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Backward Mean-Field Linear-Quadratic-Gaussian (LQG) Games: Full and Partial Information

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Abstract

This paper introduces the *backward* mean-field (MF) linear-quadratic-Gaussian (LQG) games (for short, BMFLQG) of weakly coupled stochastic large-population system. In contrast to the well-studied *forward* mean-field LQG games, the individual state in our large-population system follows the backward stochastic differential equation (BSDE) whose *terminal* instead *initial* condition should be prescribed. Two classes of BMFLQG games are discussed here and their decentralized strategies are derived through the consistency condition. In the first class, the individual agents of large-population system are weakly coupled in their state dynamics and the full information can be accessible to all agents. In the second class, the coupling structure lies in the cost functional with only partial information structure. In both classes, the asymptotic near-optimality property (namely, ϵ-Nash equilibrium) of decentralized strategies are verified. To this end, some estimates to BSDE, are presented in the large-population setting.

Index Terms

BSDE, decentralized control, ϵ-Nash equilibrium, full information, large-population system, mean-field LQG games, partial information

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I. Introduction

In recent years, the dynamic optimization or control of stochastic large-population (also called multi-agent) system has attracted consistent and intense attentions by research communities. The agents (or players) in large population system are individually negligible but their collective behaviors will impose some significant impact on all agents. This feature can be captured by the weakly-coupling structure in the individual dynamics and cost functionals through the state-average. In this way, the individual behaviors of all agents in micro-scale, can be connected to their mass effects in the macro-scale. The large population systems arise naturally in various different fields (e.g., engineering, social science, economics and finance, operational research and management, etc.). The interested readers may refer [15], [16], [18] and the reference therein for more details of their solid backgrounds and real applications. In the controlled large population system, it is intractable for a given agent to collect the "central" or "global" information of all agents due to the highly complex interactions among its peers. Consequently, the centralized controls, which are built upon the full information of all agents' states, are not implementable and not efficient in large population framework. Alternatively, it is more reasonable and effective to study the decentralized strategies which depend on the local information only. By "local information", we mean the optimal control regulator for a given agent, is designed on its own individual state and some quantity which can be computed in off-line manner. In this regard, one powerful technique is the so-called mean-field games (see, e.g., [21]). Its main idea is to approximate the initial large population control problem by its limiting problem through some mean-field term (i.e., the asymptotic limit of state-average). Some recent literature can be found in [3], [5], [13], [16], [19], [20], [22] for the study of mean-field games; [17] for cooperative social optimization; [15], [28] and [29] and references therein for models with a major player; [1], [7] and [34] for optimal control with a mean term in the dynamics and cost, etc.

The main novelty of this paper is to study the *backward* mean-field LQG games of large population systems for which the individual states follow some backward stochastic differential equations (BSDEs). This feature makes our setting very different to existing works of mean-field LQG games wherein the individual states evolve by some forward stochastic differential equations (SDEs). Different to SDE, the terminal instead initial condition of BSDE should be specified as the priori. As a consequence, the BSDE will admit one adapted solution pair (y_t, z_t)

where the second solution component z_t (it is also called the diffusion component) is naturally presented here due to the martingale representation and the adaptiveness requirement. The linear BSDEs are introduced in [6] and the general nonlinear BSDEs are first introduced in [30]. Based on them, the study of BSDE has experienced intense discussions and it has been found many applications in different areas. For instance, the BSDE has been found to be very important to characterize the nonlinear expectation in decision making, or the stochastic differential recursive utility (say, [10]). Later, [11] presents many applications of BSDE in mathematical finance and optimal control theory.

