturbation theory. The second goal of this paper is to show the nonlinear stability of traveling wave solutions. Roughly speaking, we prove that all the traveling wave solutions obtained above are asymptotically stable if the initial data are sufficiently close to them. Our proof is based on the technique of taking antiderivative originally developed in [21,22], owing to the conservative structure of the Boussinesq–Burgers system, alongside the method of energy estimates in spatially weighted Sobolev spaces.

The rest of this paper is organized as follows. In Section 2, we state our main results on the existence and stability of traveling wave solutions to the Boussinesq–Burgers system (1.4)–(1.5). Then we prove the existence results in Section 3 and stability results in Section 4. In Section 5, we use numerical simulations to illustrate the wave profiles generated by the Boussinesq–Burgers system (1.4)–(1.5) and discuss the implications of the numerical results.

2. Statement of main results

We shall prove there are three possible different traveling wave profiles generated by the Boussinesq–Burgers system (1.4)–(1.5), which connect critical points between O, A and B. Thereof a natural question is under what conditions a specific traveling wave profile will be produced. The answer relies on the initial profile (w_0, u_0) in which the asymptotic states $(w_\pm, u_\pm)$ play dominant roles. Since $w(x, t)$ and $u(x, t)$ represent the height and speed of the free surface of the fluid, we assume without loss of generality that $w_\pm \ge 0$ and $u_\pm \ge 0$. The first results on the existence of traveling wave solutions are stated below.

Theorem 2.1. *Consider the Cauchy problem (1.4)–(1.5) with fixed constant $u_- > 0$. If $\epsilon = 0$ or $\epsilon > 0$ is sufficiently small, then the following results hold.*

(i) If $u_+ = 0$ and $w_- > 0$, then there is a unique wave speed $c = u_-$ such that the system (1.4)–(1.5) has a unique traveling wave solution (W, U) up to a translation satisfying (1.8) with $w_- = \frac{u_-^2}{2}$ and $w_+ = 0$. Moreover, the solution satisfies $W_z < 0$ and $U_z < 0$.

(ii) If $u_+ > 0$, then there is a wave speed $c = \frac{u_-}{2}$ such that the system (1.4)–(1.5) has a unique traveling wave solution (W, U) up to a translation satisfying (1.8) with $w_- = 0$, $w_+ = \frac{u_-^2}{8}$ and $u_+ = \frac{u_-}{2}$. Moreover, the solution satisfies $W_z > 0$ and $U_z < 0$.

(iii) If $u_+ = 0$ and $w_- = 0$, then there is a unique wave speed $c = \frac{u_-}{2}$ such that the system (1.4)–(1.5) has a traveling wave solution (W, U) satisfying (1.8) with $w_+ = 0$ and $U_z < 0$, where the wave profile W is non-monotone and there is a point $z_0 \in \mathbb{R}$ such that $W_z > 0$ when $z \in (-\infty, z_0)$ and $W_z < 0$ when $z \in (z_0, \infty)$.

The second result of this paper is concerned with the nonlinear local stability of the traveling wave solutions obtained in Theorem 2.1. As asserted in Theorem 2.1, there are three different types of traveling wave profiles. Hence naturally we ask which of them is stable or unstable. Interestingly we are able to show that they are all locally stable by a unified approach — weighted energy estimates. In the results of Theorem 2.1, the traveling wave profile W could be monotone increasing or decreasing or non-monotone, while U is always monotone decreasing. Fortunately our stability analysis depends only on the monotonicity of U, which offers us a chance to employ a unified approach to prove the stability for different traveling wave profiles.

Our energy estimates cover the singularities at $w_+ = 0$ and/or $w_- = 0$. In this paper, we shall introduce a weight function $\omega(z)$ to overcome the singularities and perform weighted energy estimates to prove the nonlinear stability of traveling wave solutions in the case $w_+ = 0$ and/or $w_- = 0$. The weight function is chosen as

$$\omega(z) = W(z)^{-1}, \quad z \in \mathbb{R}. \tag{2.1}$$

In what follows, by $H^k_\omega(\mathbb{R})$ we denote the space of measurable functions f so that $\sqrt{\omega}\partial_x^j f \in L^2(\mathbb{R})$ for $0 \le j \le k$ with norm $\|f\|_{H^k_\omega(\mathbb{R})} := \left(\sum_{j=0}^k \int_{\mathbb{R}} \omega(x)|\partial_x^j f|^2 dx\right)^{1/2}$. For simplicity, the convention $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$ and $\|\cdot\|_{k,\omega} := \|\cdot\|_{H^k_\omega(\mathbb{R})}$ will be adopted.

The nonlinear local stability of traveling wave solutions to (1.4)–(1.5) is given below.

Theorem 2.2. *Let the assumptions in Theorem 2.1 hold, and let $(W, U)(z)$ be the traveling wave solutions obtained in Theorem 2.1. Assume that there exists a constant x_0 such that the initial perturbation from the spatially shifted traveling waves with shift x_0 is of integral zero, namely $\phi_0(\infty) = \psi_0(\infty) = 0$, where*

$$(\phi_0, \psi_0)(x) := \int_{-\infty}^{x} (w_0(y) - W(y + x_0), u_0(y) - U(y + x_0))dy.$$

Then there exists a constant $\delta_0 > 0$ such that if $\|\phi_0\|_w + \|\psi_0\| + \|w_0 - W\|_{1,w} + \|u_0 - U\|_2 \le \delta_0$, then the Cauchy problem (1.4)–(1.5) with sufficient small $\epsilon > 0$ has a unique solution $(w, u)(x, t)$ satisfying

$$w - W \in C([0, \infty), H^1_w) \cap L^2((0, \infty), H^2_w),$$
$$u - U \in C([0, \infty), H^2) \cap L^2((0, \infty), H^2)$$

with the following asymptotic stability:

$$\sup_{x\in\mathbb{R}} |(w, u, u_x)(x, t) - (W, U, U_x)(x + x_0 - st)| \to 0 \quad \text{as} \quad t \to \infty.$$

3. Existence: Proof of Theorem 2.1

As mentioned previously, we shall prove the existence of traveling wave solutions to (1.4)–(1.5) for small $\epsilon > 0$ by the geometric singular perturbation theory. Integrating the equations in (1.8) once, we get

$$\begin{cases} -cW + WU = W_z + \rho_1, \\ -cU + W + \frac{U^2}{2} = U_z - c\epsilon U_{zz} + \rho_2, \end{cases} \tag{3.1}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are constants of integration. Evaluating equations in (3.1) at $z = \pm\infty$, we find

$$\begin{cases} \rho_1 = -cw_+ + w_+u_+ = -cw_- + w_-u_-, \\ \rho_2 = -cu_+ + w_+ + \frac{u_+^2}{2} = -cu_- + w_- + \frac{u_-^2}{2}. \end{cases} \tag{3.2}$$

To avoid excessive technicalities, we shall only consider the case $\rho_1 = \rho_2 = 0$ in this paper. The extensions to the case $\rho_1 \ne 0$ or $\rho_2 \ne 0$ are possible but will not be considered in this paper. Then we get

$$\begin{cases} w_-(c - u_-) = 0, \\ -cu_- + w_- + \frac{u_-^2}{2} = 0, \end{cases} \quad \text{and} \quad \begin{cases} w_+(c - u_+) = 0, \\ -cu_+ + w_+ + \frac{u_+^2}{2} = 0. \end{cases} \tag{3.3}$$

With $\rho_1 = \rho_2 = 0$, (3.1) becomes

$$\begin{cases} W_z = -cW + WU, \\ U_z - c\epsilon U_{zz} = -cU + W + \frac{U^2}{2}. \end{cases} \tag{3.4}$$

We first consider the case $\epsilon = 0$. For convenience, we rewrite (3.4) with $\epsilon = 0$ as

$$\begin{cases} U_z = -cU + W + \frac{U^2}{2}, \\ W_z = -cW + WU, \end{cases} \tag{3.5}$$

and study its dynamics in the UW-plane. Let

$$F(U, W) = \begin{pmatrix} -cU + W + \frac{U^2}{2} \\ -cW + WU \end{pmatrix} := \begin{pmatrix} f(U, W) \\ g(U, W) \end{pmatrix}. \tag{3.6}$$

Table 1
Classification of critical points of (3.5).

Critical points	Type	Eigenvalues
$O(0, 0)$	Stable node	$\lambda_1 = \lambda_2 = -c$
$A(2c, 0)$	Unstable node	$\lambda_1 = \lambda_2 = c$
$B(c, \frac{c^2}{2})$	Saddle	$\lambda_1 = -\frac{c}{\sqrt{2}}$, $\lambda_2 = \frac{c}{\sqrt{2}}$

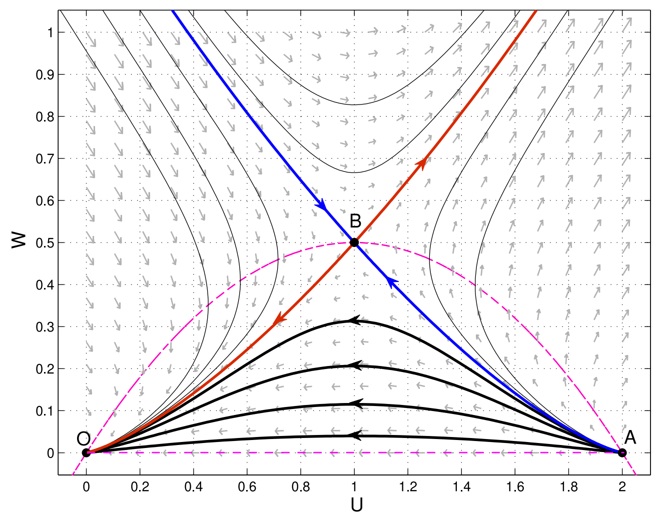


Fig. 1. Phase portrait of system (3.5) with $c = 1$.

Solving $F(U, W) = (0, 0)^{\mathrm{T}}$, we find that (3.5) has only three critical points

$$O(0, 0), \quad A(2c, 0) \quad \text{and} \quad B(c, c^2/2).$$

By $J(u_c, w_c)$ we denote the Jacobian of $F(U, W)$ at (u_c, w_c). Then it follows that

$$J(u_c, w_c) = \begin{pmatrix} -c + u_c & 1 \\ w_c & -c + u_c \end{pmatrix}. \tag{3.7}$$

The eigenvalues of $J(u_c, w_c)$ are denoted by λ_1 and λ_2. Then the critical points of (3.5) can be classified based on λ_1 and λ_2, and we summarize the results in Table 1.

