Optimal Strategy for Limit Order Book Submissions in High Frequency Trading

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Abstract. An optimal selection problem for bid and ask quotes subject to a stock inventory constraint is investigated, formulated as a constrained utility maximisation problem over a finite time horizon. The arrivals of buy and sell orders are governed by Poisson processes, and a diffusion approximation is employed on assuming the Poisson arrivals intensity is sufficiently large. Using the dynamic programming principle, we adopt an efficient numerical procedure to solve this constrained utility maximisation problem based on a successive approximation algorithm, and conduct numerical experiments to analyse the impacts of the inventory constraint on a dealer's terminal profit and stock inventory level. It is found that the stock inventory constraint has material effects on the terminal stock inventory level.

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1. Introduction

High-frequency trading uses complex algorithms to analyse multiple markets and execute orders rapidly. Unlike in regular trading, an investment position in high-frequency trading may be held from fractions of a second to several hours, and it is characterised by short portfolio holding periods. There has been a rapid growth in high-frequency trading since the U.S. Securities and Exchange Commission (SEC) authorised electronic exchanges in 1998. Thus in the early 2000s high-frequency trading accounted for fewer than 10% of equity orders, but by 2010 an estimated 73% of exchange volume came from high-frequency trading orders in the United States. In Europe, high-frequency trading accounts for about 40% of equity orders volume, and in Asia about 5% to 10% with potential for rapid growth [6]. One high-frequency trading strategy involves placing a limit order to sell (or ask) or a buy limit order (or bid) to earn the bid-ask spread. To maximise the terminal profit, the dealer faces an inventory risk arising from uncertainty in the stock's price and a transactions risk due to Poisson arrivals of market buy and sell orders. To consider these two sources of risk, Ho & Stoll [8] developed a model to analyse the optimal prices for a monopolistic dealer in a single stock. Their results indicate that the optimal bid and ask quotes are around the "true" price of the stock, and Ho & Stoll [8] also pointed out that the bid and ask quotes are related to the reservation prices of the dealers when they are in competition. Based on these results, Avellaneda & Stoikov [1] studied optimal submission strategies by assuming the "true" price of the stock is modelled as a Brownian motion.

High-frequency traders moving in and out of short-term positions at high volumes aim to capture sometimes only a fraction of a cent in profit on every trade. High-frequency trading firms do not consume significant amounts of capital, accumulate positions or hold their portfolios overnight [11]. The risk of holding an overnight position is relatively high in a high volatility market, and in any case the payment of interest from margin for an overnight position reduces the profit of high-frequency trading. Thus in practice high-frequency traders may not hold their portfolios overnight or the stock inventory level of the trader may be very low at end of the trading day, corresponding to a stock inventory constraint on the optimal selection problem for bid and ask quotes.

Song *et al.* [16] employed a diffusion approximation to Poisson arrivals of market orders, such that the stock inventory level and the wealth dynamics can be approximated by Wiener processes. For practical applications in various fields, normal distributions often provide a reasonable approximation to a variety of data. Feller [3] pointed out that the Poisson distribution $Poi(\lambda)$ can be well approximated by the normal distribution $N(\lambda, \lambda)$, when the intensity is large enough. In this article, a diffusion approximation to Poisson arrivals of the market buy and sell orders is adopted.

There are many applications of a diffusion approximation. For example, Kobayashi [10] pointed out the queueing processes of various service stations that interact with each other can be approximated by a vector-valued Wiener process. Nagaev *et al.* [13] assumed that stock price evolution is described by a Markov chain, and obtained a simple but powerful approximation formula for the evolution by applying a diffusion approximation to the Markov chain. In insurance risk theory, a diffusion approximation has been used to approximate insurance surplus processes described by compound Poisson processes — e.g. see Grandell [4], where as in Song et al. [16] the dynamic programming principle is used to derive an Hamilton-Jacobi-Bellman (HJB) equation. The solution of the optimal submission problem of bid and ask quotes can then be obtained by solving the HJB equation; and we employ the successive approximation algorithm introduced by Chang & Krishna [2] to solve this second-order partial differential equation (PDE), coupled with an optimisation problem. The successive approximation algorithm separates the optimisation problem from the boundary value PDE problem, making the problem solvable by some standard numerical techniques. This article is an extension to the conference paper by [16], with the stock price now assumed to be governed by Geometric Brownian motion rather than Brownian motion.

