A-STABLE TIME DISCRETIZATIONS PRESERVE MAXIMAL PARABOLIC REGULARITY[*]

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Abstract. It is shown that for a parabolic problem with maximal L^p-regularity (for $1 < p < \infty$), the time discretization by a linear multistep method or Runge–Kutta method has maximal ℓ^p-regularity uniformly in the stepsize if the method is A-stable (and satisfies minor additional conditions). In particular, the implicit Euler method, the Crank–Nicolson method, the second-order backward difference formula (BDF), and the Radau IIA and Gauss Runge–Kutta methods of all orders preserve maximal regularity. The proof uses Weis' characterization of maximal L^p-regularity in terms of R-boundedness of the resolvent, a discrete operator-valued Fourier multiplier theorem by Blunck, and generating function techniques that have been familiar in the stability analysis of time discretization methods since the work of Dahlquist. The $A(\alpha)$-stable higher-order BDF methods have maximal ℓ^p-regularity under an R-boundedness condition in a larger sector. As an illustration of the use of maximal regularity in the error analysis of discretized nonlinear parabolic equations, it is shown how error bounds are obtained without using any growth condition on the nonlinearity or for nonlinearities having singularities.

Key words. maximal regularity, A-stability, multistep methods, Runge–Kutta methods, parabolic equations

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1. Introduction. Maximal regularity is an important mathematical tool in studying existence, uniqueness, and regularity of the solution of nonlinear parabolic partial differential equations (PDEs) [6, 27, 28, 34, 38]. A generator A of an analytic semigroup on a Banach space X is said to have *maximal L^p-regularity* if the solution of the evolution equation

$$(1.1) \qquad \begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = 0, \end{cases}$$

satisfies

$$(1.2) \qquad \|u'\|_{L^p(\mathbb{R}_+;X)} + \|Au\|_{L^p(\mathbb{R}_+;X)} \le C_{p,X}\|f\|_{L^p(\mathbb{R}_+;X)} \quad \forall\, f \in L^p(\mathbb{R}_+;X)$$

for some (or, as it turns out, for all) $1 < p < \infty$. On a Hilbert space, every generator of a bounded analytic semigroup has maximal L^p-regularity [18], and Hilbert spaces are the only Banach spaces for which this holds true [25]. Beyond Hilbert spaces, a characterization of the maximal L^p-regularity was given by Weis [42, 43] on $X = L^q(\Omega)$ (with $1 < q < \infty$ and Ω a region in $\mathbb{R}^d$) and more generally on UMD spaces in terms of the R-boundedness of the resolvent operator. Operators having maximal L^p-regularity

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include elliptic differential operators on $L^q(\Omega)$ with general boundary conditions, and operators that generate a positive and contractive semigroup on $L^q(\Omega, \mathrm{d}\mu)$ spaces for an arbitrary measure space $(\Omega, \mathrm{d}\mu)$, as do many generators of stochastic processes; see [27] and references therein.

In this paper we address the following question: *Given an operator A that has maximal L^p-regularity, for which time discretization methods for* (1.1) *is the maximal L^p-regularity preserved in the discrete ℓ^p-setting, uniformly in the stepsize?*

We will show that this holds for *A-stable* multistep and Runge–Kutta methods, under minor additional conditions. In particular, the implicit Euler method, the Crank–Nicolson method, the second-order backward difference formula (BDF), and higher-order A-stable implicit Runge–Kutta methods such as the Radau IIA and Gauss methods all preserve maximal regularity.

We recall that a numerical time discretization method is called A-stable if for every complex λ with $\mathrm{Re}\,\lambda \leq 0$, for every stepsize $\tau > 0$, and for arbitrary starting values, the numerical solution of the scalar linear differential equation $y' = \lambda y$ remains bounded as the discrete time goes to $+\infty$. It is remarkable that this deceivingly simple and well-studied concept, which was introduced by Dahlquist [16], essentially suffices to yield maximal ℓ^p-regularity, uniformly in the stepsize for every operator A that has maximal L^p-regularity (1.2), not only on Hilbert spaces but on a large class of Banach spaces known as UMD spaces, which include in particular $L^q(\Omega)$ spaces for $1 < q < \infty$.

Our proofs rely on Weis' characterization of maximal L^p-regularity on UMD spaces [42], on a discrete operator-valued Fourier multiplier theorem of Blunck [11], and on generating function techniques for time discretization methods, which have been familiar for linear multistep methods since the work of Dahlquist [15, 16], but are less common for Runge–Kutta methods [35].

The question of discrete maximal ℓ^p-regularity has received attention in a number of publications previously. In the following we give a brief overview of this literature:

- Discrete maximal ℓ^2-regularity on Hilbert spaces X for the implicit Euler method applied to (1.1) is obtained directly from Parseval's identity and a resolvent bound. In the book by Ashyralyev and Sobolevskiĭ [8], which is largely based on earlier work of the authors going back to the early 1970s, this was nontrivially extended to discrete maximal ℓ^p-regularity (there called ℓ^p coercivity inequality) for $1 < p < \infty$ on Hilbert spaces X for the implicit Euler method and for Padé schemes.
- Discrete maximal ℓ^p-regularity restricted to a subspace that interpolates between a Banach space X and the domain of the operator A on X was also shown in [8].
- A characterization of discrete maximal ℓ^p-regularity for recurrence relations $u_{n+1} = Tu_n + f_n$ $(n \geq 0)$ on UMD spaces was given by Blunck [11, 12]. However, this characterization does not lend itself directly to studying the question of stepsize-uniform discrete maximal ℓ^p-regularity of numerical time integration methods for (1.1).
- Ashyralyev, Piskarev, and Weis [7] showed discrete maximal ℓ^p-regularity for the implicit Euler method on UMD spaces. They also gave a variant of discrete maximal ℓ^p-regularity for the Crank–Nicolson method.
- A generalization of the results in [11, 12] to the explicit Euler scheme with certain nonconstant time step sequences was given in [40]; also see [2].
- ℓ^p-bounds for $1 \leq p \leq \infty$ on general Banach spaces X, which show quasi-maximal regularity up to a factor that is logarithmic in the time step, were given in [7] for the implicit Euler and Crank–Nicolson methods and in [29] for discontinuous Galerkin time-stepping methods.

In another line of research, discrete maximal regularity in temporally weighted Hölder spaces has been studied using semigroup techniques in numerous papers from the 1970s onward; see again the book by Ashyralyev and Sobolevskiĭ [8] and also the review in [7] and [21, 22, 23].

Maximal L^p-regularity of finite element spatial semidiscretizations of parabolic PDEs has been used in the analysis of numerical methods for PDEs with minimal regularity assumption on the solution [19, 20, 30] or on the diffusion coefficient [33]. In order to prove the convergence of fully discrete solutions of some nonlinear PDEs, e.g., the dynamic Ginzburg–Landau equations [31], maximal L^p-regularity in the time-discrete setting as given here is needed. Discrete maximal L^p-regularity has also proved fundamental in the study of time discretizations of quasi-linear parabolic equations [3] and large classes of fully nonlinear parabolic equations [26].

The paper is organized as follows.

In section 2 we recall important notions and results from the theory of maximal parabolic regularity: Weis' characterization of maximal L^p-regularity on UMD spaces, R-boundedness, and operator-valued Fourier multipliers in a Banach space setting [11, 27, 42, 43].

In section 3 we give discrete maximal ℓ^p-regularity estimates for two simple one-step methods, the backward Euler method and the Crank–Nicolson scheme. This allows us to show basic arguments in a technically simpler setting than for the other methods considered in later sections.

In sections 4 and 5 maximal ℓ^p-regularity results are shown for higher order methods, BDFs, and A-stable Runge–Kutta methods, respectively. While linear multistep methods have a scalar differentiation symbol in the appearing generating functions, the differentiation symbol of Runge–Kutta methods is matrix-valued, which makes the analysis more complicated. All our results of discrete maximal ℓ^p-regularity are stated and proved for constant stepsizes.

In section 6 we briefly discuss maximal regularity of full discretizations and show how uniformity of the bounds in both the spatial gridsize h and the temporal stepsize τ can be obtained.

In section 7 we give ℓ^p-bounds for $1 \leq p \leq \infty$ on general Banach spaces, which show maximal regularity up to a factor that is logarithmic in the number of time steps considered. These bounds are obtained for a subclass of methods that includes the BDF and Radau IIA methods but not the Crank–Nicolson and Gauss methods. These bounds rely on the convolution quadrature interpretation of linear multistep methods [37] and Runge–Kutta methods [35].

Finally, in section 8 we use discrete maximal ℓ^p-regularity for deriving error bounds for discretizations of *nonlinear* parabolic differential equations. We illustrate this for a class of semilinear equations with the Laplacian as a dominant linear term and a pointwise nonlinear term $f(u, \nabla u)$. We show that in contrast to previously existing techniques, the approach via discrete maximal regularity enables us to obtain optimal-order error bounds without any growth condition on the nonlinearity $f(u, \nabla u)$ and also for nonlinearities having singularities, in arbitrary space dimension. This becomes possible because via the discrete maximal ℓ^p-regularity we can control the $\ell^\infty(W^{1,\infty})$-norm of the error, provided the exact solution of the parabolic problem has sufficient regularity. This proof requires the full discrete maximal ℓ^p-regularity for the given operator A and cannot be used with the logarithmically quasi-maximal ℓ^p-regularity of section 7. The latter permits us, however, to further refine the error bounds.

2. Preliminaries. Here we collect basic results on maximal L^p-regularity and related concepts, which will be needed later on. For further background and details, proofs, and references we refer to the excellent lecture notes by Kunstmann and Weis [27].

2.1. Characterization of maximal L^p-regularity in terms of the resolvent. As was shown by Weis [43], maximal L^p-regularity of an operator A on a Banach space X can be characterized in terms of its resolvent $(\lambda - A)^{-1} = R(\lambda, A)$ for a large class of Banach spaces that includes Hilbert spaces and $L^q(\Omega, \mathrm{d}\mu)$-spaces with $1 < q < \infty$. We begin with formulating the notions that permit us to state this fundamental result.

A Banach space X is said to be a *UMD space* if the Hilbert transform

$$Hf(t) = \text{P.V.} \int_{\mathbb{R}} \frac{1}{t-s} f(s) \, \mathrm{d}s$$

is bounded on $L^p(\mathbb{R}; X)$ for all $1 < p < \infty$; see [27]. From [9, 10] we know that this definition is equivalent to the definition by using the unconditional martingale differences approach, which explains the abbreviation UMD. Examples of UMD spaces include Hilbert spaces and $L^q(\Omega, \mathrm{d}\mu)$ and its closed subspaces, where $(\Omega, \mathrm{d}\mu)$ is any measure space and $1 < q < \infty$. Throughout the paper X always denotes a UMD space, unless otherwise stated.

