Journal of Industrial and Management Optimization
Vol. 19, No. 7, July 2023, pp. 4726–4739
doi:10.3934/jimo.2022147

MEAN-VARIANCE PORTFOLIO SELECTION WITH RANDOM INVESTMENT HORIZON

Jingzhen Liu

China Institute for Actuarial Science
Central University of Finance and Economics
Beijing, 100081, China

Ka-Fai Cedric Yiu*

Department of Applied Mathematics
The Hong Kong Polytechnic University
Hunghom, Hong Kong, China

Xun Li

Department of Applied Mathematics
The Hong Kong Polytechnic University
Hunghom, Hong Kong, China

Tak Kuen Siu

Department of Actuarial Studies and Business Analytics
Macquarie Business School, Macquarie University
Sydney, NSW 2109, Australia

Kok Lay Teo

School of Mathematical Sciences
Sunway University
Danul Ehsan, Selangor, Malaysia

(Communicated by Kai Zhang)

Abstract. This paper studies a continuous-time securities market where an agent, having a random investment horizon and a targeted terminal mean return, seeks to minimize the variance of a portfolio's return. Two situations are discussed, namely a deterministic time-varying density process and a stochastic density process. In contrast to [18], the variance of an investment portfolio is no longer minimal when all assets are invested in a risk-free security. Furthermore, the random investment horizon has a material effect on the efficient frontier. This provides some insights into the classical mutual fund theorem.

2020 *Mathematics Subject Classification.* Primary: 91G10, 49L12; Secondary: 93E20.

Key words and phrases. Mean variance, random time horizon, HJB equations, efficient frontier.

This work is supported by RGC Grant PolyU 15223419, PolyU grant UAHF, and Project 11771466 supported by National Natural Science Foundation of China.

*Corresponding author: Ka-Fai Cedric Yiu.

1. **Introduction.** Asset allocation is one of the major topics in financial economics. A prominent approach for quantitative asset allocation is the mean-variance portfolio optimization which was pioneered by the seminal work of the Nobel Laureate in Economics, Harry Markowitz, (see [13]). The Markowitz mean-variance portfolio selection model is a static single-period model, where an economic agent seeks to minimize the risk of his investment, measured by the variance of a portfolio's return, subject to a given level of the mean return[1] Extension to multi-period for different applications are still actively developed [10, 5]. A dynamic extension of the Markowitz model, especially in continuous time, has been widely studied in the literature (see, for example, [6, 8, 9, 16, 18]). Analytical forms of efficient portfolios were often obtained in these works.

Since the original work of Markowitz, much attention has been paid to those mean-variance portfolio problems where economic agents know exactly their time horizons when making investment decisions. In many situations, however, an investor typically does not know exactly the time of exiting from the market which may possibly be due to some unexpected events (see, for example, [7]). These events may be "defaults", which force an investor to leave a market liquidating his/her assets. Furthermore, in some situations, the rate of arrival of a default event at a future date, which is called a default intensity, is not known. Consequently, it is questionable if the default intensity should be assumed deterministic. A stochastic intensity model may provide a more suitable way than the deterministic one to describe the real market situation. Finally, it may often be the case that an investor is not certain about his/her investment horizon when an initial investment decision is made. An investment model with random time horizons provides the flexibility in incorporating this feature and may be used to describe the risk from an uncertain investment horizon, which is also known as the timing risk (see, for example, [1]).

In this paper, we study a continuous-time mean-variance problem with a random investment horizon. A closely related problem was considered by Martellini and Urosevic [14] on the Markowitz mean-variance analysis in the situation where an uncertain independent time of exit was incorporated. They studied the problem in a static setting. More recently, Blanchet-Scallient et al. [1] considered an optimal investment-consumption problem with utility maximization and uncertain exit time by taking into account uncertainties in future default intensities and developed a stochastic density process of the default time, which was modeled as a geometric Brownian motion. Blanchet-Scallient et al. [1] pointed out the difficulty in using the Hamilton-Jacobi-Bellman (HJB) dynamic programming approach and adopted the martingale approach to derive an explicit characterization of the optimal wealth process. Random model parameters and random horizon was considered by Yu [17] for continuous time mean-variance portfolio selection problem. Here, as in [1], both deterministic and stochastic functions of the exit time (in particular, the default time) are discussed. In addition to the geometric Brownian motion model, we also develop a switching regime model to describe the stochastic density process of the exit time. Furthermore, we model the relationship between the intensity process of

[1] In Markowitz's original setup, the model is formulated as a *multi*-objective optimization problem, namely, to maximize the mean return and minimize the variance of the return. There are multiple solutions to this problem, leading to the known *efficient frontier*. Mathematically, each solution can be recovered by solving a *single*-objective optimization problem where the variance is minimized while the mean return is constrained at a given level.

the random time horizon and the economic uncertainty described by a continuous-time, finite-state, Markov chain (see Section 4.2). Indeed, the uncertainty described by the Markov chain may be possibly interpreted more generally as the environmental uncertainty. In the literature, both dynamic programming and linear quadratic control are two common methods to tackle the mean-variance problem, which can be reduced to solving the Hamilton-Jacobi-Bellman (HJB) equations or the Riccati equations, respectively. Here we adopt the dynamic programming approach. An interesting finding is that even in the deterministic case, the variance is no longer minimal with the risk-free investment only and the minimal variance is positive. Moreover, the efficient frontier is an upper part of a hyperbolic curve. Due to the timing risk, two mutual funds are needed to replicate the portfolio on the efficient frontier. These results are in contrast with Zhou and Li [18]. Indeed our results seem to refute the conventional wisdom appearing in some of the previous literature that an optimal portfolio strategy is not affected by the presence of an uncertain time horizon.

The rest of the paper is organized as follows. The next section formulates the mean-variance portfolio selection problem with random horizon. In Section 3, optimal portfolios and efficient frontiers from the mean-variance portfolio problem are derived under two models for the conditional density process of the random exit time given the price information. A particular case with one risky asset and constant coefficients is considered in Section 4. Numerical analysis of the theoretical results is provided in Section 5. Concluding remarks are given in Section 6.