In Section 2, the optimal submission problem is formulated as a Limit Order Book (LOB) with the stock inventory constraint, and the Poisson arrival of market orders is approximated by a Wiener process. The HJB equation is obtained from the dynamic programming principle, and in Section 3 the successive approximation algorithm is used to solve this equation with the inventory constraint. Numerical experiments are presented in Section 4, and we summarise our main results in Section 5.

2. The Model

2.1. Problem formulation

Due to rapid advancements in Information Technology (IT), it is possible for dealers to post limit orders at prices they choose, ensuring high frequency data on the LOB. A limit order is an order to buy a security at no more than a specific price p_b, or to sell a security at no less than a specific price p_a. A buy limit order can only be executed at the bid price p_b or lower, and a sell limit order can only be executed at the ask price p_a or higher. To describe the price discrepancies, the following distances may be used:

$$\delta_b = S_t - p_b \quad \text{and} \quad \delta_a = S_t - p_a \,,$$

where S_t is the stock price at any time t. In a security market, the execution of limit orders is determined by the dealer's submission of the limit orders and arrival of market orders. As elsewhere in the literature, we assume that a dealer's buy limit order is executed at a Poisson rate $\lambda_b(\delta_b)$, since a buy limit order can only be executed at the bid price p_b or lower. The Poisson rate $\lambda_b(\delta_b)$ should be a decreasing function in δ_b, and similarly the Poisson rate $\lambda_a(\delta_a)$ for an executed sell limit order is a decreasing function in δ_a. Intuitively, the further away from the mid-price the agent positions his quotes, the less often he will receive buy and sell orders. According to Ref. [1] and results in the econophysics literature (e.g. Refs. [5], [12] and [15]), the Poisson intensity can be expressed as

$$\lambda_a(\delta_a) = Ae^{-d\delta_a} \quad \text{and} \quad \lambda_b(\delta_b) = Ae^{-d\delta_b} \,.$$

The agent controls the bid and ask prices and therefore indirectly influences the flow of orders he or she receives, so the inventory level of the stock at time t becomes

$$q_t = N_t^b - N_t^a ,$$

where N_t^b and N_t^a are the respective Poisson processes with intensities λ_b and λ_a, N_t^b is the amount of the stock bought by the dealer, and N_t^a is the amount of the stock sold. Consequently, the corresponding wealth is also a stochastic process determined by the executed limit orders:

$$dX_t = rX_t + p_a dN_t^a - p_b dN_t^b ,$$

where r is the risk-free interest rate. Here we model the stock price in the market as a Geometric Brownian motion:

$$dS_t = S_t \mu dt + S_t \sigma dW_t .$$

As the dealer wishes to maximise the expected exponential utility of his terminal wealth, the optimal submission problem in a LOB can be formulated as follows:

$$\max_{\delta_a, \delta_b} E_t[-e^{-\gamma X_T}], \tag{2.1}$$

subject to

$$\begin{cases} dS_t & = & S_t \mu dt + S_t \sigma dW_t, \quad S_0 = s , \\ dX_t & = & rX_t + p_a dN_t^a - p_b dN_t^b , \\ q_t & = & N_t^b - N_t^a, \\ Prob(|\, q_T \,| < R) & \geq & 98\% . \end{cases}$$

Here γ is the coefficient of the exponential utility that represents the degree of risk aversion, and R is the upper bound for stock inventory level at end of the trading day. The rationale for this dealer optimisation problem is that the expected utility on terminal wealth at the terminal time T is maximised without too much stock inventory held at the terminal time T.

2.2. The diffusion approximation

From Ref. [3], we know that if $X \sim Poi(\lambda)$ then $X \approx N(\lambda, \lambda)$ for $\lambda > 20$, and the approximation improves as λ increases. Here the diffusion approximation is applied to Poisson arrivals of market orders, so the constraints for the optimal submission in the LOB can be rewritten

$$\begin{cases} dS_t & = & S_t \mu dt + S_t \sigma dW_t , \quad S_0 = s , \\ dX_t & = & (rX_t + p_a \lambda_a - p_b \lambda_b) dt + p_a \sqrt{\lambda_a} dW_t^2 - p_b \sqrt{\lambda_b} dW_t^3 , \\ dq_t & = & (\lambda_b - \lambda_a) dt + \sqrt{\lambda_b} dW_t^3 - \sqrt{\lambda_a} dW_t^2 . \end{cases}$$

