Chi Xie, Yanjie Wan, Min Xu, Xiqun Chen & Travis Waller (2023) On the primal and dual formulations of traffic assignment problems with perception stochasticity and demand elasticity, Transportation Letters, 15:6, 537-552, DOI: 10.1080/19427867.2022.2071534

On the primal and dual formulations of stochastic traffic assignment with elastic demand

Chi Xie [a, b, c, d], Yanjie Wan [e], Min Xu [f], Xiqun Chen [g], Travis Waller [h]

[a] School of Economics, Hainan University, Haikou, Hainan, China;

[b] Key Laboratory of Road and Traffic Engineering of the Ministry of Education, Tongji University, Shanghai, China;

[c] School of Transportation Engineering, Tongji University, Shanghai, China;

[d] Jiangsu Province Collaborative Innovation Center of Modern Urban Traffic Technologies, Southeast University, Nanjing, Jiangsu, China;

[e] School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiaotong University, Shanghai, China;

[f] Department of Industrial and Systems Engineering, Hong Kong Polytechnic University, Hong Kong, Hung Hom, China;

[g] College of Civil Engineering and Architecture, Zhejiang University, Hangzhou, Zhejiang, China;

[h] School of Civil and Environmental Engineering, University of New South Wales, Sydney, NSW, Australia

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Abstract

In a traffic assignment problem with elastic demand, the equilibrium conditions are achieved on two levels, namely, the stochastic user equilibrium on the path level and the supply-demand equilibrium on the origin-destination level. By recognizing both equilibrium levels imply random individual perceptions and decision makings, this paper reinvestigates the mathematical formulations of this kind of problems in both the system optimum and user equilibrium principles. Different from previous research that was devoted to the development of solution methods for some specific versions of elastic-demand traffic assignment problems, our focus is given to a pair of new general formulations that pose a duality relationship to each other. The primal formulation has a convex programming form with nonlinear constraints, while the dual one poses a concave programming problem. In this primal-dual modeling framework, we found that the equilibrium or optimality conditions of a traffic assignment problem with elastic demand can be redefined as a combination of three sets of equations and an arbitrary feasible solution of either the primal or dual formulation satisfies only two of them. We further rigorously proved the solution equivalency and uniqueness of both the primal and dual formulations, by using derivative-based techniques. We also found that when a problem of this type collapses to a logit-based case, both formulations can be conveniently written into a tractable (i.e., analytically evaluable) path flow-based form and the primal formulation becomes a convex programming problem with linear constraints. While the two formulations pose their respective modeling advantages and drawbacks, our preliminary algorithmic analysis and numerical test results indicate that the dual formulation-based algorithm, i.e., the Cauchy algorithm, can be more readily implemented for large-scale problems and converge evidently faster than the primal formulation-based one, i.e., the Frank-Wolfe algorithm, at least in the logit-based case.

Keywords: Traffic assignment, demand elasticity, stochastic user equilibrium, supply-demand equilibrium, unconstrained optimization, Frank-Wolfe algorithm, Cauchy algorithm

1. Introduction

Traffic assignment, one of the fundamental research problems in urban transportation planning, has been widely and intensively studied by many scholars. Since Beckman et al. (1956) first stated the classic deterministic traffic assignment problems through two assignment principles, user equilibrium (UE) and system optimum (SO), a variety of traffic assignment problems, following different network settings and behavioral assumptions, have been proposed and investigated in past decades. Stochastic traffic assignment problems were first introduced by extending the Wardropian principles (Wardrop, 1952) to accommodate stochastic travel cost perceptions, such as stochastic user equilibrium (SUE) problem (Sheffi and Powell, 1982) and stochastic system optimum (SSO) problem (Maher et al., 2005). The models for these problems are mathematical products made from embedding the widely acclaimed random utility theory into the network equilibrium paradigm and offer a theoretically sound and practically tractable approach for describing individual route choice behaviors in a more realistic manner. Other types of stochastic traffic assignment problems also incrementally appeared in the literature and were tackled in different mathematical frameworks, for accommodating different types of supply and demand uncertainties caused by various factors (Xie and Liu, 2014).

On the other hand, demand elasticity is a fundamental concept in travel demand analysis and has been a focused subject in travel demand forecasting practices for a long period (Graham and Glaister, 2004; Goodwin et al., 2004; Litman, 2013). It aggregately characterizes a prevailing individual response behavior within any travel population and provides a concise way to quantify the size and constitution of travel demands as well as their variations to the system congestion. By nature, travel demand elasticity is a synthetic result of individual discrete choices with uncertain travel and non-travel disutilities between making a trip and not making[1]. It also reflects the traffic routing and diversion phenomenon when the target network behaves as a subnetwork in travel demand analysis (see Carey et al., 1981; McNeil and Hendrickson, 1985; Zhou et al., 2006; Xie et al., 2011). While it is used in many different contexts and for different purposes, a basic form of elastic-demand traffic assignment problems was proposed in the literature by assuming that the demand rate between an origin-destination (O-D) pair is negatively (linearly or nonlinearly) influenced by the perceived travel costs by travelers who potentially demand travels between this O-D pair. Example problems of this type include the elastic-demand deterministic user equilibrium

[1] Demand elasticity also reflects the stochasticity or heterogeneity of travel-related attributes and other stochastic factors of individual travelers (Parkin, 2002; Mankiw, 2014).

(EDDUE) problem (Beckman et al., 1956; Carey, 1985) and elastic-demand deterministic system optimum (EDDSO) problem (Gartner, 1980; Carey, 1985).

Not surprisingly, incorporating travel demand elasticity and random route choices into a traffic assignment process would engender a more behaviorally consistent and realistic modeling tool, since such a model implies travel cost perception randomness in both trip choice and route choice (De Jong et al., 2005). In fact, this type of problems has attracted increasing awareness in recent years. These efforts at least include: An elastic-demand stochastic user equilibrium problem proposed by Maher (2001), a probit-based elastic-demand stochastic user equilibrium problem by Connors et al. (2007), a probit-based asymmetric elastic-demand stochastic user equilibrium problem by Meng and Liu (2012), a logit-based elastic-demand stochastic user equilibrium problem by Yu et al. (2014), a weibit-based elastic-demand combined stochastic modal split and traffic assignment problem by Kitthamkesorn et al. (2015), and a literature review on this type of problems (Cantarella et al., 2013). While these researchers employed different mathematical formulations (e.g., optimization problems vs. fixed-point problems), behavioral mechanisms (e.g., logit-based vs. probit-based vs. weibit-based route choice models), and relationships of modeling components (e.g., arc cost independence vs. arc cost interaction), the main purpose of these studies was devoted to the development of solution methods for some specific versions of elastic-demand traffic assignment problems. In contrast, our interest in this paper is paid to a fundamental understanding and analysis on the mathematical programming formulations for a general version[2] of this type of problems.

The main purpose in conducting this research is to capture the common features of those traffic assignment problems with travel demand elasticity and travel cost perception stochasticity and to identify the values of these features in solution development from a primal-dual perspective. For this purpose, we make use of the primal-dual modeling method in Xie and Waller (2012), which was originally proposed for modeling fixed-demand stochastic traffic assignment problems, and show how the primal and dual formulations are derived in this framework and how they behave in term of solution equivalency and uniqueness. In the meantime, we also sketch two formulation-based solution algorithms for the primal and dual formulations, respectively, to further distinguish the performance of these formulations in algorithm development. In this given analysis framework, we focus our attention in this

[2] By *general*, we do not mean that the proposed mathematical programming formulations can include non-separable travel cost functions, link flow capacity constraints, and other extra constraints or settings; instead, we mean the inclusion of different route and trip choice principles/objectives and models.

paper to the above development and investigation work for two representative problems, namely, elastic-demand stochastic system optimum (EDSSO) and elastic-demand stochastic user equilibrium (EDSUE) problems. Unless specified, all the findings and conclusions presented in this paper are equivalently applicable to the two problems.

The remaining part is arranged in such a manner. Section 2 introduces the notation, modeling assumptions, and optimality conditions for both the EDSSO and EDSUE problems. Section 3 presents problem formulations in the primal-dual modeling framework, discusses the duality relationships between the formulations, and identifies the difference of the proposed dual formulation from a previous one in the literature. Section 4 and Section 5 prove the solution equivalency and uniqueness of both formulations. Section 6 then presents two solution algorithms, based on the primal and dual formulations, respectively, for both the EDSSO and EDSUE problems, and implements and compares the algorithms through a numerical example. Section 7 presents a more general version of the EDSSO and EDSUE problems, which allows for spatial Markovian routing behaviors and can be accommodated by the aforementioned modeling framework. Finally, Section 8 concludes the paper.

2. Problem Definitions and Optimality Conditions

Prior to a discussion of problem formulations, it is necessary to provide some prerequisite and necessary information, including the notation for models and solution methods, problem assumptions and specifications, and optimality conditions pertaining to the EDSSO and EDSUE problems.

2.1 Notation

The notation of sets, parameters, functions and variables used for the EDSSO and EDSUE problems is given in Table 1.

Table 1 Notation

Sets and parameters	
N	Set of nodes, $N = \{n\}$
R	Set of origin nodes, $R = \{r\}$
S	Set of destination nodes, $S = \{s\}$
A	Set of arcs, $A = \{a\}$

C	Set of arc cost functions, $C = \{c_a\}$
K	Set of paths, $K = \{k\}$
K_{rs}	Set of paths between O-D pair r-s, $K_{rs} \subset K$
$\delta_{a,k}^{rs}$	Arc-path incidence indicator, where $\delta_{a,k}^{rs} = 1$ if arc a is on path k, $\delta_{a,k}^{rs} = 0$ otherwise
Functions and variables	
f_k^{rs}	Traffic flow rate on path k between O-D pair r-s
q_{rs}	Travel demand rate between O-D pair r-s, where $q_{rs} = \sum_k f_k^{rs}$
g_k^{rs}	Gradient of the objective function with respect to f_k^{rs}
x_a	Traffic flow rate on arc a, where $x_a = \sum_{rs} \sum_k f_k^{rs} \, \delta_{a,k}^{rs}$
t_a	Travel cost on arc a, where $t_a = t_a(x_a)$ is a positive, strictly increasing, continuously differentiable function of arc flow x_a
c_a	Basic travel cost on arc a, where $c_a = t_a$ in the EDSSO problem and $c_a = \int_0^{x_a} t_a(\omega) \mathrm{d}\omega / x_a$ in the EDSUE problem
b_a	Supplementary travel cost on arc a in the dual formulation, where $b_a = x_a \, \mathrm{d}t_a / \mathrm{d}x_a$ in the EDSSO problem and $b_a = t_a - \int_0^{x_a} t_a(\omega) \mathrm{d}\omega / x_a$ in the EDSUE problem
ξ_a	Random perceived travel cost on arc a, where $E(\xi_a) = 0$ and $VAR(\xi_a) = \theta_a$
c_k^{rs}	Basic travel cost on path k between O-D pair r-s, where $c_k^{rs} = \sum_a c_a \delta_{a,k}^{rs}$
d_k^{rs}	Supplementary travel cost on path k between O-D pair r-s in the primal formulation
b_k^{rs}	Supplementary travel cost on path k between O-D pair r-s, where $b_k^{rs} = \sum_a b_a \delta_{a,k}^{rs}$, in the dual formulation
ξ_k^{rs}	Random perceived travel cost on path k between O-D pair r-s, where $E(\xi_k^{rs}) = 0$ and $VAR(\xi_k^{rs}) = \theta_k^{rs} = \sum_a \theta_a \delta_{a,k}^{rs}$, which implies $COV(\theta_a, \theta_b) = 0$, where $\delta_{a,k}^{rs} = 1$ and $\delta_{b,k}^{rs} = 1$
u_k^{rs}	Generalized travel cost on path k between O-D pair r-s, where $u_k^{rs} = c_k^{rs} + d_k^{rs}$, in the primal formulation
w_k^{rs}	Generalized travel cost on path k between O-D pair r-s, where $w_k^{rs} = c_k^{rs} + b_k^{rs}$, in the dual formulation

U_k^{rs}	Perceived generalized travel cost on path k between O-D pair r-s, where $U_k^{rs} = u_k^{rs} + \xi_k^{rs}$, in the primal formulation
W_k^{rs}	Perceived generalized travel cost on path k between O-D pair r-s, where $W_k^{rs} = w_k^{rs} + \xi_k^{rs}$, in the dual formulation
P_k^{rs}	Probability of choosing path k between O-D pair r-s, where $P_k^{rs}(\cdot) = \mathrm{P_r}(U_k^{rs} < U_l^{rs}, l \in K_{rs}, l \neq k)$, where $P_k^{rs} > 0$ and $\sum_{k \in K_{rs}} P_k^{rs} = 1$
S_{rs}	Satisfaction or potential function of O-D pair r-s, where $S_{rs} = E\left(\min_{k \in K_{rs}}\{U_k^{rs}\} \middle| \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}\right)$ in the primal formulation or $S_{rs} = E\left(\min_{k \in K_{rs}}\{W_k^{rs}\} \middle| \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{b}^{rs}\right)$ in the dual formulation
D_{rs}	Demand function of O-D pair r-s, where $q_{rs} = D_{rs}(S_{rs})$ is a bounded, positive, strictly decreasing, continuously differentiable function

2.2 Problem Assumptions and Specifications

By using the notation in Table 1, we then can define the type of traffic assignment problems of interest in the following manner. Suppose that we are given a transportation network $G = (N, A)$, where N is the set of nodes, A is the set of arcs. There are two subsets of N, namely, R and S, which are the sets of origin nodes and destination nodes, respectively. For each origin-destination (O-D) pair r-s, we assume that its demand rate D_{rs} is determined by a demand function $D_{rs}(S_{rs})$, where S_{rs} is the expected minimum travel cost among all paths connecting O-D pair r-s. It is further assumed that $D_{rs}(S_{rs})$ is a bounded, strictly decreasing, continuously differentiable, convex function of S_{rs}. S_{rs} is the satisfaction function (Daganzo, 1982) or potential function (Nesterov, 2007) of O-D pair r-s, a function of the perceived travel costs of all paths connecting this O-D pair, as described earlier in Table 1. The perceived travel cost along path k connecting O-D pair r-s, U_k^{rs} or W_k^{rs}, consists of two parts, a deterministic part, $c_k^{rs} + d_k^{rs}$ or $c_k^{rs} + b_k^{rs}$, which is a function of all relevant arc flow rates, and a stochastic part, ξ_k^{rs}, which is assumed to be a flow-independent random parameter with its mean equal to zero and variance equal to a constant θ_k^{rs}. The deterministic part of the travel cost on arc a, c_a, is a strictly increasing, continuously differentiable, convex function of the traffic flow rate on arc a, x_a.