It is clear that traveling wave solutions will not exist for $c = 0$ since all critical points are centers then. In what follows we shall assume $c > 0$ without further reminder. We further use the "pplane" program [23] in Matlab to generate the phase portrait of (3.5) shown in Fig. 1, which is consistent with the results given in Table 1. From the phase portrait shown in Fig. 1, we see there are three different types of orbits connecting $B(c, \frac{c^2}{2})$ to $O(0, 0)$, $A(2c, 0)$ to $B(c, \frac{c^2}{2})$ and $A(2c, 0)$ to $O(0, 0)$. We shall rigorously prove the existence of these orbits below by phase plane analysis.

3.1. Heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $O(0, 0)$

Since $B(c, \frac{c^2}{2})$ is a saddle point containing an unstable manifold and $O(0, 0)$ is a stable node, we expect there is a heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $O(0, 0)$. We shall construct a triangular region OBC where $C(c, 0)$ is the middle point of OA (see the schematic in Fig. 2), and show it constitutes an invariant set for the flow of (3.5). Next we shall show that the triangular region OBC is an invariant set for the system (3.5) by studying the flow direction on each side of the triangle.

First, we check the segment OB which can be represented by the equation $L_1(U, W) = 0$, where

$$L_1(U, W) = \frac{c}{2}U - W, \quad \text{for} \quad 0 \le U \le c, \quad 0 \le W \le \frac{c^2}{2}.$$

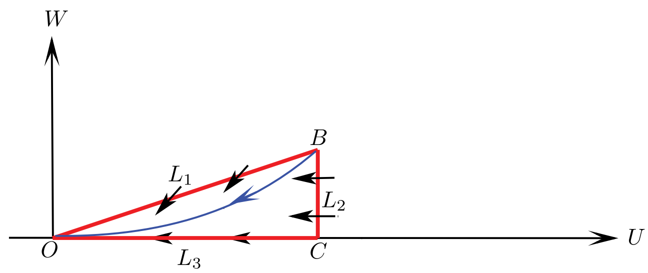


Fig. 2. A schematic of invariant set for the heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $O(0, 0)$.

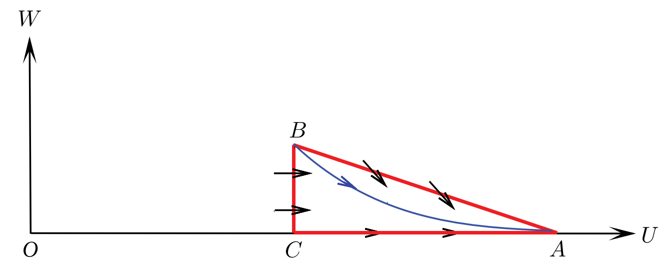


Fig. 3. A schematic of invariant set for the heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $A(2c, 0)$.

It is clear that $L_1 < 0$ on the left side of the line and $L_1 > 0$ on the right side of the line. If there is a flow/orbit leaving OB, there must exist a point p on OB such that $\frac{dL_1}{dz}\big|_p < 0$. We calculate $\frac{dL_1}{dz}$ by substituting (3.5) and $W = \frac{c}{2}U$ into the differentiation and get

$$\begin{aligned}\frac{dL_1}{dz} &= \frac{c}{2}U_z - W_z\\ &= \frac{c}{2}(-cU + W + \frac{U^2}{2}) - (-cW + WU) = \frac{c}{4}U(c-U).\end{aligned}$$

Since $0 < U < c$, $\frac{dL_1}{dz} > 0$ and hence a contradiction arises. Therefore, there are no orbits that can leave the triangular region OBC through the hypotenuse OB.

Next, along the line BC: $U = c$, $0 < w < \frac{c^2}{2}$, we have

$$W_z = W(-c+U) = 0, \quad U_z = -cU + W + \frac{U^2}{2} = W - \frac{c^2}{2} < 0.$$

Hence the flow of (3.5) will point to the left across BC.

Lastly, for the line OC: $W = 0$, $0 < U < c$, we have

$$W_z = W(-c+U) = 0,\ U_z = -cU + W + \frac{U^2}{2} = -cU - \frac{c^2}{2} = U(-c + \frac{U}{2}) < 0.$$

This indicates that the orbits inside the triangular region OBC cannot cross the segment OC to leave the region. Once the flow touches OC, it will move towards O along OC as indicated by the arrows in Fig. 2.

In summary we have shown that the closed bounded triangle OBC is an invariant set, namely once a flow of (3.5) enters the region OBC, it cannot leave it. Next we show there is an orbit emanating from B and entering the triangular region OBC. Since $B(c, \frac{c^2}{2})$ is a saddle, the unstable manifold of (3.5) at B is tangential to the eigenvector associated with the positive eigenvalue $\lambda_2 = \frac{c}{\sqrt{2}}$ (see Table 1) of the Jacobian at $B(c, \frac{c^2}{2})$. By $(J - \lambda_2 I)\vec{v} = 0$, we get the eigenvector $\vec{v} = (1, \frac{c}{\sqrt{2}})^{\mathrm{T}}$ associated with the positive eigenvalue $\lambda_2 = \frac{c}{\sqrt{2}}$. In the following, we denote the slope of a straight line L by $m|_L$. Note that the eigenvector $\vec{v}$ has the slope $\frac{c}{\sqrt{2}}$. Since $m|_{OB} = \frac{c}{2}$ and $m|_{BC} = \infty$, we have $\frac{c}{2} < \frac{c}{\sqrt{2}} < \infty$. This indicates that the unstable manifold (orbit) emanating from B will enter the triangular region OBC. We further show that this orbit has to converge to the critical point O by the Poincaré–Bendixson theorem. To this end, it remains to prove there is no periodic (closed) orbit inside the triangular region OBC. We shall show this by Bendixson's criterion. Indeed it follows from (3.5) and (3.6) that

$$\begin{aligned}&\frac{\partial f}{\partial U} + \frac{\partial g}{\partial W}\\ &= \frac{\partial}{\partial U}(-cU + W + \frac{U^2}{2}) + \frac{\partial}{\partial W}(-cW + WU) = 2(U - c).\end{aligned} \tag{3.8}$$

Inside the triangle OBC, it has that $U \in (0, c)$ and $W \in (0, \frac{c^2}{2})$. Therefore

$$\frac{\partial f}{\partial U} + \frac{\partial g}{\partial W} \neq 0.$$

Thus, by the Bendixson's criterion (cf. [24,25]), there is no closed orbit inside the triangle OBC. Further by the Poincaré–Bendixson theorem (cf. [24,25]), the above orbit emanating from B and entering the closed bounded triangular region OBC will converge to the critical point $O(0, 0)$ as $z \to \infty$. This generates a heteroclinic orbit connecting B to O, which corresponds to a traveling wave solution (U, V) satisfying (3.5). Since the traveling wave ODE system (3.5) is autonomous, if $(U, W)(z)$ is a solution, then so is $(U, W)(z) = (U, W)(z - z_0)$ for any constant z_0, which has the same orbit as $(U, W)(z)$ and corresponds to a traveling wave solution of the same speed that is translated by a constant distance z_0. Noticing that B is a saddle and there is only one unstable manifold emanating from B and entering the triangular region OBC, this heteroclinic orbit is unique up to a translation. Since inside the invariant set OBC, $U < c$ and $W < \frac{c}{2}U$, we must have $U_z = -cU + W + \frac{U^2}{2} < -cU + \frac{c}{2}U + \frac{U^2}{2} = \frac{U}{2}(U - c) < 0$ and $W_z = W(U - c) < 0$. This implies that both profiles U and W are monotone decreasing and completes the proof of Theorem 2.1-(i) for $\epsilon = 0$.

3.2. Heteroclinic orbit connecting $A(2c, 0)$ to $B(c, \frac{c^2}{2})$

Since $A(2c, 0)$ is an unstable node, we cannot construct an invariant region containing A as a vertex for the system (3.5) to show the existence of a heteroclinic orbit connecting $A(2c, 0)$ to $B(c, \frac{c^2}{2})$. Here we shall achieve our goal by reversing the flow direction. That is, we define $\xi = -z = -x + ct$ and show the existence of a heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $A(2c, 0)$ in the phase plane by using ξ as the independent variable. With $\xi = -z = -x + ct$ and $(u, w)(x, t) = (U, W)(\xi)$, we get from (3.5) that

$$\begin{cases} U_\xi = cU - W - \frac{U^2}{2}, \\ W_\xi = cW - WU. \end{cases} \tag{3.9}$$

The critical points of (3.9) are the same as those for the system (3.5), which are $O(0, 0)$, $A(2c, 0)$, and $B(c, \frac{c^2}{2})$. Now we are looking for a heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $A(2c, 0)$. It can be directly checked that $A(2c, 0)$ is a stable node of (3.9) and $B(c, \frac{c^2}{2})$ is a saddle point of (3.9).

Let $A = (2c, 0)$, $B = (c, \frac{c^2}{2})$ and $C = (c, 0)$. Now, we prove the triangular region ABC, as plotted in Fig. 3, is an invariant set of the system (3.9).

First, we check the side BA, which is given by the equation

$$W = -\frac{c}{2}U + c^2, \quad c \le U \le 2c, \quad 0 \le W \le \frac{c^2}{2}.$$

Then the inner product between the inward normal vector $(-\frac{c}{2}, -1)$ to BA and the vector field of (3.9) along the side BA is given by

$$\begin{aligned} p_1 =: (-\frac{c}{2}, -1)\cdot(U_\xi, W_\xi) &= -\frac{c}{2}U_\xi - W_\xi \\ &= -(cW - WU) - \frac{c}{2}(cU - W - \frac{U^2}{2}) \\ &= -\frac{c}{4}\left(U^2 - 3cU + 2c^2\right). \end{aligned}$$

We let $h(U) = U^2 - 3cU + 2c^2$. Since $h(\cdot)$ is a quadratic function and $h(c) = h(2c) = 0$, we know that $h(U) < 0$ for $c < U < 2c$. This gives $p_1 > 0$ for $c < U < 2c$, which means that the angle between the inward normal vector $(-\frac{c}{2}, -1)$ to BA and the vector field of (3.9) at the boundary BA is acute and hence the vector field of (3.9) will flow into the triangular region ABC across the hypotenuse BA.

Next, we check the side BC, which is given by $U = c$ for $0 \le W \le \frac{c^2}{2}$. Along the side BC, $W_\xi = W(c - U) = 0$ and $U_\xi = cU - W - \frac{U^2}{2} = c^2 - W - \frac{c^2}{2} = \frac{c^2}{2} - W > 0$ since $W \in (0, \frac{c^2}{2})$. This implies that the vector field of (3.9) will flow into the triangular region ABC across the side BC and is perpendicular to BC, as sketched in Fig. 3.

