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Inspection Policy for Inventory System with Deteriorating Products

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Abstract

Typical inventory models usually focus on screening out the deteriorating items, not much attention is paid to the situation that a proportion of deteriorated items are sold to consumers together with serviceable items during a replenishment cycle. This paper first presents a model to formulate the process of such kind of mixed sales. A simple closed-form solution is obtained for maximizing the net profit per unit time under the assumption of small deterioration rate. The effect of shortage back-ordering is then considered and closed-form solutions of optimal ordering quantity and replenishment time are obtained under similar assumption. We further study the effect of inspection policies on optimal decisions for a deteriorating inventory system. A dynamic programming approach is proposed for solving the optimal policy. Numerical examples are then provided to demonstrate our conclusions and results.

Keywords: Back-ordering, Deteriorating items, Dynamic programming, Inspecting policy, Ordering quantity, Replenishment time.

1 Introduction

In the existing models of inventory systems, one implicit assumption is that items can be stored indefinitely to meet the future market demand. However, it is common yet unrealistic to assume that the quality of all items is always perfect during their normal storage period. Indeed, certain types of items, such as food items, metal parts and medicines, undergo deterioration while in storage and hence they become unfit to use with time. In general, deterioration refers to decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of utility or loss of marginal value of stored commodities, and interested readers can find its definition in [22].

The idea of deterioration was first introduced by Whitin [24] in the study of fashion products deteriorating after a storage period. Since then many researchers have focused on modeling the processes of inventory problems for deteriorating items by considering different assumptions on the patterns of deterioration. For example, Covert and Philip extended the idea of constant deterioration rate in [4, 5] to a two-parameter Weibull distribution. Misra [13] proposed the production lot size model with varying and constant deterioration rates for deteriorating items. Tadikamalla [20] applied Gamma distribution to model the distribution of the time to deterioration. Hung [7] extended the inventory model from ramp type demand rate and Weibull deterioration rate in [17] to a more general type demand, deterioration and backorder rates.

Furthermore, other topics such as delayed payment, time value of money, expiration date, demand rate and shortage can be found in recent inventory studies of deteriorate items. For instance, Ouyang et al. [14] established an inventory control model for deteriorating items under partially permissible delay in payments. Chung and Lin [3] proposed an Economic Order Quantity (EOQ) model for deteriorating items, where the time value of money is taken into account. Yang [26] and Lee et al. [9] developed two-warehouse inventory models for deteriorating products. The difference is that the demand rate is assumed to be a constant in [26] while the demand rate in [9] is time-dependent. Huang et al. [8] introduced a lead-time discount coordination strategy for supply chains with deteriorating products. The profit of the entire supply chain was maximized by appropriately determining the optimal order quantity and lead-time. Wu et al. [23] proposed an EOQ model for the retailer by taking the expiration date of a deteriorating item into consideration. They showed that the retailer's optimal credit period and cycle time exist and are also unique. Related discussions can be found in Wang et al. [25] and Pahl et al. [15]. We refer interested readers to the reviews by Raafat [16], Goyal et al. [6] and Bakker et al. [1].

However, most of the research works considered screening out deteriorated items once they deteriorated, see for instance, [2, 11, 12, 19, 21]. This will require continuous monitoring or inspection on the items in the inventory and it will be costly. In fact, inventory holders may perform a complete inspection or inspect samples to make sure that the product quality is perfect when the order is received, before stocking it for immediate or later use. During the storage period, the stored items may gradually deteriorate and may not be identified in time and eventually sold to consumers with serviceable items together. Once a consumer unfortunately receives a defective commodity, he/she will return it and may choose to buy from other inventory holders. Then the sales volume which attributes to successful trading (consumers receive serviceable commodities) would be less than the market demand. In this case, the traditional inventory models for deteriorating items, in which deteriorated items are screened out, will therefore be ineffective.

Based on above discussions, this paper presents a general inventory model for mixed sales which means deteriorated items are sold to consumers together with serviceable items. This is an important and unique contribution because only a few studies have considered this situation. The optimal ordering quantity and optimal replenishment time are provided with regard to the situation of that shortage back-ordering is allowed during the replenishment cycle. Then, in order to reduce the chance of delivering deteriorated items to customers, this paper includes an inspection policy in the model. The number of inspections carried out in the inventory during the replenishment cycle will also be discussed.

The remainder of this paper is structured as follows. Section 2 discusses the inventory models for mixed sales and the method for finding the optimal ordering quantity and optimal replenishment time. In Section 3, we investigate the effect of shortage back-ordering in the inventory system. Section 4 describes an inspection policy which aims at maximizing the net profit per unit time. In Section 5, numerical examples are given with discussions. Finally, concluding remarks are given in Section 6.

2 The General Model

In this section, we begin with a general inventory model for mixed sales which means deteriorated items are sold to consumers together with serviceable items. The notations in Table 1 are adopted throughout the discussion. The following assumptions are adopted to formulate the problem:

1. The deterioration rate of the whole inventory system is deterministic and constant.

2. The demand rate for the item is deterministic and constant.

3. The deteriorated items are sold to customers together with the serviceable items.

4. If a customer receives a deteriorated item, he/she gets full refund. The deteriorated item has no salvage value.

Suppose that a replenishment cycle starts at time 0 and ends at time T. The inventory level is Q when the replenishment cycle starts and the replenishment cycle ends when all items are sold to customers. The demand rate in the inventory system is a constant $\lambda > 0$ and the deterioration rate of the inventory system is a constant $\theta > 0$. Let $I(t)$ and

Table 1: List of Notations

Q	inventory level;
$I(t)$	the inventory level of serviceable items at time t;
$J(t)$	the inventory level of deteriorated items at time t;
p	the selling price of an item;
λ	the demand rate;
θ	the deterioration rate of the inventory system;
T	the period of a replenishment cycle;
K	the ordering cost per order;
c	the purchasing cost per unit;
h	the holding cost of an item per unit time;
B	the back-ordering quantity;
b	the back-ordering cost per unit;
τ_j	the jth inspection time;
D	the booking cost per inspection;
d	the inspecting cost per unit.