In a traffic assignment problem with elastic demand, the equilibrium conditions are achieved on two levels. On the path level, travelers make their routing decisions in response to prevailing network congestion conditions in such a way as to minimize their personal costs until they gain no further improvement by altering any choices. On the O-D level, the network or part of the network behaves as a whole and uses the point of intersection of an upward-sloping supply curve and a downward-sloping demand curve in the demand-cost coordinate system to determine an aggregate demand-supply consistency state.

It should be noted here that *both* levels of equilibrium or optimality are constructed on the basis of stochastic individual perceptions and decision makings. On the path level, travelers perceive the travel costs on the path set with random errors and thus make different routing decisions when everyone aims at minimizing his or her perceived travel cost; as a whole, the choice proportions over the path set are specified by a probability function. This equilibrium or optimality is the aforementioned SUE or SSO conditions. On the O-D level, travelers make trip-making decisions based on their stochastic perceptions on the network congestion or their stochastic willingness to make trips (De Jong et al., 2005); as an aggregate result, the trip-making demand is typically a decreasing curve (instead of a step-wise curve if all travelers perceive precisely) specified by a demand function. The latter equilibrium is often named the supply-demand equilibrium (SDE) in the literature. In this paper, we determine the trip rate between an O-D pair under SDE by the demand function of that O-D pair.

All aforementioned equilibrium or optimality conditions are then specified below in detail, for both the EDSSO[3] and EDSUE problems.

2.3 The Optimality Conditions

It is readily known that the optimality conditions of the EDSSO or EDSUE problem is a combination of the SSO or SUE and SDE conditions, on the path and O-D levels, respectively. Mathematically, the EDSSO or EDSUE conditions are specified by the following set of equations:

$$D_{rs}\left(S_{rs}^{*}(\mathbf{c}^{rs}(\mathbf{f}^{*})+\mathbf{d}^{rs*})\right)=q_{rs}^{*} \qquad \forall r,s \tag{1.1}$$

[3] Unlike the EDSUE problem (see Maher, 2001; Connors et al., 2007; Meng and Liu, 2012; Yu et al., 2014, for example), previous research has not yet defined the optimality conditions of the EDSSO problem. From the perspective of individual travel behaviors, its optimality conditions can be described as such a network flow pattern: Anyone traveling between any O-D pair aims to achieve a goal of minimizing his or her *perceived total travel cost* over the *augmented network* by choosing whether or not to make a trip and choosing which route to take if he or she makes a trip, where the trip choice could be regarded as a choice between the whole path set connecting the O-D pair and a dummy non-trip path connecting this O-D pair. The augmented network is formed by adding a dummy non-trip path to each O-D pair with positive demand and the perceived total travel cost is defined as the sum of the perceived travel cost associated with the arcs in the original network and the perceived not-make-a-trip cost associated with dummy arcs.

$$q_{rs}^* P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}^*) + \mathbf{d}^{rs*}) = f_k^{rs*} \qquad \forall k, r, s \qquad (1.2)$$

$$d_k^{rs*} = b_k^{rs}(\mathbf{f}^*) \qquad \forall k, r, s \qquad (1.3)$$

In the above system of equations, $P_k^{rs}(\cdot) = \mathrm{P_r}(U_k^{rs} < U_l^{rs}, l \in K_{rs}, l \neq k)$ is the probability function used to evaluate the probability that the perceived generalized travel cost along path k is less than or equal to that along any other path connecting O-D pair r-s; $S_{rs}(\cdot) = E\left(\min_{k \in K_{rs}}\{U_k^{rs}\} \middle| \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}\right)$ is the satisfaction or potential function deriving the expected minimum perceived generalized travel cost among all paths connecting O-D pair r-s; $b_k^{rs}(\cdot)$ is a path cost function, defined as $b_k^{rs}(\cdot) = \sum_a x_a \, \partial t_a/\partial x_a \, \delta_{a,k}^{rs}$ for the EDSSO problem and $b_k^{rs}(\cdot) = \sum_a \left(t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a\right)\delta_{a,k}^{rs}$ for the EDSUE problem. In the above equations, we use the notation of some prime variables such as d_k^{rs} and U_k^{rs} for writing the above optimality conditions. It should be noted that they can be equivalently expressed by using dual variables b_k^{rs} and W_k^{rs} as well.

It is readily known that Equation (1.1) presents the SDE conditions[4], and Equations (1.2) and (1.3) jointly describe the SSO or SUE conditions. Specifically, if $c_k^{rs} = \sum_a t_a \delta_{a,k}^{rs}$ and $b_k^{rs} = \sum_a x_a \, \partial t_a/\partial x_a \, \delta_{a,k}^{rs}$, the conditions in Equations (1.2) and (1.3) specifies an SSO solution and the overall conditions specify an EDSSO solution; if $c_k^{rs} = \sum_a \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a \, \delta_{a,k}^{rs}$ and $b_k^{rs} = \sum_a \left(t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a\right)\delta_{a,k}^{rs}$, the overall conditions specify an EDSUE solution. It should be noted that Equations (1.2) and (1.3) could be combined to form such an equation: $q_{rs}^* P_k^{rs}\left(\mathbf{c}^{rs}(\mathbf{f}^*) + \mathbf{b}^{rs}(\mathbf{f}^*)\right) = f_k^{rs*}$, $\forall k, r, s$. However, writing the equilibrium or optimality conditions into such three sets of equations as (1) eases the subsequent discussion on the solution feasibility of the formulations. It is noted here that we do not include an explicit trip choice function in either the EDSSO or EDSUE model for specifying the elastic demand; instead, we use a supply-demand function, the parameters of which can be estimated by aggregating individual system optimum or user equilibrium trip choice behaviors[5].

Nevertheless, the equilibrium or optimality conditions presented in (1) completely characterizes an optimal solution of the EDSSO or EDSUE problem. Any solution fully satisfying the above conditions must be an equilibrium or optimal solution of the EDSSO or EDSUE problem, and vice versa. In other words, if a solution satisfies only part of the

[4] This condition implicitly implies $\bar{q}_{rs} > q_{rs}^*$, where $\bar{q}_{rs}$ is an upper bound of q_{rs}.
[5] It must be noted that the elastic demand functions of the EDSSO and EDSUE models for the same network must be different. The difference is rooted from the difference between the system optimum and user equilibrium trip choice behaviors and can be evaluated by aggregating the difference between individual marginal social costs and marginal private costs.

conditions (i.e., one or more of the conditions are violated), then it must not be an equilibrium or optimal solution. In particular, we will see below that a solution satisfying Equations (1.1) and (1.2) is only a feasible solution of the proposed primal problem while a solution satisfying Equations (1.1) and (1.3) is merely a feasible solution of the dual problem.

3. Problem Formulations

Now we propose a pair of mathematical programming formulations for the EDSSO and EDSUE problems by using the modeling framework in Xie and Waller (2012) and present some of their mathematical properties. The pair of formulations are given below in both the primal and dual forms, respectively. We will also discuss the relationship between this pair of formulations and a formulation proposed by Maher (2001) in this section.

3.1 Primal Formulation

The primal formulation for the EDSSO and EDSUE problems is such a convex programming problem:

$$\min \quad z^p(\mathbf{f}, \mathbf{d}) = \sum_{rs} \int_0^{S_{rs}} D_{rs}(t)\,\mathrm{d}t \; - \sum_{rs} \sum_k d_k^{rs} f_k^{rs} \tag{2.1}$$

$$\text{subject to} \quad q_{rs} P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}) = f_k^{rs} \qquad \forall k, r, s \tag{2.2}$$

$$\text{where} \quad S_{rs} = E\left(\min_{k \in K_{rs}} \{U_k^{rs}\} \middle| \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}\right) \qquad \forall r, s \tag{2.3}$$

$$q_{rs} = D_{rs}(S_{rs}) \qquad \forall r, s \tag{2.4}$$

The above formulation is written in the path-based form, in which path flows $\mathbf{f} = [f_k^{rs}]$ and path supplementary costs $\mathbf{d} = [d_k^{rs}]$ are the decision variables. While all other variables and parameters are defined in the same way for both the EDSSO and EDSUE problems, it is noted that the basic travel costs, $\mathbf{c}^{rs} = [c_k^{rs}]$, where $c_k^{rs} = \sum_a c_a \delta_{a,k}^{rs}$, have different functional forms and economic implications between the two, i.e., $c_a = t_a$ for the EDSSO problem and $c_a = \int_0^{x_a} t_a(\omega)\mathrm{d}\omega / x_a$ for the EDSUE problem. The supplementary travel costs, $\mathbf{d}^{rs} = [d_k^{rs}]$, do not have an explicit mathematical expression; instead, they are so set as to make constraints (2.2) and (2.4) always hold.

It should be noted that, according to Kolmogorov's first and second axioms, i.e., $P_k^{rs}(\cdot) \geq 0$ and $\sum_{k\in K_{rs}} P_k^{rs}(\cdot) = 1$, the path flow nonnegativity and reservation relationships are implied by constraint (2.2):

$$f_k^{rs} \geq 0 \qquad \forall k, r, s \tag{3.1}$$

$$q_{rs} = \sum_k f_k^{rs} \qquad \forall r, s \tag{3.2}$$

The idea of using supplementary costs for constructing primal formulations originally comes from Maher et al. (2005), who defined in their research $\mathbf{d}^{rs}$ as such an implicit cost vector that guarantees $q_{rs}P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}) = f_k^{rs}$, $\forall k$, in a primal formulation for the SSO problem, in which q_{rs} is a given constant. They also justified that, for each O-D pair r-s, only the relative values of the $\mathbf{d}^{rs}$ vector affect the value of each probability function $P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs})$, $\forall k$. As a result, there could be numerous $\mathbf{d}^{rs}$ values corresponding to a path flow pattern $\mathbf{f}$, as long as the $\mathbf{d}^{rs}$ values satisfy the relationship: $d_k^{rs} - d_l^{rs} = d_k^{rs\prime} - d_l^{rs\prime}$, where $\forall k, l \in K_{rs}$ and $k \neq l$. In this study, we still define $\mathbf{d}^{rs}$ as an implicit cost vector with O-D pair r-s, but it simultaneously satisfies both the flow-cost consistency relationship $q_{rs}P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}) = f_k^{rs}$, $\forall k$, and the supply-demand consistency relationship $q_{rs} = D_{rs}(S_{rs})$, $\forall k$. For this reason, its values uniquely correspond to a given path flow pattern. This conclusion is written in Proposition 1.

Proposition 1. For the given value of a feasible path flow pattern $\mathbf{f} = [f_k^{rs}]$, the value of a corresponding supplementary cost vector $\mathbf{d}^{rs} = [d_k^{rs}]$ in the primal formulation given in (2) uniquely exists for each O-D pair r-s.

Proof. We can prove this conclusion by contradiction. Suppose that we have two supplementary cost vectors $\mathbf{d}_1^{rs}$ and $\mathbf{d}_2^{rs}$, where $\mathbf{d}_1^{rs} \neq \mathbf{d}_2^{rs}$, which implies $\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_1^{rs} \neq \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_2^{rs}$. Since $\mathbf{f}$ satisfies condition (1.2), we know

$$P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_1^{rs}) = P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_2^{rs}) \qquad \forall k, r, s$$

Then we obtain

$$\left(c_k^{rs} + d_{k,1}^{rs}\right) - \left(c_l^{rs} + d_{l,1}^{rs}\right) = \left(c_k^{rs} + d_{k,2}^{rs}\right) - \left(c_l^{rs} + d_{l,2}^{rs}\right) \qquad \forall k \neq l, r, s$$

where paths k and l are two arbitrary paths in the path set connecting O-D pair r-s, i.e., k, $l \in K_{rs}$, $k \neq l$.