We assume that the three standard Brownian motions W_t, W_t^2 and W_t^3 are independent, and recall that the dealer's objective is given by the value function

$$v(s, x, t) = \max_{\delta_a, \delta_b} E_t[-e^{-\gamma X_T}].$$

According to the standard dynamic programming principle, the following Hamilton-Jacobi-Bellman (HJB) equation can readily be obtained:

$$\begin{aligned} 0 &= v_t + \frac{1}{2}s^2\sigma^2\frac{\partial^2 v}{\partial s^2} + s\mu\frac{\partial v}{\partial s} + rx\frac{\partial v}{\partial x} \\ &\quad + \max_{\delta_a,\delta_b}\left\{\frac{1}{2}(p_a^2\lambda_a + p_b^2\lambda_b)\frac{\partial^2 v}{\partial x^2} + (p_a\lambda_a - p_b\lambda_b)\frac{\partial v}{\partial x}\right\}, \end{aligned} \tag{2.2}$$

with the terminal condition

$$v(T,\cdot) = -e^{-\gamma x}.$$

The HJB equation (2.2) is a second order PDE, which is coupled with an optimisation over δ_a and δ_b. Analytical solutions of the HJB equation are only known for some special cases with simple state equations and cost functions, so we employ an iterative algorithm to solve it numerically. From the standard verification theorem of the HJB equation (e.g. see Ref. [14], Chapter 11), the optimal bid and ask prices in a limit order can be obtained from the first-order conditions in its Hamiltonian term.

2.3. The stock inventory constraint

For the stock inventory constraint on the optimal selection problem in a LOB, it is reasonable to assume that the dealer submits limit orders at the beginning of small time intervals discretely — i.e. the execution volume of the dealer is determined by the dealer's submission at the beginning of any small time interval $[t, t+h]$. Thus the stock amounts bought and sold by a dealer in the time interval $[t, t+h]$ are decided by the bid price and ask price, respectively. We assume that the dealer submits LOBs at discrete-time point $t_0, t_1, \ldots, t_k (t_0 = 0, t_k = T)$, and divide the time interval $[0, T]$ into $k+1$ small time intervals. Let $\Delta N_{t_i}^a$ denote the amount of stock bought by the dealer and $\Delta N_{t_i}^b$ the amount of stock sold by the dealer in the time interval $[t_i, t_i + h], i = 0, \ldots, k$, which follow Poisson distributions with the respective associated parameters $\lambda_a^i h$ and $\lambda_b^i(\delta_b^i)$ — i.e. $\Delta N_{t_i}^a \sim$ Poi$(\lambda_a^i h)$, and similarly $\Delta N_{t_i}^b \sim Poi(\lambda_b^i h)$ where

$$\lambda_a^i(\delta_a^i) = Ae^{-d\delta_a^i} \quad \text{and} \quad \lambda_b^i(\delta_b^i) = Ae^{-d\delta_b^i}, \quad \text{for } i = 0, \cdots, k.$$

We adopt the diffusion approximation

$$\Delta N_{t_i}^a \sim N(\lambda_a^i h, \lambda_a^i h)$$

— i.e. $\Delta N_{t_i}^a = \sqrt{\lambda_a^i h}X_i + \lambda_a^i h$, where $\{X_i\}_{i=0,\cdots,k}$ are independent identically distributed (i.i.d.) random variables with standard normal distribution. Similarly, we assume that $\Delta N_{t_i}^b = \sqrt{\lambda_b^i h}Y_i + \lambda_b^i h$, where $\{Y_i\}_{i=0,\cdots,k}$ are i.i.d. random variables with standard normal distribution. Thus we have

$$\begin{aligned} N_T^a &= \sum_{i=0}^{k} \Delta N_{t_i}^a \\ &= \sum_{i=0}^{k} \sqrt{\lambda_a^i h} X_i + \sum_{i=0}^{k} \lambda_a^i h \\ &= \left(\sum_{i=0}^{k} \sqrt{\lambda_a^i} \right) \sqrt{h} X + \left(\sum_{i=0}^{k} \lambda_a^i \right) h \end{aligned}$$