A collection of operators $\{M(\lambda) : \lambda \in \Lambda\}$ is said to be *R-bounded* if there is a positive constant C_R, called the *R-bound* of the collection, such that any finite subcollection of operators $M(\lambda_1), M(\lambda_2), \dots, M(\lambda_l)$ satisfies (see [27, section 1.9, p. 75])

$$\int_0^1 \left\| \sum_{j=1}^l r_j(s) M(\lambda_j) v_j \right\|_X^2 \mathrm{d}s \leq C_R^2 \int_0^1 \left\| \sum_{j=1}^l r_j(s) v_j \right\|_X^2 \mathrm{d}s \quad \forall\, v_1, v_2, \dots, v_l \in X,$$

where $r_j(s) = \text{sign } \sin(2j\pi s)$, for $j = 1, 2, \dots$, are the Rademacher functions defined on the interval $[0,1]$. In view of the Kahane–Khintchine inequality [5, Theorem 6.25, p. 134], the definition above is equivalent to the definition in [42, 43].

In the special case $X = L^q(\Omega, \mathrm{d}\mu)$ a simpler condition suffices: there, a collection of operators $\{M(\lambda) : \lambda \in \Lambda\}$ is *R*-bounded if and only if there is a positive constant C_R^* such that any finite subcollection of operators $M(\lambda_1), M(\lambda_2), \dots, M(\lambda_l)$ satisfies

$$\left\| \left(\sum_{j=1}^l |M(\lambda_j) v_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq C_R^* \left\| \left(\sum_{j=1}^l |v_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \quad \forall\, v_1, v_2, \dots, v_l \in L^q(\Omega, \mathrm{d}\mu).$$

For Hilbert spaces, a collection of operators is *R*-bounded if and only if it is bounded, and the *R*-bound equals its bound.

We can now state Weis' characterization of maximal L^p-regularity. Here Σ_ϑ denotes the sector $\Sigma_\vartheta = \{z \in \mathbb{C} \backslash \{0\} : |\arg z| < \vartheta\}$.

THEOREM 2.1 (Weis [43, Theorem 4.2]). *Let X be a UMD space and let A be the generator of a bounded analytic semigroup on X. Then A has maximal L^p-regularity if and only if for some $\vartheta > \pi/2$ the set of operators $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\vartheta\}$ is R-bounded.*

2.2. Operator-valued multiplier theorems. The "if" direction of Theorem 2.1 is obtained from the following result, which extends a scalar-valued Fourier

multiplier theorem of Mikhlin. Here, $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}$: for appropriate f,

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi t} f(t)\,dt, \qquad \xi \in \mathbb{R}.$$

$B(X)$ denotes the space of bounded linear operators on X.

THEOREM 2.2 (Weis [43, Theorem 3.4]).   *Let X be a UMD space. Let $M : \mathbb{R} \setminus \{0\} \to B(X)$ be a differentiable function such that the set*

$$\{M(\xi) : \xi \in \mathbb{R} \setminus \{0\}\} \ \cup \ \{\xi M'(\xi) : \xi \in \mathbb{R} \setminus \{0\}\} \quad \text{is R-bounded,}$$

with R-bound C_R. Then, $\mathcal{M}f = \mathcal{F}^{-1}\big(M(\cdot)(\mathcal{F}f)(\cdot)\big)$ extends to a bounded operator

$$\mathcal{M} : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, X) \quad \text{for } 1 < p < \infty.$$

Moreover, there exists a constant $C_{p,X}$ independent of M such that the operator norm of $\mathcal{M}$ is bounded by $C_{p,X} C_R$.

On noting that the resolvent is the Laplace transform of the semigroup, with $M(\xi) = i\xi(i\xi - A)^{-1}$ it is seen that $\mathcal{M}f = u'$, for the solution u of (1.1). By Theorem 2.2, R-boundedness of $\lambda(\lambda - A)^{-1}$ on the imaginary axis therefore yields maximal L^p-regularity of A.

In this paper we will use the discrete version of Theorem 2.2. Here, $\mathcal{F}$ denotes the Fourier transform on $\mathbb{Z}$, which maps a sequence to its Fourier series on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$: for appropriate $f = (f_n)_{n\in\mathbb{Z}}$,

$$\mathcal{F}f(\theta) = \sum_{n\in\mathbb{Z}} e^{i\theta n} f_n, \qquad \theta \in \mathbb{T}.$$

THEOREM 2.3 (Blunck [11, Theorem 1.3]).   *Let X be a UMD space. Let $\widetilde{M} : (-\pi, 0) \cup (0, \pi) \to B(X)$ be a differentiable function such that the set*

$$(2.1) \quad \big\{\widetilde{M}(\theta) : \theta \in (-\pi,0)\cup(0,\pi)\big\} \ \cup \ \big\{(1-e^{i\theta})(1+e^{i\theta})\widetilde{M}'(\theta) : \theta \in (-\pi,0)\cup(0,\pi)\big\}$$

is R-bounded, with R-bound C_R. Then, $\mathcal{M}f = \mathcal{F}^{-1}\big(\widetilde{M}(\cdot)(\mathcal{F}f)(\cdot)\big)$ extends to a bounded operator

$$\mathcal{M} : \ell^p(\mathbb{Z}, X) \to \ell^p(\mathbb{Z}, X) \quad \text{for } 1 < p < \infty.$$

Moreover, there exists a constant $C_{p,X}$ independent of $\widetilde{M}$ such that the operator norm of $\mathcal{M}$ is bounded by $C_{p,X} C_R$.

We will encounter the situation where the generating function of a sequence $\{M_n\}_{n\geq 0}$ of operators on X converges on the complex unit disk:

$$M(\zeta) = \sum_{n=0}^{\infty} M_n \zeta^n, \qquad |\zeta| < 1,$$

and the radial limits

$$(2.2) \qquad\qquad \widetilde{M}(\theta) = \lim_{r \nearrow 1} M(re^{i\theta})$$

exist for $\theta \neq 0, \pi$ and satisfy the conditions of Theorem 2.3. For a sequence $f = (f_n)_{n\geq 0} \in \ell^p(X) := \ell^p(\mathbb{N}, X)$, extended to negative subscripts n by 0, the operator $\mathcal{M}$ is then given by the discrete convolution

$$(\mathcal{M}f)_n = \sum_{j=0}^{n} M_{n-j} f_j, \qquad n = 0, 1, 2, \ldots.$$

2.3. Enlarging R-bounded sets of operators. By the definition of R-boundedness, it is clear that if $\{M_1(\lambda) : \lambda \in \Lambda\}$ and $\{M_2(\lambda) : \lambda \in \Lambda\}$ are two R-bounded collections of operators on X, then $\{M_1(\lambda) + M_2(\lambda) : \lambda \in \Lambda\}$ and $\{M_1(\lambda)M_2(\lambda) : \lambda \in \Lambda\}$ are also R-bounded. Moreover, the union of two R-bounded collections is R-bounded, and the closure of an R-bounded collection of operators in the strong topology of $B(X)$ is again R-bounded.

The following lemma is often used to prove the R-boundedness of a collection of operators.

LEMMA 2.4 (see [13, Lemma 3.2]). *Let $\mathcal{T}$ be an R-bounded set of linear operators on X, with R-bound C_R. Then the absolute convex hull of $\mathcal{T}$, that is, the collection of all finite linear combinations of operators in $\mathcal{T}$ with complex coefficients whose absolute sum is bounded by 1, is also R-bounded, with R-bound at most $2C_R$.*

A simple consequence of this lemma is the following, which we will use later.

LEMMA 2.5. *Let $\{M(z) \; : \; z \in \Gamma\} \subset B(X)$ be an R-bounded collection of operators, with R-bound C_R, where Γ is a contour in the complex plane. Let $f(\lambda, z)$ be a complex-valued function of $z \in \Gamma$ and $\lambda \in \Lambda$. If*

$$\int_\Gamma |f(\lambda, z)| \cdot |\mathrm{d}z| \le C_0,$$

where C_0 is independent of $\lambda \in \Lambda$, then the collection of operators in the closure of the absolute convex hull of $\{M(z) : z \in \Gamma\}$ in the strong topology of $B(X)$,

$$\left\{ \frac{1}{C_0} \int_\Gamma f(\lambda, z) M(z) \, \mathrm{d}z \, : \, \lambda \in \Lambda \right\},$$

is R-bounded with R-bound at most $2C_R$.

3. Implicit Euler and Crank–Nicolson method. We first present basic ideas to prove discrete maximal parabolic regularity on two simple methods. Later these ideas will be carried over to higher-order BDF and Runge–Kutta methods, where the key properties remain R-boundedness and A- or A(α)-stability.

We consider the backward Euler and Crank–Nicolson method applied with stepsize $\tau > 0$,

$$(3.1) \qquad \frac{u_n - u_{n-1}}{\tau} = A u_n + f_n, \qquad\qquad n \ge 1, \qquad u_0 = 0,$$

and

$$(3.2) \qquad \frac{u_n - u_{n-1}}{\tau} = A \frac{u_n + u_{n-1}}{2} + \frac{f_n + f_{n-1}}{2}, \qquad n \ge 1, \qquad u_0 = 0.$$

In this section, we use the following notation for the backward difference:

$$\dot{u}_n = \frac{u_n - u_{n-1}}{\tau}.$$

The following result for the implicit Euler method is given in [7]. We include a proof that shows basic ideas which will later be used for higher-order methods.

THEOREM 3.1 (Ashyralyev, Piskarev, and Weis [7, Remark 5.2]). *If A has maximal L^p-regularity, for $1 < p < \infty$, then the numerical solution $(u_n)_{n=1}^N$ of (3.1),*

obtained by the backward Euler method with stepsize τ, satisfies the discrete maximal regularity estimate

$$\left\|(\dot{u}_n)_{n=1}^N\right\|_{\ell^p(X)} + \left\|(Au_n)_{n=1}^N\right\|_{\ell^p(X)} \le C_{p,X}\left\|(f_n)_{n=1}^N\right\|_{\ell^p(X)},$$

where the constant is independent of N and τ.