On the other hand, from $\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_1^{rs} \neq \mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_2^{rs}$, i.e., $c_k^{rs} + d_{k,1}^{rs} \neq c_k^{rs} + d_{k,2}^{rs}$, $\exists k \in K_{rs}$, we get

$$q_{rs,1} = D_{rs}\big(S_{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_1^{rs})\big) \neq q_{rs,2} = D_{rs}\big(S_{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_2^{rs})\big) \qquad \forall r, s$$

Then we different two different f_k^{rs} values, $q_{rs,1}P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_1^{rs})$ and $q_{rs,2}P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}_2^{rs})$, which contradicts the given condition. Hence the proposition is proved. ■

While the values of the $\mathbf{d}^{rs}$ cost vector can be determined if a path flow pattern $\mathbf{f}$ is given, the value of each d_k^{rs} in the vector may not be calculated directly, except in some special cases or for very small networks. Such a special case arises when the logit model is used as the route choice model in the EDSSO and EDSUE problems. Specifically, the satisfaction or potential function for O-D pair r-s can be written as, if the logit model is applied,

$$S_{rs}(\mathbf{c}^{rs} + \mathbf{d}^{rs}) = -\frac{1}{\rho}\ln\sum_k \exp\big(-\rho(c_k^{rs} + d_k^{rs})\big) \qquad \forall r, s \tag{4}$$

where ρ is the dispersion parameter of the logit model. Given $q_{rs} = D_{rs}(S_{rs})$, which implies $S_{rs} = D_{rs}^{-1}(q_{rs})$, we then readily know that,

$$D_{rs}^{-1}(q_{rs}) = -\frac{1}{\rho}\ln\sum_k \exp\big(-\rho(c_k^{rs} + d_k^{rs})\big) \qquad \forall r, s \tag{5.1}$$

$$\Rightarrow \ \exp\big(-\rho D_{rs}^{-1}(q_{rs})\big) = \sum_k \exp\big(-\rho(c_k^{rs} + d_k^{rs})\big) \qquad \forall r, s \tag{5.2}$$

Inserting this expression into the probability function defined by the logit model for choosing path k connecting O-D pair r-s results in:

$$P_k^{rs} = \frac{\exp\big(-\rho(c_k^{rs} + d_k^{rs})\big)}{\exp\big(-\rho D_{rs}^{-1}(q_{rs})\big)} \qquad \forall k, r, s \tag{6.1}$$

$$\Rightarrow \ c_k^{rs} + d_k^{rs} = -\frac{1}{\rho}\ln\Big(P_k^{rs}\exp\big(-\rho D_{rs}^{-1}(q_{rs})\big)\Big) \qquad \forall k, r, s \tag{6.2}$$

In such a way, we finally obtain the function for calculating d_k^{rs} as follows,

$$d_k^{rs} = -\frac{1}{\rho}\ln\frac{f_k^{rs}}{q_{rs}} + D_{rs}^{-1}(q_{rs}) - c_k^{rs} \qquad \forall k, r, s \tag{7}$$

where $q_{rs} = \sum_k f_k^{rs}$, which is readily obtained from constraint (2.2) and Kolmogorov's second axiom.

3.2 Dual Formulation

We then propose the dual formulation for the EDSSO and EDSUE problem as,

$$\max \ z^d(\mathbf{f}) = \sum_{rs}\int_0^{S_{rs}} D_{rs}(t)\,\mathrm{d}t \ - \sum_{rs}\sum_k b_k^{rs} f_k^{rs} \tag{8.1}$$

$$\text{where} \ S_{rs} = E\left(\min_{k\in K_{rs}}\{W_k^{rs}\}\middle|\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{b}^{rs}(\mathbf{f})\right) \qquad \forall r, s \tag{8.2}$$

$$q_{rs} = D_{rs}(S_{rs}) \qquad \forall r, s \tag{8.3}$$

The above formulation is also written in the path-based form, in which path flows $\mathbf{f} = [f_k^{rs}]$ are the decision variables. Different from the primal formulation, the basic and supplementary travel costs, $\mathbf{c}^{rs} = [c_k^{rs}] = \left[\sum_a c_a \delta_{a,k}^{rs}\right]$ and $\mathbf{b}^{rs} = [b_k^{rs}]$, where $c_k^{rs} = \sum_a c_a \delta_{a,k}^{rs}$ and $b_k^{rs} = \sum_a b_a \delta_{a,k}^{rs}$, in the dual formulation, both have an explicit functional form of path flows. Specifically, $c_a = t_a$ and $b_a = x_a\,\mathrm{d}t_a/\mathrm{d}x_a$ are defined in the EDSSO problem, while $c_a = \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a$ and $b_a = t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a$ are given in the EDSUE problem. All other variables and parameters appearing in the EDSSO and EDSUE problems are defined in the same way.

Other distinguishing features between the primal and dual formulations include: 1) the primal formulation poses a minimization problem, while dual formulation is a maximization problem; 2) the primal formulation is a constrained optimization problem, while the dual formulation is an unconstrained optimization problem. The connection between the two formulations, however, is that they imply a duality relationship. This conclusion is formally recorded in Proposition 2.

Proposition 2. The dual formulation presented in (8) is a Lagrangian dual to the primal formulation presented in (2).

Proof. This proposition can be proved by the same method used for Theorem 1 in Xie and Waller (2012). ■

We noted in the literature that Maher (2001) proposed an alternative dual formation for the EDSUE problem. This formulation can be written as, when the notation given in this paper is used,

$$\max \sum_{rs}\left(D_{rs}(S_{rs})\left(S_{rs}-D_{rs}^{-1}(q_{rs})\right)-\int_0^{q_{rs}} D_{rs}^{-1}(v)\mathrm{d}v+q_{rs}D_{rs}^{-1}(q_{rs})\right) \\ -\sum_a\left(x_a t_a-\int_0^{x_a} t_a(\omega)\mathrm{d}\omega\right) \tag{9.1}$$

$$\text{where } \; S_{rs}=E\left(\min_{k\in K_{rs}}\{U_k^{rs}\}\middle|\mathbf{c}^{rs}(\mathbf{f})+\mathbf{b}^{rs}(\mathbf{f})\right) \qquad \forall r,s \tag{9.2}$$

It is noted here that the supply-demand consistency relationship $q_{rs}=D_{rs}(S_{rs})$ for each O-D pair r-s does not exist in Maher's formulation. The solution properties such as solution equivalency and uniqueness can be referred to in Maher (2001). By a direct comparison, we can readily verify that our dual formulation is different from Maher's. As a result, we found that a feasible solution of our dual formulation satisfies (1.1) and (1.3), while a feasible solution of Maher's formulation satisfies only (1.3). However, at their optimal points, these problem formulations present the same solution. We prove this conclusion in Proposition 3 below.

Proposition 3. While the formulation proposed in (8) and the formulation presented in (9) from Maher (2001) for the EDSUE problem have different functional forms, their optimal solutions coincide.

Proof. To see this, let us first assume that Maher's (2001) formulation reaches its optimal solution, which implies $q_{rs}=D_{rs}(S_{rs})$. Then, its objective function becomes,

$$\sum_{rs}\left(-\int_0^{q_{rs}} D_{rs}^{-1}(v)\mathrm{d}v+q_{rs}D_{rs}^{-1}(q_{rs})\right)-\sum_a\left(x_a t_a-\int_0^{x_a} t_a(\omega)\mathrm{d}\omega\right) \tag{10}$$

On the other hand, the formulation in (8) can be rewritten as,

$$\sum_{rs}\left(\int_{q_{rs}}^{+\infty} D_{rs}^{-1}(v)\mathrm{d}v + q_{rs}D_{rs}^{-1}(q_{rs})\right) - \sum_{a} x_a\left(t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega / x_a\right)$$

$$\Rightarrow \sum_{rs}\left(\int_0^{+\infty} D_{rs}^{-1}(v)\mathrm{d}v - \int_0^{q_{rs}} D_{rs}^{-1}(v)\mathrm{d}v + q_{rs}D_{rs}^{-1}(q_{rs})\right) - \sum_{a}\left(x_a t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega\right) \quad (11)$$

By comparing the functions in (10) and (11), we readily know that their difference is just a constant term, $\sum_{rs}\int_0^{+\infty} D_{rs}^{-1}(v)\mathrm{d}v$. Thus, the optimal solutions of the two formulations are the same. ■

3.3 An Illustrative Example for Duality and Coincidence

Here we provide a toy network example to numerically illustrate the duality relationship between the proposed primal and dual formulations and the coincidence relationship of the optimal solutions of the dual formulation and Maher's (2001) dual formulation. The network is a two-arc network with one O-D pair. Its topology as well as the arc cost and O-D demand functions are shown in Figure 1. For the illustration purpose, we assume that the route choice model used in this network example is the binomial logit model with a scale parameter $\rho = 1$.

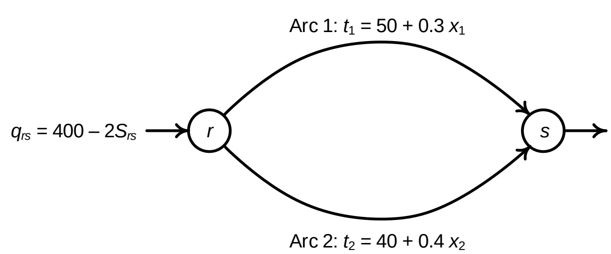


Figure 1 A toy network for illustrating problem duality and solution coincidence

Given that Maher's (2001) formulation has three independent decision variables, i.e., arc flow rates x_1 and x_2 and O-D demand rate q_{rs}, while we can display at most two variables in a 3-dimensional space (in addition to the objective function), we decided to use three figures to illustrate the numerical relationship between the objection function and decision variables, as shown in Figure 2. Specifically, we set the value of arc flow rate x_1 always equal to its optimal value in Figure 2(a), the value of arc flow rate x_2 equal to its optimal value in Figure 2(b), and the value of demand rate q_{rs} equal to its optimal value in Figure 2(c). Thus, we reduced the number of variables that are shown in each of these figures. As a result, for example, the feasible region of the proposed primal or dual formulation is represented by a

2-dimensional curve supported the objection function value and arc flow rate x_1 or x_2, and the feasible region of Maher's (2001) formulation reduces to a 3-dimensional curved surface supported by the objective function value and arc flow rates x_1 and x_2.

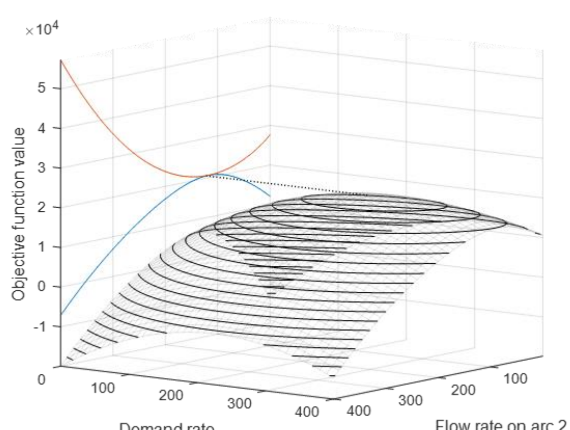
Following the above numerical and graphical settings, we solved a logit-based EDSUE problem, as an example, for the given network. The obtained equilibrium solution is: $x_1^* = 122.4$, $x_2^* = 117.7$, $q_{rs}^* = 240.1$, $t_1^* = 86.7$, $t_2^* = 87.1$, and $S_{rs}^* = 80.0$. From Figure 2(a) and 2(b), we can see that the objective function curves of the primal and dual formulations show convex and concave shapes, respectively, and meet exactly at their optimal points. The latter phenomenon clearly shows the zero duality gap between the primal and dual formulations at the optimal solution of the problem and hence exemplifies the result of Proposition 2.

According to Proposition 3, we know that at the optimal point of the problem the difference between the objective function values of our dual formulation and Maher's (2001) dual formulation is a constant, $\sum_{rs}\int_0^{+\infty} D_{rs}^{-1}(v)\mathrm{d}v$. For a better visualization of the coincidence of their optimal solutions, we added this constant into the objective function of Maher's (2001) dual formulation. As a result from this addition, we can see that the coincided optimal solutions pose the same objective function value, as shown in Figure 2(a), 2(b), and 2(c). However, any other solutions of the two formulations give different objective function values, which clearly exhibits the difference between the two formulations, as shown in Figure 2(c).