and

$$\begin{aligned} N_T^b &= \sum_{i=0}^{k} \Delta N_{t_i}^b \\ &= \sum_{i=0}^{k} \sqrt{\lambda_b^i h} Y_i + \sum_{i=0}^{k} \lambda_b^i h \\ &= \left(\sum_{i=0}^{k} \sqrt{\lambda_b^i} \right) \sqrt{h} Y + \left(\sum_{i=0}^{k} \lambda_b^i \right) h \end{aligned}$$

where X and Y are two independent random variables with standard normal distribution, such that

$$\begin{aligned} | q_T | &= \left| N_T^a - N_T^b \right| \\ &= \left| \left(\sum_{i=0}^{k} \sqrt{\lambda_a^i} - \sum_{i=0}^{k} \sqrt{\lambda_b^i} \right) \sqrt{h} Z + \left(\sum_{i=0}^{k} \lambda_a^i - \sum_{i=0}^{k} \lambda_b^i \right) h \right| \end{aligned}$$

where Z is a random variable with standard normal distribution — i.e. writing

$$\Lambda_1 = \left(\sum_{i=0}^{k} \lambda_a^i - \sum_{i=0}^{k} \lambda_b^i \right) h \quad \text{and} \quad \Lambda_2 = \left(\sum_{i=0}^{k} \sqrt{\lambda_a^i} - \sum_{i=0}^{k} \sqrt{\lambda_b^i} \right) \sqrt{h}$$

we have

$$| q_T | = | \Lambda_2 Z + \Lambda_1 | .$$

Consequently, the stock inventory constraint can be rewritten as

$$\begin{aligned} \text{Prob}(| q_T | < R) &= \text{Prob}\left(\frac{-R - \Lambda_1}{\Lambda_2} < Z < \frac{R - \Lambda_1}{\Lambda_2} \right) \\ &= \Phi\left(\frac{R - \Lambda_1}{\Lambda_2} \right) - \Phi\left(\frac{-R - \Lambda_1}{\Lambda_2} \right) \geq 98\% \end{aligned}$$

provided that $\Lambda_2 > 0$, where Φ is the cumulative distribution function of the standard normal distribution. Thus the optimal submission problem in a LOB with the stock inventory constraint can be summarised as

$$\max_{\delta_a, \delta_b} E_t[-e^{-\gamma X_T}]$$

subject to

$$\begin{cases} dS_t &= S_t\mu dt + S_t\sigma dW_t\,, \quad S_0 = s\,, \\ dX_t &= (rX_t + p_a\lambda_a - p_b\lambda_b)dt + p_a\sqrt{\lambda_a}dW_t^2 - p_b\sqrt{\lambda_b}dW_t^3\,, \\ dq_t &= (\lambda_b - \lambda_a)dt + \sqrt{\lambda_b}dW_t^3 - \sqrt{\lambda_a}dW_t^2\,, \end{cases}$$

and

$$\mathrm{Prob}(\mid q_T \mid < R) = \Phi\left(\frac{R-\Lambda_1}{\Lambda_2}\right) - \Phi\left(\frac{-R-\Lambda_1}{\Lambda_2}\right) \geq 98\%\,. \tag{2.3}$$

3. Numerical Experiments and Discussion

In this section, we first present the iterative algorithm for the numerical solution of the HJB equation with the stock inventory constraint. Using red to denote the optimal selection of the bid and ask prices under the stock inventory constraint, we can find how the stock inventory constraint affects the dealer's choices. We also conduct numerical experiments to analyse the influence of the stock inventory constraint on the dealer's terminal profit and inventory level.

3.1. The iterative algorithm

The HJB equation is a second-order nonlinear PDE, for which analytical solutions have only been obtained in some special cases with simple state equations. Consequently, here we apply the successive approximation algorithm to solve the HJB equation numerically, where the procedure leads to two separate sub-problems [2] — i.e.

1. solve the PDE numerically, and
2. optimise the nonlinear function with respect to δ_a and δ_b.