Proof. We use the generating functions $u(\zeta) = \sum_{n=0}^\infty u_n\zeta^n$ and $f(\zeta) = \sum_{n=0}^\infty f_n\zeta^n$. Since the initial value is zero, we obtain

$$\left(\frac{1-\zeta}{\tau} - A\right)u(\zeta) = f(\zeta)$$

and hence $\dot{u}(\zeta) = \sum_{n=0}^\infty \dot{u}_n\zeta^n$ is given by

$$\dot{u}(\zeta) = \frac{1-\zeta}{\tau}u(\zeta) = M(\zeta)f(\zeta) \quad \text{with} \quad M(\zeta) = \frac{1-\zeta}{\tau}\left(\frac{1-\zeta}{\tau} - A\right)^{-1}.$$

In view of Theorem 2.3, we only have to show analyticity of $M(\zeta)$ in the open unit disk $|\zeta| < 1$ and the R-boundedness of the set (2.1), with $\widetilde{M}(\theta) = M(e^{i\theta})$ for $\theta \ne 0$ modulo 2π. To this end we show that the set

(3.3) $\left\{M(\zeta) : |\zeta| \le 1, \zeta \ne 1\right\} \cup \left\{(1-\zeta)M'(\zeta) : |\zeta| \le 1, \zeta \ne 1\right\}$ is R-bounded

with an R-bound independent of τ. Since $\mathrm{Re}\,(1-\zeta) \ge 0$ for $|\zeta| \le 1$, with strict inequality for $\zeta \ne 1$, we have that

$$\{M(\zeta) : |\zeta| \le 1, \zeta \ne 1\} \subset \{\lambda(\lambda - A)^{-1} : \mathrm{Re}\,\lambda > 0\},$$

where the latter set is R-bounded by the "only if" direction of Theorem 2.1. Since

$$(1-\zeta)M'(\zeta) = -M(\zeta) + M(\zeta)^2,$$

we obtain (3.3), with an R-bound independent of the stepsize τ. The stated result therefore follows from Theorem 2.3.   □

The following result for the Crank–Nicolson method improves on [7, Remark 5.2], where a bound for $\left\|\left(A(u_n + u_{n-1})/2\right)_{n=1}^N\right\|_{\ell^p(X)}$ was given.

THEOREM 3.2. *If A has maximal L^p-regularity, for $1 < p < \infty$, then the numerical solution $(u_n)_{n=1}^N$ of (3.2), obtained by the Crank–Nicolson method with stepsize τ, is bounded by*

$$\left\|(\dot{u}_n)_{n=1}^N\right\|_{\ell^p(X)} + \left\|(Au_n)_{n=1}^N\right\|_{\ell^p(X)} \le C_{p,X}\left\|(f_n)_{n=0}^N\right\|_{\ell^p(X)},$$

where the constant is independent of N and τ.

Proof. We only have to slightly modify the previous proof. In contrast to before, now the factor $1 + e^{i\theta}$ in condition (2.1) becomes important. Using the generating functions we obtain

$$\left(\frac{1-\zeta}{\tau} - A\frac{1+\zeta}{2}\right)u(\zeta) = \frac{1+\zeta}{2}\,f(\zeta),$$

which can be rewritten as

$$\left(\frac{2}{\tau} \frac{1 - \zeta}{1 + \zeta} - A \right) u(\zeta) = f(\zeta).$$

Introducing $\delta(\zeta) = 2(1 - \zeta)/(1 + \zeta)$, we arrive at

$$\dot{u}(\zeta) = \frac{1 - \zeta}{\tau} u(\zeta) = \frac{1 + \zeta}{2} M(\zeta) f(\zeta) \quad \text{with} \quad M(\zeta) = \frac{\delta(\zeta)}{\tau} \left(\frac{\delta(\zeta)}{\tau} - A \right)^{-1}$$

and

$$A u(\zeta) = \big(M(\zeta) - 1 \big) f(\zeta).$$

To apply Theorem 2.3, it suffices to show that the set

(3.4)
$$\big\{ M(\zeta) \, : \, |\zeta| \le 1, \zeta \ne \pm 1 \big\} \cup \big\{ (1 + \zeta)(1 - \zeta) M'(\zeta) \, : \, |\zeta| \le 1, \zeta \ne \pm 1 \big\} \quad \text{is } R\text{-bounded.}$$

For the Crank–Nicolson method, we have $\operatorname{Re} \delta(\zeta) \ge 0$ for $|\zeta| \le 1, \zeta \ne -1$, and $\delta(\zeta) \ne 0$ for $\zeta \ne 1$, so that

$$\big\{ M(\zeta) \, : \, |\zeta| \le 1, \zeta \ne \pm 1 \big\} \subset \big\{ \lambda(\lambda - A)^{-1} \, : \, \operatorname{Re} \lambda \ge 0, \lambda \ne 0 \big\},$$

where the latter set is again R-bounded by Theorem 2.1. Since

$$(1 - \zeta)(1 + \zeta) M'(\zeta) = -2 M(\zeta) + 2 M(\zeta)^2,$$

we then obtain (3.4), and hence Theorem 2.3 yields the stated result. $\qquad \square$

4. Backward difference formulae. We consider general k-step BDFs for the discretization of (1.1):

(4.1)
$$\frac{1}{\tau} \sum_{j=0}^{k} \delta_j u_{n-j} = A u_n + f_n, \qquad n \ge k,$$

where the coefficients of the method are given by

$$\delta(\zeta) = \sum_{j=0}^{k} \delta_j \zeta^j = \sum_{\ell=1}^{k} \frac{1}{\ell} (1 - \zeta)^\ell.$$

The method is known to have order k for $k \le 6$ and to be A(α)-stable with angle $\alpha = 90°, 90°, 86.03°, 73.35°, 51.84°, 17.84°$ for $k = 1, \dots, 6$, respectively; see [24, Chapter V]. A(α)-stability is equivalent to $|\arg \delta(\zeta)| \le \pi - \alpha$ for $|\zeta| \le 1$. Note that the first- and second-order BDF methods are A-stable, that is, $\operatorname{Re} \delta(\zeta) \ge 0$ for $|\zeta| \le 1$.

In this section, we use the notation

$$\dot{u}_n = \frac{1}{\tau} \sum_{j=0}^{k} \delta_j u_{n-j}$$

for the approximation to the time derivative. We consider the method with zero starting values,

(4.2)
$$u_0 = \cdots = u_{k-1} = 0.$$

Like for the continuous problem, the effect of nonzero starting or initial values needs to be studied separately, but this is not related to the notion of maximal L^p- or ℓ^p-regularity. We will discuss the case of an initial value $u_0 = 0$ and possibly nonzero starting values $u_1, \dots, u_{k-1}$ in Remark 4.3.

4.1. BDF method of order 2. We obtain preservation of maximal L^p-regularity also for time discretization by the A-stable second-order BDF method.

THEOREM 4.1. *If A has maximal L^p-regularity, for $1 < p < \infty$, then the numerical solution $(u_n)_{n=k}^N$ of (4.1) with (4.2), obtained by the two-step BDF method with stepsize τ, is bounded by*

$$\big\|(\dot{u}_n)_{n=k}^N\big\|_{\ell^p(X)} + \big\|(Au_n)_{n=k}^N\big\|_{\ell^p(X)} \leq C_{p,X}\big\|(f_n)_{n=k}^N\big\|_{\ell^p(X)},$$

where the constant is independent of N and τ.

Proof. We consider the generating function of both sides of (4.1) and obtain

$$(4.3) \qquad\qquad u(\zeta) = \left(\frac{\delta(\zeta)}{\tau} - A\right)^{-1} f(\zeta)$$

so that

$$\frac{\delta(\zeta)}{\tau} u(\zeta) = M(\zeta) f(\zeta) \quad \text{with} \quad M(\zeta) = \frac{\delta(\zeta)}{\tau}\left(\frac{\delta(\zeta)}{\tau} - A\right)^{-1}.$$

Since $\operatorname{Re}\delta(\zeta) \geq 0$ for $|\zeta| \leq 1$ (this expresses the A-stability of the method) and $\delta(\zeta) \neq 0$ for $\zeta \neq 1$, it follows as before from Theorem 2.1 that the set

$$\{M(\zeta) : |\zeta| \leq 1, \zeta \neq 1\} \ \text{ is } R\text{-bounded.}$$

We also have that

$$\{(1-\zeta)M'(\zeta) : |\zeta| \leq 1, \zeta \neq 1\} \ \text{ is } R\text{-bounded,}$$

because, with $\mu(\zeta) = \delta(\zeta)/(1-\zeta) = \frac{1}{2}(3-\zeta)$,

$$\begin{aligned}
(1-\zeta)M'(\zeta) &= -(1-\zeta)\frac{\delta'(\zeta)}{\tau} A\left(\frac{\delta(\zeta)}{\tau} - A\right)^{-2} \\
&= -\frac{1-\zeta}{\tau}\big(-\mu(\zeta) + (1-\zeta)\mu'(\zeta)\big) A\left(\frac{\delta(\zeta)}{\tau} - A\right)^{-2} \\
&= \left(1 - (1-\zeta)\frac{\mu'(\zeta)}{\mu(\zeta)}\right)\frac{\delta(\zeta)}{\tau} A\left(\frac{\delta(\zeta)}{\tau} - A\right)^{-2} \\
&= \left(1 - (1-\zeta)\frac{\mu'(\zeta)}{\mu(\zeta)}\right) M(\zeta)\big(1 - M(\zeta)\big),
\end{aligned}$$

where $(1-\zeta)\mu'(\zeta)/\mu(\zeta)$ is a bounded scalar function, since $\mu(\zeta) \neq 0$ for $|\zeta| \leq 1$. Therefore, Theorem 2.3 yields the result. $\qquad\square$

Remark 4.1. The above proof extends in a direct way to yield discrete maximal ℓ^p-regularity for A-stable linear multistep methods,

$$\sum_{j=0}^k \alpha_j u_{n+j} = \tau \sum_{j=0}^k \beta_j(Au_{n+j} + f_{n+j}), \qquad n \geq 0,$$

that have the further property that the quotient of the generating polynomials,

$$\delta(\zeta) = \frac{\alpha_0\zeta^k + \alpha_1\zeta^{k-1} + \cdots + \alpha_k\zeta^0}{\beta_0\zeta^k + \beta_1\zeta^{k-1} + \cdots + \beta_k\zeta^0},$$

has no poles or zeros in the closed unit disk $|\zeta| \leq 1$, with the exception of a zero at 1. We note that here A-stability is equivalent to $\operatorname{Re} \delta(\zeta) \geq 0$ for $|\zeta| \leq 1$, and the requirement that $\delta(\zeta)$ has no pole for $|\zeta| \leq 1$ is equivalent to stating that ∞ is an interior point of the stability region on the Riemann sphere; cf., e.g., [24]. Note, however, that by Dahlquist's order barrier [16], A-stable linear multistep methods have at most order 2, and the practically used A-stable multistep methods are the second-order BDF method and the Crank–Nicolson method.