As the BSDE are well-defined stochastic systems with broad-range applications, it is very natural to study its dynamic optimization in large-population setup. Indeed, the dynamic optimization of backward large population system is inspired by a variety of scenarios. For example, the dynamic economic models for which the participants are of some recursive utilities or nonlinear expectations, or some production planning problems with some tracking terminal objectives but affected by the market price via production average. Another example arises from the risk management when considering the relative or comparable criteria based on the average performance of all other peers through the whole sector. This is the case for a given pension fund to evaluate its own performance by setting the average performance (say, average hedging cost or initial deposit, surplus) as its benchmark. In addition, the controlled forward large population systems, which are subjected to some terminal constraints, can be reformulated by some backward large population systems, as motivated by [24]. Inspired by above mentioned motivations, this paper studies the backward mean-field linear-quadratic-Gaussian (BMFLQG) games. In particular, two classes of backward large population systems are formulated: in the first class, the agents are coupled in their state dynamics and the full information structure is assumed; in the second class, the agents are coupled via their cost functionals to be minimized and only partial information is accessible. Moreover, the state-average limit in partial information setup turns out to be some stochastic process.

The rest of this paper is organized as follows. In Section II, we introduce two classes of mean-field backward differential games. As to the first class, the individual state dynamics are coupled through the state-average and the full information (FI) structure is assumed thus the individual agent can access the central information of all other agents. In the second class of backward mean-field games, the individual agents are coupled through their cost functionals and the partial information (PI) structure is formulated there. Section III aims to study the explicit form of the limiting process and ϵ-Nash equilibrium of the decentralized control strategy in the full information (FI) case. Section IV will give the explicit form of limiting process and ϵ-Nash equilibrium of the decentralized control strategy in the partial information (PI) case. Section V is the conclusion of our work.

II. Formulation of Backward Mean-Field LQG Games

Throughout this paper, $\mathbb{R}^m$ denotes the m-dimensional Euclidean space, $\|\cdot\|$ its norm. Let $C(0,T;\mathbb{R}^m)$ be the space of all continuous functions defined on $[0,T]$ with values in $\mathbb{R}^m$; $L^2(0,T;\mathbb{R}^m)$ the space of all deterministic functions on $[0,T]$ with values in $\mathbb{R}^m$ satisfying $\int_0^T |x(t)|^2 dt < \infty$; $L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ ($L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times n})$) the space of all $\mathcal{F}_t$-progressively measurable processes with values in $\mathbb{R}^m$ ($\mathbb{R}^{m\times n}$) satisfying $\mathbb{E}\int_0^T |x(t)|^2 dt < \infty$. Here $\mathcal{F}_t$ is some filtration depending on the (full or partial) information structure we set.

The information structure of our large population system can be described as follows. First, introduce $(\Omega, \mathcal{F}, P)$ the complete probability space on which a standard $(d+m\times N)$-dimensional Brownian motion $\{W(t), W_i(t),\ 1 \le i \le N\}_{0\le t\le T}$ is defined. Here, N stands for the population size of our large population system. Depending on which problems to be addressed, we have different setup to the information structure. In case of full information (see Section II.A), we denote by $\mathcal{F}_t = \bigvee_{i=1}^N \mathcal{F}_t^{w_i}$ the full information of large population system where $\mathcal{F}_t^{w_i} = \sigma\{W_i(s); 0 \le s \le t\}$ is the natural filtration generated by i^{th} Brownian motion W_i but augmented by all P-null sets. In case of partial information (see Section II.B), we let $\mathcal{G}_t = \mathcal{F}_t \bigvee \mathcal{F}_t^w$ denote the complete information of large population system. In particular, $\mathcal{F}_t = \bigvee_{i=1}^N \mathcal{F}_t^{w_i}$ the information accessible to all agents but $\mathcal{F}_t^w = \sigma\{W(s); 0 \le s \le t\}$ the information of some underlying process which can't be directly observed by our agents (say, some latent marco-economic process, or hidden action process). Now we are ready to formulate our backward mean-field LQG games.