To solve the PDE, we employ a finite difference scheme from Ref. [17]. Accordingly, we divide the domain of the computation into a grid of $N_t \times N_q \times N_X$ mesh points, where N_t denotes the number of mesh points in the time domain and (N_q, N_X) the numbers in the space domain. For a function v defined on the grid, $v_{l,m,n}$ is the value of v at the grid point (t_l, q_m, X_n), so the steps in the iterative algorithm may be represented as follows:

Step I: For each $l = N_t - 1, \ldots, 0$, $m = 1, \ldots, N_q$, $n = 1, \ldots, N_X$, assume $k = 0$ and choose an arbitrary initial control law δ_a^0, δ_b^0. Then for the fixed control law δ_a^k, δ_b^k, obtain the value function v^k from

$$\begin{aligned} v_t^k + \frac{1}{2}s^2\sigma^2\frac{\partial^2 v^k}{\partial s^2} + s\mu\frac{\partial v^k}{\partial s} + rx\frac{\partial v^k}{\partial x} \\ + \frac{1}{2}\left(p_a^2\lambda_a + p_b^2\lambda_b\right)\frac{\partial^2 v^k}{\partial x^2} + (p_a\lambda_a - p_b\lambda_b)\frac{\partial v^k}{\partial x} \;=\; 0\,, \end{aligned}$$

with the terminal condition $v^k(T,\cdot) = -e^{-\gamma X_T}$.

Step II: Compute the succeeding control law δ_a^{k+1}, δ_b^{k+1} from the optimisation problem

$$\max_{\delta_a^{k+1},\delta_b^{k+1}} \left\{ \frac{1}{2}(p_a^2\lambda_a + p_b^2\lambda_b)\frac{\partial^2 v^k}{\partial x^2} + (p_a\lambda_a - p_b\lambda_b)\frac{\partial v^k}{\partial x} \right\}$$

subject to

$$\Phi\left(\frac{R-\Lambda_1}{\Lambda_2}\right) - \Phi\left(\frac{-R-\Lambda_1}{\Lambda_2}\right) \geq 98\% \,,$$

where $$\Lambda_1 = \left(\sum_{i=0}^{k}\lambda_a^i - \sum_{i=0}^{k}\lambda_b^i\right)h \quad \text{and} \quad \Lambda_2 = \left(\sum_{i=0}^{k}\sqrt{\lambda_a^i} - \sum_{i=0}^{k}\sqrt{\lambda_b^i}\right)\sqrt{h}\,.$$

Step III: Return to Step I with $k = k+1$ until

$$\|v^{k-1} - v^k\| < \epsilon \,,$$

where ϵ is a small positive number. The proof of the convergence of this Successive Approximation Algorithm has been provided in Ref. [7].

3.2. Optimal bid and ask prices with the stock inventory constraint

We conducted numerical experiments to compare the optimal submission in an LOB arising from the model with the stock inventory constraint to that obtained from the model without that constraint, on implementing the above iterative algorithm using MATLAB.

Fig. 1 shows the simulated path for the parameter set $\{T=1\,,\ \gamma=0.1\,,\ r=0.02\,,\ s=100\,,\ \sigma=2\,,\ \mu=0.9\,,\ h=0.01\,,\ d=1.5\,,\ A=140\,,\ R=0.01\}$ from the model without the stock inventory constraint, and we observe that the stock price dynamic has a significant impact on the dealer's choice. For example, in the time interval $(73,75)$ where the stock price is increasing, both the optimal bid and ask prices are increasing because the dealer holds more stocks to increase the inventory value at a higher bid price. To increase the profit by selling the stock at a higher price, the dealer should submit a higher ask price, and as the stock price increases the increasing rate of the ask price exceeds that of the bid price. This makes intuitive sense as the dynamic of the stock price affects the dealer's submission from two perspectives. On the one hand, to minimise the inventory risk the dealer should submit a higher ask price if his inventory level is positive, but lower the bid price if the dealer is in a short position. On the other hand, with regard to transaction risk and maximising terminal wealth the dealer should sell the stock at a higher price and buy in at a lower price, to earn a greater bid-ask spread. The dealer's incentive to hold more stock is seen to result in a higher intensity of market buy orders until $t = 75$, where the stock price starts to decrease.