2. **Dynamic mean-variance problem formulation and preliminaries.**

We consider a standard continuous-time multi-dimensional financial market with a (locally) risk-free bond and n risky shares, where these securities can be traded continuously over time in the horizon $[0,\infty)$. As usual, uncertainty and its resolution over time are modeled by a complete, filtered probability space $(\Omega,\mathcal{G},\mathbb{G},P)$, where $\mathbb{G}:=\{\mathcal{G}_t|t\geq 0\}$ is a filtration satisfying the usual conditions of P-completeness and right-continuity and P is a real-world probability measure. Similar to Blanchet-Scallient et al. [1], Page 1102, it is supposed that $\mathcal{G}_\infty\subset\mathcal{G}$. We also impose the following standard assumptions about the market structure such as a perfect frictionless market, (see, for example, Merton [15]).

For each $t\geq 0$, let $r(t)$ be the instantaneous continuously compounded rate of interest at time t, where $r(t)>0$. To simplify the discussion, we suppose that $r(t)$ is a deterministic function of time t. Then the value of the risk-free bond evolves over time as:

$$dB(t)=r(t)B(t)dt\ ,\quad B(0)=1\ .$$

Let $\{W(t)|t\geq 0\}$ be an n-dimensional standard Brownian motion with respect to the filtration $\mathbb{F}$ under the measure P, where $W(t):=(W_1(t),W_2(t),\cdots,W_n(t))^\top$ and $M^\top$ is the transposition of a matrix, (or a vector), M. For each $t\geq 0$ and each $i,j=1,2,\cdots,N$, let $\mu_i(t)$ be the appreciation rate of the i^{th} risky share at time t and $\sigma_{ij}(t)$ be the volatility component of the i^{th} risky share attributed to the j^{th} random shock described by the Brownian motion component $W_j(t)$. We suppose that for each $i,j=1,2,\cdots,N$, $\{\mu_i(t)|t\geq 0\}$ and $\{\sigma_{ij}(t)|t\geq 0\}$ are deterministic functions of time t. Write, for each $t \geq 0$,

$$\begin{aligned} S(t) &:= (S_1(t), S_2(t), \cdots, S_n(t))^\top , \\ \mu(t) &:= (\mu_1(t), \mu_2(t), \cdots, \mu_n(t))^\top , \\ \sigma(t) &:= [\sigma_{ij}(t)]_{i.j=1,\ldots,n} , \end{aligned}$$

and $D(S(t)) := \mathrm{d}iag[S(t)]$, the diagonal matrix with diagonal elements being the components in the vector $S(t)$. Then we assume that the evolution of the prices of the n risky shares over time is described by the following n-dimensional geometric Brownian motion:

$$dS(t) = D(S(t))(\mu(t)dt + \sigma(t)dW(t)) , \quad S(0) = s_0 ,$$

where $s_0 := (s_{10}, s_{20}, \cdots, s_{n0})^\top$ with $s_{i0} > 0$ for each $i = 1, 2, \cdots, n$.

For each $t \geq 0$, let $\mathcal{F}_t := \sigma\{S(u)|u \in [0,t]\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of all the P-null subsets in $\mathcal{F}$ and $\mathcal{A} \vee \mathcal{B}$ is the minimal σ-algebra containing both the σ-algebras $\mathcal{A}$ and $\mathcal{B}$. That is, $\mathcal{F}_t$ represents the price information up to and including time t. The assumption that $\mathcal{F}_\infty \subset \mathcal{F}$ means that there are other sources of information besides the price information. This assumption is imposed to accommodate the uncertainty about the exit time of an economic agent from the market. In the sequel, we impose the following two technical assumptions on the market coefficients, (see, for example, [1]):

Assumption (A1) $\{r(t)|t \in [0,T]\}$, $\{\mu(t)|t \in [0,T]\}$ *and* $\{\sigma(t)|t \in [0,T]\}$ *are bounded.*

Assumption (A2) *For each* $t \in [0,T]$, $\sigma(t)$ *is bounded, invertible, and its inverse* $[\sigma(t)]^{-1}$ *is also bounded.*

We now introduce the following notation:

$$B(t) := (\mu_1(t) - r(t), \mu_2(t) - r(t), \cdots, \mu_n(t) - r(t))^\top , \tag{1}$$

and

$$\theta(t) \equiv (\theta_1(t), \theta_2(t), \cdots, \theta_n(t))^\top := [\sigma(t)]^{-1} B(t) . \tag{2}$$

Note that the existence and boundedness of $\theta(t)$ is guarranteed by the assumptions (A1) and (A2). Consider now an economic agent, with an initial endowment $x_0 > 0$, who invests continuously over time the bond and the n risky shares in the horizon $[0, \infty)$. To simplify our discussion, we assume that there is no consumption and that investment portfolios are self-financing. For each $i = 1, 2, \cdots, n$ and $t \geq 0$, let $\pi_i(t)$ be the amount of money invested in the i^{th}-risky share at time t and $\{X^\pi(t)|t \geq 0\}$ be the corresponding wealth process. Then $X^\pi(t) - \sum_{i=1}^n \pi_i(t)$ is the amount of money invested in the bond at time t. To simplify the notation, we suppress π in $X^\pi(t)$ and write $X(t)$ for $X^\pi(t)$ unless otherwise stated. Then it is easy to see that

$$\begin{aligned} dX(t) &= [r(t)X(t) + \pi(t)^\top(\mu(t) - r(t)\underline{1})]dt + \pi(t)^\top \sigma(t) dW_t, \quad t \geq 0, \\ X(0) &= x_0, \end{aligned} \tag{3}$$

where $\underline{1} = (1, 1, \cdots, 1)^\top \in \Re^n$; $\pi(t) := (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))^\top \in \Re^n$; $\{\pi(t)|t \geq 0\}$ is called a portfolio process.

For a given finite horizon T, (i.e., $T < \infty$), let $L^2_{\mathcal{F}}(0, T; \Re^n)$ be the space of $\Re^n$-valued, $\mathbb{F}$-progressively measurable processes $\{\phi(t)|t \in [0,T]\}$ such that

$$\int_0^T |\phi(t)|^2 dt < \infty.$$

A portfolio $\{\pi(t)|t \in [0,T]\}$ is said to be admissible if (1) it is self-financing; (2) it belongs to $L^2_{\mathcal{F}}(0,T;\Re^n)$; (3) the solution of the SDE (3) has a unique strong solution. Denote Π as the space of all admissible portfolio processes.