$J(t)$ be the inventory levels of serviceable and deteriorated items at time t, respectively, $0 \leq t \leq T$. The inventory level in a replenishment cycle is illustrated in Figure 1, and the rates of change of the inventory levels of serviceable and deteriorated items at time t can be obtained by solving the following differential equations:

$$\begin{cases} I'(t) + \theta I(t) & = & -\dfrac{I(t)}{I(t)+J(t)}\lambda, \\ J'(t) & = & \theta I(t) - \dfrac{J(t)}{I(t)+J(t)}\lambda. \end{cases} \tag{2.1}$$

The sum of the above two rates is simply the demand rate:

$$I'(t) + J'(t) = -\lambda.$$

With the boundary conditions $I(0) = Q$ and $J(0) = 0$, the solution of the above differential equation is given by

$$I(t) + J(t) = Q - \lambda t.$$

Combining the equations in Eq. (2.1), we have

$$\begin{cases} I(t) & = & (Q-\lambda t)e^{-\theta t}, \\ J(t) & = & (Q-\lambda t)(1-e^{-\theta t}). \end{cases}$$

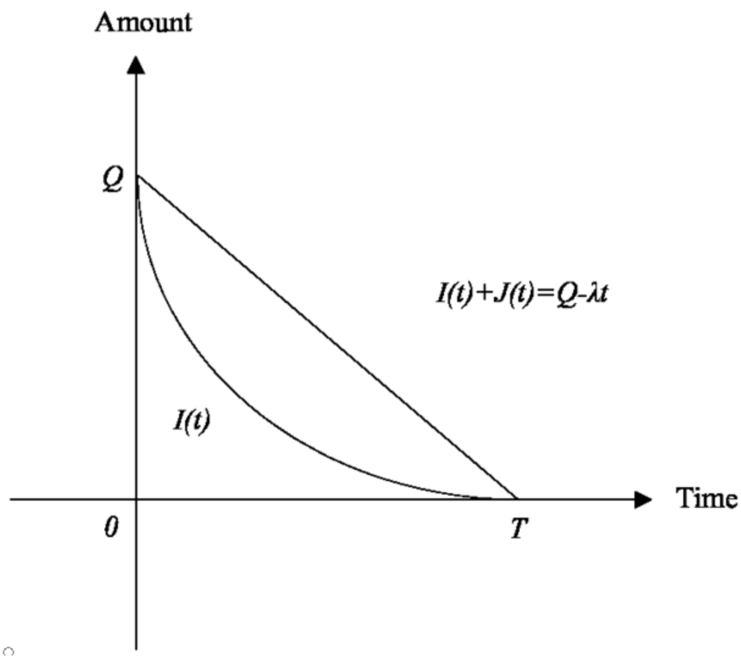

Figure 1: The inventory level in a replenishment cycle where deteriorated items are sold to customers together with serviceable items.

The amount of serviceable items sold in a replenishment cycle is given by

$$\int_0^T \frac{I(t)}{I(t)+J(t)}\lambda\, dt = \lambda \int_0^T e^{-\theta t}\, dt = \frac{\lambda}{\theta}(1-e^{-\theta T}).$$

The net profit per unit time, $V(Q,T)$, is then given by

$$V(Q,T) = \frac{p\Big[\frac{\lambda}{\theta}(1-e^{-\theta T})\Big] - \Big(K + cQ + h\frac{QT}{2}\Big)}{T}.$$

Since the replenishment cycle length $T = Q/\lambda$, one has

$$V(Q) = \Big[\frac{p\lambda^2}{\theta Q}(1-e^{-\theta Q/\lambda})\Big] - \Big(\frac{K\lambda}{Q} + c\lambda + h\frac{Q}{2}\Big).$$

Our objective is to maximize the net profit per unit time. Setting $V'(Q) = 0$ yields

$$\frac{p\lambda^2}{\theta}\Big[\frac{(e^{-\theta Q/\lambda})\theta Q/\lambda - (1-e^{-\theta Q/\lambda})}{Q^2}\Big] + \frac{K\lambda}{Q^2} - \frac{h}{2} = 0. \tag{2.2}$$

Let

$$g(Q) := \frac{p\lambda^2}{\theta}\Big[\Big(1+\frac{\theta Q}{\lambda}\Big)(e^{-\theta Q/\lambda}) - 1\Big] + K\lambda - \frac{hQ^2}{2},$$

which means

$$V'(Q) = \frac{g(Q)}{Q^2}.$$

Since $g(0) = K\lambda > 0$, $g(Q) \to -\infty$ as $Q \to \infty$ and $g'(Q) < 0$ for $Q > 0$, the equation $g(Q) = 0$ has exactly one positive root.

We then differentiate $V(Q)$ twice and get

$$V''(Q) = \frac{p\lambda^2}{\theta Q^3}\Big\{-\Big(\frac{\theta^2 Q^2}{\lambda^2}\Big)(e^{-\theta Q/\lambda}) - 2\Big[\Big(1+\frac{\theta Q}{\lambda}\Big)(e^{-\theta Q/\lambda}) - 1\Big]\Big\} - \frac{2K\lambda}{Q^3}. \tag{2.3}$$

Substituting Eq. (2.2) into Eq. (2.3) yields

$$V''(Q) = \frac{1}{Q}\big(-p\theta(e^{-\theta Q/\lambda}) - h\big) < 0 \quad \text{for } Q > 0.$$

Therefore, the optimal solution for maximizing $V(Q)$ can be obtained by solving Eq. (2.2) for its positive root Q.

For the case of small θ, Taylor series expansion can be applied to obtain an approximation in closed-form.

Proposition 2.1. *Suppose θ is small then the optimal ordering quantity and the optimal replenishment time are given, respectively, by*

$$Q^* = \sqrt{\frac{2K\lambda}{h+p\theta}} \quad \text{and} \quad T^* = \sqrt{\frac{2K}{\lambda(h+p\theta)}}.$$

3 Shortage Back-ordering

We consider the situation that when the inventory level reaches zero at time T_1 ($T_1 < T$), shortage occurs and is able to completely back-ordering at the instant. We assume that the back-ordering quantity is B and the back-ordering cost per unit is b. The demand rate during the shortage period remain the same as before. The net profit per unit time, $V(Q,T)$, is then given by

$$V(Q,T) = \frac{p\Big[\frac{\lambda}{\theta}(1-e^{-\theta T_1}) + B\Big] - \Big[K + c(Q+B) + \frac{hQT_1}{2} + \frac{bB(T-T_1)}{2}\Big]}{T}$$

where $T_1 = Q/\lambda$ and $B = \lambda(T - T_1) = \lambda T - Q$.

Hence, one has

$$V(Q,T) = \Big[\frac{p\lambda}{\theta}\frac{\big(1-\exp(-\theta Q/\lambda)\big)}{T} + p\lambda - \frac{pQ}{T}\Big] - \Big[\frac{K}{T} + c\lambda + \frac{hQ^2}{2\lambda T} + \frac{b(\lambda T - Q)^2}{2\lambda T}\Big].$$

Proposition 3.1. *When θ is small enough, the optimal ordering quantity and the optimal replenishment time are given, respectively, by*

$$Q^* \approx \sqrt{\frac{2K\lambda}{h+p\theta}\Big(\frac{b}{h+b+p\theta}\Big)} \quad \text{and} \quad T^* \approx \frac{Q^*}{\lambda}\left(1+\frac{h+p\theta}{b}-\frac{p\theta^2 Q^*}{2b\lambda}\right).$$

We remark that when the unit back-ordering cost b is large, The optimal pair of Q^* and T^* obtained in the back-ordering case will be close to the pair in Section 2.