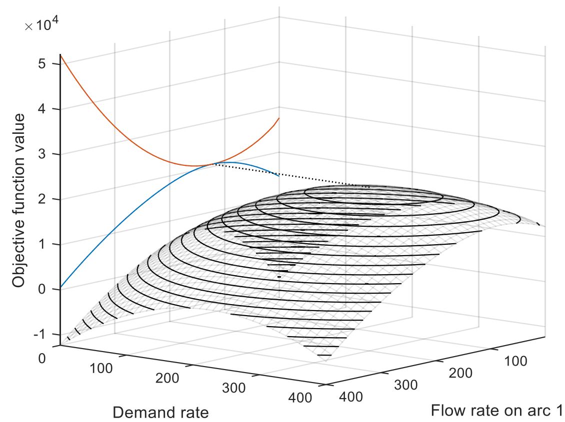
4. Solution Properties

Given that the feasible solution space of the primal formulation is a polyhedron and its objective function is limited, the existence of optimal solutions can be readily obtained. In this regard, the focus of this section is placed on proving the equivalency between the optimal solutions of the primal and dual formulations and the defined equilibrium or optimality conditions and the uniqueness of these optimal solutions.

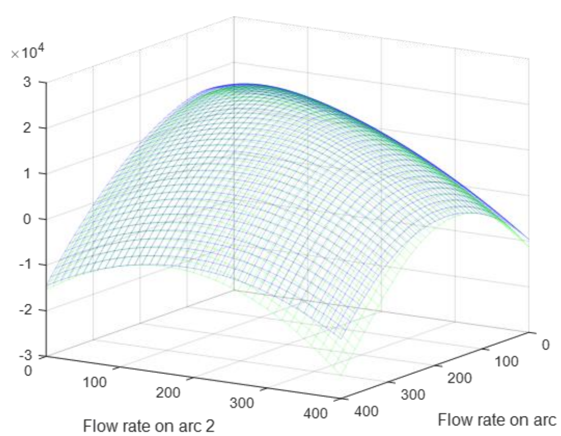
4.1 Solution Equivalency



(a) The objective function curves with fixing arc flow rate x_1 = 122.4



(b) The objective function curves with fixing arc flow rate x_2 = 117.7



(c) The objective function curves with fixing O-D demand rate q_{rs} = 240.1

Figure 2 The objective function curves of the primal and dual formulations

The solution equivalency of the primal and dual formulations (i.e., (2) and (8), respectively) of the EDSSO and EDSUE problems can be established by proving that an optimal solution of these problem formulations satisfy the optimality conditions given in (1). We provide below such a proof for the primal and dual formulations, which is formally written in Propositions 4 and 5, respectively.

Proposition 4. An optimal solution of the primal formulation fully satisfies the conditions in (1), for both the EDSSO and EDSUE problems.

Proof: Since a feasible solution of the primal formulation has already satisfied (1.1) and (1.2), we only need to prove that an optimal solution of the formulation satisfies (1.3). This can be accomplished by checking the first-order condition of the objective function with respect to an arbitrary path flow variable f_l^{mn}:

$$\begin{aligned}\frac{\partial z^p}{\partial f_l^{mn}} &= \sum_{rs} D_{rs} \frac{\partial S_{rs}}{\partial f_l^{mn}} - d_l^{mn} \\ &= \sum_{rs} D_{rs} \sum_k P_k^{rs} \frac{\partial (c_k^{rs} + d_k^{rs})}{\partial f_l^{mn}} - d_l^{mn} \\ &= \sum_{rs} \sum_k q_{rs} P_k^{rs} \sum_a \frac{\partial c_a}{\partial f_l^{mn}} \delta_{a,k}^{rs} - d_l^{mn} \\ &= \sum_a x_a \frac{\partial c_a}{\partial x_a} \delta_{a,l}^{mn} - d_l^{mn} \qquad \forall l, m, n \end{aligned} \tag{12}$$

According to Kolmogorov's first axiom and no possibility for $P_l^{mn*} = 0$, we know that $P_l^{mn*} > 0$ and hence $f_l^{mn*} > 0$, which implies $\partial z^p / \partial f_l^{mn*} = 0$, i.e., $d_l^{mn*} = \sum_a x_a^* (\partial c_a^* / \partial x_a^*) \delta_{a,l}^{mn}$.

Then, for the EDSSO problem, given $c_a = t_a$, we can obtain $d_l^{mn} = \sum_a x_a (\partial c_a / \partial x_a) \delta_{a,l}^{mn}$; for the EDSUE problem, given $c_a = \int_0^{x_a} t_a(\omega) \mathrm{d}\omega / x_a$, we obtain $\sum_a \left(t_a - \int_0^{x_a} t_a(\omega) \mathrm{d}\omega / x_a\right) \delta_{a,k}^{rs}$. Thus, the proposition is proved for both the EDSSO and EDSUE problems. ■

Proposition 5. An optimal solution of the dual formulation fully satisfies the conditions in (1), for both the EDSSO and EDSUE problems.

Proof: Given that a feasible solution of the dual formulation has already satisfied (1.1) and (1.3), we only need to prove that an optimal solution of the formulation satisfies (1.2). For this purpose, we derive the first-order condition of the objective function with respect to an arbitrary path flow variable f_l^{mn}:

$$\frac{\partial z^d}{\partial f_l^{mn}} = \sum_{rs} D_{rs} \frac{\partial S_{rs}}{\partial f_l^{mn}} - \sum_{rs} \sum_k \frac{\partial b_k^{rs}}{\partial f_l^{mn}} f_k^{rs} - b_l^{mn}$$

$$
\begin{aligned}
&= \sum_{rs} D_{rs} \sum_{k} P_k^{rs} \frac{\partial (c_k^{rs} + b_k^{rs})}{\partial f_l^{mn}} - \sum_{rs} \sum_{k} \frac{\partial b_k^{rs}}{\partial f_l^{mn}} f_k^{rs} - b_l^{mn} \\
&= \sum_{rs} \sum_{k} q_{rs} P_k^{rs} \sum_{a} \left(2 \frac{\partial c_a}{\partial x_a} + x_a \frac{\partial^2 c_a}{\partial x_a^2} \right) \delta_{a,l}^{mn} \delta_{a,k}^{rs} \\
&\qquad - \sum_{rs} \sum_{k} \sum_{a} \left(\frac{\partial c_a}{\partial x_a} + x_a \frac{\partial^2 c_a}{\partial x_a^2} \right) \delta_{a,l}^{mn} \delta_{a,k}^{rs} f_k^{rs} - \sum_{a} x_a \frac{\partial c_a}{\partial x_a} \delta_{a,l}^{mn} \\
&= \sum_{a} \left(2 \frac{\partial c_a}{\partial x_a} + x_a \frac{\partial^2 c_a}{\partial x_a^2} \right) \delta_{a,l}^{mn} \sum_{rs} \sum_{k} (q_{rs} P_k^{rs} - f_k^{rs}) \delta_{a,k}^{rs} \\
&\qquad\qquad \forall l, m, n
\end{aligned}
\tag{13}
$$

Again, for the EDSSO problem, $c_a = t_a$; for the EDSUE problem, $c_a = \int_0^{x_a} t_a(\omega)\, \mathrm{d}\omega / x_a$. In either of the problem instances, it is readily known that $2\, \partial c_a / \partial x_a + x_a\, \partial^2 c_a / \partial x_a^2 > 0$, given that t_a is an increasing, convex and continuously differentiable function of x_a. At the optimal point, we know that $\partial z^d / \partial f_l^{mn*} = 0$, which implies $q_{rs}^* p_k^{rs*} = f_k^{rs*}$. Thus, the proposition is proved for both the EDSSO and EDSUE problems. ■

The above conclusions for the primal and dual formulations justify that a local optimal solution of these formulations is a solution of the proposed EDSSO or EDSUE problem. To assess the global optimality of these solutions, we need to further investigate their uniqueness.

4.2 Solution Uniqueness

The solution uniqueness can be proved by contradiction, as shown in detail below in Proposition 6. For the sake of discussion convenience, we defined in Table 2 a set of matrices and vectors of variables and parameters, as used in the proof when appropriate.

Table 2 Notation of matrices and vectors

$\mathbf{\Delta}^{rs} = \left[\delta_{a,k}^{rs}\right]_{\lvert A\rvert \times \lvert K_{rs}\rvert}$	Matrix of arc-path incidence indicators
$\mathbf{x} = [x_a]_{\lvert A\rvert \times 1}$	Vector of arc flow rates
$\mathbf{c} = [c_a]_{1 \times \lvert A\rvert}$	Vector of basic arc cost functions
$\mathbf{f}^{rs} = [f_k^{rs}]_{\lvert K_{rs}\rvert \times 1}$	Vector of path flow rates of O-D pair r-s
$\mathbf{c}^{rs} = [c_k^{rs}]_{1 \times \lvert K_{rs}\rvert}$	Vector of basic path costs of O-D pair r-s, where $\mathbf{c}^{rs} = \mathbf{c}\mathbf{\Delta}^{rs}$

$\mathbf{d}^{rs} = [d_k^{rs}]_{1\times\lvert K_{rs}\rvert}$	Vector of supplementary path costs of O-D pair r-s in the primal formulation
$\mathbf{b}^{rs} = [b_k^{rs}]_{1\times\lvert K_{rs}\rvert}$	Vector of supplementary path costs of O-D pair r-s in the dual formulation
$\mathbf{P}_K^{rs} = [P_k^{rs}]_{\lvert K_{rs}\rvert\times 1}$	Vector of path choice probabilities for O-D pair r-s
$\mathbf{P}_A^{rs} = [P_a^{rs}]_{\lvert A\rvert\times 1}$	Vector of arc choice probabilities for O-D pair r-s, where $\mathbf{P}_A^{rs} = \boldsymbol{\Delta}^{rs}\mathbf{P}_K^{rs}$

In the above table, P_k^{rs} and P_a^{rs} are the probabilities of a traveler pertaining to O-D pair r-s choosing path k and arc a, respectively. Moreover, the arc flow pattern $\mathbf{x}$ can also be written as $\mathbf{x} = \sum_{rs} \boldsymbol{\Delta}^{rs}\mathbf{f}^{rs}$, or $\mathbf{x} = \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{c}^{rs} + \mathbf{d}^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{c}^{rs} + \mathbf{d}^{rs})$ in the primal formulation or $\mathbf{x} = \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{c}^{rs} + \mathbf{b}^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{c}^{rs} + \mathbf{b}^{rs})$ in the dual formulation. However, if a solution $\mathbf{x}$ is at the optimal point, then we have $\mathbf{x} = \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{c}^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{c}^{rs})$.

Proposition 6. Both the primal and dual formulations have only a single solution satisfying the optimality conditions in (1), for both the EDSSO and EDSUE problems.

Proof. Suppose that we have two different arc flow solutions satisfying the optimality conditions in (1), $\mathbf{x}_1$ and $\mathbf{x}_2$, where $\mathbf{x}_1 \neq \mathbf{x}_2$, and two corresponding arc cost patterns, $\mathbf{e}_1$ and $\mathbf{e}_2$, where $\mathbf{e}_1 = \mathbf{c}(\mathbf{x}_1) + \mathbf{b}(\mathbf{x}_1)$ and $\mathbf{e}_2 = \mathbf{c}(\mathbf{x}_2) + \mathbf{b}(\mathbf{x}_2)$. (Note that when the condition in (1.1) is always held, we do not need to explicitly consider O-D demand solutions here. When an arc flow solution is given, the O-D demand solution can be analytically determined.) Then, we have

$$(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2) = (\mathbf{e}_1 - \mathbf{e}_2)\left(\sum_{rs} D_{rs}\big(S_{rs}(\mathbf{c}_1^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_1^{rs}) - \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{c}_2^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_2^{rs})\right) \tag{14}$$

Then we can prove $(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2) < 0$ and $(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2) \geq 0$ (see Appendix A for details). To this end, we know that the hypothesis is not valid and we must have $\mathbf{x}_1 = \mathbf{x}_2$. ■

5. Solution Algorithms

In the literature, stochastic traffic assignment problems, particularly the SUE and EDSUE problems, have been solved by various solution algorithms, such as those presented in Sheffi and Powell (1982), Chen and Alfa (1991), Maher and Hughes (1997), Maher et al. (1998), Maher (2001), Liu et al. (2009), Lee et al. (2010), Meng and Liu (2012), and Yu et al. (2014), Kitthamkesorn et al. (2015). Some of them are formulation-based while many others are not.

These algorithms could be applied directly or used with slight modifications for solving the EDSSO and EDSUE problems.

As a further effort in analyzing the proposed formulations in this paper, we present below two formulation-based solution algorithms, namely, the Frank-Wolfe algorithm and the Cauchy algorithm, which are based on the primal and dual formulations, respectively. The purpose of developing and implementing these algorithms is twofold: 1) to supplement the family of aforementioned solution algorithms for stochastic traffic assignment problems; 2) to further justify the usefulness of the proposed formulations in solution development and evaluate these formulations from the perspective of solution efficiency.

For simplicity and convenience, these algorithms are specified below exclusively for logit-based elastic-demand traffic assignment problems. The application of these algorithms, especially the primal formulation-based algorithm, for other types of distribution of perceived travel costs is very technically difficult[6].