4.2. Higher-order BDF methods. We obtain maximal regularity for the BDF methods up to order 6 under a R-boundedness condition in a larger sector.

THEOREM 4.2. *Suppose that the set $\{\lambda(\lambda - A)^{-1} : |\arg \lambda| < \vartheta\}$ is R-bounded for an angle $\vartheta > \pi - \alpha$, where α is the angle of $A(\alpha)$-stability of the k-step BDF method, for $3 \leq k \leq 6$. Then the numerical solution $(u_n)_{n=k}^N$ of (4.1) with (4.2), obtained by the k-step BDF method with stepsize τ, is bounded by*

$$\left\| (\dot{u}_n)_{n=k}^N \right\|_{\ell^p(X)} + \left\| (Au_n)_{n=k}^N \right\|_{\ell^p(X)} \leq C_{p,X} \left\| (f_n)_{n=k}^N \right\|_{\ell^p(X)}$$

for $1 < p < \infty$, where the constant is independent of N and τ.

Proof. For the $A(\alpha)$-stable k-step BDF method, $|\arg \delta(\zeta)| \leq \pi - \alpha < \vartheta$ for $|\zeta| \leq 1$, $\zeta \neq 1$, and so the set

$$\left\{ \frac{\delta(\zeta)}{\tau} \left(\frac{\delta(\zeta)}{\tau} - A \right)^{-1} : |\zeta| \leq 1, \ \zeta \neq 1 \right\} \subset \left\{ \lambda(\lambda - A)^{-1} : |\arg \lambda| < \vartheta \right\}$$

is R-bounded, with an R-bound independent of τ. The rest of the proof is the same as for the two-step BDF method. $\qquad\square$

Remark 4.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and consider the parabolic problem

$$(4.4) \quad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u(x,t)}{\partial x_j} \right) = 0 & \text{for } (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{for } (t,x) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

where the real symmetric coefficients $a_{ij}(x)$, $i,j = 1, \ldots, d$, satisfy the ellipticity condition

(4.5)

$$K_0^{-1} \sum_{,j=1}^d |\xi_j|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq K_0 \sum_{,j=1}^d |\xi_j|^2 \quad \text{for some positive constant } K_0.$$

Let $E_2(t) : L^2(\Omega) \to L^2(\Omega)$ be the map from u_0 to $u(\cdot, t)$ given by (4.4). Then $\{E_2(t)\}_{t>0}$ is a semigroup of operators on $L^2(\Omega)$, which has analytic extension to the sector $\Sigma_{\pi/2} = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \pi/2\}$ (see [17, 39]), and the kernel $G(z,x,y)$ of the analytic semigroup $\{E_2(z)\}_{z \in \Sigma_{\pi/2}}$ satisfies (see [17, p. 103])

$$(4.6) \qquad |G(z,x,y)| \leq C_\vartheta |z|^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x-y|^2}{C_\vartheta |z|}} \quad \forall z \in \Sigma_\vartheta, \ \forall x,y \in \Omega, \quad \forall \vartheta \in (0, \pi/2).$$

The operator $A = -e^{i\vartheta} A_2$ then satisfies the condition of [27, Theorem 8.6] with $m = 2$ and $g(s) = C_\vartheta e^{-s^2/C_\vartheta}$. As a consequence of [27, Theorem 8.6], $E_2(z)$ extends

compatibly to an analytic semigroup $E_q(z)$ on $L^q(\Omega)$, $1 < q < \infty$, which is R-bounded in the sector Σ_ϑ for all $\vartheta \in (0, \pi/2)$. If we denote by A_q the generator of the semigroup $\{E_q(t)\}_{t>0}$, then Weis' characterization of maximal L^p-regularity [43, Theorem 4.2] implies that the set of operators $\{\lambda(\lambda - A_q)^{-1} : \lambda \in \Sigma_\vartheta\}$ is R-bounded in the sector Σ_ϑ for all $\vartheta \in (0, \pi)$, and the R-bound depends only on K_0, ϑ, and q.

In view of the above results on the angle of R-boundedness, for $X = L^q(\Omega)$ and $A = A_q$, the condition of Theorem 4.2 is satisfied for the BDF methods of orders $1 \le k \le 6$.

Remark 4.3. If $u_0, \ldots, u_{k-1}$ may be different from zero, then we define $\widetilde{f}_n = f_n$ for $n \ge k$ and

$$\widetilde{f}_n := \frac{1}{\tau} \sum_{j=0}^{n} \delta_j u_{n-j} - Au_n \quad \text{for } n = 0, \ldots, k-1,$$

so that

$$\frac{1}{\tau} \sum_{j=0}^{k} \delta_j u_{n-j} - Au_n = \widetilde{f}_n \quad \text{for } n \ge 0.$$

Then we obtain that

$$\left\|(\dot{u}_n)_{n=1}^{N}\right\|_{\ell^p(X)} + \left\|(Au_n)_{n=1}^{N}\right\|_{\ell^p(X)} \le C\left\|(\widetilde{f}_n)_{n=1}^{N}\right\|_{\ell^p(X)}$$

$$\le C\left\|(f_n)_{n=k}^{N}\right\|_{\ell^p(X)} + C\left(\sum_{i=0}^{k-1} \left\|u_i/\tau\right\|_X^p\right)^{\frac{1}{p}} + C\left(\sum_{i=0}^{k-1} \left\|Au_i\right\|_X^p\right)^{\frac{1}{p}},$$

where the constant C does not depend on N and τ. In the next section, we shall see that if $u_0 = 0$ and the starting values $u_1, \ldots, u_{k-1}$ are computed by an A-stable Runge–Kutta method with invertible coefficient matrix $\mathcal{O}$, with s stages and nodes $c_1, \ldots, c_s$, then we have

$$\left(\sum_{i=0}^{k-1} \left\|u_i/\tau\right\|_X^p\right)^{\frac{1}{p}} + \left(\sum_{i=0}^{k-1} \left\|Au_i\right\|_X^p\right)^{\frac{1}{p}} \le C_{p,X} \left(\sum_{i=0}^{k-1}\sum_{j=1}^{s} \left\|f(t_i + c_j\tau)\right\|_X^p\right)^{\frac{1}{p}}.$$

5. A-stable Runge–Kutta methods. We consider an implicit Runge–Kutta method with s stages for the time discretization of the evolution equation (1.1). We refer to Hairer and Wanner [24] for the basic notions related to such methods.

The coefficients of the method are given by the Butcher tableau

$$\begin{array}{c|c} c & \mathcal{O} \\ \hline & b^T \end{array} \quad = \quad \begin{array}{c|c} (c_i) & (a_{ij}) \\ \hline & (b_j) \end{array} \qquad (i, j = 1, \ldots, s).$$

Applied to the evolution equation (1.1), a step of the method with stepsize $\tau > 0$ reads

(5.1)
$$U_{ni} = u_n + \tau \sum_{j=1}^{s} a_{ij} \dot{U}_{nj} \qquad \text{for} \quad i = 1, 2, \ldots, s,$$

$$u_{n+1} = u_n + \tau \sum_{i=1}^{s} b_i \dot{U}_{ni}, \qquad n \ge 1,$$

$$\dot{U}_{ni} = AU_{ni} + f(t_n + c_i\tau) \qquad \text{for} \quad i = 1, 2, \ldots, s.$$

Here $u_n \in X$ is the solution approximation at the nth time step, $U_{ni} \in X$ are the internal stages, and $\dot{U}_{ni} \in X$ is again not a continuous derivative but a suggestive notation for the increments.

The *stability function* of the Runge–Kutta method is the rational function

$$R(z) = 1 + zb^T(I - z\mathcal{Q})^{-1}\mathbb{1},$$

where $\mathbb{1} = (1, \ldots, 1)^T \in \mathbb{R}^s$. The stability function is a rational approximation to the exponential function, $R(z) = e^z + O(z^{r+1})$ for $z \to 0$, where r is greater than or equal to the order of the Runge–Kutta method, which we always assume to be at least 1. Note that if $\mathcal{Q}$ is invertible, then $R(\infty) = 1 - b^T\mathcal{Q}^{-1}\mathbb{1}$.

The Runge–Kutta method is *A-stable* if $I - z\mathcal{Q}$ is nonsingular for $\operatorname{Re} z \leq 0$ and the stability function satisfies

$$|R(z)| \leq 1 \quad \text{for} \quad \operatorname{Re} z \leq 0.$$

Example 5.1. Radau IIA methods are an important class of Runge–Kutta methods that are A-stable, have an invertible matrix $\mathcal{Q}$, and have $R(\infty) = 0$ for an arbitrary number of stages $s \geq 1$; see [24, section IV.5]. For these methods, $b_j = a_{sj}$, so that $u_{n+1} = U_{ns}$. The s-stage method has classical order $2s - 1$, that is, the error on a finite time interval is bounded by $O(\tau^{2s-1})$ when the method is applied to smooth ordinary differential equations. For parabolic problems as considered in this paper, the order of approximation is studied in [35] and is typically a noninteger number between $s + 1$ and $2s - 1$. Radau IIA methods can be viewed as collocation methods on the Radau quadrature nodes. For linear evolution equations they can alternatively be viewed as fully discretized discontinuous Galerkin methods with Radau quadrature on the integral terms; see [4]. The s-stage method has classical order $2s$. The 1-stage Radau IIA method is the implicit Euler method.

Example 5.2. Gauss methods are a class of Runge–Kutta methods that are A-stable for all stage numbers $s \geq 1$, have an invertible matrix $\mathcal{Q}$, and have $R(\infty) = (-1)^s$; see [24, section IV.5]. The s-stage method has classical order $2s$. The 1-stage Gauss method is the implicit midpoint rule (Crank–Nicolson method).

We have the following result on discrete maximal regularity.

THEOREM 5.1. *Consider an A-stable Runge–Kutta method with an invertible coefficient matrix $\mathcal{Q}$. If the operator A has maximal L^p-regularity, for $1 < p < \infty$, then the numerical solution* (5.1), *obtained by the Runge–Kutta method with stepsize τ, is bounded by*

$$\sum_{i=1}^{s} \left\|(\dot{U}_{ni})_{n=0}^{N}\right\|_{\ell^p(X)} + \sum_{i=1}^{s} \left\|(AU_{ni})_{n=0}^{N}\right\|_{\ell^p(X)} \leq C_{p,X} \sum_{i=1}^{s} \left\|(f(t_n + c_i\tau))_{n=0}^{N}\right\|_{\ell^p(X)},$$

where the constant is independent of N and τ.