A. Full information with coupling in state dynamics

Now, we first introduce the backward mean-field LQG games in which the large population system is weakly-coupled in the states of individual agents. For short, the problem is given by

$$\textbf{(FI)}\begin{cases}\text{state}:\begin{cases}-dy_i(t)=\big[Ay_i(t)+Bu_i(t)+Cy^{(N)}(t)\big]dt-z_i(t)dW_i(t)\\ \qquad\qquad -\sum\limits_{j=1,j\neq i}^{N}z_{ij}(t)dW_j(t),\\ y_i(T)=\xi_i,\end{cases}\\ \text{cost functional}:\mathcal{J}_i\big(u_i(\cdot),u_{-i}(\cdot)\big)=\mathbb{E}\left[\int_0^T Ru_i^2(t)dt+Hy_i^2(0)\right].\end{cases}\tag{1}$$

Here, we assume the full information (hence (FI) for short) structure. That is, each agent can access the states of all other agents; the dynamics of agent $\mathcal{A}_i$ is denoted by y_i which satisfies the above controlled linear backward stochastic differential equation (LBSDE). It is remarkable that $(z_i, z_{ij}, 1\leq j\leq N, j\neq i)$ is also part of our solution of (1) which are introduced here to enable y_i to satisfy the adaptation requirement; A, B, C are scalar constants, $R>0, H\geq 0$; $y^{(N)}(t)=\frac{1}{N}\sum\limits_{i=1}^{N}y_i(t)$ is the state average across the whole population. It stands for the global population effects in macro-scale. $\xi_i\in\mathcal{F}_T,\ i=1,2,\cdots,N$, are the terminal conditions for individual agents which stand for the future objective or tracking target. Let $U_i,\ i=1,2,\cdots,N$ be subsets of $\mathbb{R}$. The admissible control $u_i\in\mathcal{U}_i$ where the admissible control set $\mathcal{U}_i$ is defined as

$$\mathcal{U}_i:=\Big\{u_i\big|u_i(t)\in U_i, 0\leq t\leq T;\ u_i(\cdot)\in L^2_{\mathcal{F}_t}(0,T;\mathbb{R})\Big\},\ 1\leq i\leq N.$$

Let $u=(u_1,\cdots,u_i,\cdots,u_N)$ denote the set of control strategies of all N agents; $u_{-i}=(u_1,\cdots,u_{i-1},\ u_{i+1},\cdots u_N)$ the control strategies except the i^{th} agent $\mathcal{A}_i$. Here, we write the cost functional as $\mathcal{J}_i(u_i,u_{-i})$ to emphasize that it depends on both u_i and u_{-i} due to the weakly coupling structure in dynamics.

In full information structure, we make the following assumption:

(H1) The terminal conditions $\{\xi_i\}_{i=1}^N$ are independent identically distributed (i.i.d) with $\mathbb{E}|\xi_i|^2<+\infty$.

It follows that under (H1), the state equation in (1) admits a unique solution for all $u_i\in\mathcal{U}_i$. In fact, if we denote by

$$Y=\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, U=\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, Z=\begin{pmatrix} z_1 & z_{12} & \cdots & z_{1,N-1} & z_{1N} \\ z_{21} & z_2 & \cdots & z_{2,N-1} & z_{2N} \\ \vdots & \vdots & & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{N,N-1} & z_N \end{pmatrix}, \tilde{W}=\begin{pmatrix} W_1 \\ \vdots \\ W_N \end{pmatrix},$$

$$\Xi=\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, J_N=\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

Then the state equation in (1) can be rewritten as

$$-dY(t)=\Big[AY(t)+BU(t)+\frac{C}{N}J_NY(t)\Big]dt-Z(t)d\tilde{W}(t), \quad Y(T)=\Xi$$

which is a LBSDE of vector value and admits a unique solution $(Y,Z)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^N)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{N\times N})$ for $U\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^N)$, (see [30]). Thus, for any $1\le i\le N$, the state equation in (1) admits a unique solution $\big(y_i,z_i,z_{ij}(j\neq i)\big)\in L^2_{\mathcal{F}}(0,T;\mathbb{R})\times\cdots\times L^2_{\mathcal{F}}(0,T;\mathbb{R})$.