5.1 Primal Algorithm

Embedding the logit model into the primal formulation leads to a closed-form formulation as shown below and makes it possible to evaluate all algorithmic components in the solution process in an analytical way, which greatly eases the implementation of the algorithm. Specifically, by making use of the expressions of S_{rs} in (4) and d_k^{rs} in (7), we can derive the logit-based primal formulation as:

$$\min \;\; z^p(\mathbf{f}) = \sum_{rs} \int_0^{S_{rs}} D_{rs}(t)\,\mathrm{d}t + \frac{1}{\rho} \sum_{rs} \sum_k f_k^{rs} \ln \frac{f_k^{rs}}{q_{rs}} - \sum_{rs} D_{rs}^{-1}(q_{rs}) q_{rs} + \sum_{rs} \sum_k c_k^{rs} f_k^{rs} \quad (15.1)$$

$$\text{subject to} \;\; \sum_k f_k^{rs} = q_{rs} \qquad \forall r, s \quad (15.2)$$

$$f_k^{rs} \geq 0 \qquad \forall k, r, s \quad (15.3)$$

[6] For example, when the distribution of perception errors follows the normal distribution, the resulting problem is a probit-based elastic-demand traffic assignment problem. In this case, the primal formulation still contains nonlinear constraints (see the equation in (2.2)); to evaluate the objective function and its derivative with respect to any path flow variable in the Frank-Wolfe framework, we need to calculate the value of d_k^{rs}, $\forall k, r, s$. This latter task, even if to be conducted by numerical methods, involves enumerating paths and solving a nonlinear system of equations (in the form of (2.2)) for each O-D pair. The size of this system is equivalent to the number of paths connecting the O-D pair, $|K_{rs}|$. Evidently, such computational requirements pose a very challenging technical difficulty.

$$\text{where} \quad S_{rs} = -\frac{1}{\rho}\ln\sum_k \exp\left(-\rho(c_k^{rs} + d_k^{rs})\right) \qquad \forall r, s \tag{15.4}$$

$$q_{rs} = D_{rs}(S_{rs}) \qquad \forall r, s \tag{15.5}$$

Note that in this logit-based formulation, the constraint in (2.2) in the original primal formulation is written into the objective function by introducing the expression of d_k^{rs} in (6) into the formulation. However, the path flow nonnegativity and conservation constraints, which are implied in constraint (2.2) by Kolmogorov's first and second axioms, are not preserved in the resulting objective function. Due to this reason, constraints (15.2) and (15.3) must be supplemented into the logit-based primal formulation.

Based on the above primal formulation, an adaptation of the Frank-Wolfe algorithm is then sketched as follows:

Step 0 (Initialization): Calculate $\mathbf{q}^0 = \mathbf{D}\left(\mathbf{S}(\mathbf{c}^0)\right)$ and perform a stochastic traffic network loading to obtain the initial path flow pattern $\mathbf{f}^0$ based on $\mathbf{c}^0 = \mathbf{c}(\mathbf{0})$, where $\mathbf{c} = [\mathbf{c}_k^{rs}]$, $\mathbf{q} = [\mathbf{q}_{rs}]$, $\mathbf{D} = [\mathbf{D}_{rs}]$, and $\mathbf{S} = [\mathbf{S}_{rs}]$. Set counter $i \coloneqq 1$.

Step 1 (Cost update): Calculate $\mathbf{c}^i = \mathbf{c}(\mathbf{f}^i)$.

Step 2 (Direction determination): Find a path flow pattern $\mathbf{g}^i = \left[g_k^{rs,i}\right]$ that minimizes

$$\min \sum_{rs}\sum_k \frac{\partial z^p(\mathbf{f})}{\partial f_k^{rs,i}} g_k^{rs,i} = \sum_{rs}\sum_k \left(\sum_a x_a^i \frac{\partial c_a^i}{\partial x_a^i}\delta_{a,k}^{rs} + \frac{1}{\rho}\ln\frac{f_k^{rs,i}}{q_{rs}^i} - D_{rs}^{-1}\left(q_{rs}^i\right) + c_k^{rs,i}\right) g_k^{rs,i} \tag{16.1}$$

$$\text{subject to} \quad \sum_k g_k^{rs,i} = q_{rs}^i \qquad \forall r, s \tag{16.2}$$

$$g_k^{rs,i} \geq 0 \qquad \forall k, r, s \tag{16.3}$$

Note that the above linear programming problem can be decomposed by O-D pairs. For O-D pair r-s, the problem can be solved in such simple rules: if $\min_{k\in K_{rs}} \partial z^p(\mathbf{f})/\partial f_k^{rs,i-1} \geq 0$, set $g_k^{rs,i} = 0$, $\forall k \in K_{rs}$; if $\min_{k\in K_{rs}} \partial z^p(\mathbf{f})/\partial f_k^{rs,i-1} < 0$, set $g_{k^*}^{rs,i} = \bar{q}_{rs}$ and $g_k^{rs,i} = 0$, $\forall k \in K_{rs}$, $k \neq k^*$, where path k^* is the path that achieves $\min_{k\in K_{rs}} \partial z^p(\mathbf{f})/\partial f_k^{rs,i-1}$ and $\bar{q}_{rs}$ is a prespecified upper bound on q_{rs}^i.

Step 3 (Line search): Find λ^i that minimizes $z^p\left(\lambda\mathbf{f}^i + (1-\lambda)\mathbf{g}^i\right)$ subject to $0 \le \lambda \le 1$. This one-dimensional optimization problem can be solved by, for example, the bi-section method or golden section method.

Step 4 (Flow update): Set $\mathbf{f}^{i+1} = \lambda^i\mathbf{f}^i + \left(1-\lambda^i\right)\mathbf{g}^i$.

Step 5 (Convergence test): If the convergence criterion is met, stop the algorithm; otherwise, set $i \coloneqq i+1$ and go to step 1.

It is noted that in the direction determination step of the primal algorithm, solving the problem $\min_{k\in K_{rs}} \partial z^p(\mathbf{f})/\partial f_k^{rs,i}$ poses a shortest path problem for O-D pair r-s. Given that the value of the term $\ln\left(f_k^{rs,i-1}/q_{rs}^{i-1}\right)$ is path-specific and not additive along the path, we know that the task of finding a shortest path here requires path enumeration.

5.2 Dual Algorithm

On the other hand, an adaptation of the Cauchy algorithm for the dual formulation presents the following algorithmic procedure:

Step 0 (Initialization): Calculate $\mathbf{q}^0 = \mathbf{D}\left(\mathbf{S}(\mathbf{c}^0 + \mathbf{b}^0)\right)$ and perform a traffic network loading to obtain the initial arc flow pattern $\mathbf{x}^0$ based on $\mathbf{c}^0 + \mathbf{b}^0 = \mathbf{c}(\mathbf{0}) + \mathbf{b}(\mathbf{0})$, where $\mathbf{c} = [c_a]$, $\mathbf{b} = [b_a]$, $\mathbf{q} = [\mathbf{q}_{rs}]$, $\mathbf{D} = [\mathbf{d}_{rs}]$, and $\mathbf{S} = [\mathbf{S}_{rs}]$. Set counter $i \coloneqq 1$.

Step 1 (Cost update): Calculate $\mathbf{c}^i = \mathbf{c}\left(\mathbf{x}^i\right)$ and $\mathbf{b}^i = \mathbf{b}\left(\mathbf{x}^i\right)$.

Step 2 (Direction determination): Find the search direction $\mathbf{w}^i = \left[w_a^i\right]$ by employing the gradient direction of the objective function with respect to arc flow rates such as,

$$
\begin{aligned}
w_a^i &= \frac{\partial z^d(\mathbf{f})}{\partial x_a^i} \\
&= \left(2\frac{\partial c_a^i}{\partial x_a^i} + x_a^i \frac{\partial^2 c_a^i}{\partial {x_a^i}^2}\right) \sum_{rs}\sum_{k} \left(D_{rs}\left(S_{rs}^i\right)P_k^{rs,i} - f_k^{rs,i}\right)\delta_{a,k}^{rs} \\
&= \left(2\frac{\partial c_a^i}{\partial x_a^i} + x_a^i \frac{\partial^2 c_a^i}{\partial {x_a^i}^2}\right) \left(y_a^i - x_a^i\right) \qquad \forall k, r, s
\end{aligned}
$$

where $\mathbf{y}^i = \left[y_a^i\right] = \left[\sum_{rs}\sum_k D_{rs}\left(S_{rs}^i\right)P_k^{rs,i}\delta_{a,k}^{rs}\right]$ is the arc flow pattern obtained from a stochastic network loading, which implies solving such an unconstrained optimization problem:

$$\max \sum_{rs} \int_0^{S_{rs}^i} D_{rs}(t)\, \mathrm{d}t - \sum_{rs} \sum_k b_k^{rs,i} f_k^{rs} \tag{17.1}$$

$$\text{where } S_{rs}^i = E\left(\min_{k \in K_{rs}} \{U_k^{rs}\} \middle| \mathbf{c}^{rs,i} + \mathbf{b}^{rs,i} \right) \qquad \forall r, s \tag{17.2}$$

This stochastic network loading can be achieved in an excess-demand network (see, for example, Sheffi, 1985; Gartner, 1980; Connors et al., 2007; Xie and Duthie, 2015), which is a slightly modified procedure of the stochastic network loading for the stochastic traffic assignment problems with fixed demand (Dial, 1971; Bell, 1995; Akamatsu, 1996; Maher and Hughes, 1997).

Step 3 (Line search): Find λ^i that maximizes $z^d(\mathbf{x}^i + \lambda \mathbf{w}^i)$ subject to $\lambda \geq 0$.

Step 4 (Flow update): Set $\mathbf{x}^{i+1} = \mathbf{x}^i + \lambda^i \mathbf{w}^i$.

Step 5 (Convergence test): If the convergence criterion is met, stop the algorithm; otherwise, set $i \coloneqq i + 1$ and go to step 1.

By comparing the primal and dual algorithms presented above, we made the following observations: 1) while both algorithmic procedures involves iteratively solving an approximation subproblem to gradually approach the optimal point, the primal algorithm constructs a linear subproblem by using gradients and solves it by searching each O-D pair for its shortest path (see the optimization problem in (16)) while the dual algorithm solves a nonlinear subproblem (see the optimization problem in (17)) by a stochastic network loading for each O-D pair; 2) in the direction determination step, the primal algorithm solves its subproblem on the basis of path flow variables, while the dual algorithm can solve its subproblem by dealing with arc flow variables, where the latter greatly reduces the number of variables in the problem and operational complexity of the solution process; 3) in the line search step, the move size λ in the primal algorithm ranges in $0 \leq \lambda \leq 1$ (due to the constraints), while in the dual algorithm λ ranges in $\lambda \geq 0$ (due to no constraint). It should be noted that it may be difficult to implement in the primal algorithm an analogous procedure to the stochastic network loading in the dual algorithm for finding an auxiliary flow pattern, because of the existence of $f_k^{rs} \ln(f_k^{rs}/q_{rs})$ in the objective function.

5.3 A Comparative Evaluation for Algorithm Implementation and Convergence Performance

For a comparative evaluation on the presented primal and dual algorithms, we applied them for solving the logit-based EDSSO and EDSUE problems. The evaluation conducted below, however, is rather illustrative than comprehensive, in that we merely intend to make an assessment on the relative convergence performance between the two algorithms instead of a thorough investigation of their solution efficiency in solving a variety of real problems. For this simple task, we coded the algorithms in MATLAB and selected a small network shown in Figure 3 as the test problem.

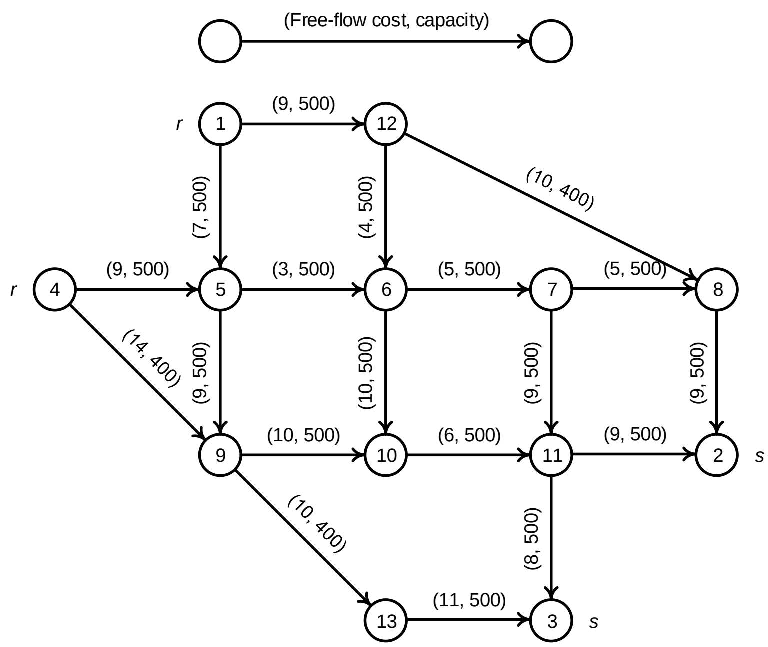


Figure 3 The Nguyen-Dupuis network

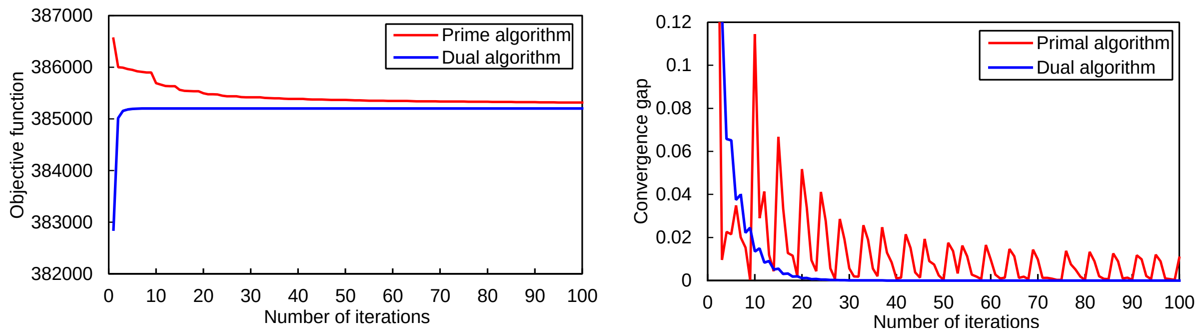
The arc cost function and O-D demand function used in the EDSSO and EDSUE problems over the test network are of the following forms:

$$t_a = t_a^0\left(1 + \alpha\left(\frac{x_a}{c_a}\right)^\beta\right) \qquad \forall a \qquad (18)$$