Last, along the side CA: $W = 0$, $c < U < 2c$, we find $W_\xi = W(c - U) = 0$ and $U_\xi = cU - \frac{U^2}{2} = U(c - \frac{U}{2}) > 0$ since $U \in (c, 2c)$. This indicates that the side CA is an invariant set of (3.9) and the orbit of (3.9) must stay on CA whenever it touches the side CA.

The above results have collectively shown that the triangular region ABC is an invariant set for the system (3.9). We proceed to show that (3.9) has no periodic orbit contained inside the triangle ABC where $U \in (c, 2c)$ and $W \in (0, \frac{c^2}{2})$. Indeed with $\tilde{f}(U, W) = cU - W - \frac{U^2}{2}$ and $\tilde{g} = cW - WU$, we can check that

$$\begin{aligned} &\frac{\partial \tilde{f}}{\partial U} + \frac{\partial \tilde{g}}{\partial W} \\ &= \frac{\partial}{\partial U}(cU - W - \frac{U^2}{2}) + \frac{\partial}{\partial W}(cW - WU) = 2(c - U) \neq 0 \end{aligned}$$

which alongside the Bendixson's criterion entails that there is not a periodic (closed) orbit contained within the triangular region ABC. Finally we prove there is an orbit departing from B and entering the triangular region ABC. To this end, we check the direction of the unstable manifold emanating from the saddle point B, which is tangential to the eigenvector associated with the positive eigenvalue of the Jacobian matrix of (3.9) at B. It is easy to find that the Jacobian matrix of (3.9) at B has two eigenvalues $\lambda_\pm = \pm\frac{c}{\sqrt{2}}$ and the eigenvector associated with the positive eigenvalue $\lambda_+ = \frac{c}{\sqrt{2}}$ is $\vec{v} = (1, -\frac{c}{\sqrt{2}})^T$, which has slope $-\frac{c}{\sqrt{2}}$. Note that the slope of BA is $-\frac{c}{2}$ and the slope of BC is ∞. Hence the fact $-\frac{c}{2} < -\frac{c}{\sqrt{2}} < \infty$ asserts that the unstable manifold emanating from the saddle point B will enter the triangular region ABC. Since there is not a periodic orbit contained within the triangular region ABC, this orbit will converge to the critical point A as $\xi \to +\infty$, which forms a heteroclinic orbit connecting B to A that is unique up to a translation. If we reverse the direction of this orbit (i.e. changing $-\xi$ to z), we obtain a heteroclinic orbit connecting A to B, which generates a traveling wave solution for the system (3.5). This finishes the proof of Theorem 2.1-(ii) for $\epsilon = 0$.

3.3. Heteroclinic orbit connecting $A(2c, 0)$ to $O(0, 0)$

Now we prove that the system (3.5) admits heteroclinic orbits connecting $A(2c, 0)$ to $O(0, 0)$. In Section 3.1, we showed that

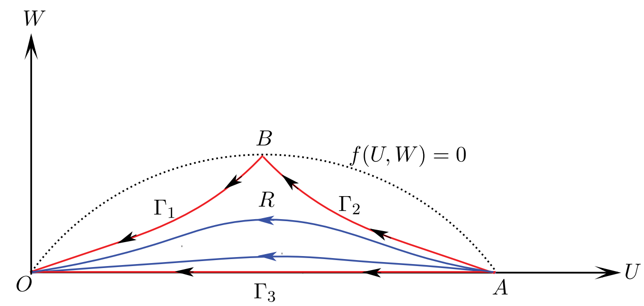


Fig. 4. A schematic of invariant set for the heteroclinic orbit connecting $A(2c, 0)$ to $O(0, 0)$.

there is a heteroclinic orbit connecting $B(c, \frac{c^2}{2})$ to $O(0, 0)$, which is indeed a separatrix denoted by Γ_1 in the following. In Section 3.2, we showed that there is a heteroclinic orbit connecting $A(2c, 0)$ to $B(c, \frac{c^2}{2})$, which is another separatrix denoted by Γ_2. Now we consider a region, denoted by R, bounded by Γ_1, Γ_2 and Γ_3 denoting the segment OA, as shown in Fig. 4. We have shown previously that A is an unstable node and O is a stable node. Hence any unstable manifold emanating from A and entering the region R cannot touch or intersect with Γ_i $(i = 1, 2, 3)$, since otherwise the uniqueness of solutions of the ODE system (3.5) is violated. Next we shall use Bendixson–Dulac theorem to show that there is no closed (periodic) orbit within the region R. To this end, we construct a Dulac function: $\varphi(U, W) = U^{-\alpha}$, where $\alpha \gg 1$. By (3.5) and (3.8), we have

$$\begin{aligned} \frac{\partial}{\partial U}(\varphi f) + \frac{\partial}{\partial W}(\varphi g) &= \frac{\partial}{\partial U}(U^{-\alpha} f) + \frac{\partial}{\partial W}(U^{-\alpha} g) \\ &= U^{-\alpha}\left(\frac{\partial f}{\partial U} + \frac{\partial g}{\partial W}\right) - \alpha U^{-\alpha-1} f \\ &= U^{-\alpha}(2(U - c)) - \alpha U^{-\alpha-1}(-cU + W + \frac{U^2}{2}) \\ &= (2 - \frac{\alpha}{2})U^{-\alpha-1}\left(U^2 + \frac{\alpha - 2}{2 - \frac{\alpha}{2}}cU - \frac{\alpha W}{2 - \frac{\alpha}{2}}\right). \end{aligned}$$

Noting that $\alpha \gg 1$, one can easily check that

$$\lim_{\alpha\to\infty}\left(U^2 + \frac{\alpha - 2}{2 - \frac{\alpha}{2}}cU - \frac{\alpha W}{2 - \frac{\alpha}{2}}\right) = 2(-cU + W + \frac{U^2}{2}).$$

Since the region R is always below the curve $f(U, W) = -cU + W + \frac{U^2}{2} = 0$ except the vertex B, we have $2(-cU + W + \frac{U^2}{2}) < 0$ for any (U, W) within the region R. Therefore, when $\alpha > 0$ is sufficiently large, we have $2 - \frac{\alpha}{2} < 0$ and consequently

$$\frac{\partial}{\partial U}(\varphi f) + \frac{\partial}{\partial W}(\varphi g) > 0.$$

That means $\frac{\partial}{\partial U}(\varphi f) + \frac{\partial}{\partial W}(\varphi g) \neq 0$ within the region R. Hence, by the Bendixson–Dulac theorem, there are no periodic orbits lying within the region R. Furthermore it follows from Poincaré–Bendixson theorem that the orbits emanating from the unstable manifolds of A and entering the region R will converge to the critical points O or B as $z \to +\infty$. But the convergence to the critical point B is impossible since B is a saddle point and Γ_2 is a separatrix. Therefore such orbits have to converge to the stable node O as $z \to +\infty$. Since there are infinitely many outgoing unstable manifolds from A entering the region R, there will be infinitely many such heteroclinic orbits connecting A to O, which generate infinitely many traveling wave solutions connecting $A(2c, 0)$ to $O(0, 0)$. Note that all orbits within the region R are under the parabola $f(U, W) = 0$. Hence $U_z = -cU + W + \frac{U^2}{2} < 0$ and there is a unique point $z_0 \in \mathbb{R}$ such that $U(z_0) = c$. Therefore

$W_z = W(U-c) > 0$ if $z < z_0$ while $W_z = W(U-c) < 0$ if $z > z_0$. This completes the proof of Theorem 2.1-(iii) for $\epsilon = 0$.

3.4. Geometric singular perturbation

In this section, we apply the geometric singular perturbation theory to show the existence of traveling wave solutions to the Boussinesq–Burgers system (1.4)–(1.5) when $\epsilon > 0$ is small and hence complete the proof of Theorem 2.1.

From the analysis in Section 2, we know that the traveling wave profile $(W, U)(z)$ of the original Boussinesq–Burgers system (1.4) satisfies

$$\begin{cases} W_z = -cW + WU, \\ -\epsilon c U_{zz} + U_z = -cU + W + \frac{U^2}{2} \end{cases} \tag{3.10}$$

where the constants of integration have been assumed to be zero, see also (3.4). With $V = U_z$, we rewrite the above equations as a system of first order ODEs:

$$\begin{cases} U_z = V, \\ W_z = -cW + WU, \\ \epsilon V_z = U + \frac{1}{c}(V - W - \frac{U^2}{2}) \end{cases} \tag{3.11}$$

which is a so-called slow system (cf. [26–28]). We see that the critical manifold $\mathcal{M}_0$ as a set of critical points is a compact subset contained in the following manifold

$$\mathscr{L} = \left\{ (U, W, V) \middle| V = -cU + W + \frac{U^2}{2} =: h^0(U, W) \right\}.$$

Hence on the manifold $\mathcal{M}_0$, (U, W) satisfies the following reduced system

$$\begin{cases} U_z = -cU + W + \frac{U^2}{2}, \\ W_z = -cW + WU \end{cases} \tag{3.12}$$

which is nothing but the system (3.5). We have proved in the preceding subsections that the system (3.5) subject to (1.7) admits three different types of heteroclinic orbits (or solutions) for which we denote by $(U^0, W^0)(z)$. That is, the slow system with $\epsilon = 0$ has solutions (U^0, W^0, V^0) with $V^0 = h^0(U^0, W^0) = -cU^0 + W^0 + \frac{(U^0)^2}{2}$.