4 Inspection Policy

In this section, we present the model which includes an inspection policy such that the net profit per unit time is maximized. Assume that the deterioration rate of the inventory system is θ during the whole replenishment cycle time $[0, T]$. We remark that similar assumption can be found in [7, 18]. Inspections can be carried out in the inventory at certain time during the replenishment cycle. Let k be the number of inspections preformed in the inventory in one replenishment cycle. Then the deteriorated items, which emerge at a rate of θ during the time between two inspections, can be screened out and the chance of delivering deteriorated items to customers will become lower after inspection. Our aim is to develop a mathematical model for obtaining the optimal inspection time, given Q, T and k. Based on this model, one can readily find the optimal values of Q, T and k by some numerical optimization method.

The inspection process may be imperfect. In general, two types of inspection errors may appear. One is Type 1 error, where a serviceable item is diagnosed as deteriorated. Another one is Type 2 error, where a deteriorated item is not detected. In Lee and Rosenblatt [10], they considered the case that each inspection is perfect and then gave some discussions of Type 1 error. Given the current advances in the technologies used for inspection, in this paper, we assume that there is no error in inspection. Imperfect inspection with errors will be an interesting issue for further study in our future research.

4.1 The Objective Function when $k = 1$

In this subsection, we analyze the special case when $k = 1$. The inventory system evolves as follows. We have Q units of serviceable items arrive at the beginning of the replenishment cycle. The demand rate in the inventory system is a constant λ and the deterioration rate of the inventory system is a constant θ over the whole replenishment cycle $[0, T]$. The inventory level declines only due to the demand rate over time interval $[0, \tau)$. At time τ $(0 < \tau < T)$, one inspection is conducted and the inventory level reduces since the items which have deteriorated during the time interval $[0, \tau)$ are screened out.

Then the inventory level reduces to zero owing to the demand during $(\tau, T]$. The inventory level in a replenishment cycle is illustrated in Figure 2.

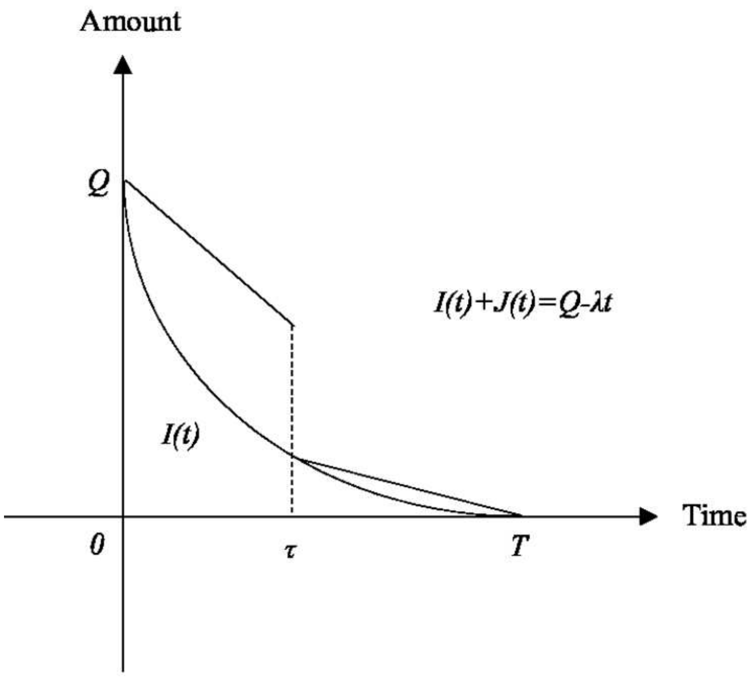

Figure 2: The inventory level in a replenishment cycle when there is one inspection during the replenishment cycle.

For different values of Q and T, we have the following proposition.

Proposition 4.1. *Suppose that there is one inspection during the replenishment cycle.*

(i) If $T > \frac{Q}{\lambda}$ *then shortage occurs before the inventory is replenished.*

(ii) If

$$T < \tau^* + \frac{Q}{\lambda}\exp(-\theta\tau^*) - \tau^*\exp(-\theta\tau^*),$$

where

$$\tau^* = \frac{2\lambda + \theta Q - \sqrt{4\lambda^2 - 2\lambda\theta Q + \theta^2 Q^2}}{3\lambda\theta}$$

then there is an excessive supply of the product items when time reaches T. *Here* τ^* *is the inspection time that minimizes the length of replenishment cycle.*

The proof of Proposition 4.1 can be found in Appendix. We remark that in Case (ii), since there is excess supply of the products at time T, the value of Q may be reduced so that the inventory cost incurred during a replenishment cycle can be reduced. Hence, if we choose $Q > \lambda T$, then there is an excess supply of the product for any value of inspection time τ. To save the holding cost of the inventory, the replenishment quantity may be reduced from Q to λT or less.

Assume that one inspection is conducted at time τ during the replenishment cycle, then the optimal net profit per unit time for fixed Q and T is given by

$$\begin{aligned} V(Q,T) &= \sup_{\tau} \frac{1}{T}\Big\{ -K - cQ + p\Big[\frac{\lambda}{\theta}\big(1-\exp(-\theta\tau)\big)\Big] \\ &\quad -\Big[\frac{h(2Q-\lambda\tau)\tau}{2} + D + d(Q-\lambda\tau)\Big] \\ &\quad +v((Q-\lambda\tau)\exp(-\theta\tau), T-\tau)\Big\}. \end{aligned} \tag{4.4}$$

In the above expression, $v((Q-\lambda\tau)\exp(-\theta\tau), T-\tau)$ is the net profit gained over $[\tau, T]$ where the inventory level of serviceable items is $(Q-\lambda\tau)\exp(-\theta\tau)$ at time τ. To simply the notations, denote $q = (Q-\lambda\tau)\exp(-\theta\tau)$. Then according to the values of q and τ, $v(q, T-\tau)$ can be expressed under the following three cases:

1. Case 1: $\lambda(T-\tau) < q$. The inventory level at time T is $q - \lambda(T-\tau)$, in which $(q-\lambda(T-\tau))\exp(-\theta(T-\tau))$ are serviceable. Here we assume that the supplier provides product return service for the unsold serviceable products at the original price (or the ordering quantity for next cycle can be $(q-\lambda(T-\tau))\exp(-\theta(T-\tau))$ items less). Hence, we have

$$\begin{aligned} v(q,T-\tau) &= p\Big[\frac{\lambda}{\theta}\big(1-\exp(-\theta(T-\tau))\big)\Big] - \frac{h(2q-\lambda(T-\tau))(T-\tau)}{2} \\ &\quad +c(q-\lambda(T-\tau))\exp(-\theta(T-\tau)). \end{aligned}$$

2. Case 2: $\lambda(T-\tau) = q$. The inventory level reaches 0 when the replenishment cycle ends. Hence, we have

$$v(q,T-\tau) = p\Big[\frac{\lambda}{\theta}\big(1-\exp(-\theta(T-\tau))\big)\Big] - \frac{hq(T-\tau)}{2}.$$