Let τ be the random exit time of the agent from the market, which is also a stopping time. However, as in Blanchet-Scallient et al. [1], Page 1102, it is not supposed that τ is an $\mathbb{F}$-stopping time. Instead, τ is a stopping time with respect to a large filtration than $\mathbb{F}$. In this case, given the price information up to and including time t, it cannot be decided whether the event $\{\tau < t\}$ has occurred or not. Write, for each $t \in [0,\infty)$,

$$F(t) := P(\tau \leq t|\mathcal{F}_t), \quad t \geq 0,$$

so that $F(t)$ is the conditional probability distribution function of τ given $\mathcal{F}_t$.

By definition, $\{F(t)|t \geq 0\}$ is an increasing, $\mathbb{F}$-adapted, stochastic process. Assume, as in Blanchet-Scallient et al. [1], Page 1102, that for each $t \geq 0$, $F(t)$ is an increasing and absolutely continuous with respect to the Lebesgue measure dt on $[0,\infty)$ so that there exists a conditional density function $f(s)$ such that

$$F(t) = \int_0^t f(s)ds, \quad t \geq 0.$$

This is called the **G** assumption and holds true if $P(\tau \leq t|\mathcal{F}_t) = P(\tau \leq t|\mathcal{F}_\infty)$ (see Blanchet-Scallient et al. [1], Remark 1). The sufficient condition for the **G** assumption may be interpreted as the filtered and smoothed estimates for $1_{\{\tau \leq t\}}$ being the same. By definition, $\{f(t)|t \geq 0\}$ is itself an $\mathbb{F}$-adapted, non-negative, stochastic process.

If τ is an $\mathbb{F}$-stopping time (which is not assumed here), $\{\tau \geq t\} \in \mathcal{F}_t$ (i.e., $\{\tau \geq t\}$ is $\mathcal{F}_t$-measurable), then $F(t)$ is equal to either zero or one, and so $F(t)$ is singular with respect to the Lebesgue measure dt on $[0,\infty)$. In this case, the conditional density process $\{f(t)|t \geq 0\}$ does not exist.

Some works in the literature, for example, [1, 2, 3] considered optimal investment and/or consumption problems with random horizon. Here we also consider the mean variance problem where the investment time horizon is random. There may be different interpretations for the random exit time. Some interpretations may be the random life time of an investor, the random market timing, and the default of the agent.

Suppose now that the finite horizon T is the terminal investment horizon of an economic agent. The object of the agent is to select an admissible portfolio process $\{\pi(t)|t \in [0,T]\} \in \Pi$ so that the risk described by the variance of the terminal wealth:

$$\mathrm{Var}[X(T\wedge\tau)] := E[X(T\wedge\tau)-E(X(T\wedge\tau))]^2 \equiv E[X(T\wedge\tau)-d]^2 \equiv E[X(T\wedge\tau)]^2-d^2 \,, \tag{4}$$

is minimized subject to the following constraint on the expected terminal wealth:

$$E[X(T \wedge \tau)] = d \,.$$

Here E is the expectation with respect to the measure P; d is a given mean level. Note that the agent will exit from the market at the random time τ, so we call either τ or T a terminal time. The above optimization problem of the agent is called the mean-variance portfolio selection problem with random exit time. Mathematically, it can be formulated as follows.

Problem. The mean-variance portfolio selection problem with an initial wealth x_0 is formulated as the following constrained stochastic optimization problem parameterized by $d \geq x_0 E[e^{\int_0^{T\wedge\tau} r(s)ds}]$:

$$\begin{cases} \textit{minimize} & J(\pi) := E[X(T\wedge\tau)]^2 - d^2, \\ \textit{subject to} & \begin{cases} X(0) = x_0 \ , \quad E[X(T\wedge\tau)] = d \ , \\ (X(\cdot), \pi(\cdot)) \ \ \textit{admissible.} \end{cases} \end{cases} \tag{5}$$

The problem is said to be feasible (with respect to d) if there is at least one admissible portfolio process π satisfying $E[X^\pi(T\wedge\tau)] = d$.

Remark 1. In the above formulation, the parameter d is restricted to be no less than d_0. In Zhou and Li (2004), it was assumed that $d_0 = x_0 e^{\int_0^T r(s)ds}$, which is the terminal payoff if the agent invests all of his/her wealth in the bond at time 0 and re-invest the proceeds continuously over time up to time T. This is a (locally) risk-free portfolio and has the minimal expected return d_0. In our current situation, due to the presence of the random exit time τ, we can no longer assert that $d_0 = x_0 E[e^{\int_0^{T\wedge\tau} r(s)ds}]$ which corresponds to the minimal variance.

As in Blanchet-Scalliet et al.[1], **Problem 2** is equivalent to the following fixed time-horizon problem.

Problem.

$$\min_{\pi\in\Pi} E\left[\int_0^T f(s)(X(s)-d)^2 ds + (1-F(T))(X(T)-d)^2\right] ,$$

subject to the static budget constraint:

$$E\left[\int_0^T f(s)X(s)ds + (1-F(T))X(T)\right] = d,$$

and the dynamic budget constraint (2.3) for the wealth process $\{X(t)|t\in[0,T]\}$.

Since (5) is a convex optimization problem, the equality constraint $E[X(T\wedge\tau)] = d$ can be handled using a Lagrange multiplier $\gamma \in \Re$.
Define

$$\begin{aligned} \bar{J}(\pi,\gamma) \quad := \quad & E\left[\int_0^T f(s)\Big((X(s)-d)^2 - 2\gamma(X(s)-d)\Big)ds\right. \\ & \left. +(1-F(T))((X(T)-d)^2 - 2\gamma(X(T)-d))\right] , \end{aligned} \tag{6}$$

with $(\pi(t), X(t))$ satisfying (3).
By the Lagrange duality theorem, (see, for example, Luenberger [12]),

$$\min_\pi J(\pi) = \max_\gamma \min_\pi \bar{J}(\pi,\gamma) = \min_\pi \max_\gamma \bar{J}(\pi,\gamma).$$