Now we define the so-called fast variable $\tau = \frac{z}{\epsilon}$ and write the slow system (3.11) into a fast system

$$\begin{cases} \dot{U} = \epsilon V, \\ \dot{W} = \epsilon(-cW + WU), \\ \dot{V} = U + \frac{1}{c}(V - W - \frac{U^2}{2}) \end{cases} \tag{3.13}$$

where $\dot{f} = \frac{df}{d\tau}$. It is clear that

$$\frac{\partial}{\partial V} G(U, W, V)\big|_{\mathcal{M}_0} = \frac{1}{c}, \quad \text{where} \quad G(U, W, V) = \dot{V} = U + \frac{1}{c}\left(V - W - \frac{U^2}{2}\right),$$

which implies that the manifold $\mathcal{M}_0$ is normally hyperbolic for the fast system (3.13) (cf. [26,27]). Then by the Fenichel's invariant manifold theorem [27,29], for $\epsilon > 0$ sufficiently small, there is a slow manifold $\mathcal{M}_\epsilon$ that lies within $O(\epsilon)$ neighborhood of $\mathcal{M}_0$ and is diffeomorphic to $\mathcal{M}_0$. Moreover it is locally invariant under the flow of (3.13) and can be written as

$$\mathcal{M}_\epsilon = \{(U, W, V) \mid V = h^\epsilon(U, W) = h^0(U, W) + O(\epsilon)\}.$$

Then the slow system (3.11) on $\mathcal{M}_\epsilon$ can be written as

$$\begin{cases} U_z = -cU + W + \frac{U^2}{2} + O(\epsilon), \\ W_z = -cW + WU, \end{cases}$$

which implies that the manifold $\mathcal{M}_0$ is normally hyperbolic for the fast system (3.13) (cf. [26,27]). Then by the Fenichel's invariant manifold theorem [27,29], for $\epsilon > 0$ sufficiently small, there is a slow manifold $\mathcal{M}_\epsilon$ that lies within $O(\epsilon)$ neighborhood of $\mathcal{M}_0$ and is diffeomorphic to $\mathcal{M}_0$. Moreover it is locally invariant under the flow of (3.13) and can be written as

$$\mathcal{M}_\epsilon = \{(U, W, V) \mid V = h^\epsilon(U, W) = h^0(U, W) + O(\epsilon)\}.$$

Then the slow system (3.11) on $\mathcal{M}_\epsilon$ can be written as

$$\begin{cases} U' = -cU + W + \frac{U^2}{2} + O(\epsilon), \\ W' = -cW + WU, \end{cases}$$

which is a regular perturbation of (3.12). Next we show that the heteroclinic orbits connecting B to O, A to B, and A to O for $\epsilon = 0$ on $\mathcal{M}_0$ persist to small $\epsilon > 0$ on $\mathcal{M}_\epsilon$. Below we shall discuss the case of heteroclinic orbits connecting B to O only while other two cases are similar. By [30, Theorem 3.1], we just need to verify that the unstable manifold of the critical point $B(c, c^2/2)$ (denoted by $\mathcal{N}_B^u$) and the stable manifold of another critical point $O(0, 0)$ (denoted by $\mathcal{N}_O^s$) intersect transversally along the so-called singular heteroclinic orbit. Since the manifold $\mathscr{L}$ consists of only one branch, namely the critical points B and O lie in the same branch of the manifold of the reduced problem (i.e. the heteroclinic orbits of (3.12)), then the transversal intersection of $\mathcal{N}_B^u$ and $\mathcal{N}_O^s$ is completely determined by the reduced problem (see the discussion in section 3 of [30]). However we have shown in Section 3.1 that the reduced problem admits a heteroclinic orbit connecting B to O along which $\mathcal{N}_B^u$ and $\mathcal{N}_O^s$ intersect transversally. Thus by [30, Theorem 3.1], for sufficiently small $\epsilon > 0$, there is a transversal heteroclinic orbit $(U^\epsilon, W^\epsilon, V^\epsilon)(z)$ of the singularly perturbed system (3.11) on $\mathcal{M}_\epsilon$, which is a small perturbation of (U^0, W^0, V^0) and connects the two critical points B and O as illustrated in Fig. 1 where the reduced system (3.12) on $\mathcal{M}_0$ and the singularly perturbed system (3.11) share the same critical points. This heteroclinic orbit $(U^\epsilon, W^\epsilon, V^\epsilon)(z)$ gives a traveling wave solution to the singularly perturbed system (3.11) with small $\epsilon > 0$. Alongside the results established for the reduced problem (3.12) in preceding sections, we complete the proof of Theorem 2.1.

4. Nonlinear asymptotic stability

In this section, we prove the nonlinear asymptotic stability of the traveling wave solutions obtained in Theorem 2.1. Specifically, we show that the solution of (1.4)–(1.5) approaches the traveling wave solution $(W, U)(x - ct)$, properly translated by an amount x_0, i.e.,

$$\sup_{x\in\mathbb{R}} |(w, u)(x, t) - (W, U)(x + x_0 - ct)| \to 0, \quad \text{as} \quad t \to +\infty,$$

where x_0 satisfies the following identity derived from the principle of "conservation of mass" (see details in [31]):

$$\int_{-\infty}^{+\infty} \begin{pmatrix} w_0(x) - W(x) \\ u_0(x) - U(x) \end{pmatrix} dx = x_0 \begin{pmatrix} w_+ - w_- \\ u_+ - u_- \end{pmatrix} + \beta r_1(w_-, u_-), \tag{4.1}$$

where $r_1(w_-, u_-)$ denotes the first right eigenvector of the Jacobian matrix of (1.4) with $\mu = \epsilon = 0$ evaluated at (w_-, u_-). The coefficient β yields the diffusion wave in general [32]. Both β and x_0 are uniquely determined by the initial data (w_0, u_0). For the stability of small-amplitude viscous shock waves of conservation laws with diffusion wave (i.e. $\beta \neq 0$), we refer to [33,34] for details.

In this paper, we neglect the diffusion wave by assuming $\beta = 0$ (equivalent to the assumption in Theorem 2.2), but consider the stability of large-amplitude waves (meaning that the wave strengths $|w_- - w_+|$ and $|u_- - u_+|$ are allowed to be arbitrarily large). It is worth mentioning that the stability of large-amplitude traveling waves of conservation laws is a prominent question and there is no result for general conservation laws (cf. [21,33,35,36]), except a few for particular systems (e.g., some system of conserved equations derived from chemotaxis model [32,37–39]). Then using the conservative property of the equations in (1.4) and (1.8), along with the boundary conditions and (4.1), we can show that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\begin{pmatrix} w(x,t)-W(x+x_0-ct)\\ u(x,t)-U(x+x_0-ct)\end{pmatrix}dx\\ &=\int_{-\infty}^{+\infty}\begin{pmatrix} w_0(x)-W(x+x_0)\\ u_0(x)-U(x+x_0)\end{pmatrix}dx\\ &=\int_{-\infty}^{+\infty}\begin{pmatrix} w_0(x)-W(x)\\ u_0(x)-U(x)\end{pmatrix}dx+\int_{-\infty}^{+\infty}\begin{pmatrix} W(x)-W(x+x_0)\\ U(x)-U(x+x_0)\end{pmatrix}dx\\ &=\int_{-\infty}^{+\infty}\begin{pmatrix} w_0(x)-W(x)\\ u_0(x)-U(x)\end{pmatrix}dx-x_0\begin{pmatrix} w_+-w_-\\ u_+-u_-\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.\end{aligned} \tag{4.2}$$

This allows us to make use of the technique of taking antiderivative to decompose the solution of (1.4) as

$$(w,u)(x,t)=(W,U)(x+x_0-ct)+(\phi_z,\psi_z)(z,t), \tag{4.3}$$

where $z=x-ct$. That is

$$\begin{aligned}(\phi(z,t),\psi(z,t))=&\int_{-\infty}^{z}(w(y,t)-W(y+x_0-ct),u(y,t)\\ &-U(y+x_0-ct))dy\end{aligned}$$

for all $z\in\mathbb{R}$ and $t\ge 0$. It then follows from (4.2) that

$$\phi(\pm\infty,t)=\psi(\pm\infty,t)=0,\quad \text{for all}\quad t\ge 0.$$

Next we derive the equations for (ϕ,ψ). Indeed, substituting (4.3) into (1.4) and using (3.4), we integrate the resulting equations with respect to z and get

$$\begin{cases}\phi_t=\phi_{zz}+(c-U)\phi_z-W\psi_z-\phi_z\psi_z,\\ \psi_t=\psi_{zz}+(c-U)\psi_z-\phi_z-\dfrac{1}{2}\psi_z^2-\epsilon c\psi_{zzz}+\epsilon\psi_{zzt},\\ \phi(\pm\infty)=\psi(\pm\infty)=0,\end{cases} \tag{4.4}$$

where the initial value of (ϕ,ψ) satisfies

$$(\phi_0,\psi_0)(z)=\int_{-\infty}^{z}(w_0(y)-W(y+x_0),u_0(y)-U(y+x_0))dy \tag{4.5}$$

with $(\phi_0,\psi_0)(\pm\infty)=(0,0)$.

We look for solutions of system (4.4) in the following weighted functional space:

$$\begin{aligned}X(0,T):=\{&(\phi(z,t),\psi(z,t)):\phi\in C([0,T];H^2_\omega),\phi_z\in L^2((0,T);H^2_\omega),\\ &\psi\in C([0,T];H^3),\psi_z\in L^2((0,T);H^2)\},\end{aligned}$$

where $\omega(z)=1/W(z)$. From the analysis in Section 3 (see Fig. 1 also), we know that $0<W(z)<u_-^2/2$ for all $x\in\mathbb{R}$ with given $u_->0$ and hence there is constant $m=u_-^2/2$ such that $\omega>m>0$. Define

$$N(t):=\sup_{\tau\in[0,t]}(\|\phi(\tau)\|_{2,\omega}+\|\psi(\tau)\|_3),$$

where $\|f(\cdot)\|_N=\sum_{k=0}^N\|\partial_x^k f\|_{L^2(\mathbb{R})}$ and $\|f(\cdot)\|_{N,\omega}=\sum_{k=0}^N\|\sqrt{\omega}\,\partial_x^k f\|_{L^2(\mathbb{R})}$. In the sequel, we shall abbreviate $\|f\|_{L^2(\mathbb{R})}$ as $\|f\|$ if there is no danger of confusion and denote $\|f\|_\omega=\|\sqrt{\omega}f\|$. Thanks to the first equation of (3.4) and monotonicity of U, one has

$$\begin{aligned}&\frac{|W_z|}{W}=|U-c|\le u_-,\quad\text{and}\quad\left|\left(\frac{\phi}{\sqrt{W}}\right)_z\right|\\ &=\left|\frac{\phi_z}{\sqrt{W}}-\frac{\phi W_z}{2W\sqrt{W}}\right|\le\frac{|\phi_z|}{\sqrt{W}}+\frac{u_-}{2}\cdot\frac{|\phi|}{\sqrt{W}}.\end{aligned} \tag{4.6}$$

Hence by the basic inequality: $\|f\|^2_{L^\infty}\le 2\|f\|_{L^2}\|f_x\|_{L^2}$, for all $f\in W^{1,2}(\mathbb{R})$, it holds that

$$\begin{aligned}&\sup_{\tau\in[0,t]}\left\{\left\|\frac{\phi(\tau)}{\sqrt{W}}\right\|_{L^\infty},\left\|\frac{\phi_z(\tau)}{\sqrt{W}}\right\|_{L^\infty},\|\psi(\tau)\|_{L^\infty},\|\psi_z(\tau)\|_{L^\infty}\right\}\\ &\le C_0N(t),\end{aligned} \tag{4.7}$$

where $C_0=[2(1+\frac{u_-}{2})]^{\frac{1}{2}}$ is a positive constant. Owing to (4.3), Theorem 2.2 is a consequence of the following result for the reformulated system (4.4).