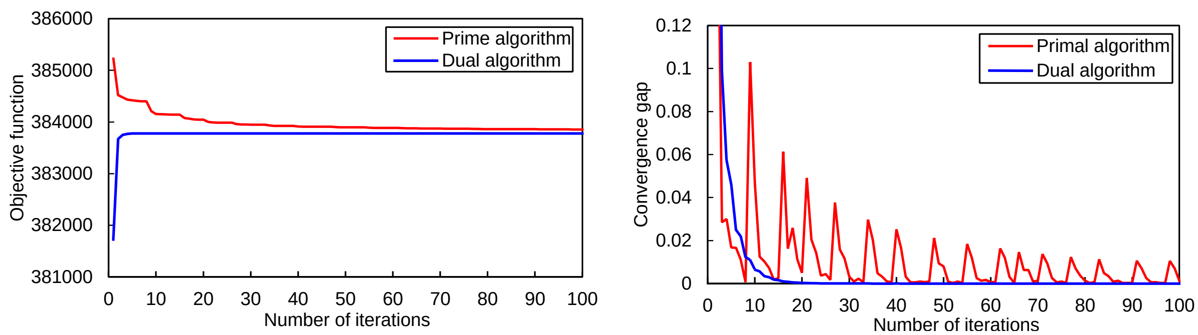
$$q_{rs} = \frac{\bar{q}_{rs}}{\exp(\theta S_{rs})} \qquad \forall r,\, s \qquad (19)$$

where t_a^0 is the free-flow travel cost on arc a, c_a is the nominal capacity of arc a, and $\bar{q}_{rs}$ is the upper bound of the demand rate of O-D pair r-s, i.e., $D_{rs}(0) = \bar{q}_{rs}$. The values of t_a^0 and c_a can be found in the network and $\bar{q}_{rs}$ is set as 10,000 for all O-D pairs. Moreover, we set different parameters in the range of $0.06 \le \alpha \le 0.24$ for different arc cost functions, and apply $\beta = 4$ for all arc cost functions and $\theta = 0.1$ for all O-D demand functions.

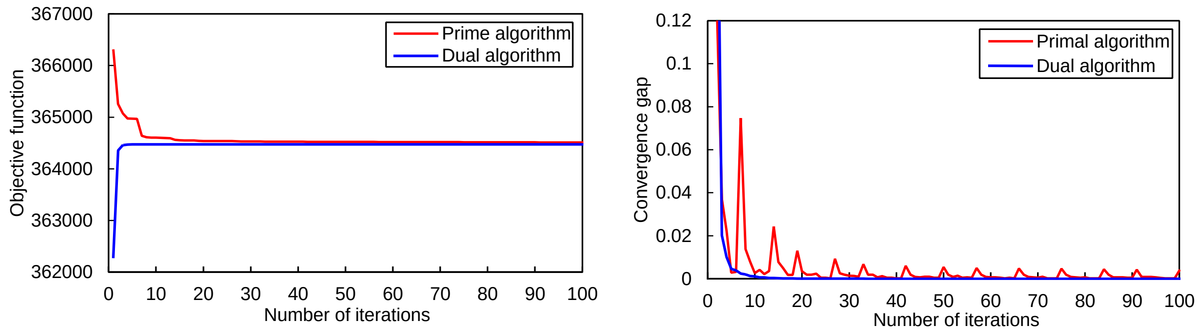
The numerical experiments of solving the EDSSO and EDSUE problems were then conducted with the scale parameter $\rho = 1$, 0.5, and 0.1. The computational results are exhibited in Figure 4 and Figure 5, for the EDSSO and EDSUE problems, respectively. The convergence curves of both objective function values and convergence gap values over iterations (where each iteration contains a traffic network loading) are used to graphically exhibit the convergence performance of these algorithms. Moreover, the convergence gap ϵ is calculated as $\epsilon = \sum_{rs} \sum_k \left| f_k^{rs,i} - g_k^{rs,i} \right| / \sum_{rs} \sum_k f_k^{rs,i}$, where i is the iteration index.



(a) $\rho = 1$

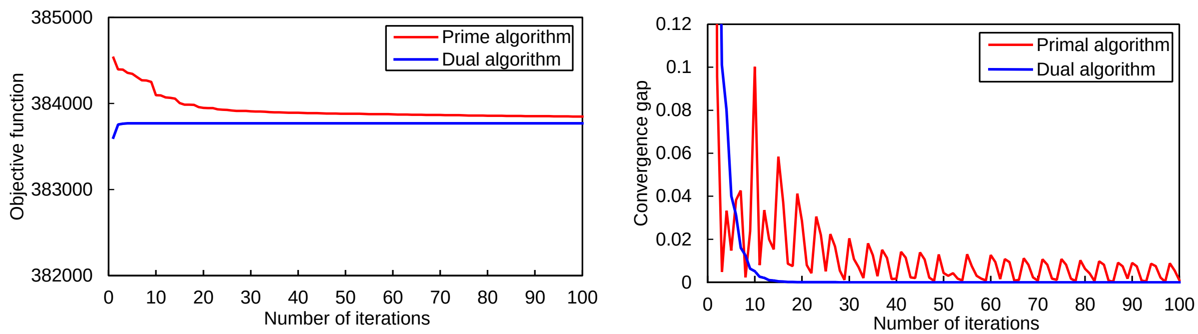


(b) $\rho = 0.5$

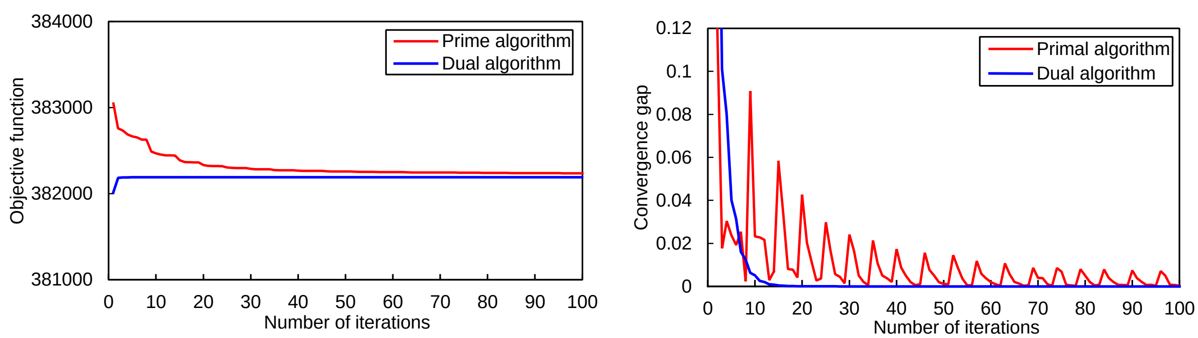


(c) $\rho = 0.1$

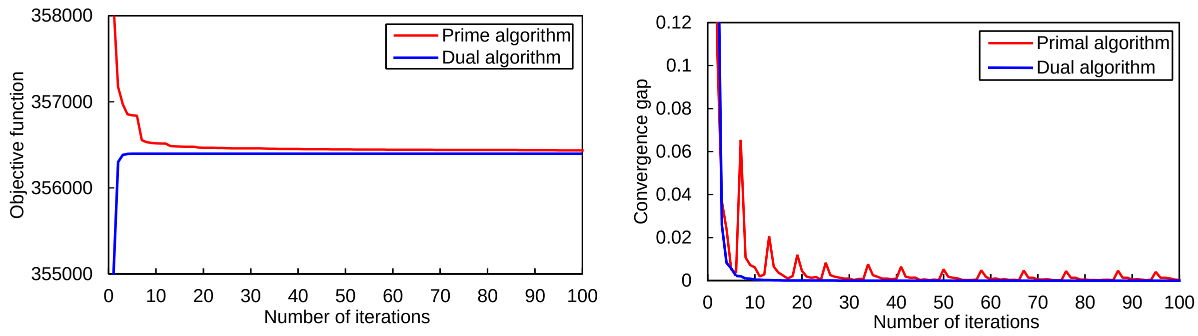
Figure 4 Convergence curves from solving the EDSSO problem in the Nguyen-Dupuis network



(a) $\rho = 1$



(b) $\rho = 0.5$



(c) $\rho = 0.1$

Figure 5 Convergence curves from solving the EDSUE problem in the Nguyen-Dupuis network

Table 3 The converged solutions from solving EDSSO in the Nguyen-Dupuis network

		Primal algorithm			Dual algorithm		
Scale parameter		1	0.5	0.1	1	0.5	0.1
Arc	(1, 5)	305.8	336.0	494.2	306.3	336.4	494.3
	(1, 12)	298.1	323.9	460.9	295.0	322.9	461.2
	(4, 5)	277.0	306.6	424.7	276.0	306.5	424.4
	(4, 9)	206.2	207.7	257.9	206.2	207.8	257.8
	(5, 6)	554.3	553.6	622.0	554.8	554.6	621.9
	(5, 9)	28.5	89.1	296.9	27.6	88.3	296.8
	(6, 7)	422.6	447.2	598.0	421.6	446.9	597.8
	(6, 10)	178.0	208.4	294.4	174.5	207.4	294.5
	(7, 8)	164.0	184.8	258.3	162.6	184.0	258.0
	(7, 11)	258.6	262.4	339.7	259.0	263.0	339.8
	(8, 2)	415.7	406.7	448.7	416.2	407.2	448.9
	(9, 10)	26.5	81.0	250.1	25.9	80.4	250.1
	(9, 13)	208.2	215.8	304.7	207.8	215.7	304.4
	(10, 11)	204.5	289.4	544.5	200.5	287.8	544.6

	(11, 2)	151.0	216.8	425.8	147.6	215.7	425.9
	(11, 3)	312.1	335.0	458.4	311.9	335.1	458.5
	(12, 6)	46.3	102.0	270.5	41.4	99.7	270.4
	(12, 8)	251.8	221.9	190.4	253.7	223.2	190.8
	(13, 3)	208.2	215.8	304.7	207.8	215.7	304.4
O-D	1-2	334.0	377.0	607.0	331.8	376.3	607.5
	1-3	269.8	282.9	348.1	269.6	283.0	348.0
	4-2	232.8	246.5	267.6	232.1	246.5	267.2
	4-3	250.4	267.8	415.0	250.1	267.8	415.0

Table 4 The converged solutions from solving EDSUE in the Nguyen-Dupuis network

		Primal algorithm			Dual algorithm		
Scale parameter		1	0.5	0.1	1	0.5	0.1
Arc	(1, 5)	373.2	418.0	696.4	373.0	418.2	696.5
	(1, 12)	361.3	398.7	642.9	359.4	397.8	643.0
	(4, 5)	355.9	398.9	595.9	354.6	398.3	595.4
	(4, 9)	221.2	227.3	345.4	221.4	227.6	345.6
	(5, 6)	710.2	736.7	884.4	710.9	737.4	884.5
	(5, 9)	18.9	80.2	407.9	16.7	79.1	407.4
	(6, 7)	588.2	614.2	852.1	587.7	614.4	852.2
	(6, 10)	159.1	239.6	420.2	156.1	237.6	420.3
	(7, 8)	256.5	274.0	369.9	255.0	273.6	370.1
	(7, 11)	331.7	340.2	482.1	332.7	340.9	482.1
	(8, 2)	580.6	555.6	624.9	581.5	556.7	625.2
	(9, 10)	18.8	76.0	347.3	16.4	75.3	347.1
	(9, 13)	221.3	231.5	406.0	221.6	231.4	405.9
	(10, 11)	177.9	315.6	767.5	172.5	312.9	767.4

	(11, 2)	111.4	219.8	601.4	107.7	217.7	601.0
	(11, 3)	398.3	436.0	648.2	397.5	436.1	648.5
	(12, 6)	37.2	117.1	387.9	32.9	114.7	387.9
	(12, 8)	324.1	281.6	255.0	326.5	283.1	255.1
	(13, 3)	221.3	231.5	406.0	221.6	231.4	405.9
O-D	1-2	410.7	470.9	849.9	408.8	470.0	850.3
	1-3	323.8	345.8	489.3	323.6	345.9	489.2
	4-2	281.2	304.5	376.4	280.5	304.4	375.9
	4-3	295.8	321.6	564.9	295.5	321.6	565.1

From the above convergence curves, the following observations can be made: First, it is observed that both algorithms show an overall convergence pattern for different problem instances and under different scale parameter values. Second, the variations of objective function values clearly show a decreasing duality gap between the primal and dual formulations in the convergence process. Third, we can see that the dual algorithm converges significantly faster with a more stable convergence tendency. The result of different convergence speeds of the two algorithms are consistent with our identification on the search directions in these algorithms: The dual algorithm uses a stochastic network loading while the primal algorithm can use only an all-or-nothing loading. Moreover, as we can see, the primal algorithm frequently shows suddenly increasing convergence errors from time to time in its convergence process, which may be due to the sign change of the solution of $\min_{k \in K_{rs}} \partial z^p(\mathbf{f}) / \partial f_k^{rs,i-1}$ for some O-D pairs.