Fig. 2 shows the simulated paths with the same parameters for the model with the stock inventory constraint, when the upper bound for the stock inventory level R is 0.01. Comparing Figs. 1 and 2, we find that the dealer behaves more conservatively to reduce the inventory risk, when the stock inventory constraint is present. For example, in the

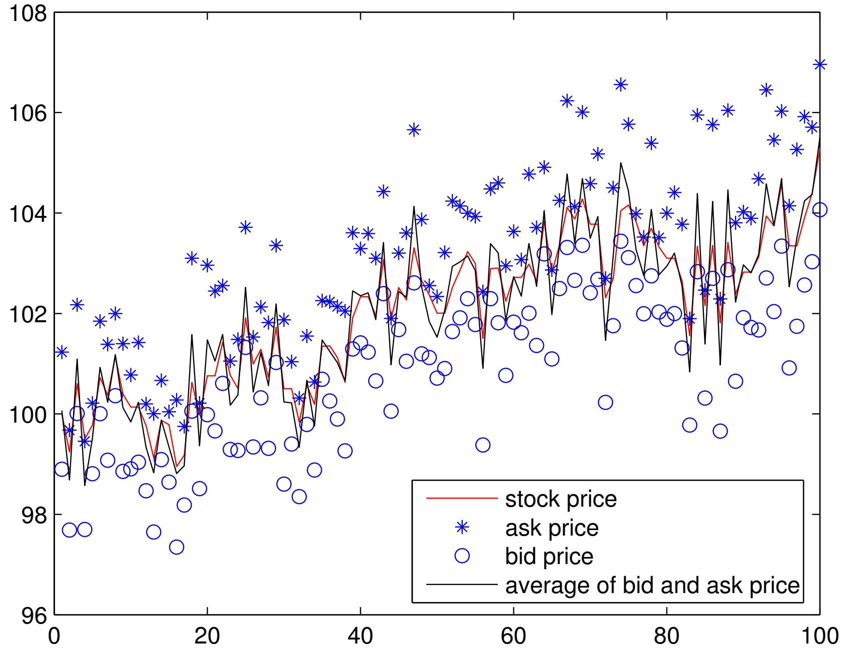

Figure 1: Simulation results without the stock inventory constraint.

time interval $(73, 75)$ the optimal ask price p_a at $t = 75$ in Fig. 1 is 106.6, while in Fig. 2 $p_a = 105.9$ at the same time. When the stock price moves drastically, the dealer's strategy is seen to be more modest when the stock inventory constraint is in force. This seems reasonable since maximising terminal wealth is one objective, and the dealer also needs to minimize the stock inventory level at the end of a trading day. We also note that the dealer's bid/ask quotes look more like a symmetric strategy as the terminal time is approached, when any adjustment in the stock inventory level may be unlikely.

From 200 simulations we obtained the averages and standard deviations of the profits for the two models when $\gamma = 0.1$. As shown in Table 1, the strategy with the inventory constraint has a lower profit (\$62.45 versus \$66.38) and a lower standard deviation (\$3.13 versus \$7.82), because the dealer behaves more conservatively to reduce the inventory risk. The model with the inventory constraint results in a lower final inventory level (final q of \$0.006 versus \$0.53) and a lower standard deviation (\$0.42 versus \$1.79). These results show that the stock inventory constraint has a pronounced effect on the stock inventory level at the end of a trading day. We also observe that the strategy without the inventory constraint yields a slightly higher return, but with the inventory constraint the profit and loss (P&L) profile has a much smaller variance — cf. the histogram in Fig. 3.

The results of the simulations with and without the inventory constraint are displayed in Table 2 for $\gamma = 0.01$, a small value for a strategy close to risk neutral. As shown in Fig. 4, the inventory effect is then much smaller and the P&L profiles of the two strategies are very