Proposition 4.1. *There exists a positive constant δ_0, such that if $N(0)\le\delta_0$, then the Cauchy problem (4.4)–(4.5) has a unique global solution $(\phi,\psi)\in X(0,+\infty)$ satisfying*

$$\begin{aligned}&\|\phi(t)\|^2_{2,\omega}+\|\psi(t)\|^2_3+\int_0^t(\|\phi_z(\tau)\|^2_{2,\omega}+\|\psi_z(\tau)\|^2_2)d\tau\\ &\le C(\|\phi_0\|^2_{2,\omega}+\|\psi_0\|^2_3)\end{aligned} \tag{4.8}$$

for all $t\in[0,+\infty)$, where the constant $C>0$ is independent of t. Moreover, it holds that

$$\sup_{z\in\mathbb{R}}|(\phi_z,\psi_z,\psi_{zz})(z,t)|\to 0\quad as\quad t\to\infty. \tag{4.9}$$

To prove Proposition 4.1, we first present the local existence and uniqueness of solutions to the Cauchy problem (4.4)–(4.5).

Proposition 4.2 (*Local Existence*). *For any $\delta_1>0$, there exists a positive constant T_0 depending on δ_1, such that if $(\phi_0,\psi_0)\in H^2_\omega\times H^3$ and $N(0)\le\delta_1$, then (4.4)–(4.5) has a unique solution $(\phi,\psi)\in X(0,T_0)$ satisfying $N(t)\le 2N(0)$ for any $t\in[0,T_0]$.*

In [40] (see also [41,42]), the local existence and uniqueness of solutions (in the H^s space) to the Cauchy problem of the Boussinesq-*abcd* system was proved in detail. The approach utilized therein consists of well-known reasonings, such as mollification of initial data, integral representation of solutions by Fourier transform, local existence and uniqueness of smooth approximate solutions by contraction mapping principle, *a priori* estimate, and compactness argument. Since the Boussinesq–Burgers equations (1.3) is an appended version of a specific member of the Boussinesq-*abcd* system by adding diffusion, and since the traveling wave solution is a smooth function of its arguments, we can perform the similar procedures as in [40] to prove the local existence and uniqueness of solutions to (4.4)–(4.5) by modifying some computations. As usual, the essence of the proof is the *a priori* estimates of the solution. Hence, we omit most of the standard technical details for brevity, while focus on deriving the *a priori* estimates. Moreover, as in the typical situation (see e.g. [40]), the *a priori* estimates are indeed performed on the smooth approximate solutions obtained from the mollified initial data and contraction mapping principle. Hence the required regularity of the solution in our subsequent estimates is warranted and we shall derive the requisite *a priori* estimates below.

Proposition 4.3 (*A priori Estimates*). *Suppose that $(\phi,\psi)\in X(0,T)$ is a solution to (4.4)–(4.5) obtained in Proposition 4.2 for some $T>0$. Then there exists a constant $\delta_2>0$ independent of T, such that if $N(t)\le\delta_2$ for any $t\in[0,T]$, then the solution (ϕ,ψ) satisfies (4.8) for any $t\in[0,T]$.*

After this proposition is established, we can extend the local solution obtained in Proposition 4.2 by repeating the procedures mentioned above. More importantly, since the *a priori* estimates are independent of time, the extension can be made on consecutive time intervals with equal length which is the lifespan of the local solution. Hence, the construction of a unique global solution is complete.

The proof of Proposition 4.3 is based on the following series of lemmas. In what follows, for the sake of simplicity we shall abbreviate $\int_{\mathbb{R}} f(x,t)dx$ and $\int_0^t\int_{\mathbb{R}} f(x,\tau)dxd\tau$ as $\int_{\mathbb{R}} f(x,t)$ and $\int_0^t\int_{\mathbb{R}} f(x,\tau)$, respectively.

Lemma 4.4. *Under the assumptions of Proposition 4.3, there exists a constant $C>0$, independent of t, such that if $C_0N(t)\le 1/2$, then*

$$\|\phi(t)\|_\omega^2+\|\psi(t)\|_1^2 + \int_0^t(\|\phi_z(\tau)\|_\omega^2+\|\psi_z(\tau)\|^2)d\tau \le C(\|\phi_0\|_\omega^2+\|\psi_0\|_1^2). \tag{4.10}$$

Proof. Multiplying the first equation of (4.4) by $\frac{2\phi}{W}$ and the second one by 2ψ, and adding the results, we obtain

$$\begin{aligned}&\left(\frac{\phi^2}{W}+\psi^2+\epsilon\psi_z^2\right)_t+\frac{2\phi_z^2}{W}+2\psi_z^2\\&\quad+\left[-\left(\frac{1}{W}\right)_{zz}+\left(\frac{c-U}{W}\right)_z\right]\phi^2-U_z\psi^2\\&=\Big[\frac{2\phi\phi_z}{W}-\left(\frac{1}{W}\right)_z\phi^2+\frac{c-U}{W}\phi^2-2\phi\psi\\&\quad+2\epsilon\psi\psi_{zt}+2\psi_z\psi+(c-U)\psi^2\Big]_z\\&\quad-\frac{2\phi\phi_z\psi_z}{W}-\psi\psi_z^2-2\epsilon c\psi_{zzz}\psi.\end{aligned}\tag{4.11}$$

By the first equation of (3.4), a direct calculation yields

$$-\left(\frac{1}{W}\right)_{zz}+\left(\frac{c-U}{W}\right)_z=0. \tag{4.12}$$

Integrating (4.11) in z and t, noticing $U_z<0$, $2\epsilon c\psi_{zzz}\psi=2\epsilon c(\psi_{zz}\psi)_z-\epsilon c(\psi_z^2)_z$, and utilizing (4.12), we can show that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{\phi^2}{W}+\psi^2+\epsilon\psi_z^2\right)+2\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2\right)\\&\le 2\int_0^t\int_{\mathbb{R}}\frac{|\phi\phi_z\psi_z|}{W}+\int_0^t\int_{\mathbb{R}}|\psi|\psi_z^2+\int_{\mathbb{R}}\left(\frac{\phi_0^2}{W}+\psi_0^2+\epsilon\psi_{0z}^2\right).\end{aligned}\tag{4.13}$$

The two terms on the right hand side of (4.13) can be estimated by Young's inequality alongside (4.7) as follows:

$$\begin{aligned}2\int_0^t\int_{\mathbb{R}}\frac{|\phi\phi_z\psi_z|}{W}&\le 2\int_0^t\left\|\frac{\phi(\tau)}{\sqrt{W}}\right\|_{L^\infty}\int_{\mathbb{R}}\frac{|\phi_z\psi_z|}{\sqrt{W}}\\&\le C_0N(t)\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2\right)\end{aligned}$$

and

$$\int_0^t\int_{\mathbb{R}}|\psi|\psi_z^2\le\int_0^t\|\psi(\tau)\|_{L^\infty}\int_{\mathbb{R}}\psi_z^2\le C_0N(t)\int_{\mathbb{R}}\psi_z^2.$$

Then substituting the above two inequalities into (4.13) yields that

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{\phi^2}{W}+\psi^2+\epsilon\psi_z^2\right)+2(1-C_0N(t))\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2\right)\\&\le\int_{\mathbb{R}}\left(\frac{\phi_0^2}{W}+\psi_0^2+\epsilon\psi_{0z}^2\right),\end{aligned}$$

which gives (4.10) under our assumption $C_0N(t)\le\frac{1}{2}$. □

We proceed to derive the estimate of the first order derivatives of (ϕ,ψ).

Lemma 4.5. *Under the assumptions of Proposition 4.3, there exists a constant $C>0$, independent of t, such that if $C_0N(t)\le 1/2$, then the following a priori estimate holds:*

$$\begin{aligned}&\|\phi_z(t)\|_\omega^2+\|\psi_z(t)\|_1^2+\int_0^t(\|\phi_{zz}(\tau)\|_\omega^2+\|\psi_{zz}(\tau)\|^2)d\tau\\&\le C(\|\phi_0\|_{1,\omega}^2+\|\psi_0\|_2^2).\end{aligned}\tag{4.14}$$

Proof. Differentiating (4.4) with respect to z yields

$$\begin{cases}\phi_{zt}=\phi_{zzz}+(c-U)\phi_{zz}-U_z\phi_z-W\psi_{zz}-W_z\psi_z-(\phi_z\psi_z)_z,\\ \psi_{zt}=\psi_{zzz}+(c-U)\psi_{zz}-U_z\psi_z-\phi_{zz}-\psi_z\psi_{zz}-c\epsilon\psi_{zzzz}\\ \qquad+\epsilon\psi_{zzzt}.\end{cases}\tag{4.15}$$

Multiplying the first equation of (4.15) by $\frac{2\phi_z}{W}$ and the second equation by $2\psi_z$, after some careful calculations, we end up with

$$\begin{aligned}&\left(\frac{\phi_z^2}{W}+\psi_z^2+\epsilon\psi_{zz}^2\right)_t\\&\quad+\frac{2\phi_{zz}^2}{W}+2\psi_{zz}^2+\left[-\left(\frac{1}{W}\right)_{zz}+\left(\frac{c-U}{W}\right)_z\right]\phi_z^2\\&=\Big[\frac{2\phi_{zz}\phi_z}{W}-\left(\frac{1}{W}\right)_z\phi_z^2+\frac{(c-U)}{W}\phi_z^2-2\psi_z\phi_z\\&\quad+2\psi_z\psi_{zz}+2\epsilon\psi_z\psi_{zzt}+(c-U)\psi_z^2-\frac{2}{3}\psi_z^3\Big]_z\\&\quad-\frac{2U_z\phi_z^2}{W}-\frac{2W_z\phi_z\psi_z}{W}-U_z\psi_z^2-\frac{2\phi_z(\phi_z\psi_z)_z}{W}-2c\epsilon\psi_z\psi_{zzzz}.\end{aligned}\tag{4.16}$$