We also examined intermediate solutions in the course of executing the coded computer programs. It is found that the primal algorithm always generates a feasible flow pattern, in terms of the path flow conservation and nonnegativity constraints, while the dual algorithm does not. More specifically, a solution generated by the primal algorithm always satisfies condition (1.2): $q_{rs} P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}) = f_k^{rs}$, $\forall k$, r, s, but not condition (1.3): $d_k^{rs} = b_k^{rs}(\mathbf{f})$, $\forall k$, r, s; a solution generated by the dual algorithm always holds condition (1.3): $d_k^{rs} = b_k^{rs}(\mathbf{f})$, $\forall k$, r, s, but does not satisfy condition (1.2): $q_{rs} P_k^{rs}(\mathbf{c}^{rs}(\mathbf{f}) + \mathbf{d}^{rs}) = f_k^{rs}$, $\forall k$, r, s. Till the optimal point, the solutions from the two algorithmic procedures then get close to satisfying both conditions. This can be seen from examining the computing results in Table 3 and 4, which contain a complete set of converged solutions derived from the two algorithms for the EDSSO and EDSUE problems, respectively, with different scale parameter values.

5.4 Application of the Dual Algorithm for a Larger Network

The above comparative evaluation clearly shows that, in terms of convergence speed and stability, the dual algorithm is evidently the winner between the two candidates, which, in our opinion, is mainly due to the much higher efficiency of its network loading method. This feature makes it attractive in solving network problems of larger size. As an initial attempt and demonstration, we applied the dual algorithm for solving the EDSSO and EDSUE problems in the Sioux Falls network, which contains 24 nodes, 76 links, and 552 O-D pairs. For this computational experiment, we used the same arc cost function and O-D demand function as the ones shown in (18) and (19). The scale parameter ρ is arbitrarily set as 0.5 while the demand parameter θ is set as 0.02 for all O-D pairs. The values of other parameters are all from this transportation network test problem website: http://www.bgu.ac.il/~bargera/tntp/.

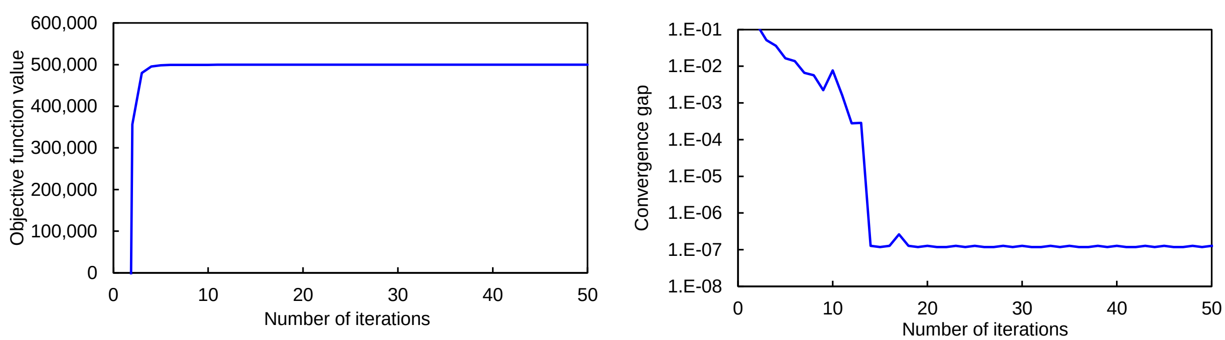


Figure 6 Convergence curves of the dual algorithm solving the EDSSO problem in the Sioux Falls network

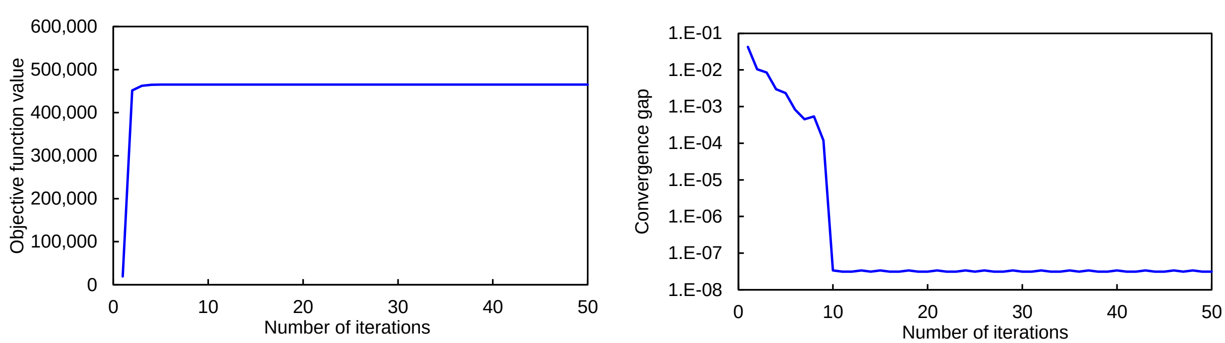


Figure 7 Convergence curves of the dual algorithm solving the EDSUE problem in the Sioux Falls network

The convergence processes from solving the EDSSO and EDSUE problems in the Sioux Falls network are given in Figure 6 and 7, the implied computational performance of which looks very promising: The dual algorithm approximately achieves the convergence precision of 10^{-7} in less than 15 iterations. Specifically, it takes 132.4 sec (14 iterations) and 94.1 sec (10 iterations), respectively, in solving the EDSSO and EDSUE problems.

As we mentioned earlier, the algorithm was written in MATLAB. The above computing times are the reported result of running the MATLAB code on a computer with Intel Core i7 CPU and 4G RAM. So far we have tested the dual algorithms on two synthetic networks with a hypothetical demand function. For a more thorough investigation on this algorithm's performance, a number of real-world example problems with different sizes, topologies and demand levels under different scale parameter values should be used. This poses an important empirical analysis task in the future.

6. A More General Version

As similar to the classic stochastic traffic assignment problems such as the SUE problem, the Markovian user equilibrium (MUE) traffic assignment problem[7] by Baillon and Cominetti (2008) was also developed for traffic network environments with random travel cost perceptions, in which the routing behavior in the MUE problem can be specified by the discrete choice theory and the expected minimum perceived travel cost be evaluated by the satisfaction function or potential function. The distinctive behavioral feature of the MUE problem from the SUE problem is that a traveler is allowed to make a new arc choice when they arrive at any node in the network, by which the route choice of the traveler is a combination of a series of arc choices at intermediate nodes; in contrast, in the SUE problem, the underlying assumption is that any traveler makes his or her route choice only once at the origin node. In other words, the route choice result in the SUE problem is a one-time or one-stage decision-making process, while the route choice result in the MUE problem is a multi-stage decision-making result. For the MUE problem, the arc choice chain setting permits different choice models or mechanisms used at different nodes and for different O-D pairs. If

[7] The Markovian traffic assignment problem we discuss here are the type of spatial Markovian traffic assignment problems that imply an individual Markovian routing behavior with its state space consisting of nodes along paths, but not the temporal type of Markovian problems that imply a Markovian behavior with its state space consisting of periods along the time dimension.

we enforce a single route choice model, for example, the logit model, with the same parameter values, used for the whole network, the MUE problem then collapses to the SUE problem, as shown in Akamatsu (1996). In this case, the result from the one-time route choices at origin nodes in a network is equivalent to the result from a series of arc choices at all passing nodes over the network.

In the same sense, we may sketch as follows the elastic-demand Markovian system optimum (EDMSO) and elastic-demand Markovian user equilibrium (EDMUE) traffic assignment problems, as a more general version of the EDSSO and EDSUE problems, respectively. Following the Markov chain rule, the set of satisfaction functions or potential functions along a path in the dual formulation of the EDMSO or EDMUE problem can be written in the following order:

$$S_{rs}\big(\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f}),\boldsymbol{\theta}^{r}(\mathbf{f})\big)=E\left(\min_{a\in\{(r,i_1):\,i_1\in N\}}\{c_a+b_a+\xi_a\}\middle|\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f})\right)+S_{i_1 s}(\cdot) \quad (18.1)$$

$$S_{i_1 s}\left(\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f}),\boldsymbol{\theta}^{i_1}(\mathbf{f})\right)=E\left(\min_{a\in\{(i_1,i_2):\,i_2\in N\}}\{c_a+b_a+\xi_a\}\middle|\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f})\right)+S_{i_2 s}(\cdot) \quad (18.2)$$

$\dots\dots$

$$S_{i_n s}\left(\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f}),\boldsymbol{\theta}^{i_n}(\mathbf{f})\right)=E\left(\min_{a\in\{(i_n,s)\}}\{c_a+b_a+\xi_a\}\middle|\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f})\right)+S_{ss}(\cdot) \quad (18.3)$$

$$S_{ss}\big(\mathbf{c}(\mathbf{f})+\mathbf{b}(\mathbf{f}),\boldsymbol{\theta}^{s}(\mathbf{f})\big)=0 \quad (18.4)$$

where the arc chain, (r,i_1), (i_1,i_2), …, (i_n,s), specifies a path consisting of $n+1$ arcs from origin node r to destination node s. It is clear that the satisfaction function of any O-D pair r-s is a function of a series of recursive satisfaction functions, each of which evaluates the satisfaction function value between an intermediate node and the destination node s. Also, note that for notation convenience, we use arc cost vectors, $\mathbf{c}(\mathbf{f})=[c_a]$ and $\mathbf{b}(\mathbf{f})=[b_a]$, instead of path cost vectors, $\mathbf{c}^{rs}(\mathbf{f})$ and $\mathbf{b}^{rs}(\mathbf{f})$, and use node-specific arc variance vector $\boldsymbol{\theta}^{i}(\mathbf{f})=\left[\theta_a^i\right]$ instead of O-D-specific path variance vector $\boldsymbol{\theta}^{rs}(\mathbf{f})$ or network-applicable path variance vector $\boldsymbol{\theta}(\mathbf{f})$.

Following this notation resetting, the dual formulation for the EDMSO and EDMUE problems can be simply written as,

$$\max \;\; z^d(\mathbf{f}) = \sum_{rs} \int_0^{S_{rs}} D_{rs}(t)\,\mathrm{d}t \;\; - \sum_{rs}\sum_{k} b_k^{rs} f_k^{rs} \tag{19.1}$$

$$\text{where} \;\; S_{rs} \text{ defined in (28)} \qquad \forall r, s \tag{19.2}$$

$$q_{rs} = D_{rs}(S_{rs}) \qquad \forall r, s \tag{19.3}$$

Following the definition of c_k^{rs} and b_k^{rs} given earlier for the EDSSO and EDSUE problem, if we set $c_a = t_a$ and $b_a = x_a \,\mathrm{d}t_a/\mathrm{d}x_a$, the formulation given in (19) represents the EDMSO problem; if we set $c_a = \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a$ and $b_a = t_a - \int_0^{x_a} t_a(\omega)\mathrm{d}\omega/x_a$, the formulation in (19) specifies the EDMUE problem. The solution equivalency and uniqueness of the above formulation can be proved by the same methods we used in this paper, so we omit them here.