Proof. We use the generating functions

$$u(\zeta) = \sum_{n=0}^{\infty} u_n\zeta^n, \qquad U(\zeta) = \sum_{n=0}^{\infty} U_n\zeta^n, \quad \text{and} \quad F(\zeta) = \sum_{n=0}^{\infty} f_n\zeta^n,$$

where $U_n = (U_{ni})_{i=1}^{s} \in X^s$ and $F_n = (f(t_n + c_i\tau))_{i=1}^{s} \in X^s$. We write $AU_n = (AU_{ni})_{i=1}^{s}$ and in this way consider A in an obvious way as an operator on X^s, that is, we write A instead of the Kronecker product $I_s \otimes A$ for brevity.

Following [35], we define the $s \times s$ matrix-valued function

$$\Delta(\zeta) = \left(\mathcal{O} + \frac{\zeta}{1-\zeta} \mathbb{1} b^T \right)^{-1},$$

which will be a key object in our discrete maximal regularity analysis for Runge–Kutta methods. It will play a similar role as $\delta(\zeta)$ in sections 3 and 4 but is now matrix-valued instead of scalar-valued. The formula of [35, Lemma 2.4],

$$(5.2) \qquad (\Delta(\zeta) - z)^{-1} = \mathcal{O}(I - z\mathcal{O})^{-1} + (I - z\mathcal{O})^{-1} \mathbb{1} b^T (I - z\mathcal{O})^{-1} \frac{\zeta}{1 - R(z)\zeta},$$

shows that for a Runge–Kutta method with invertible matrix $\mathcal{O}$, the spectrum of $\Delta(\zeta)$ satisfies

$$\sigma(\Delta(\zeta)) \subseteq \sigma(\mathcal{O}^{-1}) \cup \{z \in \mathbb{C} \ : \ R(z)\zeta = 1\}.$$

Hence, for an A-stable method the spectrum of $\Delta(\zeta)$ is contained in the closed right half-plane without 0 for $|\zeta| \leq 1$ with $\zeta \neq 1$, since $|R(z)| \geq 1$ requires $\operatorname{Re} z \geq 0$ by A-stability and $R(0) = 1$.

It was shown in [35, Proposition 2.1, equation (2.9)] that

$$(5.3) \qquad U(\zeta) = \left(\frac{\Delta(\zeta)}{\tau} - A \right)^{-1} F(\zeta).$$

Hence,

$$\dot{U}(\zeta) = AU(\zeta) + F(\zeta) = M(\zeta)F(\zeta) \quad \text{with} \quad M(\zeta) = \frac{\Delta(\zeta)}{\tau} \left(\frac{\Delta(\zeta)}{\tau} - A \right)^{-1}.$$

In view of Theorem 2.3 on the space X^s instead of X, it suffices to prove that
(5.4)
$$\left\{ M(\zeta) \ : \ |\zeta| \leq 1, \zeta \neq 1 \right\} \cup \left\{ (1+\zeta)(1-\zeta)M'(\zeta) \ : \ |\zeta| \leq 1, \zeta \neq \pm 1 \right\} \text{ is } R\text{-bounded.}$$

We use the Cauchy-type integral formula

$$M(\zeta) = \frac{1}{2\pi i} \int_\Gamma (z - \Delta(\zeta))^{-1} \otimes \frac{z}{\tau} \left(\frac{z}{\tau} - A \right)^{-1} \mathrm{d}z,$$

where Γ is a union of circles centered at the eigenvalues of $\Delta(\zeta)$ and lying in the sector Σ_ϑ of R-boundedness of $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\vartheta\}$ with $\vartheta > \frac{\pi}{2}$. We also use the integral formula differentiated with respect to ζ. We insert formula (5.2) and its derivative with respect to ζ in the integrands. The estimates required for proving (5.4) are different in the three cases $|R(\infty)| < 1$, $R(\infty) = -1$, and $R(\infty) = +1$, which in the following are studied in items (a), (b), and (c), respectively.

(a) We consider first the case where $|R(\infty)| < 1$. We distinguish two situations:

(i) If ζ with $|\zeta| \leq 1$ is bounded away from 1, $|\zeta - 1| \geq c > 0$, then all eigenvalues of $\Delta(\zeta)$ have nonnegative real part and are bounded away from 0. Therefore the radii of all circles can be chosen to have a fixed lower bound (depending on $c > 0$ and $\vartheta > \frac{\pi}{2}$), and we then have for $|\zeta| \leq 1$ with $|\zeta - 1| \geq c > 0$ that $|1 - R(z)\zeta| \geq c' > 0$ uniformly for z on each circle. This yields

$$(5.5) \qquad \int_\Gamma \|(z - \Delta(\zeta))^{-1}\| \, |\mathrm{d}z| \leq C,$$
$$\int_\Gamma \left\| (1+\zeta)(1-\zeta) \frac{\partial}{\partial \zeta} (z - \Delta(\zeta))^{-1} \right\| \, |\mathrm{d}z| \leq C,$$

where $\|\cdot\|$ denotes an arbitrary matrix norm.

(ii) If ζ with $|\zeta| \leq 1$ is close to 1, then the implicit function theorem yields that there is a unique $z_0(\zeta)$ near 0 with $R(z_0(\zeta))\zeta = 1$, and we obtain $z_0(\zeta) = 1 - \zeta + O((1-\zeta)^2)$, so that for sufficiently small $|1-\zeta|$ we have $|z_0(\zeta)| \geq \frac{1}{2}|1-\zeta|$. By A-stability, we further have $\operatorname{Re} z_0(\zeta) \geq 0$ for $|\zeta| \leq 1$. The radius r of the circle in Σ_ϑ around $z_0(\zeta)$ can be chosen proportional to $|z_0(\zeta)|$, and hence to $|1-\zeta|$, depending on $\vartheta > \frac{\pi}{2}$. For z on this circle we have

$$1 - R(z)\zeta = R(z_0(\zeta))\zeta - R(z)\zeta = \big((z_0(\zeta) - z) + O((z_0(\zeta) - z)^2)\big)\zeta,$$

so that $|1 - R(z)\zeta| \geq r/2$ on this circle. This yields again the bounds (5.5), uniformly for ζ in a small neighbourhood of 1 with $|\zeta| \leq 1$.

We thus have proved the bounds (5.5) uniformly for $|\zeta| \leq 1$, $\zeta \neq 1$. By Theorem 2.1 and Lemma 2.5, the bounds (5.5) yield (5.4), in the considered case where $|R(\infty)| < 1$.

(b) We now consider the case $R(\infty) = -1$.

(i) If ζ is bounded away from both 1 and -1, the proof is the same as part (i) of (a).

(ii) If ζ is close to 1, the proof is the same as part (ii) of (a).

(iii) If ζ is close to -1, we proceed as follows. A-stability and $R(\infty) = -1$ imply that

$$R(z) = -1 - cz^{-1} + O(z^{-2}) \quad \text{for } z \to \infty, \quad \text{with } c > 0.$$

For ζ close to -1, there exists therefore a unique $z_\infty(\zeta)$ of large absolute value and with nonnegative real part such that $R(z_\infty(\zeta))\zeta = 1$. The Cauchy-type integrals then contain a contribution from a circle around $z_\infty(\zeta)$, contained in Σ_ϑ, with a radius that can be chosen proportional to $|z_\infty(\zeta)|$. The distance of this circle from the origin can also be chosen proportional to $|z_\infty(\zeta)|$. For z on this circle we then have

$$1 - R(z)\zeta = R(z_\infty(\zeta))\zeta - R(z)\zeta = -c(z_\infty(\zeta)^{-1} - z^{-1})\zeta + O(z_\infty(\zeta)^{-2})$$

and therefore $|1 - R(z)\zeta|$ is bounded from below by a positive constant times $|z_\infty(\zeta)^{-1}|$, which in turn is bounded from below by a positive constant times $|1+\zeta|$. With (5.2) it follows that on this circle,

$$\|(\Delta(\zeta) - z)^{-1}\| + \left\|(1+\zeta)\frac{\partial}{\partial\zeta}(\Delta(\zeta) - z)^{-1}\right\| \leq C|z|^{-2}|1+\zeta|^{-1} \leq C|1+\zeta|.$$

This yields (5.5) (note that the factor $1 + \zeta$ in the second integral of (5.5) is now needed). When ζ is away from -1, the proof is the same as part (i) of (a). We therefore obtain (5.4) also in the case $R(\infty) = -1$.

(c) The remaining case $R(\infty) = 1$ can be dealt with in the same way. We now have

$$R(z) = 1 + cz^{-1} + O(z^{-2}) \quad \text{for } z \to \infty, \quad \text{with } c > 0,$$

and the bounds (5.5) can be obtained by the same arguments as in the case $R(\infty) = -1$, replacing $1 + \zeta$ by $1 - \zeta$ on every occurrence.

We have thus obtained (5.4) for every A-stable Runge–Kutta method with invertible coefficient matrix $\mathcal{Q}$. Theorem 2.3 now yields the stated result. $\qquad\square$

By [35, Lemma 3.1], we have for a Runge–Kutta method with invertible coefficient matrix $\mathcal{Q}$ that

$$(5.6) \qquad u_{n+1} = b^T \mathcal{Q}^{-1} \sum_{k=0}^{n} R(\infty)^{n-k} U_k,$$

and so we obtain the following corollary.

COROLLARY 5.2. *Under the assumptions of Theorem* 5.1, *and if* $|R(\infty)| < 1$, *we have*

$$\left\|\left(\frac{u_n - u_{n-1}}{\tau}\right)_{n=1}^N\right\|_{\ell^p(X)} + \left\|(Au_n)_{n=1}^N\right\|_{\ell^p(X)} \leq C_{p,X} \sum_{i=1}^s \left\|(f(t_n + c_i\tau))_{n=0}^N\right\|_{\ell^p(X)},$$

where the constant is independent of N and τ.

6. Space-time full discretizations. Let X be a UMD space and let X_h, $h > 0$, be a family of closed subspaces of X such that there exist linear projection operators $P_h : X \to X_h$ satisfying

(6.1) $$\|P_h u\|_X \leq C_0 \|u\|_X \qquad \forall\, u \in X,$$

where the constant C_0 is independent of h. Consider the problem

(6.2) $$\begin{cases} u_h'(t) = A_h u_h(t) + f_h(t), & t > 0, \\ u_h(0) = 0, \end{cases}$$

where A_h is the generator of a bounded analytic semigroup on X_h and $f_h(t)$, $u_h(t) \in X_h$ for all $t > 0$. We have the following result.