Using the first piece of information in (4.6) and boundedness of W, we can show that

$$\begin{aligned}\left|\frac{2W_z\phi_z\psi_z}{W}\right|&\le 2u_-|\phi_z\psi_z|\le u_-\left(\frac{\phi_z^2}{W}+W\psi_z^2\right)\\&\le u_-\left(\frac{\phi_z^2}{W}+\overline{w}\psi_z^2\right),\end{aligned}\tag{4.17}$$

where $\overline{w}>0$ denotes the upper bound of W (indeed $\overline{w}=u_-^2/2$ from the analysis in Section 2). From the second equation of (3.4) we infer:

$$-U_z=\frac{1}{c\epsilon}e^{\frac{z}{c\epsilon}}\int_z^\infty e^{-\frac{\xi}{c\epsilon}}[-f(U,W)]d\xi, \tag{4.18}$$

where $f(U,W)=-cU+W+\frac{U^2}{2}$. Note that $U_z<0$ and $|f(U,W)|\le cu_-+\overline{w}+\frac{u_-^2}{2}:=\ell$. Then we update (4.18) as

$$|U_z|\le\frac{\ell}{c\epsilon}e^{\frac{z}{c\epsilon}}\int_z^\infty e^{-\frac{\xi}{c\epsilon}}d\xi=\ell. \tag{4.19}$$

With the fact $2c\epsilon\psi_z\psi_{zzzz}=2c\epsilon(\psi_z\psi_{zzz})_z-c\epsilon(\psi_{zz}^2)_z$, as well as

$$\frac{\phi_z(\phi_z\psi_z)_z}{W}=\frac{\phi_z}{\sqrt{W}}\cdot\frac{\phi_{zz}}{\sqrt{W}}\cdot\psi_z+\frac{\phi_z}{\sqrt{W}}\cdot\frac{\phi_z}{\sqrt{W}}\cdot\psi_{zz},$$

we derive from (4.16), (4.12) and (4.10):

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2+\epsilon\psi_{zz}^2\right)+2\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2\right)\\&\le C\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2\right)+\int_0^t\int_{\mathbb{R}}\frac{2|\phi_z(\phi_z\psi_z)_z|}{W}\\&\quad+\int_{\mathbb{R}}\left(\frac{\phi_{0z}^2}{W}+\psi_{0z}^2+\epsilon\psi_{0zz}^2\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_z^2}{W}+\psi_z^2\right)+C_0N(t)\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2\right)\\
&\quad+\int_{\mathbb{R}}\left(\frac{\phi_{0z}^2}{W}+\psi_{0z}^2+\epsilon\psi_{0zz}^2\right)\\
&\le C_0N(t)\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2\right)\\
&\quad+C\int_{\mathbb{R}}\left(\frac{\phi_{0z}^2}{W}+\frac{\phi_0^2}{W}+\psi_{0zz}^2+\psi_{0z}^2+\psi_0^2\right),
\end{aligned}
\tag{4.20}
$$

where we have used (4.17) and (4.19) in the first inequality, and applied (4.7) in the second inequality alongside the Cauchy–Schwarz inequality. Therefore, if $C_0N(t)\le 1/2$, the inequality (4.14) follows immediately from (4.20). □

Next we establish the estimates of the second order derivatives of (ϕ,ψ).

Lemma 4.6. *Under the assumptions of Proposition 4.3, there exists a constant $C>0$, independent of t, such that if $C_0N(t)\le 1/2$, we have the following a priori estimate:*

$$
\begin{aligned}
&\|\phi_{zz}(t)\|_\omega^2+\|\psi_{zz}(t)\|_1^2+\int_0^t(\|\phi_{zzz}(\tau)\|_\omega^2+\|\psi_{zzz}(\tau)\|^2)d\tau\\
&\quad\le C\left(\|\phi_0\|_{2,\omega}^2+\|\psi_0\|_3^2\right).
\end{aligned}
\tag{4.21}
$$

Proof. We differentiate (4.15) with respect to z to get

$$
\begin{cases}
\phi_{zzt}=\phi_{zzzz}+(c-U)\phi_{zzz}-2U_z\phi_{zz}-U_{zz}\phi_z-W_{zz}\psi_z\\
\qquad\quad-2W_z\psi_{zz}-W\psi_{zzz}-(\phi_z\psi_z)_{zz},\\
\psi_{zzt}=\psi_{zzzz}+(c-U)\psi_{zzz}-2U_z\psi_{zz}-U_{zz}\psi_z-\phi_{zzz}\\
\qquad\quad-(\psi_z\psi_z)_z-c\epsilon\psi_{zzzzz}+\epsilon\psi_{zzzzt}.
\end{cases}
\tag{4.22}
$$

Multiplying the first equation of (4.22) by $\frac{2\phi_{zz}}{W}$ and the second equation by $2\psi_{zz}$, with some tedious computations, we have

$$
\begin{aligned}
&\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2+\epsilon\psi_{zzz}^2\right)_t+\frac{2\phi_{zzz}^2}{W}+2\psi_{zzz}^2\\
&+\left[-\left(\frac{1}{W}\right)_{zz}+\left(\frac{c-U}{W}\right)_z\right]\phi_{zz}^2\\
&=\Bigg[\frac{2\phi_{zzz}\phi_{zz}}{W}-\left(\frac{1}{W}\right)_z\phi_{zz}^2+\frac{(c-U)}{W}\phi_{zz}^2-2\psi_{zz}\phi_{zz}\\
&+2\psi_{zz}\psi_{zzz}+(c-U)\psi_{zz}^2+2\epsilon\psi_{zz}\psi_{zzzt}\\
&-2\psi_z\psi_{zz}^2-\frac{2(\phi_z\psi_z)_z\phi_{zz}}{W}\Bigg]_z\\
&-\frac{4U_z\phi_{zz}^2}{W}-\frac{2U_{zz}\phi_z\phi_{zz}}{W}-\frac{2W_{zz}\phi_{zz}\psi_z}{W}-\frac{4W_z\phi_{zz}\psi_{zz}}{W}\\
&-3U_z\psi_{zz}^2-2U_{zz}\psi_z\psi_{zz}+2(\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{W}\right)_z\\
&+2\psi_z\psi_{zz}\psi_{zzz}-2c\epsilon\psi_{zz}\psi_{zzzzz}.
\end{aligned}
\tag{4.23}
$$

From (3.4) and (4.19), one can find a constant $C>0$ such that

$$
|U_{zz}|\le C,\quad\text{and}\quad\frac{|W_{zz}|}{W}\le(U-c)^2+|U_z|\le C.
\tag{4.24}
$$

Thus, by (4.24) and the boundedness of W, one can show that

$$
\frac{2|W_{zz}\phi_{zz}\psi_z|}{W}\le C\left(\frac{\phi_{zz}^2}{W}+\psi_z^2\right),\quad\text{and}\quad\frac{4|W_z\phi_{zz}\psi_{zz}|}{W}\le C\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2\right).
$$

Moreover, a direct calculation gives us

$$
\begin{aligned}
2\left|(\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{W}\right)_z\right|&\le 2\left|\psi_z\frac{\phi_{zz}}{\sqrt{W}}\frac{\phi_{zzz}}{\sqrt{W}}\right|+2\left|\frac{\phi_z}{\sqrt{W}}\psi_{zz}\frac{\phi_{zzz}}{\sqrt{W}}\right|\\
&\quad+2\left|\frac{W_z}{W}\frac{\phi_z}{\sqrt{W}}\psi_{zz}\frac{\phi_{zz}}{\sqrt{W}}\right|+2\left|\frac{W_z}{W}\psi_z\frac{\phi_{zz}^2}{W}\right|.
\end{aligned}
$$

Note that $2c\epsilon\psi_{zz}\psi_{zzzzz}=2c\epsilon(\psi_{zz}\psi_{zzzz})_z-c\epsilon(\psi_{zzz}^2)_z$. Then integrating (4.23) in z and t, we can show that

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{\phi_{zz}^2}{W}+\psi_{zz}^2+\epsilon\psi_{zzz}^2\right)+2\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zzz}^2}{W}+\psi_{zzz}^2\right)\\
&\le C_0N(t)\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zzz}^2}{W}+\psi_{zzz}^2\right)\\
&\quad+C\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{zz}^2}{W}+\frac{\phi_z^2}{W}+\psi_{zz}^2+\psi_z^2\right)\\
&\quad+\int_{\mathbb{R}}\left(\frac{\phi_{0zz}^2}{W}+\psi_{0zz}^2+\epsilon\psi_{0zzz}^2\right),
\end{aligned}
$$

where we have used (4.6), (4.7), (4.12) and (4.24). Therefore, we get the desired inequality (4.21) from (4.10) and (4.14) under the assumption $C_0N(t)\le 1/2$. □

Proof of Proposition 4.1. As mentioned before, the global existence of a unique solution follows from Proposition 4.2 and Proposition 4.3. The global estimate (4.8) follows directly from Lemmas 4.4–4.6. It remains to derive (4.9) which is obtained in the following two steps.

Step 1. From the global estimate (4.8) alongside the following well-known fact (cf. [43])

$$
\text{if } f\in W^{1,1}(0,\infty)\text{ and } f\ge 0,\text{ then } f\to 0\text{ as } t\to\infty,
\tag{4.25}
$$

we claim that

$$
\|\phi_z(t)\|\to 0\text{ and }\|\psi_z(t)\|_1\to 0\quad\text{as}\quad t\to\infty.
\tag{4.26}
$$

Case 1. We first show that $\|\phi_z(t)\|\to 0$ as $t\to\infty$. Note that $\|\phi_z\|^2\in L^1(0,\infty)$ was given by (4.8) directly. Hence it remains to show $\frac{d}{dt}\|\phi_z\|^2\in L^1(0,\infty)$. To this end, we remark that the results of Theorem 2.1 and the first equation of (3.10) directly entail that U, W and W_z are bounded. Furthermore from the second equation of (3.10), we obtain