Rewrite $\bar{J}(\pi,\gamma)$ as follows:

$$E\left[\int_0^T f(s)(X(s)-\beta)^2 ds + (1-F(T))(X(T)-\beta)^2 - (\beta-d)^2\right],$$

with $\beta = d+\gamma$. Then

$$\min_\pi \bar{J}(\pi,\gamma),$$

with $\{X^\pi(t)|t\in[0,T]\}$ satisfying (3), is equivalent to solving:

Problem. *For each fixed $\beta \in \Re$, the problem is*

$$\begin{cases} \text{minimize } \hat{J}(\pi,\beta) := E\left[\int_0^T f(s)(X(s)-\beta)^2 ds + (1-F(T))(X(T)-\beta)^2\right], \\ \text{subject to } \pi \in \Pi, \text{ and } (\pi, X^\pi(t)) \text{ satisfying } (3). \end{cases}$$

3. **Optimal portfolios and efficient frontiers.** Recall that the conditional density process $\{f(t)|t \geq 0\}$ is an $\mathbb{F}$-adapted, non-negative, stochastic process, where $f(t) := \frac{dP(\tau \leq t|\mathcal{F}_t)}{dt}$. Specifically, for each $t \geq 0$, $f(t)$ is interpreted as the conditional density function of the random exit time τ evaluated at time t given $\mathcal{F}_t$. In order to incorporate more realistic settings, we consider two models for the conditional density process $\{f(t)|t \geq 0\}$. First, it is supposed that the conditional density process $\{f(t)|t \geq 0\}$ is governed by a geometric Brownian motion. This situation was considered in Blanchet-Scallient et al. [1] Section 6 therein. Second, it is assumed that the conditional density process $\{f(t)|t \geq 0\}$ is specified by the probability density function of sojourn times of a Cox process with its intensity process being modulated by a continuous-time, finite-state, Markov chain. The first model can reflect random perturbations to economic conditions, while the second model can be used to capture structural changes in (macro)-economic conditions. Both models have found a wide range of applications in economics and finance.

3.1. **Geometric Brownian motion.** The situation considered in this subsection is related to those in Blanchet-Scalliet et al.[1] and Yu [17]. Let $\{W^e(t)|t \in [0,T]\}$ be another standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$, where $\{W^e(t)|t \in [0,T]\}$ and $\{W(t)|t \in [0,T]\}$ are stochastically independent under P. Following Blanchet-Scalliet et al. [1], we suppose that the random density process $\{f(t)|t \in [0,T]\}$ evolves over time as the following geometric Brownian motion:

$$df(t) = f(t)(a(t)dt + b(t)dW^e(t)), \quad f(0) = y_0, \quad t \in [0,T], \tag{3.1}$$

where $a(t)$ and $b(t)$ are deterministic functions of time t with $a(t) \in \Re_-$ and $b(t) \in \Re_+$; $\Re_+$ is the set of positive real numbers. In Blanchet-Scalliet et al. [1], $f(t)$ is also represented as the following explicit form:

$$f(t) = f_0 \exp\left(\int_0^t a(u)du\right)\xi(t),$$

where $\xi(t) := \exp\left(\int_0^t b(u)dW^e(u) - \frac{1}{2}\int_0^t b^2(u)du\right)$. The representation form resembles the form of a state price density in asset pricing theory. For each $(t,x,y) \in [0,T] \times \Re \times \Re_+$, we define the value function by

$$V(t,x,y;\beta) := \min_{\pi \in \Pi} E\left[\int_t^T f(s)(X(s)-\beta)^2 ds \right.$$
$$\left. + \int_T^{+\infty} f(t)dt(X(T)-\beta)^2 \middle| X(t)=x, f(t)=y\right].$$

Now Problem 2 is reduced to solving the following HJB equation:

$$\frac{\partial V}{\partial t} + y(x-\beta)^2 + \inf_{\pi \in \Re^n}\left\{\left[r(t)x + \pi^\top(\mu(t) - r(t)\mathbf{1})\right]\frac{\partial V}{\partial x} + \frac{1}{2}\pi^\top\sigma(t)\sigma(t)^\top\pi\frac{\partial^2 V}{\partial x^2}\right\}$$
$$+a(t)y\frac{\partial V}{\partial y} + \frac{1}{2}b^2(t)y^2\frac{\partial^2 V}{\partial y^2} = 0\ , \tag{3.2}$$

with terminal condition

$$V(T,x,y;\beta) = y(x-\beta)^2 \int_T^{+\infty} \exp\left(\int_T^u a(v)dv\right)du. \tag{3.3}$$

Theorem 3.1. (Verification Theorem) *Let $\tilde{W}(t,x,y;\beta) \in \mathcal{C}^{1,2}([0,T]\times\Re\times\Re_+)$, where $\mathcal{C}^{1,2}([0,T]\times\Re\times\Re_+)$ is the space of functions which are continuously differentiable in $t\in[0,T]$ and twice continuously differentiable in $(x,y)\in\Re\times\Re_+$. Suppose $\tilde{W}(t,x,y;\beta)$ is a solution of the HJB equation (3.2)-(3.3). Define the following partial differential operator $\mathcal{L}_1^\pi$ on the function space $\mathcal{C}^{1,2}([0,T]\times\Re\times\Re_+)$:*

$$\begin{aligned}\mathcal{L}_1^\pi[\tilde{W}(t,x,y;\beta)] &:= \frac{\partial \tilde{W}}{\partial t} + y(x-\beta)^2 + \Bigg[(r(t)x+\pi^\top(\mu(t)-r(t)\mathbf{1}))\frac{\partial \tilde{W}}{\partial x} \\ &+\frac{1}{2}\pi^\top\sigma(t)\sigma(t)^\top\pi\frac{\partial^2\tilde{W}}{\partial x^2} + a(t)y\frac{\partial \tilde{W}}{\partial y} + \frac{1}{2}b^2(t)y^2\frac{\partial^2\tilde{W}}{\partial y^2}\Bigg].\end{aligned}$$

Assume that the following conditions hold:

1. *$\mathcal{L}_1^\pi[\tilde{W}(t,x,y;\beta)] \geq 0$;*
2. *there exists an admissible portfolio process $\pi^*\in\Pi$ satisfying*

$$\mathcal{L}_1^{\pi^*}[\tilde{W}(t,x,y;\beta)] = 0\ , \quad (t,x,y)\in[0,T]\times\Re\times\Re_+\ .$$

Then π^* is optimal and

$$\tilde{W}(t,x,y;\beta) = V(t,x,y;\beta)\ , \quad (t,x,y)\in[0,T]\times\Re\times\Re_+\ .$$

We state without proof the following theorem, which gives a solution to the HJB equation (3.2)-(3.3). In particular, the solution is expressed as a quadratic form in x. The result can be proven by standard mathematical arguments based on substitution.