THEOREM 6.1. *Assume* (6.1) *and that the set of operators* $\{\lambda(\lambda - A_h)^{-1} : \lambda \in \Sigma_\vartheta\}$ *is R-bounded in $B(X_h)$ with an R-bound C_R that is independent of h. Let $\vartheta > \pi - \alpha$, where α is the angle of $A(\alpha)$-stability of the time discretization method considered. Then, all theorems of sections 3 to 5, with A replaced by A_h and X by X_h, hold for the numerical methods applied to* (6.2), *with constants C_{p,X_h} that are independent of both τ and h.*

Proof. Since $\lambda(\lambda - A_h)^{-1}$ is R-bounded in $B(X_h)$ for $\lambda \in \Sigma_\vartheta$, it follows that the collection of operators

(6.3) $$\{\lambda(\lambda - A_h)^{-1} P_h : \lambda \in \Sigma_\vartheta\} \quad \text{is R-bounded in } B(X)$$

and the R-bound is at most $C_0 C_R$. The numerical solution given by the backward Euler scheme satisfies

$$\frac{u_{h,n} - u_{h,n-1}}{\tau} = A_h u_{h,n} + P_h f_{h,n},$$

and so it follows that the generating functions are related by

$$\frac{1-\zeta}{\tau} u_h(\zeta) = M_h(\zeta) f_h(\zeta) \quad \text{with} \quad M_h(\zeta) = \frac{\delta(\zeta)}{\tau}\left(\frac{\delta(\zeta)}{\tau} - A_h\right)^{-1} P_h.$$

In the same way as in the proof of Theorem 3.1, it is concluded from (6.3) that $M_h(\zeta)$ satisfies the R-boundedness condition (3.3) with an R-bound that is independent of τ and h, and then Theorem 2.3 yields the desired discrete maximal ℓ^p-regularity bound, uniformly in τ and h.

The results for the other methods (Crank–Nicolson, BDF, and A-stable Runge–Kutta) are proved in the same way. $\qquad\square$

Remark 6.1. If Ω is a bounded smooth domain in $\mathbb{R}^d$ ($d \geq 1$), $X = L^q(\Omega)$, X_h is the standard finite element subspace of X, A is a second-order elliptic partial differential operator, and A_h is its finite element approximation, then the R-boundedness

of $\{\lambda(\lambda - A_h)^{-1} : \lambda \in \Sigma_\vartheta\}$ in $B(X_h)$ has been proved in [30] for some $\vartheta > \pi/2$ that is independent of h (see the text between (4.10) and (4.11) in [30]). The operator P_h can be chosen as the L^2-projection operator. All the maximal regularity results for the A-stable methods studied in this paper therefore also hold for fully discrete finite element solutions. However, extension of the maximal regularity results for the BDF methods to fully discrete solutions requires further investigation on the angle ϑ.

7. Logarithmically quasi-maximal ℓ^∞-regularity. In this section we give some bounds that show maximal ℓ^∞-regularity up to a factor that is logarithmic in the number of time steps. We note that the results of this section are valid for an arbitrary complex Banach space X (not necessarily a UMD space as in the previous sections), and R-boundedness plays no role in this section. We just assume that A is the generator of analytic semigroup on X, and $\lambda(\lambda - A)^{-1}$ is uniformly bounded for $\lambda \in \Sigma_\vartheta$ with an angle $\vartheta > \pi - \alpha$ for the angle α of A(α)-stability of the numerical method. We consider again k-step BDF methods with $k \leq 6$ and A-stable Runge–Kutta methods with an invertible coefficient matrix and $|R(\infty)| < 1$.

We start with the k-step BDF method, with initial condition $u_0 = u_1 = \cdots = u_{k-1} = 0$ as in section 4. By (4.3), the numerical solution can be expressed as a discrete convolution

$$(7.1) \qquad u_n = \tau \sum_{j=k}^{n} e_{n-j}(\tau A) f_j, \qquad n \geq k,$$

with the generating function

$$\tau \sum_{n=0}^{\infty} e_n(\tau A)\zeta^n = \left(\frac{\delta(\zeta)}{\tau} - A \right)^{-1}.$$

This can be viewed as a convolution quadrature approximation of the exact solution at $t_n = n\tau$,

$$u(t_n) = \int_0^{t_n} e^{(t_n - t)A} f(t)\, \mathrm{d}t.$$

Theorem 2.1 in [37, Theorem 2.1] (used with $K(\lambda) = (\lambda - A)^{-1}$ and then with $K(\lambda) = A(\lambda - A)^{-1}$) shows that

$$\|e_n(\tau A) - e^{n\tau A}\|_{B(X)} \leq C t_{n+1}^{-k} \tau^k, \qquad n \geq 0,$$

and

$$\|A e_n(\tau A) - A e^{n\tau A}\|_{B(X)} \leq C t_n^{-1-k} \tau^k, \quad n \geq 1, \qquad \text{and} \quad \|A e_0(\tau A)\| \leq C\tau^{-1}.$$

Since $\|A e^{tA}\|_{B(X)} \leq C t^{-1}$ for $t > 0$, a direct consequence of the latter estimate is the following.

LEMMA 7.1. *Suppose that $A(\lambda - A)^{-1}$ is uniformly bounded for $\lambda \in \Sigma_\vartheta$ with an angle $\vartheta > \pi - \alpha$ for the angle of A(α)-stability of the k-step BDF method, for $1 \leq k \leq 6$. Then we have*

$$\|A e_n(\tau A)\|_{B(X)} \leq C/t_{n+1}, \qquad n \geq 0.$$

Using (7.1) and Lemma 7.1, we obtain immediately the following ℓ^∞-bound, or more generally the ℓ^p-bound uniformly for $1 \leq p \leq \infty$.

THEOREM 7.2. *Suppose that $A(\lambda - A)^{-1}$ is uniformly bounded for $\lambda \in \Sigma_\vartheta$ with an angle $\vartheta > \pi - \alpha$ for the angle of $A(\alpha)$-stability of the k-step BDF method, for $1 \le k \le 6$. Then the numerical solution $(u_n)_{n=k}^N$ of (4.1) with (4.2), obtained by the k-step BDF method with stepsize τ, is bounded by*

$$\|(Au_n)_{n=1}^N\|_{\ell^p(X)} \le C \log N \, \|(f_n)_{n=1}^N\|_{\ell^p(X)},$$

where the constant C is independent of N and τ and $1 \le p \le \infty$.

We now turn to A-stable Runge–Kutta methods. By (5.3), the vector of internal stages $U_n = (U_{ni})_{i=1}^s \in X^s$ can be expressed in terms of the vector of inhomogeneity values used in the nth step, $F_n = (f(t_n + c_i\tau))_{i=1}^s$, as a discrete block convolution

$$(7.2) \qquad U_n = \tau \sum_{j=0}^n E_{n-j}(\tau A) F_j \,,$$

with the generating function

$$\tau \sum_{n=0}^\infty E_n(\tau A)\zeta^n = \left(\frac{\Delta(\zeta)}{\tau} - A\right)^{-1}.$$

We have the following lemma.

LEMMA 7.3. *Suppose that $A(\lambda - A)^{-1}$ is uniformly bounded for $\lambda \in \Sigma_\vartheta$ with an angle $\vartheta > \pi/2$. For an A-stable Runge–Kutta method with invertible coefficient matrix $\mathcal{O}$ and $|R(\infty)| < 1$, we have*

$$\|AE_n(\tau A)\|_{B(X^s)} \le C/t_{n+1}, \qquad n \ge 0.$$

Proof. We use the Cauchy-type integral formula

$$\left(\frac{\Delta(\zeta)}{\tau} - A\right)^{-1} = \frac{1}{2\pi i}\int_\Gamma (z - \Delta(\zeta))^{-1} \otimes \left(\frac{z}{\tau} - A\right)^{-1} dz$$

with a keyhole contour $\Gamma = \Gamma_1 \cup \Gamma_2$ composed of

$$\Gamma_1 = \{re^{\pm i\vartheta} : r \ge \varepsilon\} \quad \text{and} \quad \Gamma_2 = \{\varepsilon e^{i\phi} : |\phi| \le \vartheta\}$$

with a small $\varepsilon > 0$. On inserting (5.2), using the geometric series for $(1 - R(z)\zeta)^{-1} = \sum_{n=0}^\infty R(z)^n\zeta^n$, and collecting equal powers of ζ on the left- and right-hand sides, we find

$$\tau E_0(\tau A) = \frac{1}{2\pi i}\int_\Gamma \mathcal{O}(I - z\mathcal{O})^{-1} \otimes \left(\frac{z}{\tau} - A\right)^{-1} dz = \tau \mathcal{O}(I - \mathcal{O} \otimes \tau A)^{-1}$$

and

$$(7.3) \quad \tau E_n(\tau A) = \frac{1}{2\pi i}\int_\Gamma R(z)^{n-1}(I - z\mathcal{O})^{-1}\mathbb{1}b^T(I - z\mathcal{O})^{-1} \otimes \left(\frac{z}{\tau} - A\right)^{-1} dz, \quad n \ge 1.$$

Since the eigenvalues of $\mathcal{O}$ have positive real part, we obtain

$$\|\tau AE_0(\tau A)\|_{B(X^s)} \le C.$$

Next, we estimate $\tau A E_n(\tau A)$. Since the stability function $R(z)$ satisfies $R(z) = e^z + O(z^2)$, for sufficiently small c we have

$$|R(z)| \leq e^{-\operatorname{Re} z/2} \quad \text{for } |\arg z| = \vartheta \text{ and } 0 \leq |z| \leq c,$$

and for $z \in \Gamma$ with $|z| \geq c$ we have $|R(z)| \leq \rho < 1$. Then (7.3) yields, on applying the operator A, letting $\varepsilon \to 0$ in the definition of the contour Γ, and then taking norms,

$$\|\tau A E_n(\tau A)\|_{B(X^s)} \leq C \int_0^\infty \frac{e^{-(n-1)r|\cos(\vartheta)|/2} + \rho^n}{1 + r^2} \, \mathrm{d}r \leq C/n, \qquad n \geq 1.$$

This completes the proof of Lemma 7.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The identity (7.2), Lemma 7.3, and formula (5.6) imply the following result.

THEOREM 7.4. *Suppose that* $A(\lambda - A)^{-1}$ *is uniformly bounded for* $\lambda \in \Sigma_\vartheta$ *with an angle* $\vartheta > \pi/2$. *For an A-stable Runge–Kutta method with invertible Runge–Kutta matrix* $\mathcal{Q}$ *and* $|R(\infty)| < 1$, *the numerical solution* (5.1) *is bounded by*

$$\|(Au_{n+1})_{n=0}^{N-1}\|_{\ell^p(X)} + \|(AU_n)_{n=0}^{N-1}\|_{\ell^p(X^s)} \leq C \log N \, \|(F_n)_{n=0}^{N-1}\|_{\ell^p(X^s)},$$

where the constant C *is independent of* N *and* τ *and* $1 \leq p \leq \infty$.