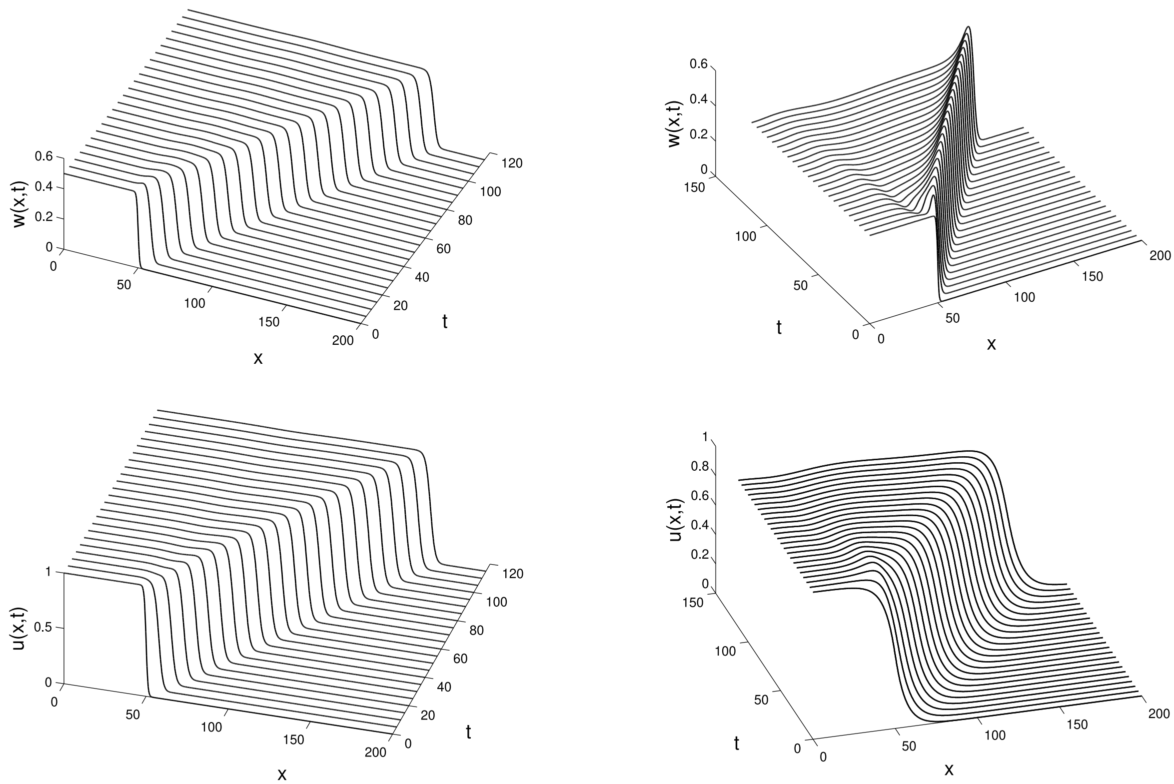
$$
U_z=\frac{1}{c\epsilon}e^{\frac{1}{c\epsilon}z}\int_z^\infty e^{-\frac{1}{c\epsilon}\xi}\left(-cU+W+\frac{U^2}{2}\right)d\xi.
$$

Simple calculation from the above identity asserts that U_z is bounded since U and W are bounded. Then from (4.15), (4.6) and (4.7), we find

$$
\begin{aligned}
\frac{d}{dt}\|\phi_z\|^2&=\frac{d}{dt}\int_{\mathbb{R}}\phi_z^2=2\int_{\mathbb{R}}\phi_z\phi_{zt}\\
&=2\int_{\mathbb{R}}\phi_z[\phi_{zzz}+(c-U)\phi_{zz}-U_z\phi_z\\
&\quad-W\psi_{zz}-W_z\psi_z-(\phi_z\psi_z)_z]\\
&\le C\int_{\mathbb{R}}(\phi_z^2+\phi_{zz}^2+\psi_z^2+\psi_{zz}^2)\\
&\le C(\|\phi_z(t)\|_{1,\omega}^2+\|\psi_z(t)\|_1^2)
\end{aligned}
$$

where we have used integration by parts and Young's inequality, along with the fact $\|\phi_z(t)\|_1\le c_1\|\phi_z(t)\|_{1,\omega}$ for some constant $c_1>0$. Then it follows from the global estimate (4.8) that $\frac{d}{dt}\|\phi_z\|^2\in L^1(0,\infty)$, which further indicates that $\|\phi_z(t)\|^2\to 0$ by (4.25) and hence $\|\phi_z(t)\|\to 0$ as $t\to\infty$.

Case 2. We next show that $\|\psi_z(t)\|_1\to 0$ as $t\to\infty$. Noticing that $\|\psi_z\|^2+\epsilon\|\psi_{zz}\|^2\in L^1(0,\infty)$ has been given by (4.8), we just need to show $\frac{d}{dt}(\|\psi_z\|^2+\epsilon\|\psi_{zz}\|^2)\in L^1(0,\infty)$. Similar to



(a) $\epsilon = 0.1$ (b) $\epsilon = 40$

Fig. 5. Numerical simulations of wave propagation and stabilization of the model (1.4) with initial data (w_0, u_0) given by (5.4).

the argument in Case 1, using integration by parts and Young's inequality, we have from (4.15), (4.6) and (4.7) that

$$\begin{aligned}\frac{d}{dt}\|\psi_z\|^2 &= 2\int_{\mathbb{R}} \psi_z\psi_{zt} \\ &= 2\int_{\mathbb{R}} \psi_z[\psi_{zzz} + (c-U)\psi_{zz} - U_z\psi_z - \phi_{zz} \\ &\quad - \psi_z\psi_{zz} - c\epsilon\psi_{zzzz} + \epsilon\psi_{zzzt}] \\ &\le C\int_{\mathbb{R}} (\phi_{zz}^2 + \psi_z^2 + \psi_{zz}^2) - \epsilon\frac{d}{dt}\int_{\mathbb{R}}\psi_{zz}^2 \\ &\le C(\|\phi(t)\|_{2,\omega}^2 + \|\psi(t)\|_2^2) - \epsilon\frac{d}{dt}\int_{\mathbb{R}}\psi_{zz}^2\end{aligned}$$

which alongside (4.8) implies $\frac{d}{dt}(\|\psi_z\|^2 + \epsilon\|\psi_{zz}\|^2) \in L^1(0,\infty)$. This shows that $\|\psi_z(t)\|^2 + \epsilon\|\psi_{zz}\|^2 \to 0$ by (4.25) and hence $\|\psi_z(t)\|_1 \to 0$ as $t \to \infty$.

Step 2. For all $z \in \mathbb{R}$, it follows from (4.8) and (4.26) that

$$\begin{aligned}\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \\ &\le 2\|\phi_z(t)\|\|\phi_{zz}(t)\| \\ &\le 2\|\phi_z(t)\|\|\phi_{zz}(t)\|_{\omega} \to 0 \quad \text{as} \quad t\to\infty.\end{aligned}$$

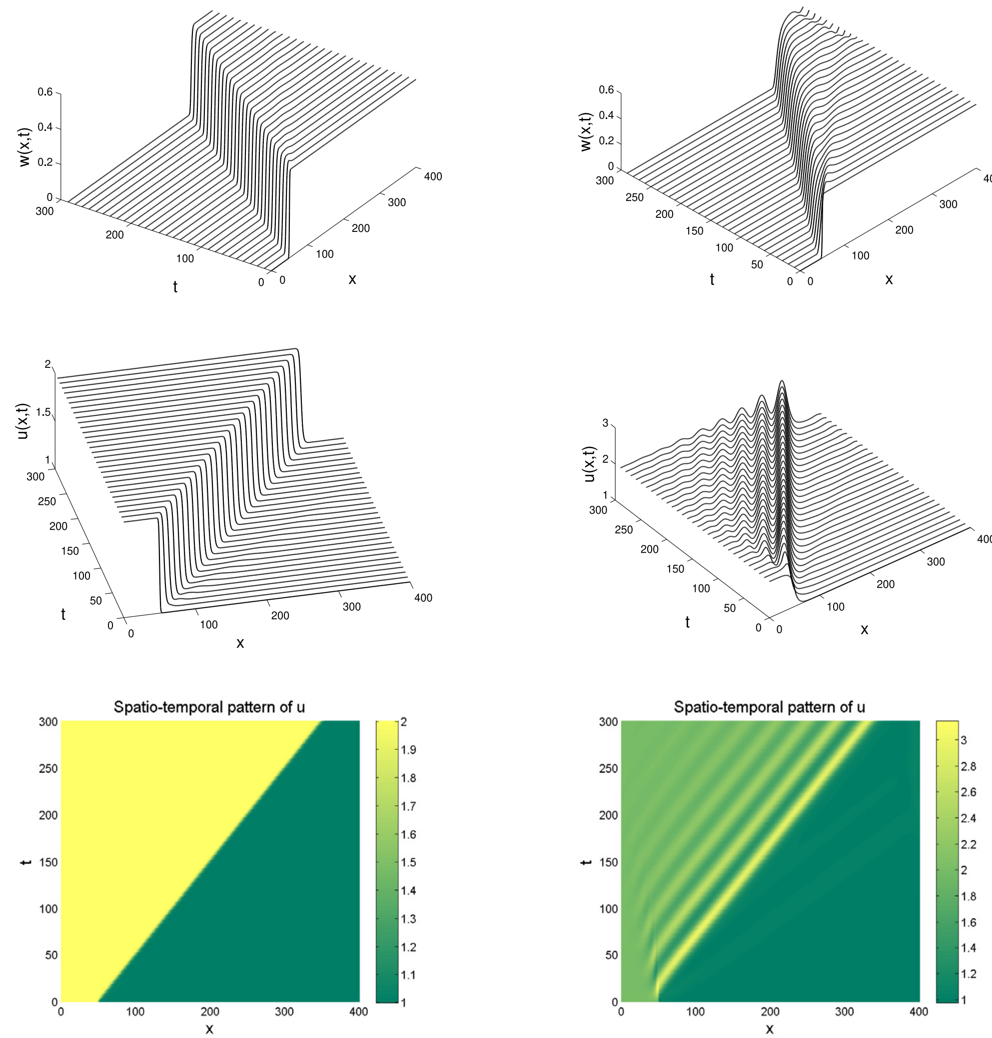
This gives that $\sup_{z\in\mathbb{R}}|\phi_z(z,t)| \to 0$ as $t \to \infty$. Applying the same procedure to ψ_z and ψ_{zz} leads to

$$\sup_{z\in\mathbb{R}}(|\psi_z(z,t)| + |\psi_{zz}(z,t)|) \to 0 \quad \text{as} \quad t\to\infty.$$

Hence (4.9) is proved and the proof of Proposition 4.1 is finished. □

5. Numerical simulations and discussion

Using the geometric singular perturbation theory alongside phase plane analysis, we have established the existence of traveling wave solutions to the Boussinesq–Burgers system (1.4)–(1.5) when the dispersion rate parameter $\epsilon > 0$ is small. Depending on the values of the asymptotic states $w_\pm$ and $u_\pm$, we showed there are three different types of traveling wave profiles connecting critical points between O, A and B as illustrated in the phase portrait diagram in Fig. 1. The detailed results are recorded in Theorem 2.1. The existence of traveling wave solutions relevant to the orbit connecting the critical point A to O has been previously shown in the paper [15, Theorem 4.2] when $4\epsilon c^2 \le 1$. Our results supplement those of [15] by showing that the Boussinesq–Burgers system (1.4)–(1.5) indeed admits more types of traveling wave profiles connecting B to O and A to B. We further showed that these three different traveling wave profiles are nonlinearly asymptotically stable when the initial data are sufficiently close to them, see the statement of Theorem 2.2. We proved the stability results by a unified approach based on the method of weighted energy estimates and the technique of taking antiderivative. The essential property of traveling wave profiles used in our stability analysis is the monotonicity of the wave profile U. Fortunately in the three different wave profiles we constructed, U has the same monotonicity property. This enables us to use a unified approach to prove the stability results without distinguishing wave profiles. As far as we know, there is not any result investigating the stability of traveling wave solutions of the Boussinesq–Burgers system (1.4) in the literature.