Theorem 3.2. *For each $(t,x,y)\in[0,T]\times\Re\times\Re_+$ and each fixed $\beta\in\Re$, we define the following function:*

$$\tilde{W}(t,x,y;\beta) := \big[\tilde{P}(t)x^2 - 2\beta\tilde{Q}(t)x + \tilde{R}(t)\beta^2\big]y\ , \tag{3.4}$$

where

$$\begin{aligned}\tilde{P}(t) &= \int_t^{+\infty} e^{\int_t^u a(v)dv} e^{\int_t^{u\wedge T}(2r(s)-\theta(s)^\top\theta(s))ds}du,\\ \tilde{Q}(t) &= \int_t^{+\infty} e^{\int_t^u a(v)dv} e^{\int_t^{u\wedge T}(r(s)-\theta(s)^\top\theta(s))ds}du,\\ \tilde{R}(t) &= \int_t^{+\infty} e^{\int_t^u a(v)dv}du - \int_t^T e^{\int_t^s a(v)dv}\frac{\tilde{Q}^2(s)}{\tilde{P}(s)}\theta(s)^\top\theta(s)ds.\end{aligned}$$

Then $\tilde{W}(t,x,y;\beta)\in\mathcal{C}^{1,2}([0,T]\times\Re\times\Re_+)$ and is a solution of the HJB equation (3.2)-(3.3).

Putting $t=0$, noting that $X(0)=x_0$, $Y(0)=y_0$ and substituting β with $d+\gamma$ into (3.4) lead to the following function with respect to the Lagrange multiplier γ,

$$\big[\tilde{P}(0)x_0^2 - 2(d+\gamma)\tilde{Q}(0)x_0 + \tilde{R}(0)(d+\gamma)^2\big]y_0 - \gamma^2\ . \tag{3.5}$$

Then (3.5) attains its maximum

$$\frac{\tilde{R}(0)y_0(d-\frac{\tilde{Q}(0)x_0^2}{\tilde{R}(0)})}{1-\tilde{R}(0)y_0}+(\tilde{P}(0)-\frac{\tilde{Q}^2(0)}{\tilde{R}(0)})x_0^2y_0$$

at

$$\gamma^*=arg\max_{\gamma}\{\tilde{W}(0,x_0,y_0;d+\gamma)-\gamma^2\}=\frac{\tilde{R}(0)dy_0-\tilde{Q}(0)x_0y_0}{1-\tilde{R}(0)y_0}. \tag{3.6}$$

Note that $\tilde{P}(0)-\frac{\tilde{Q}^2(0)}{\tilde{R}(0)}>0$. Now we set $d_0:=\frac{\tilde{Q}(0)x_0}{\tilde{R}(0)}$. Then the portfolio is efficient if and only if $d\geq d_0$. The results can be summarized as the following theorem.

Theorem 3.3. *The efficient portfolio of Problem 2, corresponding to the expected terminal wealth*

$$E[X^*(T\wedge\tau)|X(0)=x_0,f(0)=y_0]=d,\ \ d\geq d_0.$$

is given by:

$$\pi^*(t)=[\sigma(t)\sigma(t)^\top]^{-1}(\mu(t)-r(t)\underline{1})\left(\frac{\tilde{Q}(t)(d+\gamma^*)}{\tilde{P}(t)}-x_0\right), \tag{3.7}$$

and the efficient frontier is given by:

$$\sigma[X^*(T\wedge\tau)|X(0)=x_0,f(0)=y_0]=\sqrt{\frac{\tilde{R}(0)y_0(d-d_0)}{1-\tilde{R}(0)y_0}+\left(\tilde{P}(0)-\frac{\tilde{Q}^2(0)}{\tilde{R}(0)}\right)x_0^2y_0} \tag{3.8}$$

where $d_0:=\frac{\tilde{Q}(0)x_0}{\tilde{R}(0)}$.

Remark 2. An interesting observation from Theorem 3.3 is that the results are independent of the volatility $b(t)$ of the conditional density process, which describes the uncertainty about the process. Moreover, no risk free portfolio can be obtained and the volatility is not minimal when all the wealth is put into the risk-free asset due to time risk. Similar to Theorem 3.6 in ([11]), let π_1 and π_2 be the optimal investment strategy corresponding to the expected returns d_1 and d_2, respectively, where π_1 denotes the minimal-variance portfolio, then for any feasible d_3, $\pi_3=\frac{d_1-d_3}{d_1-d_2}\pi_1+(1-\frac{d_1-d_3}{d_1-d_2})\pi_2$.

3.2. **Finite state Markov chain.** For each $t\in[0,\infty]$, let $\lambda(t)$ be the hazard rate at time t, which is defined as:

$$\lambda(t):=\lim_{\Delta t\to 0}\frac{P(t\leq\tau<t+\Delta t)}{1-F(t)}=\frac{f(t)}{1-F(t)},$$

or equivalently,

$$f(t):=\lambda(t)e^{-\int_0^t\lambda(s)ds},t\geq 0,$$

so that $\lambda(t)$ and $f(t)$ are isomorphic. Here $\{\lambda(t)|t\geq 0\}$ is also called an intensity process. Note that $f(t)$ is the probability density function of sojourn times of a Cox process, or doubly stochastic process, with a Markov-modulated intensity process $\{\lambda(t)|t\geq 0\}$.