Remark 2.1: (i) We now give some remarks to the real meaning of system (1). In reality, the LBSDE in (1) stands for the dynamics of some investment behaviors such as in stocks and bonds in a self-financed market, that is, there is no infusion or withdrawal of funds over $[0,T]$. In recursive or hedging problems (finance, optimal control, etc.), the BSDE dynamics have been deeply studied in the existing literature, such as [11], [33] and so on. The cost used to be applied in some terminal hedging problems with possible nonlinear expectation, taking mean variance model as an example. Besides, the constrained forward LQ control problem with state average coupling in state dynamics can also be transferred to the backward LQ control with state given by the linear BSDE, as given in (1).

(ii) For simplicity of analysis, the state average in system (1) is coupled in dynamics only. Actually, our analysis can be extended to the problem with coupling in cost functional. Applying similar procedures, we can obtain the optimal control by virtue of the corresponding fixed point principle, and then analyze the properties of ϵ-Nash equilibrium.

(iii) In this system, there are N individual agents coupled together to be investigated for the hedging strategies. Actually, problems to get optimal strategies in forward setup with small players have been well studied by the existing literature, including [12], [16], [17], [18], etc.

In this setting, we analyze the limit when the number of players N goes to infinity where the situation considerably simplifies in the spirit of mean-field games, see [21].

B. Partial information with coupling in cost functional

In some case, it is very natural to consider the backward LQG games with coupling in their cost functionals. To this end, we formulate the following backward mean-field LQG games in which the large population system is weakly-coupled in the cost functional :

$$\textbf{(PI)}\begin{cases}\text{state}:\begin{cases}-dy_i(t)=\big[Ay_i(t)+Bu_i(t)\big]dt-z_i(t)dW_i(t)-\tilde{z}_i(t)dW(t),\\ \quad y_i(T)=\eta_i,\end{cases}\\ \text{cost functional}:\mathcal{J}_i\big(u_i(\cdot),u_{-i}(\cdot)\big)=\mathbb{E}\left[\int_0^T Ru_i^2(t)dt+2y_i(0)\big(\alpha-\beta y^{(N)}(0)\big)\right].\end{cases}\qquad(2)$$

Here, we assume the partial information structure in which each agent can not access the underlying state process driven by $\{W(t),0\le t\le T\}$; A,B are scalar constants, $R>0,\alpha\ge0,\beta\ge0$; $\eta_i\in\mathcal{G}_T,\ i=1,2,\cdots,N,$ are the terminal conditions for individual agents; $y^{(N)}(t)=\frac{1}{N}\sum\limits_{i=1}^{N}y_i(t)$ is the state average, $y^{(N)}(0)$ is its initial value. Let $V_i,\ i=1,2,\cdots,N$ be subsets of $\mathbb{R}$. The admissible control $u_i\in\mathcal{V}_i$ is defined as

$$\mathcal{V}_i:=\Big\{u_i\big|u_i(t)\in V_i,0\le t\le T;\ u_i(\cdot)\in L^2_{\mathcal{F}_t}(0,T;\mathbb{R})\Big\},\ 1\le i\le N.$$

Let $u=(u_1,\cdots,u_i,\cdots,u_N)$ denote the set of control strategies of all N agents; $u_{-i}=(u_1,\cdots,u_{i-1},\ u_{i+1},\cdots u_N)$ the control strategies except the i^{th}agent $\mathcal{A}_i$.

In partial information structure, we make the following assumption:

(H2) $\{\eta_i\}_{i=1}^N$ are conditional independent and identically conditional distributed w.r.t. $\mathcal{F}_T^w$ with $\mathbb{E}|\eta_i|^2<+\infty$. Moreover, the distribution of each η_i is not depending on i and N.