3. Case 3: $\lambda(T-\tau) > q$. In this case, the system is out of stock before time T and all shortages are back-ordered. Hence, we have

$$\begin{aligned} v(q,T-\tau) &= p\Big[\frac{\lambda}{\theta}(1-e^{-\theta q/\lambda}) + (\lambda(T-\tau)-q)\Big] \\ &\quad -\Big[\frac{hq^2}{2\lambda} + \frac{b(\lambda(T-\tau)-q)^2}{2\lambda} + c(\lambda(T-\tau)-q)\Big]. \end{aligned}$$

4.2 The Case when $k \geq 1$

For the general case, $k \geq 1$ and the jth inspection occurs at time τ_j ($j = 1, \ldots, k$), the inventory system evolves as follows. At the beginning of the replenishment cycle, there are

Q units of serviceable items in total. The deterioration rate of the inventory system is θ during the whole replenishment cycle time $[0, T]$. The inventory level declines only due to the demand rate over time interval $[0, \tau_1)$. At time τ_1 $(0 < \tau_1 < T)$, the first inspection is conducted and the inventory level reduces since the items which have deteriorated during the time interval $[0, \tau_1)$ are screened out. The aforesaid process is then repeated over the time interval $(\tau_{j-1}, \tau_j]$ $(j = 2, \ldots, k)$, and the deteriorated items which emerge during the time interval (τ_{j-1}, τ_j) are screened out at time τ_j. Finally, the inventory level reduces owing to the demand rate during $(\tau_k, T]$. In this case, the optimal net profit per unit time for fixed Q and T is

$$\begin{aligned} V_k(Q,T) &= \sup_{\tau_1,\cdots,\tau_k} \frac{1}{T}\Bigg\{ -K - cQ + p\sum_{j=1}^{k}\Big[\frac{\lambda}{\theta}\big(1-\exp(-\theta(\tau_j-\tau_{j-1}))\big)\Big] \\ &\quad -\sum_{j=1}^{k}\Big[\frac{h(2Q_{j-1}-\lambda(\tau_j-\tau_{j-1}))(\tau_j-\tau_{j-1})}{2} + D + d(Q_{j-1}-\lambda(\tau_j-\tau_{j-1}))\Big] \\ &\quad +v(Q_k, T-\tau_k)\Bigg\}, \end{aligned} \tag{4.5}$$

where $\tau_0 = 0, Q_0 = Q$ and Q_j is the inventory level after the inspection at time τ_j $(j = 1, \ldots, k)$, which is given recursively by

$$Q_j = (Q_{j-1} - \lambda(\tau_j - \tau_{j-1}))\exp(-\theta(\tau_j - \tau_{j-1})).$$

Here we assume that τ_j are chosen such that $Q_j > 0$ for $j = 1, \ldots, k$.

In what follows, we provide a recursive formula for Eq. (4.5) such that the optimization can be solved by using the Dynamic Programming (DP) approach. We first let

$$f(Q,T,\tau) = \frac{1}{T}\Bigg\{p\Big[\frac{\lambda}{\theta}\big(1-\exp(-\theta\tau)\big)\Big] - \Big[cQ + \frac{h(2Q-\lambda\tau)\tau}{2} + D + d(Q-\lambda\tau)\Big] + c\tilde{Q}(\tau)\Bigg\}$$

and $\tilde{Q}(\tau) = (Q - \lambda\tau)\big(\exp(-\theta\tau)\big)$ is the inventory level at time τ. The optimal net profit per unit time $V_k(Q, T)$ can be expressed by the following recursive formula:

$$\begin{cases} V_k(Q,T) = \sup\limits_{0<\tau<T'} \Big[f(Q,T,\tau) + \dfrac{(T-\tau)}{T}V_{k-1}(\tilde{Q}(\tau), T-\tau)\Big]; \\ V_0(Q,T) = \dfrac{-K - cQ + v(Q,T)}{T}. \end{cases} \tag{4.6}$$

Here we restrict the choices of τ to be less than $T' = \min\{T, Q/\lambda\}$ such that the inventory level remains positive after inspection.

The optimal value of $V_k(Q, T)$ and the optimal inspection policy $\mathcal{T}_k$ for k inspections can be obtained by the following iterative scheme:

Step 1. If $k = 0$, return $V_0(Q,T) = \dfrac{-K - cQ + v(Q,T)}{T}$ and $\mathcal{T}_0 = \emptyset$.
Step 2. Fix a number $n \in \mathbb{N}$.
Step 3. Set $t_i = \dfrac{iT'}{n+1}$, where $T' = \min\{T, Q/\lambda\}$ and $i = 1, \dots, n$.
Step 4. Compute $q_i = (Q - \lambda t_i)\big(\exp(-\theta t_i)\big)$, $i = 1, \dots, n$.
Step 5. For each $i = 1, \dots, n$, compute $V_{k-1}(q_i, T - t_i)$ and $\mathcal{T}_{k-1}^{(i)}$.
Step 6. Compute
$V_k(Q,T) = \max_i \Big[f(Q,T,t_i) + \dfrac{(T-t_i)}{T} V_{k-1}(q_i, T-t_i) \Big]$ and
$i^* = \operatorname{argmax}_i \Big[f(Q,T,t_i) + \dfrac{(T-t_i)}{T} V_{k-1}(q_i, T-t_i) \Big]$.
Step 7. Return $V_k(Q,T)$ and $\mathcal{T}_k = \{t_{i^*}\} \cup \{t_{i^*} + t : t \in \mathcal{T}_{k-1}^{(i^*)}\}$.

We end this section by the following proposition.

Proposition 4.2. *For any fixed $Q > 0$ and $T > 0$, let*

$$\Delta(q,t) = V_k(q, T-t) - V_{k-1}(q, T-t), \quad 0 < q < Q, \; 0 < t < T$$

and

$$\tau^* = \operatorname{argmax}_\tau \Big[f(Q,T,\tau) + \frac{(T-\tau)}{T} V_{k-1}(\tilde{Q}(\tau), T-\tau) \Big],$$

with

$$\tilde{Q}(\tau) = (Q - \lambda\tau)\big(\exp(-\theta\tau)\big).$$

We have the following.

(i) If $\Delta(\tilde{Q}(\tau^*), \tau^*) > 0$, then the optimal policy of $k+1$ inspections is better than that of k inspections.

(ii) If $\Delta(q,t) < 0$ for any $q \in (0,Q)$ and $t \in (0,T)$, then the optimal policy of k inspections is better than that of $k+1$ inspections.

The proof is given in Appendix. The contribution of Proposition 4.2 is that we can access whether $k+1$ inspections policy is preferred before $V_{k+1}(Q,T)$ is computed.

5 Numerical Results

In this section, we illustrate the effectiveness of the models proposed in Sections 2-4 through some numerical examples.