Consequently, the mathematical set up for the uncertain time-horizon is characterised by the intensity parameter $\lambda(t)$. We suppose that the intensity parameter $\lambda(t)$ depends on the state of an underlying environment at time t. In the case where the random exit time is interpreted as the random life time, the state of the environment may be interpreted as the state of the environmental risk. In the case where the random exit time is interpreted as the default time, the state of the underlying environment may be interpreted as the state of an economy. We now describe the evolution of the state of an economy over time by an observable, continuous-time, finite-state Markov chain $\mathbf{Z} := \{\mathbf{Z}(t)\}_{t\in[0,T]}$ with state space $\mathcal{Z} := (\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_N) \in \Re^N$. Without loss of generality, as in Elliott et al. [4], we identify the state space of the chain $\mathbf{Z}$ as a finite set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \cdots, N$.

Let $\mathbf{G}$ be the rate matrix, or the generator, $[g_{ij}]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{Z}$. Then, with the canonical representation of the state space of the chain, Elliott et al.[4] gave the following semimartingale dynamics for the chain $\mathbf{Z}$:

$$\mathbf{Z}(t) = \mathbf{Z}(0) + \int_0^t \mathbf{G}\mathbf{Z}(s)ds + \mathbf{M}(t)\ , \tag{3.9}$$

where $\{\mathbf{M}(t)|t \in [0, \infty]\}_{t\in\mathcal{T}}$ is an $\Re^N$-valued martingale with respect to the natural filtration generated by $\mathbf{Z}$. We assume that the intensity $\lambda(t)$ is modulated by the chain $\mathbf{Z}$ as follows:

$$\lambda(t) := \langle \lambda, \mathbf{Z}(t)\rangle\ , \tag{3.10}$$

where $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_N)^\top \in \Re^N$ with $\lambda_i > 0$, for each $i = 1, 2, \ldots, N$, λ_i represents the intensity parameter corresponding to the i^{th} state of the underlying environment. **Problem 2** is equivalent to the following fixed time-horizon problem:

$$\begin{aligned}&\min_{u\in\mathcal{U}} E\bigg[\int_0^T e^{-\int_0^s \lambda(u)du}\lambda(s)(x(s)-d)^2 ds\\&+e^{-\int_0^T \lambda(u)du}(X(T)-d)^2|X(0)=x, \mathbf{Z}(0)=\mathbf{z}\bigg],\end{aligned}$$

subject to

$$E\bigg[\int_0^T e^{-\int_0^s \lambda(u)du}\lambda(s)x(s)ds + e^{-\int_0^T \lambda(u)du}X(T)|X(0)=x, \mathbf{Z}(0)=\mathbf{z}\bigg] = d,$$

$$d \geq d_0,\ X(t) \text{ follows the wealth process in } (3).$$

With a Lagrange multiplier γ and denoting $\beta = d + \gamma$, write

$$\begin{aligned}\hat{J}(\pi,\beta) := E\bigg[&\int_0^T e^{-\int_0^s \lambda(u)du}\lambda(s)(x(s)-\beta)^2 ds + e^{-\int_0^T \lambda(u)du}(X(T)-\beta)^2\\&-(\beta-d)^2|X(0)=x, \mathbf{Z}(0)=\mathbf{z}\bigg],\end{aligned} \tag{3.11}$$

the problem becomes

$$\max_{\beta}\min_{\pi\in\Pi} J(\pi,\beta).$$

For every fixed β, $\min_{\pi\in\Pi}\hat{J}(\pi,\beta)$ is equivalent to

$$\begin{aligned}&\min_{\pi\in\Pi} E\bigg[\int_0^T e^{-\int_0^s \lambda(u)du}\lambda(s)(x(s)-\beta)^2 ds\\&+e^{-\int_0^T \lambda(u)du}(X(T)-\beta)^2|X(0)=x, \mathbf{Z}(0)=\mathbf{e}_j\bigg],\end{aligned}$$

For each $(t,x) \in [0,T] \times \Re$ and each $i = 1, 2, \cdots, N$, we define the value function by

$$V(t,x,\mathbf{e}_i;\beta) := \min_{\pi\in\Pi} E\Big[\int_t^T e^{-\int_t^s \lambda(u)du}\lambda(s)(X(s)-\beta)^2 ds$$

$$+e^{-\int_t^T \lambda(u)du}(X(T)-\beta)^2\Big| X(t)=x, \mathbf{Z}(t)=\mathbf{e}_i].$$

Suppose $V(t,x,\mathbf{e}_i;\beta) \in \mathcal{C}^{1,2}([0,T]\times\Re)$, for each $i = 1,2,\cdots,N$. Let $V_i(t,x;\beta) := V(t,x,\mathbf{e}_i;\beta)$ and $\mathbf{V} := (V_1, V_2, \ldots, V_N)^\top \in \Re^N$. Then by the standard dynamic programming principle V_i satisfies the following regime-switching HJB equation:

$$\begin{aligned}&\frac{\partial V_i}{\partial t} + \inf_{\pi\in\Re^n}\left\{\Big[r(t)x + \pi^\top(\mu(t) - r(t))\Big]\frac{\partial V_i}{\partial x} + \frac{1}{2}\pi^\top\sigma(t)\sigma(t)^\top\pi\frac{\partial^2 V_i}{\partial x^2}\right\}\\ &+\langle\lambda,\mathbf{e}_i\rangle(x-\beta)^2 - \langle\lambda,\mathbf{e}_i\rangle V_i + \langle\mathbf{V},\mathbf{G}\mathbf{e}_i\rangle = 0 \ ,\end{aligned} \tag{3.12}$$

with terminal condition

$$V_i(T,x;\beta) = (x-\beta)^2, \tag{3.13}$$

for each $i = 1,2,\cdots,N$. Define the following regime-switching partial differential operator on $\mathcal{C}^{1,2}([0,T]\times\Re)$:

$$\begin{aligned}\mathcal{L}_2^\pi[V_i(t,x;\beta)] \ := \ &\Big[r(t)x + \pi^\top(\mu(t) - r(t)\underline{1})\Big]\frac{\partial V_i}{\partial x} + \frac{1}{2}\pi^\top\sigma(t)\sigma(t)^\top\pi\frac{\partial^2 V_i}{\partial x^2}\\ &+\langle\lambda,\mathbf{e}_i\rangle(x-\beta)^2 - \langle\lambda,\mathbf{e}_i\rangle V_i + \langle\mathbf{V},\mathbf{G}\mathbf{e}_i\rangle \ .\end{aligned} \tag{3.14}$$