It follows that under (H2), the state equation in (2) admits a unique solution $(y_i,z_i,\tilde{z}_i)\in L^2_{\mathcal{G}}(0,T;\mathbb{R})\times L^2_{\mathcal{G}}(0,T;\mathbb{R})\times L^2_{\mathcal{G}}(0,T;\mathbb{R})$ for all $u_i\in\mathcal{V}_i$. In fact, the uniqueness is obtained by [30] directly in partial information framework. Noting the identically conditional distributions of $\{\eta_i\}_{i=1}^N$ in (H2), it is easy to obtain that $\mathbb{E}\big(\eta_1|\mathcal{F}_T^w\big)=\cdots=\mathbb{E}\big(\eta_N|\mathcal{F}_T^w\big)$, which is denoted by $\eta\in\mathcal{F}_T^w$. Then applying the results of [27], we get that conditionally on $\mathcal{F}_T^w$, $\frac{1}{N}\sum\limits_{i=1}^{N}\eta_i\to\eta,\ a.s.,$ as $N\to+\infty$. It is worth pointing out that if η_i has the following linear or nonlinear structure, $\{\eta_i\}_{i=1}^N$ satisfy (H2) easily: $\eta_i=\alpha_i+\beta$ or $\eta_i=\phi(\alpha_i,\beta)$, where $\alpha_i\in\mathcal{F}_T^{w_i},i=1,\cdots,N,$ $\beta\in\mathcal{F}_T^w$, and $\phi(\cdot)$ is a deterministic function. And $\frac{1}{N}\sum\limits_{i=1}^{N}\eta_i\to\mathbb{E}\alpha_1+\beta$ or $\mathbb{E}\big(\phi(\alpha_1,\beta)|\mathcal{F}_T^w\big)\ a.s.,$ as $N\to+\infty$.

Remark 2.2: (i) We now present some remarks to the real meaning of system (2). In reality, the LQ BSDE system stands for the benchmark tracking problem with portfolio selection in financial market. If a given portfolio strategy emphasizes one aspect or one product, it will be adjusted by considering the whole behaviors throughout the market.

(ii) In this system, the state average is not coupled in dynamics. There are two reasons. The first reason is from practical point: the coupling in cost functional arise naturally when we consider the relative (investment) performance (see e.g., [12]). The second reason is more technical: in partial information structure, the optimal control involves filtering equations and this always leads to considerable interrelated and complicated filter estimations. It is difficult to get similar estimated results as in the full information problem. Thus, we consider the coupled cost functional in (2) due to its financial meanings.

III. Problem (FI): Full Information and Coupling in States

Now, we study the problem **(FI)**: the backward mean-field LQG games with full information (FI). A key component in our analysis is to study the associated mean-field LQG games via limiting state average, as the number of agents tends to infinity. To obtain the desired results and the explicit feedback control, we assume $U_i = \mathbb{R}$ for $i = 1, 2, \cdots, N$.

A. The optimal control of (LFI)

We assume $y^{(N)}$ is approximated by a deterministic continuous function y^0 given by

$$\begin{cases} -dy^0(t) = [\widetilde{A}(t)y^0(t) + m(t)]dt, \\ \quad y^0(T) = \xi_0 \end{cases} \tag{3}$$

where ξ_0 is some deterministic constant, $\widetilde{A}(t)$ and $m(t)$ are some continuous functions to be determined. Actually, by (H1) and law of large numbers (LLN), $\lim\limits_{N\to+\infty} \xi^{(N)}$ exists and ξ_0 is determined by

$$\xi_0 = \lim_{N\to+\infty} \xi^{(N)} = \mathbb{E}\xi_i, \ i = 1, 2, \cdots, N \tag{4}$$

where $\xi^{(N)} = \frac{1}{N}\sum\limits_{i=1}^{N} \xi_i$. Now, we introduce the limiting full-information system

$$\begin{cases} -dy_i(t) = \big[Ay_i(t) + Bu_i(t) + Cy^0(t)\big]dt - z_i(t)dW_i(t) - \displaystyle\sum_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t), \\ \quad y_i(T) = \xi_i \end{cases} \tag{5}$$

with the cost functional

$$J_i\big(u_i(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + Hy_i^2(0)\right] \tag{6}$$

where $y^0(\cdot)$ is given by (3).