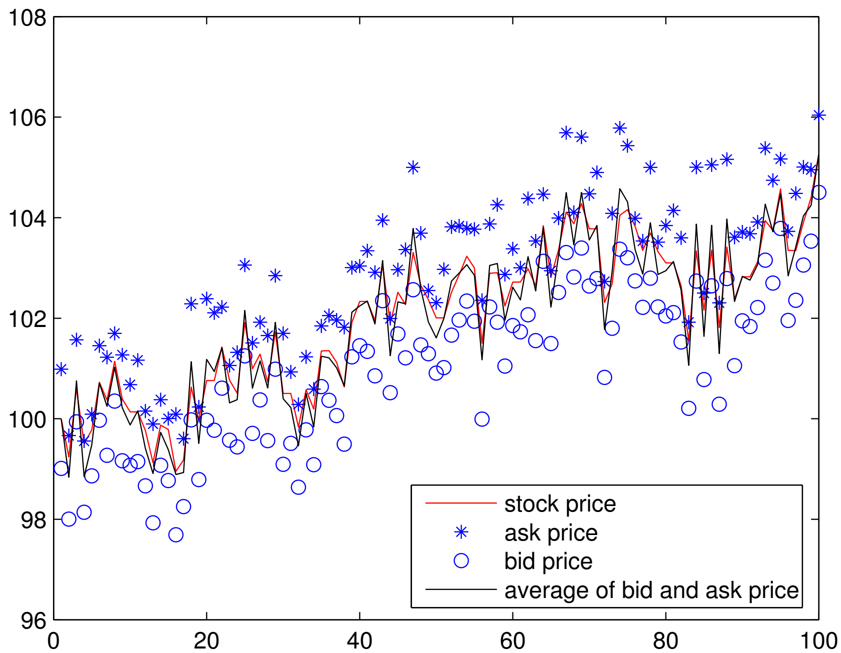

Figure 2: Simulation results with the stock inventory constraint.

Table 1: 200 simulations with $\gamma = 0.1$

Strategy	Profit	Std (Profit)	Final q	Std (Final q)
With Inventory Constraint	62.45	3.13	0.006	0.42
Without Inventory Constraint	66.38	7.82	0.53	1.79

similar. Finally, we display the performance of the two strategies for $\gamma = 0.5$ in Table 3. This choice corresponds to a very risk averse dealer, who will go to great lengths to avoid accumulating an inventory. This strategy produces low standard deviations in profit and final inventory, but generates more modest profit than the corresponding strategy without inventory constraint — cf. Fig. 5.

4. Concluding Remarks

The optimal bid and ask quotes selection problem are determined subject to the stock inventory constraint when the arrivals of the buy and sell orders are governing by Poisson process. Using a diffusion approximation to approximate the Poisson arrivals and the dynamic programming principle is used, the HJB equation is obtained. We give the numerical solution to the HJB equation. It is found that the dealer behaves more conservative when the inventory constraint is present and that the final inventory level with inventory

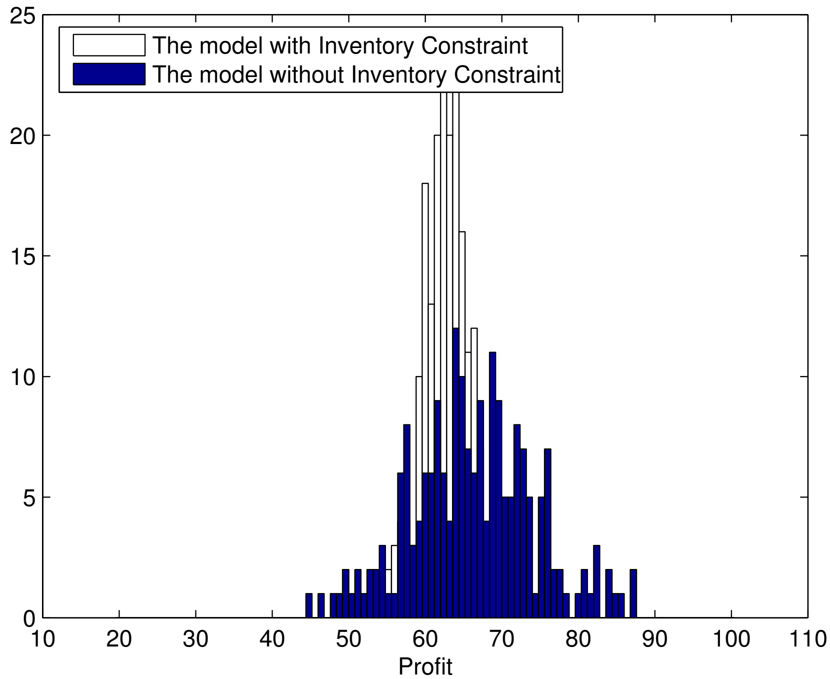

Figure 3: $\gamma = 0.1$

Table 2: 200 simulations with $\gamma = 0.01$

Strategy	Profit	Std (Profit)	Final q	Std (Final q)
With Inventory Constraint	67.23	4.73	0.004	0.63
Without Inventory Constraint	67.41	5.04	0.42	1.84

constraint has a lower expectation and a lower standard deviation than the corresponding values from the model without the inventory constraint.