Example Assume that the unit time is one week and the replenishment cycle length T is 8 weeks. The inventory level Q is assumed to be $1000m$ where 1000 is the unit ordering number of the products and m ranges from 2 to 10. The optimal net profit per unit time and the optimal number of inspections of our proposed model with respect to the given T will be investigated. Moreover, the sensitivity of some parameters will be analyzed. The parameters for finding the optimal inspection time in k inspections are given in Table 2.

Table 2: List of parameters

λ	θ	K	c	h
1000 units/week	0.02 units/week	100 /cycle	25 /unit	0.1 unit/week
T	b	D	d	p
8 weeks	1.5 unit/week	200 /cycle	0.25 unit	50 /unit

Table 3 shows the optimal results of the proposed model for given Q and k. Here we choose the inventory level $Q = 6000$, 8000 and 10000 as three cases. The columns $\tau^*(k)$ and $V_k(Q,T)$ give the optimal inspection time and optimal net profit per unit time for k inspections, respectively. We observe that $V_k(Q,T)$ is concave in shape with respect to the number of inspections k. One particular instance is shown in Figure 3. Since $n = 0$ means no inspection during the replenishment cycle, therefore, Table 3 further implies that the optimal inspection strategy is significantly better than no-inspection policy while improper inspection strategies may reduce the net profit.

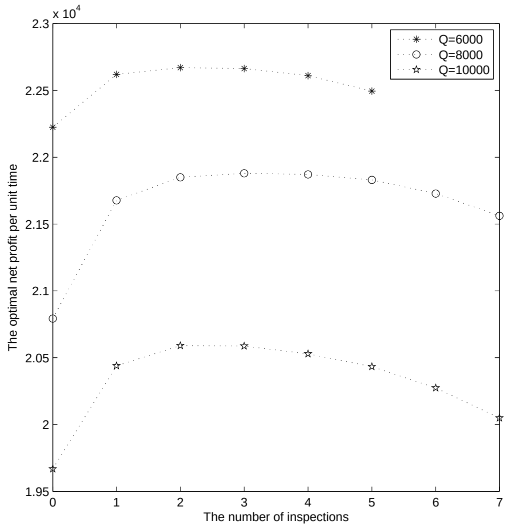

Figure 3: $V_k(Q,T)$ of three cases with respect to k.

The optimal results of our proposed model with respect to Q are indicated in Table 4. The column k^* is the optimal inspection number for any given Q, $\tau^*(k^*)$ is the corresponding optimal inspection time and $V_{k^*}(Q,T)$ is the corresponding optimal net profit

Table 3: Optimal results of the proposed model for given Q and k.

Q	k	$\tau^*(k)$	$V_k(Q,T)$
6000	0		22224.86
	1	$\tau_1 = 3$	22619.73
	2	$\tau_1 = 2$ $\tau_2 = 4$	22670.30
	3	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 5$	22663.36
	4	$\tau_1 = 2$ $\tau_2 = 3$ $\tau_3 = 4$ $\tau_4 = 5$	22609.80
	5	$\tau_1 = 1$ $\tau_2 = 2$ $\tau_3 = 3$ $\tau_4 = 4$ $\tau_5 = 5$	22494.82
8000	0		20792.57
	1	$\tau_1 = 4$	21676.97
	2	$\tau_1 = 3$ $\tau_2 = 6$	21849.38
	3	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 6$	21879.43
	4	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 6$ $\tau_4 = 7$	21870.91
	5	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 5$ $\tau_4 = 6$ $\tau_5 = 7$	21830.04
	6	$\tau_1 = 2$ $\tau_2 = 3$ $\tau_3 = 4$ $\tau_4 = 5$ $\tau_5 = 6$ $\tau_6 = 7$	21727.80
	7	$\tau_1 = 1$ $\tau_2 = 2$ $\tau_3 = 3$ $\tau_4 = 4$ $\tau_5 = 5$ $\tau_6 = 6$ $\tau_7 = 7$	21561.38
10000	0		19668.46
	1	$\tau_1 = 4$	20439.09
	2	$\tau_1 = 3$ $\tau_2 = 6$	20590.52
	3	$\tau_1 = 3$ $\tau_2 = 5$ $\tau_3 = 7$	20587.62
	4	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 6$ $\tau_4 = 7$	20528.49
	5	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 5$ $\tau_4 = 6$ $\tau_5 = 7$	20433.22
	6	$\tau_1 = 2$ $\tau_2 = 3$ $\tau_3 = 4$ $\tau_4 = 5$ $\tau_5 = 6$ $\tau_6 = 7$	20274.32
	7	$\tau_1 = 1$ $\tau_2 = 2$ $\tau_3 = 3$ $\tau_4 = 4$ $\tau_5 = 5$ $\tau_6 = 6$ $\tau_7 = 7$	20048.91

per unit time. Furthermore, Figure 4 shows that $V_{k^*}(Q,T)$ is concave with respect to the inventory level Q and reaches the maximum point at $Q = 5000$. This result implies that the optimal strategy for the inventory holder is $Q = 5000$ and $k = 2$ in this example. The suggested inspection time is in the second week and the fourth week.

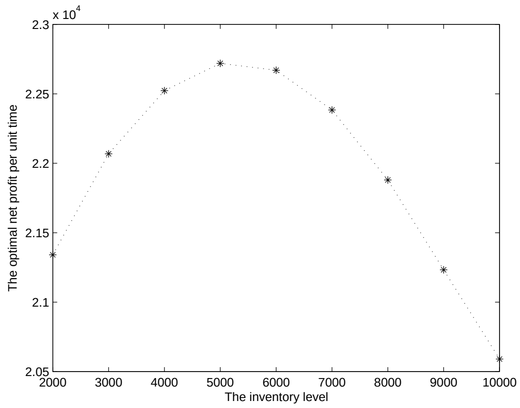

Figure 4: $V_{k^*}(Q,T)$ with respect to Q.

Table 4: Optimal results of the proposed model with respect to Q.

Q	k^*	$\tau^*(k^*)$	$V_{k^*}(Q,T)$
2000	0		21340.80
3000	1	$\tau_1 = 2$	22068.27
4000	1	$\tau_1 = 2$	22522.91
5000	2	$\tau_1 = 2$ $\tau_1 = 4$	22719.29
6000	2	$\tau_1 = 2$ $\tau_2 = 4$	22670.30
7000	2	$\tau_1 = 3$ $\tau_2 = 5$	22384.14
8000	3	$\tau_1 = 2$ $\tau_2 = 4$ $\tau_3 = 6$	21879.43
9000	3	$\tau_1 = 3$ $\tau_2 = 5$ $\tau_3 = 7$	21233.17
10000	2	$\tau_1 = 3$ $\tau_2 = 6$	20590.52

The values of parameters may change due to uncertainties in the storage environment and financial market. Thus we proceed further to study the impacts of some major parameters, such as θ and d, on the optimal solutions of the model under inspection policy. Table 5 shows that as θ increases, the optimal number of inspections k^* tends to increase while the optimal number of inventory level Q^* as well as the corresponding net profit per unit time $V_k^*(Q^*, T)$ tend to decrease. The main reason is that when θ takes large values, the inventory holder can decrease the number of deteriorated items through reducing the number of inventory level and increasing the number of inspections. However, in that situation, inspection cost, reordering cost and holding cost of reordering items all together may enhance the cost per unit time. Higher deterioration rate may result in lower net profit per unit time, which can be observed in Figure 6. Hence, finding some new methods to decreasing θ is an effective way to increase the net profit per unit time.