For illustration, we simply show here the optimality conditions of the dual formulation by analyzing its first-order condition of the objective function with respect to an arbitrary path flow variable f_l^{mn}:

$$\begin{aligned}
\frac{\partial z^d}{\partial f_l^{mn}} &= \sum_{rs} D_{rs} \frac{\partial S_{rs}}{\partial f_l^{mn}} - \sum_{rs}\sum_{k}\left(\sum_{a} \delta_{a,k}^{rs} \frac{\partial b_a}{\partial x_a} \delta_{a,l}^{mn}\right) f_k^{rs} - b_l^{mn} \\
&= \sum_{rs} D_{rs} \sum_{\{i_1:\,(r,i_1)\in A\}} P_{ri_1}^{rs} \left(\frac{\partial (c_a + b_a)}{\partial x_a} \delta_{a,l}^{mn} + \frac{\partial S_{i_1 s}(\cdot)}{\partial f_l^{mn}}\right) - \sum_{rs}\sum_{k}\left(\sum_{a} \delta_{a,k}^{rs} \frac{\partial b_a}{\partial x_a} \delta_{a,l}^{mn}\right) f_k^{rs} - b_l^{mn} \\
&= \cdots\cdots \\
&= \sum_{rs} q_{rs} \sum_{a} \frac{\partial (c_a + b_a)}{\partial x_a} \delta_{a,l}^{mn} \sum_{k} P_k^{rs} \delta_{a,k}^{rs} - \sum_{rs}\sum_{k}\left(\sum_{a} \delta_{a,k}^{rs} \frac{\partial b_a}{\partial x_a} \delta_{a,l}^{mn}\right) f_k^{rs} - b_l^{mn} \\
&= \sum_{a} x_a \frac{\partial (c_a + b_a)}{\partial x_a} \delta_{a,l}^{mn} - \sum_{a}\left(x_a \frac{\partial b_a}{\partial x_a} + b_a\right) \delta_{a,l}^{mn} \\
&= \sum_{a}\left(x_a \frac{\partial c_a}{\partial x_a} - b_a\right) \delta_{a,l}^{mn}
\end{aligned}$$

$$\forall l, m, n \tag{20}$$

where $P_{ri_1}^{rs}$ is the probability of travelers between O-D pair r-s choosing arc (r, i_1), $P_{i_1 i_2}^{rs}$ is the probability of travelers choosing arc (i_1, i_2), …, and P_k^{rs} is the probability of travelers choosing path k. Given the arc-path incidence relationship, the path and arc choice probabilities imply the following chain relationship, which is used in the derivation of the above first-order condition:

$$P_k^{rs} = P_{ri_1}^{rs} P_{i_1 i_2}^{rs} \cdots P_{i_n s}^{rs} \qquad \forall k, r, s \tag{21}$$

where it is assumed that path k comprises the arc chain, (r, i_1), (i_1, i_2), …, (i_n, s).

The above analysis and discussion is presented specifically for the dual formulation of EDMSO and EDMUE problems. It should be noted that, in a similar way, we can also construct and prove their primal formulations based on our knowledge developed earlier. From the discussion of this more general problem version with Markovian routing behaviors in the primal-dual modeling framework, we may speculate that the derived formulation-based solution algorithms could be also applied to solving the EDSMSO and EDSMUE problems, as long as the Markovian network loading process be conducted in an algorithmically tractable way. This remains as one of our future research problems.

7. Concluding Remarks

In this work, we redefined the optimality conditions of traffic assignment problems with elastic demand and proposed a pair of generalized mathematical programming formulations for these problems. The pair of formulations pose a Lagrangian duality relationship to each other. The primal formulation is a constrained minimization problem with a set of nonlinear constraints[8], where its constraint set maintains the flow-cost probability choice consistency on the path level (which implies the path flow conservation and nonnegativity relationships), while the dual formulation presents an unconstrained maximization problem. Our focus is given to an analysis on the mathematical properties of the formulations, proving their duality relationship, solution equivalency and uniqueness, and testing and comparing formulation-based solution algorithms.

From the above formulation analysis work, the following findings are identified. First, the optimality conditions can be specified by a combination of three sets of equations, namely, O-D supply-demand consistency, path flow-cost consistency, and supplementary cost equivalency. An arbitrary feasible solution of the primal formulation satisfies the first and second equations, while a feasible solution of the dual solution satisfies the first and third equations. Second, the objective functions of both the primal and dual formulations do not

[8] In the logit-based case, the set of nonlinear constraints collapse to linear constraints (see the formulation in (15)).

imply any intuitive economic meaning, but appear as a pure mathematical construct that can lead to the defined equilibrium or optimality conditions. The connection between mathematical forms and economic implications should be further discovered in the future research. Third, in the primal formulation, the values of supplementary costs of any feasible solution can be uniquely determined. Obviously, this outcome is different from what we observed previously for stochastic traffic assignment problems with fixed demand (see Maher, 2005; Xie and Waller, 2012). Fourth, the primal and dual formulations can be used to accommodate this type of problems with various randomness settings, such as different forms of perceived travel cost distributions and positive or negative covariances of perceived travel costs, as well as a more general version of elastic-demand stochastic traffic assignment problems—elastic-demand Markovian traffic assignment problems.

From the application of primal and dual formulation-based algorithms, we found that the dual algorithm (i.e., the Cauchy algorithm) can be implemented in a more convenient manner and performs in a faster convergence speed than its primal counterpart (i.e., the Frank-Wolfe algorithm). Part of the reason is that the evaluation of the dual formulation and its subproblem can be done on the arc level and a stochastic network loading can be also conducted on the arc basis. Moreover, the stochastic network loading in the dual algorithm is carried out over the entire network instead of a single path. Based on our current understanding on these solution algorithms, we speculate that only a dual formulation-based algorithm probably favors solving large-scale problems, if we limit our algorithmic choice to formulation-based algorithms. However, if we breaks this boundary, then elastic-demand stochastic traffic assignment problems could be solved more efficiently in an alternative way. It is noted that Maher (2001) showed such a possibility by developing a so-called balanced demand algorithm for the EDSUE problem, which may provide us with some useful algorithmic hints for developing a more general procedure for this type of problems.

Acknowledgements

This study is jointly supported by research grants from the China Recruitment Program of Global Experts, the National Natural Science Foundation of China (Grant No. 71471111), the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 2013-007312-0069), and the Key State Laboratory of Ocean Engineering at Shanghai Jiao Tong University (Grant No. GKZD010061).

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Appendix A

In the following, we first show $(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2) < 0$.

For this purpose, we define a continuously differentiable function $\phi(\tau)$ such as,

$$\phi(\tau) = (\mathbf{e}_1 - \mathbf{e}_2)\sum_{rs} D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)\right)\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right) \qquad \text{(A1)}$$

Then we obtain: when $\tau = 1$, $\phi(1) = (\mathbf{e}_1 - \mathbf{e}_2)\sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_1^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_1^{rs})$; when $\tau = 0$, $\phi(0) = (\mathbf{e}_1 - \mathbf{e}_2)\sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_2^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_2^{rs})$. Thus, we have

$$\phi(1) - \phi(0) = (\mathbf{e}_1 - \mathbf{e}_2)\left(\sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_1^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_1^{rs}) - \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_2^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_2^{rs})\right) \qquad \text{(A2)}$$

According to the mean value theorem, there exists such a $\tau^* \in [0,1]$ that $\phi(1) - \phi(0) = \phi'(\tau^*)(1-0)$, i.e.,

$$\phi'(\tau^*) = (\mathbf{e}_1 - \mathbf{e}_2)\left(\sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_1^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_1^{rs}) - \sum_{rs} D_{rs}\big(S_{rs}(\mathbf{e}_2^{rs})\big)\mathbf{P}_A^{rs}(\mathbf{e}_2^{rs})\right) \qquad \text{(A3)}$$

Following (14) and (A3), we readily obtain $\phi'(\tau^*) = (\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2)$.

Based on the definition of derivatives, we can derive $\phi'(\tau^*)$ as follows,

$$\begin{aligned}
\phi'(\tau^*) &= (\mathbf{e}_1 - \mathbf{e}_2)\sum_{rs} \frac{\partial D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)\right)}{\partial S_{rs}} \frac{\partial S_{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)}{\partial \tau^*} \mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs} + \tau^*(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right) \\
&\quad + \sum_{rs} D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs} + \tau^*(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)\right)\sum_a (e_{a,1} - e_{a,2}) \frac{\partial p_a^{rs}\left(e_{a,2} + \tau^*(e_{a,1} - e_{a,2})\right)}{\partial \tau^*} \\
&= \sum_{rs} \frac{\partial D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)\right)}{\partial S_{rs}} \frac{\partial S_{rs}\left(\mathbf{e}_2^{rs} + \tau(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)}{\partial \tau^*} (\mathbf{e}_1 - \mathbf{e}_2)\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs} + \tau^*(\mathbf{e}_1^{rs} - \mathbf{e}_2^{rs})\right)
\end{aligned}$$

$$+\sum_{rs} D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\right)(\mathbf{e}_1-\mathbf{e}_2)\boldsymbol{\nabla}\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)(\mathbf{e}_1-\mathbf{e}_2)^{\mathrm{T}} \tag{A4}$$

where $\boldsymbol{\nabla}\mathbf{P}_A^{rs}(\cdot)$ is the Jacobian matrix of $\mathbf{P}_A^{rs}(\cdot)$. Note that in the first term of the above derivative, we can further derive

$$\begin{aligned}\frac{\partial S_{rs}\left(\mathbf{e}_2^{rs}+\tau(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)}{\partial\tau^*} &= \sum_a \left(e_{a,1}-e_{a,2}\right)\sum_k P_k^{rs}\delta_{a,k}^{rs} \\ &= \sum_a \left(e_{a,1}-e_{a,2}\right)P_a^{rs} \\ &= (\mathbf{e}_1-\mathbf{e}_2)\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\end{aligned} \tag{A5}$$

and in the second term of the derivative, we can obtain

$$\boldsymbol{\nabla}\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)=\boldsymbol{\Delta}^{rs}\boldsymbol{\nabla}\mathbf{P}_K^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)(\boldsymbol{\Delta}^{rs})^{\mathrm{T}} \tag{A6}$$

Inserting these expressions into the derivative $\phi'(\tau^*)$, we have

$$\begin{aligned}\phi'(\tau^*) = &\sum_{rs}\frac{\partial D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs}+\tau(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\right)}{\partial S_{rs}}\left((\mathbf{e}_1-\mathbf{e}_2)\mathbf{P}_A^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\right)^2 \\ &+\sum_{rs} D_{rs}\left(S_{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\right)(\mathbf{e}_1-\mathbf{e}_2)\boldsymbol{\Delta}^{rs}\boldsymbol{\nabla}\mathbf{P}_K^{rs}\left(\mathbf{e}_2^{rs}+\tau^*(\mathbf{e}_1^{rs}-\mathbf{e}_2^{rs})\right)\left((\mathbf{e}_1-\mathbf{e}_2)\boldsymbol{\Delta}^{rs}\right)^{\mathrm{T}}\end{aligned} \tag{A7}$$

Given $\partial D_{rs}(\cdot)/\partial S_{rs}<0$, $D_{rs}(\cdot)\geq 0$, and $(\mathbf{e}_1-\mathbf{e}_2)\boldsymbol{\Delta}^{rs}\boldsymbol{\nabla}\mathbf{P}_K^{rs}(\cdot)\left((\mathbf{e}_1-\mathbf{e}_2)\boldsymbol{\Delta}^{rs}\right)^{\mathrm{T}}\leq 0$ (refer to Sheffi (1985), Page 320), we know $\phi'(\tau^*)\leq 0$, which results in $(\mathbf{e}_1-\mathbf{e}_2)(\mathbf{x}_1-\mathbf{x}_2)<0$.

We then show $(\mathbf{e}_1-\mathbf{e}_2)(\mathbf{x}_1-\mathbf{x}_2)\geq 0$.

For this purpose, we define another continuously differentiable function $\psi(\eta)$ such as,

$$\psi(\eta)=\mathbf{e}\left(\mathbf{x}_2+\eta(\mathbf{x}_1-\mathbf{x}_2)\right)(\mathbf{x}_1-\mathbf{x}_2) \tag{A8}$$

Then we obtain: when $\eta=1$, $\psi(1)=\mathbf{e}(\mathbf{x}_1)(\mathbf{x}_1-\mathbf{x}_2)$; when $\eta=0$, $\psi(1)=\mathbf{e}(\mathbf{x}_2)(\mathbf{x}_1-\mathbf{x}_2)$.

According to the mean value theorem, there is such a $\eta^* \in [0,1]$ that $\psi(1) - \psi(0) = \psi'(\eta^*)(1-0)$, i.e.,

$$\begin{aligned}\psi'(\eta^*) &= \mathbf{e}(\mathbf{x}_1)(\mathbf{x}_1 - \mathbf{x}_2) - \mathbf{e}(\mathbf{x}_2)(\mathbf{x}_1 - \mathbf{x}_2) \\ &= (\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2)\end{aligned} \tag{A9}$$

Based on the definition of derivatives, we can derive $\psi'(\eta^*)$ as follows,

$$\begin{aligned}\psi'(\eta^*) &= \sum_a \frac{\partial c_a\left(x_{a,2} + \eta^*(x_{a,1} - x_{a,2})\right)}{\partial\left(x_{a,2} + \eta^*(x_{a,1} - x_{a,2})\right)}\left(x_{a,1} - x_{a,2}\right)^2 \\ &= (\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}} \boldsymbol{\nabla}\mathbf{e}\big(\mathbf{x}_2 + \eta(\mathbf{x}_1 - \mathbf{x}_2)\big)(\mathbf{x}_1 - \mathbf{x}_2)\end{aligned} \tag{A10}$$

Given $\boldsymbol{\nabla}\mathbf{e}\big(\mathbf{x}_2 + \eta(\mathbf{x}_1 - \mathbf{x}_2)\big) \geq 0$, we know that $\psi'(\eta^*) \geq 0$, which means $(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{x}_1 - \mathbf{x}_2) \geq 0$.