Theorem 3.4. **(Verification Theorem)** *Suppose the following conditions hold:*

1. $W_i(t,x;\beta) \in \mathcal{C}^{1,2}([0,T]\times\Re)$*, for each* $i = 1,2,\cdots,N$*;*
2. $\mathcal{L}_2^\pi[W_i(t,x;\beta)] \geq 0$*;*
3. *there exists an admissible portfolio process* $\pi_i^* \in \Pi$ *satisfying* $\mathcal{L}_2^{\pi_i^*}[W_i(t,x;\beta)] = 0$*, for each fixed* $\beta\in\Re$*,* $(t,x)\in[0,T]\times\Re$ *and each* $i = 1,2,\cdots,N$*.*

Then π_i^* *is optimal, and* $W_i(t,x;\beta) = V_i(t,x;\beta)$*.*

Theorem 3.5. *For each* $i = 1,2,\cdots,N$ *and each* $(t,x)\in[0,T]\times\Re$*, let*

$$W_i(t,x;\beta) := P_i(t)x^2 - 2\beta Q_i(t)x + R_i(t)\beta^2 \ , \tag{3.15}$$

where

$$\begin{aligned}P_i(t) &= E_{t,i}\Big[e^{\int_t^T(2r(s)-\theta(s)^\top\theta(s)-\lambda(s))ds} + \int_t^T \lambda(s)e^{\int_t^s(2r(u)-\theta(u)^\top\theta(u)-\lambda(u))du}ds\Big],\\ Q_i(t) &= E_{t,i}\Big[e^{\int_t^T(r(s)-\theta(s)^\top\theta(s)-\lambda(s))ds} + \int_t^T \lambda(s)e^{\int_t^s(r(u)-\theta(u)^\top\theta(u)-\lambda(u))du}ds\Big],\\ R_i(t) &= E_{t,i}\Big[e^{-\int_t^T\lambda(s)ds} + \int_t^T e^{-\int_t^s\lambda(u)du}\Big(\lambda(s) - \frac{Q_i^2(s)}{P_i(s)}\theta(s)^\top\theta(s)\Big)ds\Big].\end{aligned} \tag{3.16}$$

Then $W_i(t,x;\beta)\in\mathcal{C}^{1,2}([0,T]\times\Re)$ *and* $V_i(t,x;\beta) = W_i(t,x;\beta)$*.* $E_{t,i}$ *is the conditional expectation given that* $\mathbf{Z}(t) = \mathbf{e}_i$*.*

Proof. When the state is in i, it follows from (3.15) that the optimal strategy

$$\begin{aligned}\pi_i^*(t,x) &= -[\sigma(t)\sigma(t)^\top]^{-1}(\mu(t)-r(t)\underline{1})\frac{\partial W_i}{\partial x}/\frac{\partial^2 W_i}{\partial x^2}\\ &= -[\sigma(t)\sigma(t)^\top]^{-1}(\mu(t)-r(t)\underline{1})\frac{P_i(t)x-\beta Q_i(t)}{P_i(t)}.\end{aligned}\tag{3.17}$$

Substituting $\pi_i^*(t,x)$ and (3.15) into (3.12), we obtain

$$\begin{aligned}&P_i'(t)x^2-2\beta Q_i'(t)x+R_i'(t)\beta^2+\theta(t)^\top\theta(t)\frac{(P_i(t)x-\beta Q_i(t))^2}{P_i(t)}\\ &+\left[r(t)x-\theta(t)^\top\theta(t)\frac{P_i(t)x-\beta Q_i(t)}{P_i(t)}\right](2P_i(t)x-2\beta Q_i(x))\\ &+\langle\lambda,\mathbf{e}_i\rangle(x-\beta)^2-\langle\lambda,\mathbf{e}_i\rangle\left[P_i(t)x^2-2\beta Q_i(t)x+R_i(t)\beta^2\right]+\langle\mathbf{V},\mathbf{G}\mathbf{e}_i\rangle\\ = &\left(P_i^{'}(t)+(2r(t)-\theta(t)^\top\theta(t)-\langle\lambda,\mathbf{e}_i\rangle)P_i(t)+\langle\lambda,\mathbf{e}_i\rangle+\langle\mathbf{P},\mathbf{G}\mathbf{e}_i\rangle\right)x^2\\ &-2\beta\left(Q_i'(t)+(r(t)-\theta(t)^\top\theta(t)-\langle\lambda,\mathbf{e}_i\rangle)Q_i(t)+\langle\lambda,\mathbf{e}_i\rangle+\langle\mathbf{Q},\mathbf{G}\mathbf{e}_i\rangle\right)x\\ &+\beta^2\left(R_i'(t)-\langle\lambda,\mathbf{e}_i\rangle R_i(t)-\frac{Q_i^2(t)}{P_i(t)}\theta(t)^\top\theta(t)+\langle\lambda,\mathbf{e}_i\rangle+\langle\mathbf{R},\mathbf{G}\mathbf{e}_i\rangle\right)=0\end{aligned}$$

where $\mathbf{P} := (P_1,P_2,\ldots,P_N)^\top \in \Re^N$, $\mathbf{Q} := (Q_1,Q_2,\ldots,Q_N)^\top \in \Re^N$ and $\mathbf{R} := (R_1,R_2,\ldots,R_N)^\top \in \Re^N$. From the Faynman-Kac formulas with regime switching (See Theorem 3.2 in Zhu et al.[20]), we obtain (3.16).

Then the following theorem gives the optimal portfolio and the efficient frontier.

Theorem 3.6. *The efficient portfolio of Problem 2 corresponding to the expected terminal wealth*

$$E\Big[X^*(T)|X(0)=x,\mathbf{Z}(0)=\mathbf{e}_j\Big]=d$$

is given by

$$\begin{aligned}&\pi_i^*(t) := \pi(t|\mathbf{Z}(t)=\mathbf{e}_i,\mathbf{Z}(0)=\mathbf{e}_j)=\\ &[\sigma(t)\sigma(t)^\top]^{-1}(t)(\mu(t)-r(t)\underline{1})\left(\frac{Q_i(t)(d+\gamma_j^*)}{P_i(t)}-x\right),\end{aligned}\tag{3.18}$$

$\gamma_j^*=\frac{R_j(0)d-Q_j(0)x_0}{1-R_j(0)}$, *and the efficient frontier is given by*

$$\begin{aligned}&\sigma\Big[X^*(T)|X(0)=x,\mathbf{Z}(0)=\mathbf{e}_j\Big]\\ &=\sqrt{\frac{R_j(0)(d-\frac{Q_j(0)x_0}{R_j(0)})^2}{1-R_j(0)}+\left(P_j(0)-\frac{Q_j^2(0)}{R_j(0)}\right)x_0^2}\end{aligned}\tag{3.19}$$

where $P_j(0)-\frac{Q_j^2(0)}{R_j(0)}>0$.