A similar logarithmically quasi-maximal regularity result was proved in [29] for the discontinuous Galerkin solutions of the heat equation with an extra logarithmic factor:

$$\|\partial_t u_\tau\|_{L^p(0,T;L^q)} + \left(\sum \tau \left\| \frac{[u_\tau]}{\tau} \right\|_{L^q}^p \right)^{\frac{1}{p}} + \|\Delta u_\tau\|_{L^p(0,T;L^q)} \leq C \ln\left(\frac{T}{\tau}\right) \|f\|_{L^p(0,T;L^q)},$$

for $1 \leq p, q \leq \infty$, where u_τ denotes the discontinuous Galerkin solution of the heat equation, $\partial_t u_\tau$ denotes the piecewise time derivative of u_τ, and the summation extends over all jumps in the time interval $[0, T]$. The discontinuous Galerkin method is closely related to the Radau IIA implicit Runge–Kutta method, but the proof given in [29] is very different from the proof of Theorem 7.4.

8. An illustration of the use of discrete maximal L^p-regularity for a nonlinear parabolic equation. In this section, we illustrate how to apply the discrete maximal L^p-regularity to derive error estimates and regularity uniform in the stepsize τ of the time-discrete solution for a nonlinear parabolic equation. In this process, we shall see the superiority of the maximal L^p-regularity approach over the widely used L^2-norm approach in the case of a strong nonlinearity.

We consider a semilinear parabolic problem for illustrating the maximal L^p-regularity approach in a comparatively simple setting. The technique can be carried further to other semilinear problems such as the Ginzburg–Landau equations [31], to quasi-linear parabolic problems [3], and even to large classes of fully nonlinear parabolic problems [26].

We illustrate our idea by considering the semilinear parabolic equation

$$(8.1) \qquad \frac{\partial u}{\partial t} - \Delta u = f(u, \nabla u) \qquad\qquad \text{in } \Omega,$$

$$(8.2) \qquad \frac{\partial u}{\partial \nu} = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega,$$

$$(8.3) \qquad u = u_0 \qquad\qquad\qquad\qquad \text{at } t = 0,$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^d$ of arbitrary dimension $d \geq 1$, where $\partial u / \partial \nu$ denotes the normal derivative on the boundary $\partial \Omega$. We assume that $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a smooth pointwise nonlinearity, appearing as $f(u(x,t), \nabla u(x,t))$ in (8.1). For example, this includes the harmonic map heat flow, where $f = u|\nabla u|^2$. We will assume that this problem has a sufficiently regular solution, but we will not impose growth conditions on the nonlinearity f.

We consider time discretization by the backward Euler scheme

$$(8.4) \qquad \frac{u_n - u_{n-1}}{\tau} - \Delta u_n = f(u_n, \nabla u_n) \qquad \text{in } \Omega, \qquad n \geq 1,$$

$$(8.5) \qquad \frac{\partial u_n}{\partial \nu} = 0 \qquad \text{on } \partial \Omega, \qquad n \geq 1,$$

$$(8.6) \qquad \text{with starting value } u_0.$$

Extensions to full space-time discretizations are also discussed below. The scope of the use of discrete maximal ℓ^p-regularity goes far beyond the implicit Euler method and the semilinear equation considered here: in [3], the time discretization of quasilinear parabolic problems by BDF methods up to order 5 is studied with the help of the results in section 4, and in [26], the time discretization of fully nonlinear parabolic equations by implicit Runge–Kutta methods such as the Radau IIA methods of arbitrary order is studied based on the results in section 5.

In the following, for any sequence $v = (v_n)_{n=1}^N$ of functions in $L^q(\Omega)$ and a given stepsize $\tau > 0$ we consider the scaled ℓ^p-norm

$$\|v\|_{\ell^p_\tau(L^q)} = \left(\sum_{n=1}^N \tau \|v_n\|_{L^q}^p \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$, which is the $L^p(0, N\tau; L^q(\Omega))$-norm of the piecewise constant function that takes the value v_n on (t_{n-1}, t_n). We write similarly $\|v\|_{\ell^\infty(L^q)} = \max_{1 \leq n \leq N} \|v_n\|_{L^q}$.

THEOREM 8.1. *If the nonlinearity $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable (here we do not assume any growth condition), and if the exact solution of (8.1)–(8.3) satisfies $\partial_{tt} u \in L^p(0, T; L^p)$ and $u \in L^p(0, T; W^{2,p})$ for some $T > 0$ and for some p with $2 + d < p < \infty$, then there exist $\tau_0 > 0$ and $C_0 > 0$ (which depend on T) such that for $0 < \tau \leq \tau_0$ and $N\tau \leq T$, the errors*

$$e_n = u_n - u(\cdot, t_n) \quad and \quad \dot{e}_n = \frac{e_n - e_{n-1}}{\tau}$$

of the time-discrete solution given by (8.4)–(8.6) are bounded by

$$(8.7) \qquad \|(\dot{e}_n)_{n=1}^N\|_{\ell^p_\tau(L^p)} + \|(\Delta e_n)_{n=1}^N\|_{\ell^p_\tau(L^p)} \leq C_0 \tau,$$

$$(8.8) \qquad \|(e_n)_{n=1}^N\|_{\ell^\infty(W^{1,\infty})} \leq C_0 \tau.$$

Proof. We rewrite (8.1)–(8.3) for the exact solution $u(t) = u(\cdot, t)$ as

$$(8.9) \qquad \frac{u(t_n) - u(t_{n-1})}{\tau} - \Delta u(t_n) = f(u(t_n), \nabla u(t_n)) + d_n \qquad \text{in } \Omega, \qquad n \geq 1,$$

$$(8.10) \qquad \frac{\partial u(t_n)}{\partial \nu} = 0 \qquad \text{on } \partial \Omega, \qquad n \geq 1,$$

$$(8.11) \qquad u(t_0) = u_0,$$

where the defect $d_n = (u(t_n) - u(t_{n-1}))/\tau - \partial_t u(t_n)$ is the consistency error of the time discretization. By Taylor expansion and our regularity assumption, $d = (d_n)_{n=1}^N$ satisfies

$$(8.12) \qquad \|d\|_{\ell_\tau^p(L^p)} \leq C\tau \, \|\partial_{tt} u\|_{L^p(0,T;L^p)}.$$

Comparing (8.4)–(8.6) with (8.9)–(8.11), we see that the error $e_n = u_n - u(t_n)$ satisfies

$$(8.13) \qquad \dot{e}_n - \Delta e_n = f(u_n, \nabla u_n) - f(u(t_n), \nabla u(t_n)) - d_n \qquad \text{in } \Omega, \qquad n \geq 1,$$

$$(8.14) \qquad \frac{\partial e_n}{\partial \nu} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \qquad n \geq 1,$$

$$(8.15) \qquad e_0 = 0.$$

Let $M = \|u\|_{L^\infty(0,T;W^{1,\infty})}$ and define the function, for $s > 0$,

$$\rho(s) = \sup_{\substack{|y| \leq s \\ |z| \leq s \\ x \in \Omega}} \left(\left| \frac{\partial f}{\partial y}(y,z,x) \right| + \left| \frac{\partial f}{\partial z}(y,z,x) \right| \right).$$

The right-hand side in (8.13) is now considered as an inhomogeneity to be used in the discrete maximal ℓ^p-regularity. Since the Neumann Laplacian Δ has maximal L^p-regularity, we can apply Theorem 3.1 with e_n here in the role of u_n there and with $b_n := f(u_n, \nabla u_n) - f(u(t_n), \nabla u(t_n)) - d_n$ in the role of f_n. This yields that $e = (e_n)_{n=1}^N$ is bounded in terms of $b = (b_n)_{n=1}^N$ by

$$\begin{aligned}
\|\dot{e}\|_{\ell_\tau^p(L^p)} + \|\Delta e\|_{\ell_\tau^p(L^p)} &\leq C \, \|b\|_{\ell_\tau^p(L^p)} \\
&\leq C \| \big(f(u_n, \nabla u_n) - f(u(t_n), \nabla u(t_n)) \big)_{n=1}^N \|_{\ell_\tau^p(L^p)} + C\|d\|_{\ell_\tau^p(L^p)} \\
&\leq C\rho(M + \|e\|_{\ell^\infty(W^{1,\infty})}) \|e\|_{\ell_\tau^p(W^{1,p})} + C\|d\|_{\ell_\tau^p(L^p)},
\end{aligned}$$

where we further estimate

$$\|e\|_{\ell_\tau^p(W^{1,p})} \leq \epsilon \|e\|_{\ell_\tau^p(W^{2,p})} + C_\epsilon \|e\|_{\ell_\tau^p(L^p)}.$$

Suppose that

$$(8.16) \qquad \|e\|_{\ell^\infty(W^{1,\infty})} \leq 1.$$

Then by choosing ϵ small enough the last three inequalities imply

$$\|\dot{e}\|_{\ell_\tau^p(L^p)} + \|\Delta e\|_{\ell_\tau^p(L^p)} \leq C\|e\|_{\ell_\tau^p(L^p)} + C\|d\|_{\ell_\tau^p(L^p)}.$$

Since

$$\|e\|_{\ell^\infty(L^p)} = \max_{1 \leq k \leq N} \|e_k\|_{L^p} \leq \sum_{k=1}^N \tau \|\dot{e}_k\|_{L^p} = \|\dot{e}\|_{L^1(L^p)} \leq T^{1-1/p} \|\dot{e}\|_{\ell_\tau^p(L^p)}$$

it follows that

$$\begin{aligned}
\|e\|_{\ell^\infty(L^p)} + \|\Delta e\|_{\ell_\tau^p(L^p)} &\leq C\|e\|_{\ell_\tau^p(L^p)} + C\|d\|_{\ell_\tau^p(L^p)} \\
&\leq \epsilon \|e\|_{\ell^\infty(L^p)} + C_\epsilon \|e\|_{L^1(L^p)} + C\|d\|_{\ell_\tau^p(L^p)}.
\end{aligned}$$