Table 6 shows that when d increases, Q^* and k^* tend to decrease. $V_{k^*}(Q^*, T)$ also tends to decrease, which can be observed in Figure 5. When d changes, k^* and $V_{k^*}(Q^*, T)$ are more sensitive to d than Q^*. The reason is that if d shows a small increase, the inventory holder can choose to adequately reduce the number of inspections to decrease the total cost. However, if d is too large, adding one inspection can be too costly while reducing one inspection can earn significant cost saving. Thus the inventory holder may tend to decrease the inventory level so as to reduce the number of deteriorated items and further reduce the demand for inspection. Hence, once d increase, the inventory holder should make an effort to decrease the cost per unit time by setting Q and k, even though the optimal net profit per unit time may decrease.

Table 5: Impact of θ on the optimal decisions.

θ	Q^*	k^*	$\tau^*(k^*)$	$V_k^*(Q^*,T)$
0.005	7000	1	$\tau_1 = 4$	23891.06
0.01	6000	1	$\tau_1 = 3$	23420.19
0.015	6000	2	$\tau_1 = 3$ $\tau_2 = 5$	23033.22
0.02	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22719.29
0.025	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22460.61
0.03	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22207.28

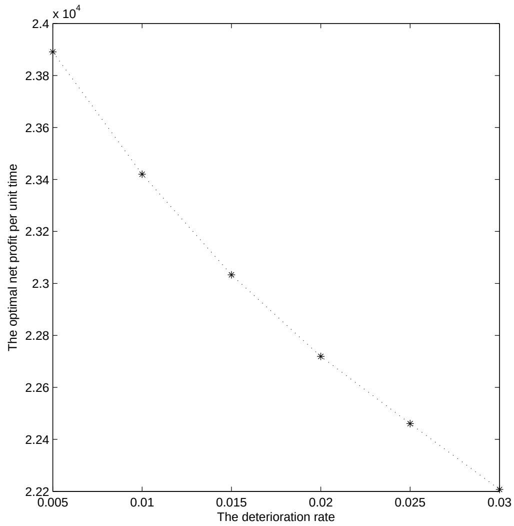

Figure 5: $V_k(Q,T)$ with respect to θ.

6 Concluding Remarks

In practice, some types of items deteriorate during their normal storage period, thereby selling some deteriorated items to consumers causing sales returns. However, most of the existing inventory models for deteriorating items have not dealt with such important practical situation of mixed sales. Therefore, this paper presents an inventory model to formulate the process of that deteriorated items are sold to consumers together with serviceable items during a replenishment cycle. Simple closed-form solutions are provided for the general model and the model including shortage back-ordering. Then we study the effect of inspections on optimal decisions for a deteriorating inventory system. An algorithm is developed to obtain the optimal inspection time which aims to maximize the net profit per unit time.

In this paper, we assume that the demand rate is a constant. For future research, we shall extend the proposed models by considering stochastic demand rate. The effect of imperfect inspections with Type 1 and Type 2 errors on the optimal inspection policy will

Table 6: Impact of d on the optimal decisions.

d	Q^*	k^*	$\tau^*(k^*)$	$V_k^*(Q^*, T)$
0.0	6000	5	$\tau_1 = 1$ $\tau_2 = 2$ $\tau_3 = 3$ $\tau_4 = 4$ $\tau_5 = 5$	22939.36
0.05	6000	5	$\tau_1 = 1$ $\tau_2 = 2$ $\tau_3 = 3$ $\tau_4 = 4$ $\tau_5 = 5$	22850.45
0.1	5000	3	$\tau_1 = 2$ $\tau_2 = 3$ $\tau_3 = 4$	22797.07
0.15	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22767.82
0.2	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22743.55
0.25	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22719.29
0.3	5000	2	$\tau_1 = 2$ $\tau_2 = 4$	22695.02
0.35	5000	1	$\tau_1 = 3$	22676.68
0.4	5000	1	$\tau_1 = 3$	22664.18
0.45	5000	1	$\tau_1 = 3$	22651.68
0.5	5000	1	$\tau_1 = 3$	22639.18

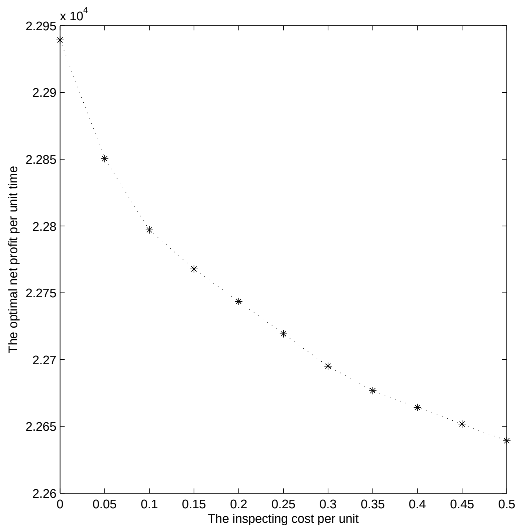

Figure 6: $V_k(Q, T)$ with respect to d.

be an interesting issue for our future research.

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References

[1] Bakker, M., Riezebos, J., Teunter, R., 2012. Review of inventory systems with deterioration since 2001. *European Journal of Operational Research*, 221, 275-284.

[2] Chung, C.J., Wee, H.M., 2011. Short life-cycle deteriorating product remanufacturing in a green supply chain inventory control system. *International Journal of Production Economics*, 129(1), 195-203.

[3] Chung, K.J., Lin, C.N., 2001. Optimal inventory replenishment models for deteriorating items taking account of time discounting. *Computers and Operations Research*, 28, 67-83.

[4] Covert, R.P., Philip, G.C., 1973. An EOQ model for items with Weibull distribution deterioration. *AIIE Transactions*, 5, 323-326.

[5] Ghare, P.M., Schrader, S.F., 1963. A model for exponentially decaying inventory. *Journal of Industrial Engineering*, 14, 238-243.

[6] Goyal, S.K., Giri, B.C., 2001. Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, 134, 1-16.

[7] Hung, K.C., 2011. An inventory model with generalized type demand, deterioration and backorder rates. *European Journal of Operational Research*, 208(3), 239-242.

[8] Huang, Y.S., Su, W.J., Lin, Z.L., 2011. A study on lead-time discount coordination for deteriorating products. *European Journal of Operational Research*, 215(2), 358-366.