Proof. Here we only give a simple explanation for $P_i(0) - \frac{Q_i^2(0)}{R_i(0)} > 0$. For every path ω from t to T, denote

$$\begin{aligned}\tilde{P}_i(t,\omega) &= E[e^{\int_t^T(2r(s)-A(s))ds} + \int_t^T \lambda(s)e^{-\int_t^s \lambda(u)du}e^{\int_t^s(2r(u)-A(u))du}ds],\\ \tilde{Q}_i(t,\omega) &= E[e^{\int_t^T(r(s)-A(s))ds} + \int_t^T \lambda(s)e^{-\int_t^s \lambda(u)du}e^{\int_t^s(r(u)-A(u))du}ds],\\ \tilde{R}_i(t,\omega) &= E[\int_t^T (\lambda(s)e^{-\int_t^s \lambda(u)} - A(s)\frac{P_i^2(s)}{Q_i(s)})ds + 1].\end{aligned} \tag{3.20}$$

As in the deterministic case, we have

$$\tilde{P}_i(0,\omega) - \frac{\tilde{Q}_i^2(0,\omega)}{\tilde{R}_i(0,\omega)} > 0.$$

Consequently,

$$P_i(0) - \frac{Q_i^2(0)}{R_i(0)} = E[\tilde{P}_i(0,\omega) - \frac{\tilde{Q}_i^2(0,\omega)}{\tilde{R}_i(0,\omega)}] > 0.$$

□

4. **Conclusion.** We investigated the mean-variance portfolio selection problem with random exit time. Both the deterministic and stochastic density processes for the random exit time were discussed. For the stochastic density process, we considered two cases, where the density process was governed by a geometric Brownian motion or was modulated by a continuous-time, finite-state, Markov chain. In the latter case, the impact of structural changes in the underlying environment on the random exit time from the market was incorporated. In contrast with that discussed in [18], we found that the risk was no longer minimal when all of the agent's wealth was invested in a (locally) risk-free bond in the presence of the timing risk. Furthermore, the efficient frontier was shown to be the upper part of a hyperbola instead of a line in [18]. To hedge the timing risk, two funds were required.

REFERENCES

[1] C. Blanchet-Scalliet, N. E. Karoui, M. Jeanblanc and L. Martellini, Optimal investment and consumption decisions when time-horizon is uncertain, *Journal of Mathematical Economics*, **44** (2008), 1100-1113.

[2] B. Bouchard and H. Pham, Wealth-path dependent utility maximization in incomplete markets, *Finance and Stochastics*, **8** (2004), 579-603.

[3] L. Delong, Optimal investment and consumption in the presence of default on a financial market driven by a L'evy noise, *Annales Universitatis Mariae Curie- Skodowska Lubl in - Polonia*, (2006), 1-15.

[4] R. J. Elliott, L. Aggoun and J. B. Moore, *Hidden Markov Models: Estimation and Control*, Springer, New York, 1994.

[5] L. Jiang and S. Wang, Robust multi-period and multi-objective portfolio selection, *Journal of Industrial & Management Optimization*, **17** (2021), 695-709.

[6] D. Li and W. L. Ng, Optimal dynamic portfolio selection: Multi-period mean- variance formulation, *Mathematical Finance*, **10** (2000), 387-406.

[7] Z. F. Li and S. X. Xie, Mean-variance portfolio optimization under stochastic income and uncertain exit time, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B*, **17** (2010), 131-147.

[8] X. Li and X. Y. Zhou, Continuous-time mean–variance efficiency: The 80 % rule, *Annals of Applied Probability*, **16** (2006), 1751-1763.

[9] X. Li, X. Y. Zhou and A. E. B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, **40** (2002), 1540-1555.

[10] C.W. Lin, L. Zeng and H.L. Wu, Multi-period portfolio optimization in a defined contribution pension plan during the decumulation phase, *Journal of Industrial & Management Optimization*, **15** (2019), 401-427.

[11] J. L. Liu, K. F. C. Yiu and A. Bensoussan, The optimal mean variance problem with inflation, *Discrete and Continuous Dynamical Systems Series B*, **21** (2016), 185-203.

[12] D. G. Luenberger, ' *Linear and Nonlinear Programming*, 2^{nd} edition, Kluwer Academic Publishers, 1984.

[13] H. Markowitz, Portfolio selection, *Journal of Finance*, **7** (1952), 77-91.

[14] L. Martellini and B. Urosevic, Static mean variance analysis with uncertain time-horizon, *Management Science*, **52** (2005), 955-964.

[15] R. C. Merton, *Continuous-Time Finance*, Blackwell Publishing, Oxford, United Kingdom, 1990.

[16] Y. Shen, X. Zhang and T. K. Siu, Mean-variance portfolio selection under a constant elasticity of variance model, *Operations Research Letters*, **42** (2014), 337-342.

[17] Z. Yu, Continuous-time mean variance portfolio selection with random horizon, *Applied Mathematics and Optimization*, **68** (2013), 333-359.

[18] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, **42** (2000), 19-33.

[19] X. Y. Zhou and G. Yin, Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model, *SIAM Journal on Control and Optimization*, **42** (2003), 1466-1482.

[20] C. Zhou, G, Yin and N. A. Baran, Feynman-Kac formulas for regime-switching jump diffusions and their applications, *Stochastics: An International Journal of Probability and Stochastic Processes*, **87** (2015), 1000-1032.

Received for publication November 2021; early access August 2022.

E-mail address: janejz.liu@hotmail.com
E-mail address: macyiu@polyu.edu.hk
E-mail address: li.xun@polyu.edu.hk
E-mail address: ken.siu@mq.edu.au
E-mail address: k.l.teo@curtin.edu.au