Since this holds for every N with $N\tau \leq T$, we derive by Gronwall's inequality that $\|e\|_{\ell^\infty(L^p)} \leq C_T \tau$, which then yields (8.7). This also implies

$$(8.17) \qquad \|\widetilde{e}\|_{W^{1,p}(0,T;L^p)} + \|\widetilde{e}\|_{L^p(0,T;W^{2,p})} \leq C\|d\|_{\ell^p_\tau(L^p)},$$

where $\widetilde{e}$ is the piecewise linear interpolation of e_n, $n = 1, \ldots, N$, at the times t_n. We have

$$
\begin{aligned}
(8.18) \qquad & W^{1,p}(0,T;L^p) \cap L^p(0,T;W^{2,p}) \\
& \hookrightarrow L^\infty(0,T;(L^p,W^{2,p})_{1-1/p,p}) \qquad \text{[38, Proposition 1.2.10]} \\
& = L^\infty(0,T;W^{2-2/p,p}) \qquad\qquad\quad \text{[41, equation (34.4)]} \\
& \hookrightarrow L^\infty(0,T;W^{1,\infty}) \qquad\qquad\quad\;\, \text{if } (1-2/p)p > d \Leftrightarrow p > d+2 \;\; \text{[1]}
\end{aligned}
$$

so that

$$(8.19) \qquad \|\widetilde{e}\|_{L^\infty(0,T;W^{1,\infty})} \leq C\big(\|\widetilde{e}\|_{W^{1,p}(0,T;L^p)} + \|\widetilde{e}\|_{L^p(0,T;W^{2,p})}\big).$$

Together with (8.17), this yields (8.8). Overall, from (8.16) one can derive (8.8). Therefore, by a fixed point argument one readily obtains that there exists a positive constant τ_0 such that when $\tau < \tau_0$ we have (8.7)–(8.8) without assuming (8.16). We thus obtain

$$(8.20) \qquad \|e\|_{\ell^p_\tau(L^p)} \leq C \, \|d\|_{\ell^p_\tau(L^p)}.$$

In view of (8.12), this completes the proof of the theorem. $\qquad\square$

Remark 8.1. The key argument of the above proof is that τ-uniform discrete maximal ℓ^p-regularity allows us to control the $\ell^\infty(W^{1,\infty})$-norm of the error, and hence of the numerical solution. In contrast, for this proof the logarithmically quasi-maximal ℓ^∞-regularity bounds of section 7 are not sufficient to control the $\ell^\infty(W^{1,\infty})$-norm of the numerical solution uniformly in τ on bounded time intervals, because the logarithmic factor harms the use of the Gronwall inequality. The ℓ^∞-regularity bounds of section 7 can, however, be used to refine the error bounds. Since we know already that (8.8) holds, we obtain from Theorem 7.2 on $X = C(\overline{\Omega})$ applied to the error equation (8.13) that

$$\|\dot{e}\|_{\ell^\infty(L^\infty)} + \|\Delta e\|_{\ell^\infty(L^\infty)} \leq C' \log N \, \big(\rho(M+1)\|e\|_{\ell^\infty(W^{1,\infty})} + \|d\|_{\ell^\infty(L^\infty)}\big),$$

which directly yields, under the additional condition that $u \in C^2([0,T], C(\overline{\Omega}))$,

$$(8.21) \qquad \|(\dot{e}_n)_{n=1}^N\|_{\ell^\infty(L^\infty)} + \|(\Delta e_n)_{n=1}^N\|_{\ell^\infty(L^\infty)} \leq C\tau \log N.$$

Remark 8.2. In the above proof we implicitly assumed the existence of a numerical solution u_n close to u_{n-1} in the $W^{1,\infty}$ norm. This can in fact be proved by a Banach fixed point argument and using the discrete maximal ℓ^p-regularity in a similar way as in the above proof over just one time step. Once we have the existence of the numerical solution, the right-hand side in (8.13) can be considered as an inhomogeneity to be used in the discrete maximal ℓ^p-regularity.

Remark 8.3. The same result as in Theorem 8.1 with the same proof also holds with a semi-implicit approach in which the term $f(u_n, \nabla u_n)$ in (8.4) is replaced by $f(u_{n-1}, \nabla u_n)$ or $f(u_{n-1}, \nabla u_{n-1})$. In the latter case, the only two things that change in the proof are the expression for the defect, which now becomes $d_n = (u(t_n) -$

$u(t_{n-1}))/\tau - \partial_t u(t_n) + f(u(t_n), \nabla u(t_n)) - f(u(t_{n-1}), \nabla u(t_{n-1}))$ and is still bounded by $C\tau$ for a sufficiently regular solution, and the first two terms in (8.13), which now become $f(u_{n-1}, \nabla u_{n-1}) - f(u(t_{n-1}), \nabla u(t_{n-1}))$ and are estimated in the same way as the analogous term in (8.13).

Given sufficient regularity of the solution, we further note that the extension of Theorem 8.1 to higher-order BDF methods and A-stable Runge–Kutta methods is just a straightforward exercise based on the results of sections 4 and 5 and on the arguments of the above proof. No new ideas are required for such an extension to higher order.

Remark 8.4. The proof of Theorem 8.1 bounds the errors (e_n) in terms of the defects (or consistency errors) (d_n). For weaker regularity conditions, one obtains weaker estimates of the consistency errors than (8.12), which by (8.20) translate into a corresponding weaker error bound. A detailed investigation of error bounds in terms of the regularity of the initial data is beyond the scope of this paper. See, e.g., [14, 36] for error bounds of time discretizations of semilinear parabolic equations for nonsmooth initial data.

Remark 8.5. Uniform regularity estimates such as (8.7) have important applications in error estimates of full discretizations, with finite element methods for the spatial discretization. In the following, let us denote for brevity $u_h^\tau = (u_{h,n})_{n=1}^N$ the fully discrete numerical solution, $u^\tau = (u_n)_{n=1}^N$ the result of the implicit Euler time discretization given by (8.4)–(8.6), and $u = (u(t_n))_{n=1}^N$ the sequence of exact solution values of the nonlinear parabolic problem (8.1)–(8.3). Typically, in order to avoid any grid-ratio condition in deriving the error estimates, the error of the fully discrete method can be decomposed into two parts (see, e.g., [32]):

$$\|u_h^\tau - u\|_{\ell_\tau^p(W^{1,p})} \le \|u_h^\tau - u^\tau\|_{\ell_\tau^p(W^{1,p})} + \|u^\tau - u\|_{\ell_\tau^p(W^{1,p})},$$

where the first part is expected to be $O(h)$, uniformly in τ. For such nonlinear problems as (8.1)–(8.3), the main difficulty in the error estimates is to prove the boundedness $\|u_h^\tau\|_{\ell^\infty(W^{1,\infty})} \le C$ for the numerical solution. Let I_h denote the Lagrange interpolation operator. Under the regularity of (8.7), the first part of the error can be proved in the following way: by assuming that

$$(8.22) \qquad \|I_h u^\tau - u_h^\tau\|_{\ell^\infty(W^{1,\infty})} \le 1,$$

via τ- and h-uniform discrete maximal ℓ^p-regularity estimates on the finite element space one can prove the τ-independent error estimate (with D_τ denoting the backward difference quotient operator and $W^{-1,p}$ denoting the dual space of $W^{1,p'}$)

$$(8.23) \qquad \|D_\tau (I_h u^\tau - u_h^\tau)\|_{\ell_\tau^p(W^{-1,p})} + \|I_h u^\tau - u_h^\tau\|_{\ell_\tau^p(W^{1,p})} \le Ch,$$

and by using the inverse inequality

$$\|I_h u^\tau - u_h^\tau\|_{\ell_\tau^p(W^{1,\infty})} \le Ch^{-d/p}\|I_h u^\tau - u_h^\tau\|_{\ell_\tau^p(W^{1,p})} \le Ch^{1-d/p},$$

$$\|D_\tau (I_h u^\tau - u_h^\tau)\|_{\ell_\tau^p(W^{1,\infty})} \le Ch^{-2-d/p}\|D_\tau (I_h u^\tau - u_h^\tau)\|_{\ell_\tau^p(W^{-1,p})} \le Ch^{-1-d/p},$$

one recovers a better $\ell^\infty(W^{1,\infty})$-estimate via the interpolation inequality

$$\|I_h u^\tau - u_h^\tau\|_{\ell^\infty(W^{1,\infty})} \le \|I_h u^\tau - u_h^\tau\|_{\ell_\tau^p(W^{1,\infty})}^{1-1/p}\|D_\tau (I_h u^\tau - u_h^\tau)\|_{\ell_\tau^p(W^{1,\infty})}^{1/p}$$

$$(8.24) \qquad\qquad\qquad\qquad \le Ch^{1-(2+d)/p}.$$

When $p > 2 + d$, one can conclude that there exists a positive constant $h_0 > 0$ such that when $h < h_0$ the inequalities (8.23)–(8.24) hold, without preassuming (8.22).

Remark 8.6. We mention that the often used $l^\infty(L^2)$-norm approach does not work when the nonlinearity is strong enough. Specifically, if one uses the standard $l^\infty(L^2)$-norm error estimate, then by assuming (8.22) one can only prove

$$\|I_h u^\tau - u_h^\tau\|_{\ell^\infty(L^2)} + h\|I_h u^\tau - u_h^\tau\|_{L^\infty(H^1)} \le Ch^2$$

for the linear finite element method. The $\ell^\infty(W^{1,\infty})$ error of the numerical solution cannot be recovered for $d \ge 2$:

$$\|I_h u^\tau - u_h^\tau\|_{\ell^\infty(W^{1,\infty})} \le Ch^{-d/2}\|I_h u^\tau - u_h^\tau\|_{L^\infty(H^1)} \le Ch^{1-d/2}.$$

This shows an advantage of the maximal L^p-regularity approach for the analysis of strongly nonlinear problems.

Of course, if the nonlinearity is not strong, then one only needs to assume (8.22) with some $\ell^\infty(W^{1,q})$ norm,

$$\|I_h u^\tau - u_h^\tau\|_{\ell^\infty(W^{1,q})} \le Ch^{d/q-d/2}\|I_h u^\tau - u_h^\tau\|_{L^\infty(H^1)} \le Ch^{1+d/q-d/2},$$

and this weaker norm can thus be recovered if $q < 2d/(d-2)$.

Remark 8.7. Since the approach via discrete maximum ℓ^p-regularity allows us to control the $\ell^\infty(W^{1,\infty})$ error of the numerical solution, it works equally well for nonlinearities $f(u, \nabla u)$ that are defined only in a subregion of $\mathbb{R} \times \mathbb{R}^d$, provided that the exact solution of the parabolic problem stays in that subregion. For example, this includes nonlinearities with singularities (e.g., rational functions) or functions that are defined only for positive u or for ∇u in a cone.

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