[9] Lee, C.C., Hsu, S.L., 2009. A two-warehouse production model for deteriorating inventory items with time-dependent demands. *European Journal of Operational Research*, 194(3), 700-710.

[10] Lee H.L., Rosenblatt M.J., 1987. Simultaneous determination of production cycle and inspection schedules in a production system. Management science, 33(9), 1125-1136.

[11] Lin, G.C., Kroll, D.E., Lin, C.J., 2006. Determining a common production cycle time for an economic lot scheduling problem with deteriorating items. *European Journal of Operational Research*, 173(2), 669-682.

[12] Liao, J., 2007. On an EPQ model for deteriorating items under permissible delay in payments. *Applied Mathematical Modelling*, 31(3), 393-403.

[13] Misra, R.B., 1975. Optimum production lot size model for a system with deteriorating inventory. *International Journal of Production Research*, 13(5), 495-505.

[14] Ouyang, L.Y., Teng, J.T., Goyal, S.K., Yang, C.T., 2009. An economic order quantity model for deteriorating items with partially permissible delay in payments linked to order quantity. *European Journal of Operational Research*, 194(2), 418-431.

[15] Pahl, J., Voβ, S., 2014. Integrating deterioration and lifetime constraints in production and supply chain planning: A survey. *European Journal of Operational Research*, 238, 654-674.

[16] Raafat, F., 1991. Survey of literature on continuously deteriorating inventory models. *Journal of the Operational Research Society*, 42, 27-37.

[17] Skouri, K., Konstantaras, I., Papachristos, S., Ganas, I., 2009. Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. *European Journal of Operational Research*, 192(1), 79-92.

[18] Shah, N. H., Soni, H. N., Patel, K. A., 2013. Optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. *Omega*, 41(2), 421-430.

[19] Teng, J. T., Chang, C. T., 2005. Economic production quantity models for deteriorating items with price- and stock-dependent demand. *Computers and Operations Research*, 32(2), 297-308.

[20] Tadikamalla, P.R., 1978. An EOQ inventory model for items with gamma distributed deterioration. *AIIE transactions*, 10(1), 100-103.

[21] Widyadana, G.A., Wee, H.M. 2012. An economic production quantity model for deteriorating items with multiple production setups and rework. *International Journal of Production Economics*, 138(1), 62-67.

[22] Wee, H.M., 1993. Economic production lot size model for deteriorating items with partial back-ordering. *Computers and Industrial Engineering*, 24, 449-458.

[23] Wu, J., Ouyang, L.Y., Cárdenas-Barrón, L.E., Goyal, S.K., 2014. Optimal credit period and lot size for deteriorating items with expiration dates under two-level trade credit financing. *European Journal of Operational Research*, 237(3), 898-908.

[24] Whitin, T.M, 1953. The theory of inventory management. Princeton University Press, Princeton, NJ, USA.

[25] Wang, W.C., Teng, J.T., Lou, K.R., 2014. Seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime. *European Journal of Operational Research*, 232(2), 315-321.

[26] Yang, H., 2004. Two-warehouse inventory models for deteriorating items with shortages under inflation. *European Journal of Operational Research*, 157, 344-356.

7 Appendix

7.1 Proof of Proposition 2.1

We adopt the approximation:

$$\exp(x) \approx 1 + x + \frac{x^2}{2}$$

by ignoring the terms with higher orders. Then Eq. (2.2) becomes

$$\frac{p\lambda^2}{\theta}\Big[\Big(1+\frac{\theta Q}{\lambda}\Big)\Big(1-\frac{\theta Q}{\lambda}+\frac{\theta^2 Q^2}{2\lambda^2}\Big)-1\Big]+K\lambda-\frac{hQ^2}{2}=0$$

or equivalently

$$-\frac{p\theta Q^2}{2}+K\lambda-\frac{hQ^2}{2}=0.$$

Thus the optimal ordering quantity is given

$$Q^* = \sqrt{\frac{2K\lambda}{h+p\theta}}, \tag{7.7}$$

and the optimal replenishment time is

$$T^* = \sqrt{\frac{2K}{\lambda(h+p\theta)}}. \tag{7.8}$$

□

7.2 Proof of Proposition 3.1

To obtain the optimal values of Q and T, we consider the following systems of equations:

$$\begin{cases} \dfrac{\partial V}{\partial Q} = \dfrac{p}{T}\left(\exp(-\theta Q/\lambda)-1\right)-\Big[\dfrac{hQ}{\lambda T}-\dfrac{b(\lambda T-Q)}{\lambda T}\Big] \\[2ex] \dfrac{\partial V}{\partial T} = \Big[\dfrac{p\lambda}{\theta}\dfrac{\big(\exp(-\theta Q/\lambda)-1\big)}{T^2}+\dfrac{pQ}{T^2}\Big]-\Big[-\dfrac{K}{T^2}-\dfrac{hQ^2}{2\lambda T^2}+\dfrac{b(\lambda^2T^2-Q^2)}{2\lambda T^2}\Big]. \end{cases}$$

Simplifying $\dfrac{\partial V}{\partial Q}=0$, we have

$$T = \frac{(h+b)Q}{b\lambda}+\frac{p}{b}\big(1-\exp(-\theta Q/\lambda)\big). \tag{7.9}$$

Substituting Eq. (7.9) into $\dfrac{\partial V}{\partial T} = 0$, one obtains

$$\begin{aligned} m(Q) &:= 2p\lambda\Big[\frac{\lambda}{\theta}\big(\exp(-\theta Q/\lambda) - 1\big) + Q\Big] + 2K\lambda + (h+b)Q^2 \\ &\quad -b\lambda^2\Big[\frac{(h+b)Q}{b\lambda} + \frac{p}{b}\big(1 - \exp(-\theta Q/\lambda)\big)\Big]^2 \\ &= 0. \end{aligned} \tag{7.10}$$

We remark that since $m(0) = 2K\lambda > 0$, $m(Q) \to -\infty$ as $Q \to \infty$ and $m'(Q) < 0$ for $Q > 0$, $m(Q) = 0$ has exactly one positive root. The value of Q^*, the solution of Eq. (7.10), can be obtained by using some numerical method. We observe from Eq. (7.9) that $T^* > 0$ as $Q^* > 0$.

Again, when θ is small, we can employ Taylor series approximation, and Eq. (7.10) becomes

$$p\theta Q^2 + 2K\lambda + (h+b)Q^2 - \frac{(h+b+p\theta)^2 Q^2}{b} = 0.$$

Hence, we have

$$Q^* = \sqrt{\frac{2K\lambda}{h+p\theta}\Big(\frac{b}{h+b+p\theta}\Big)}.$$

Then by using Eq. (7.9), we have

$$T^* = \frac{(h+b)Q^*}{b\lambda} + \frac{p}{b}\big(1 - \exp(-\theta Q^*/\lambda)\big).$$

Then by using the approximation $\exp(x) = 1 + x + \dfrac{x^2}{2}$ again, the result follows.