(a) $\epsilon = 0.1$ (b) $\epsilon = 40$

Fig. 6. Numerical simulations of wave propagation and stabilization of the model (1.4) with initial data (w_0, u_0) given by (5.5).

In the rest of this section, we shall use numerical simulations to demonstrate the wave profiles generated by the Boussinesq–Burgers system (1.4)–(1.5) and to discuss whether the Boussinesq–Burgers system is capable of describing the bore propagation. To resolve the difficulty brought by the dispersion term ϵu_{xxt}, we shall introduce a change of variable

$$v = u - \epsilon u_{xx}$$

and transform the Boussinesq–Burgers system (1.4)–(1.5) to a system of three equations

$$\begin{cases} w_t + (wu)_x = w_{xx}, \\ v_t + (w + \frac{u^2}{2})_x = \frac{1}{\epsilon}(u - v), \\ u_{xx} = \frac{1}{\epsilon}(u - v), \end{cases} \tag{5.1}$$

with initial value

$$(w, v)(x, 0) = (w_0, v_0)(x) \to \begin{cases} (w_-, v_-) & \text{as } x \to -\infty, \\ (w_+, v_+) & \text{as } x \to +\infty \end{cases} \tag{5.2}$$

where $v_\pm = u_\pm$ and

$$v_0 = u_0 - \epsilon u_{0xx}. \tag{5.3}$$

Below we shall numerically solve the transformed system (5.1)–(5.2) by the Matlab PDEPE solver based on finite difference scheme and implement three different types of initial data related to the three different wave profiles constructed in Theorem 2.1. For each type of initial data, we numerically solve (5.1)–(5.2) for weak dispersion (i.e. $\epsilon > 0$ is small) and strong dispersion (i.e. $\epsilon > 0$ is large) to illustrate the generated wave profiles and to explore the effect of dispersion on the wave propagation. In all cases, u_- is a given positive constant (see Theorem 2.1).

We first consider the generation of traveling wave profiles connecting B to O shown in Theorem 2.1-(i), where $u_+ = 0$, $w_- = \frac{u_-^2}{2}$ and $w_+ = 0$. In this case we assume the initial data (w_0, u_0) are given by

$$w_0 = \frac{w_-}{1 + e^{2(x-50)}}, \quad u_0 = \frac{u_-}{1 + e^{2(x-50)}} \tag{5.4}$$

and v_0 is given through (5.3). In the simulations, we choose $u_- = 1$, $w_- = 0.5$. The numerical solutions of (5.1) with (5.4) are plotted in Fig. 5 where we only show the numerical results for w and u which are unknowns of the original Boussinesq–Burgers system (1.4)–(1.5). From the simulations shown in Fig. 5, we observe that monotonically decreasing traveling waves are generated when the fluid dispersion is weak (see the case $\epsilon = 0.1$

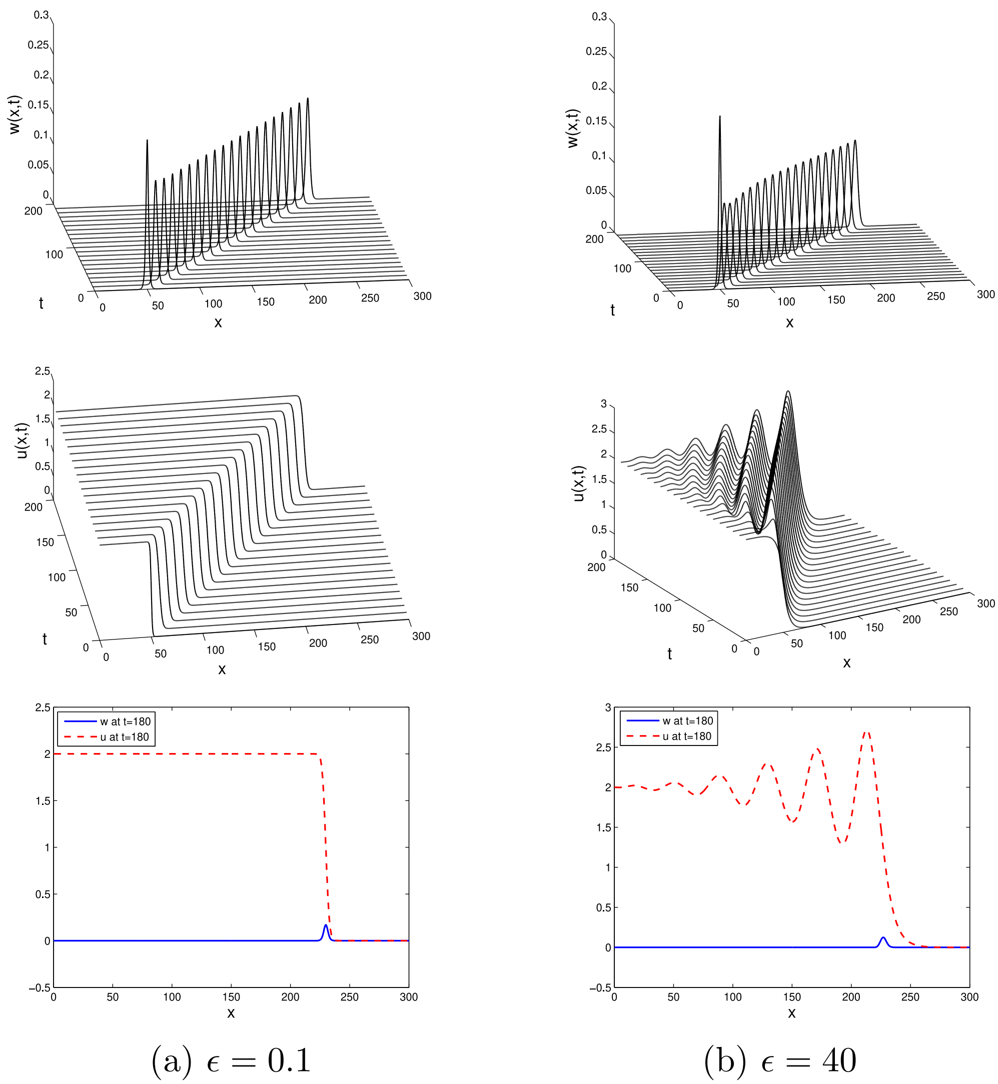


Fig. 7. Numerical simulations of wave propagation and stabilization of the model (1.4) with initial data (w_0, u_0) given by (5.6).

in column (a) of Fig. 5), which is well consistent with the results of Theorem 2.1-(i). If the dispersion is strong (see the case $\epsilon = 40$ in column (b) of Fig. 5), we find that the Boussinesq–Burgers system will generate propagating non-monotone wave profiles where in particular there is a wave surge appearing (see the profile of w in Fig. 5-(b)) which is relevant to the bore formation.

Next we consider the development of traveling wave profiles connecting B to A shown in Theorem 2.1-(ii), where $u_+ = \frac{u_-}{2} > 0$, $w_- = 0$ and $w_+ = \frac{u_-^2}{8}$. For numerical simulations, we set the initial data as

$$w_0 = \frac{w_+}{1 + e^{-2(x-50)}}, \quad u_0 = u_+ + \frac{1}{1 + e^{2(x-50)}}. \tag{5.5}$$

In our simulations, we choose $u_- = 2$ and hence $u_+ = 1$, $w_+ = 0.5$. We plot the numerical simulations in Fig. 6. On one hand, we see that the monotone propagation waves are generated for both w and u when the dispersion is weak (small ϵ), which confirms the results of Theorem 2.1-(ii). On the other hand, we also observe that the strong fluid dispersion will have large impact on the fluid speed u which becomes oscillatory at the wave trailing edge. It is straightforward to observe that the fluid dispersion has much larger influence on the fluid speed than the fluid free surface height, in contrast to the case shown in Fig. 5 where the fluid dispersion has stronger influence on the fluid free surface height instead.

Finally we explore the development of traveling wave profiles connecting A to O shown in Theorem 2.1-(iii), where $u_+ = 0$, $w_- = 0$ and $w_+ = 0$. For this case, we set the initial data for our numerical simulations as

$$w_0 = \frac{e^{x-50}}{1 + e^{2(x-50)}}, \quad u_0 = \frac{u_-}{1 + e^{2(x-50)}} \tag{5.6}$$

and v_0 is again given by (5.3). By choosing $u_- = 1$, the numerical simulations of stable propagation waves generated by (5.1)–(5.2) are shown in Fig. 7, where we see that when the fluid dispersion is weak, wave propagation appears where the profile of the fluid surface w is non-monotone (i.e. solitary wave) and changes the monotonicity once while the fluid speed profile u is monotone. However if the dispersion intensity is large, the wave speed is no longer monotone but becomes oscillatory at the trailing edge. This again shows that strong dispersion will have considerable impact on the wave propagation properties, especially on the fluid propagation speed. More importantly, the numerical simulations illustrate that the solitary wave profile w along with the wavy profile u in the case of strong dispersion is consistent with the profiles of weak bores. This implies the Boussinesq–Burgers system is capable of describing the propagation of bores.

From the above numerical simulations, we find that the Boussinesq–Burgers system (1.4)–(1.5) with weak dispersion (ϵ is small) may generate traveling wave solution pairs (W, U) where the fluid velocity U is always monotonically decreasing. This monotonicity property of U enables us to prove the nonlinear stability of traveling wave solutions as asserted in Theorem 2.2. However, if the dispersion is strong (ϵ is large), we are unable to prove the existence of traveling wave solutions to the Boussinesq–Burgers system (1.4)–(1.5). Our numerical simulations indicate that the wave profiles connecting the asymptotic states (w_-, u_-) and (w_+, u_+) still exist for the strong dispersion, but the profile U will lose monotonicity. As a consequence, the estimates of proving the stability results shown in Section 4 are no longer valid. Hence the global dynamics and asymptotic profiles of the Boussinesq–Burgers system (1.4) with large dispersion remains an interesting analytical question for future studies.

CRediT authorship contribution statement

Zhi-An Wang: Conceptualization, Investigation, Methodology, Visualization, Writing – original draft. **Anita Yang:** Formal analysis, Investigation. **Kun Zhao:** Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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