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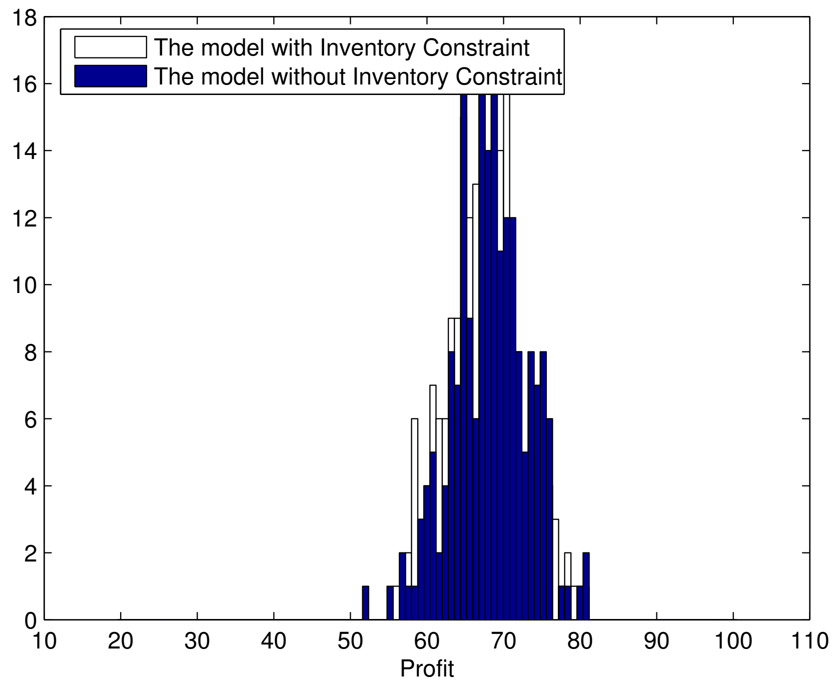

Figure 4: $\gamma = 0.01$

Table 3: 200 simulations with $\gamma = 0.5$

Strategy	Profit	Std (Profit)	Final q	Std (Final q)
With Inventory Constraint	43.80	4.88	0.003	0.12
Without Inventory Constraint	66.15	8.69	0.38	1.93

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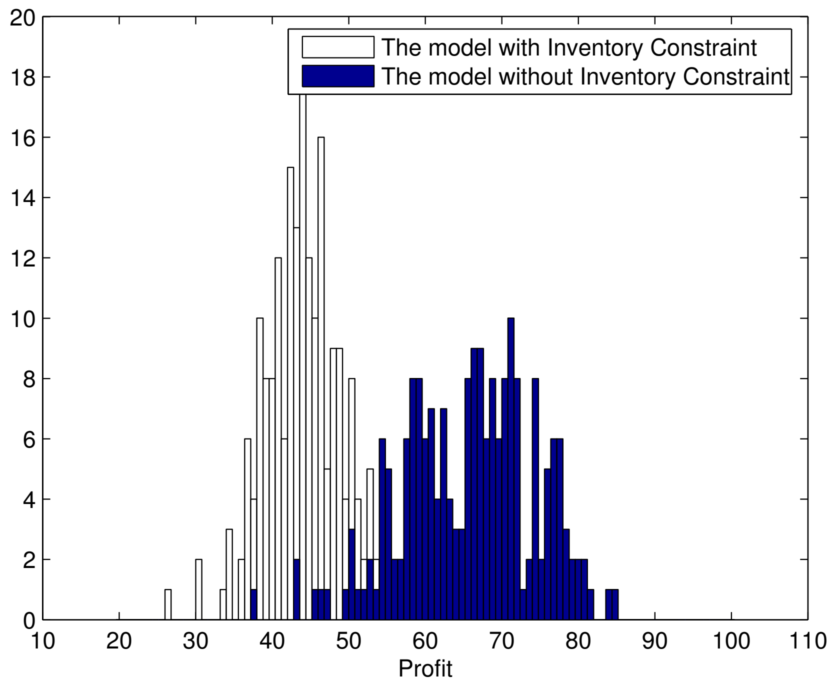

Figure 5: $\gamma = 0.5$

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