Finally, we need to consider the second-order condition. The second order partial derivatives of $V(Q,T)$ are given by

$$\begin{cases} \dfrac{\partial^2 V}{\partial Q^2} = -\dfrac{p\theta}{\lambda T}\exp(-\theta Q/\lambda) - \dfrac{h+b}{\lambda T} \\ \dfrac{\partial^2 V}{\partial T^2} = \Big[\dfrac{2p\lambda}{\theta}\dfrac{\big(1 - \exp(-\theta Q/\lambda)\big)}{T^3} - \dfrac{2pQ}{T^3}\Big] - \Big[\dfrac{2K}{T^3} + \dfrac{(h+b)Q^2}{\lambda T^3}\Big] \\ \dfrac{\partial^2 V}{\partial Q \partial T} = \dfrac{p}{T^2}\big(1 - \exp(-\theta Q/\lambda)\big) + \dfrac{(h+b)Q}{\lambda T^2}. \end{cases}$$

It is clear that $\dfrac{\partial^2 V}{\partial Q^2} < 0$ and $\dfrac{\partial^2 V}{\partial T^2} < 0$ for positive Q and T. The determinant of the

Hessian matrix is given by

$$
\begin{aligned}
& \frac{\partial^2 V}{\partial Q^2}\frac{\partial^2 V}{\partial T^2} - \Big(\frac{\partial^2 V}{\partial Q \partial T}\Big)^2 \\
= \ & \frac{2K}{\lambda T^4}\Big[(h+b) + p\theta\exp(-\theta Q/\lambda)\Big] \\
& + \frac{2p(h+b)\big(\exp(-\theta Q/\lambda)\big)}{\theta T^4}\Big[\Big(1+\frac{\theta Q}{\lambda}+\frac{\theta^2 Q^2}{2\lambda^2}\Big) - e^{\theta Q/\lambda}\Big] \\
& + \frac{p^2\big(\exp(-\theta Q/\lambda)\big)}{\theta T^4}\Big[\Big(1+\frac{\theta Q}{\lambda}+\frac{\theta^2 Q^2}{2\lambda^2}\Big) - e^{\theta Q/\lambda} + e^{-\theta Q/\lambda} - \Big(1-\frac{\theta Q}{\lambda}+\frac{\theta^2 Q^2}{2\lambda^2}\Big)\Big].
\end{aligned}
$$

If θ is small,

$$\frac{\partial^2 V}{\partial Q^2}\frac{\partial^2 V}{\partial T^2} - \Big(\frac{\partial^2 V}{\partial Q \partial T}\Big)^2 \approx \frac{2K}{\lambda T^4}\Big[(h+b) + p\theta\exp(-\theta Q/\lambda)\Big] > 0$$

for $Q, T > 0$. And the Hessian matrix is negative definite and $V(Q,T)$ is strictly concave for positive Q and T. □

7.3 Proof of Proposition 4.1

Part (i) is trivial. For Part (ii), suppose that the inspection time is τ. The inventory level at time τ is

$$I(\tau) = (Q - \lambda\tau)e^{-\theta\tau}.$$

Hence the inventory level reaches 0 at time

$$t_0 = \tau + \frac{Q}{\lambda}e^{-\theta\tau} - \tau e^{-\theta\tau}.$$

Notice that $t_0 = Q/\lambda$ when $\tau = 0$ or $\tau = Q/\lambda$.

We now find a value of τ so that t_0 is minimized. When the length of replenishment cycle T is less than the minimum value of t_0 then there is excess supply of the product items when time reaches T. Differentiating t_0 with respect to τ gives

$$\frac{dt_0}{d\tau} = 1 - \frac{Q}{\lambda}\theta e^{-\theta\tau} - \big[\tau(-\theta e^{-\theta\tau}) + e^{-\theta\tau}\big]$$

and

$$\frac{d^2 t_0}{d\tau^2} = 2e^{-\theta\tau} + \Big(\frac{Q}{\lambda} - \tau\Big)\theta^2 e^{-\theta\tau}.$$

We see that $\frac{d^2 t_0}{d\tau^2} > 0$ when $\tau \le Q/\lambda$. Hence t_0 is convex and has a unique minimum in the interval $0 \le \tau \le Q/\lambda$. To find the value of τ^* that minimizes t_0 we solve $\frac{dt_0}{d\tau} = 0$. To simplify the expression we use Taylor series approximation (for small value of θ) and we have

$$\frac{dt_0}{d\tau} \approx \Big(-\frac{Q}{\lambda} + 2\tau\Big)\theta + \Big(\frac{Q\tau}{\lambda} - \frac{3\tau^2}{2}\Big)\theta^2.$$

The result follows from the quadratic formula. □

7.4 Proof of Proposition 4.2

Let

$$\tau^* = \text{argmax}_\tau \Big[f(Q,T,\tau) + \frac{(T-\tau)}{T} V_{k-1}(\tilde{Q}(\tau), T-\tau)\Big]$$

and

$$\tau^\dagger = \text{argmax}_\tau \Big[f(Q,T,\tau) + \frac{(T-\tau)}{T} V_{k}(\tilde{Q}(\tau), T-\tau)\Big].$$

(i) Consider

$$\begin{aligned} & V_{k+1}(Q,T) - V_k(Q,T) \\ \ge\ & \Big[f(Q,T,\tau^*) + \frac{(T-\tau^*)}{T} V_k(\tilde{Q}(\tau^*), T-\tau^*)\Big] \\ & -\Big[f(Q,T,\tau^*) + \frac{(T-\tau^*)}{T} V_{k-1}(\tilde{Q}(\tau^*), T-\tau^*)\Big] \\ =\ & \frac{(T-\tau^*)}{T} \Delta(\tilde{Q}(\tau^*), \tau^*). \end{aligned}$$

If $\Delta(\tilde{Q}(\tau^*), \tau^*) > 0$, then $V_{k+1}(Q,T) > V_k(Q,T)$ and hence the optimal policy of $k+1$ inspections is better than that of k inspections.

(ii) Consider

$$\begin{aligned} V_{k+1}(Q,T) - V_k(Q,T) \ \le\ & \Big[f(Q,T,\tau^\dagger) + \frac{(T-\tau^\dagger)}{T} V_k(\tilde{Q}(\tau^\dagger), T-\tau^\dagger)\Big] \\ & -\Big[f(Q,T,\tau^\dagger) + \frac{(T-\tau^\dagger)}{T} V_{k-1}(\tilde{Q}(\tau^\dagger), T-\tau^\dagger)\Big] \\ =\ & \frac{(T-\tau^\dagger)}{T} \Delta(\tilde{Q}(\tau^\dagger), \tau^\dagger). \end{aligned}$$

If $\Delta(q,t) < 0$ for any $q \in (0,Q)$ and $t \in (0,T)$, then $V_{k+1}(Q,T) < V_k(Q,T)$ and hence the optimal policy of k inspections is better than that of $k+1$ inspections.

□