Now we formulate the limiting full information (LFI) problem of our large population system as follows.

Problem (LFI). For the i^{th} agent, $i = 1, 2, \cdots, N$, find $\bar{u}_i \in \mathcal{U}_i$ satisfying

$$J_i(\bar{u}_i) = \inf_{u_i \in \mathcal{U}_i} J_i(u_i).$$

Then $\bar{u}_i$ is called the optimal control for problem **(LFI)**.

In the following, we apply the variational method to get the optimal control $\bar{u}_i$. First, introduce the variational equation

$$\begin{cases} -d\zeta_i(t) = \big[A\zeta_i(t) + B\delta u_i(t)\big]dt - \theta_i(t)dW_i(t) - \displaystyle\sum_{j=1, j\neq i}^{N} \theta_{ij}(t)dW_j(t), \\ \zeta_i(T) = 0,\ i = 1, 2, \cdots, N \end{cases} \tag{7}$$

where $\zeta_i(t) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R})$, $\delta u_i(\cdot)$ denotes the variation of $\bar{u}_i(\cdot)$. Then the following proposition holds true.

Proposition 3.1: Let (H1) hold. Then the optimal control of **(LFI)** is

$$\bar{u}_i(t) = -R^{-1}Bp_i(t)$$

where $p_i(t) \in L^2(0, T; \mathbb{R})$ satisfies the following ordinary differential equation (ODE):

$$\begin{cases} dp_i(t) = Ap_i(t)dt, \\ p_i(0) = H\bar{y}_i(0),\ i = 1, 2, \cdots, N. \end{cases} \tag{8}$$

Proof: Suppose $(\bar{y}_i, \bar{z}_i, \bar{z}_{ij}(j \neq i), \bar{u}_i)$ is an optimal solution. Then for any variation δu_i of $\bar{u}_i$, the associated first order variation of cost functional $J_i(\bar{u}_i)$ satisfies

$$0 = \frac{1}{2}\delta J_i(\bar{u}_i) = \mathbb{E}\left[\int_0^T R\delta u_i(t)\bar{u}_i(t)dt + H\zeta_i(0)\bar{y}_i(0)\right]. \tag{9}$$

Applying Itô's formula, we have

$$
\begin{aligned}
&d\big(\zeta_i(t)p_i(t)\big)\\
=&\Bigg\{-\big[A\zeta_i(t)+B\delta u_i(t)\big]dt+\theta_i(t)dW_i(t)+\sum_{j=1,j\neq i}^{N}\theta_{ij}(t)dW_j(t)\Bigg\}p_i(t)+\zeta_i(t)Ap_i(t)dt\\
=&-B\delta u_i(t)p_i(t)dt+p_i(t)\Bigg[\theta_i(t)dW_i(t)+\sum_{j=1,j\neq i}^{N}\theta_{ij}(t)dW_j(t)\Bigg].
\end{aligned}
$$

Combining this identity with $\zeta_i(T)=0$ and $p_i(0)=H\bar{y}_i(0)$ yields

$$
\mathbb{E}\big[\zeta_i(0)H\bar{y}_i(0)\big]=\mathbb{E}\int_0^T B\delta u_i(t)p_i(t)dt. \tag{10}
$$

It follows from (9)-(10) that for any $\delta u_i(\cdot)\in L^2_{\mathcal{F}^{w_i}}(0,T;\mathbb{R})$,

$$
\mathbb{E}\int_0^T\Big(R\delta u_i(t)\bar{u}_i(t)+B\delta u_i(t)p_i(t)\Big)dt=0.
$$

This implies that $\bar{u}_i(t)=-R^{-1}Bp_i(t)$. On the other hand, the sufficiency of optimal control can be proved similarly. □

B. The fixed point principle with full information

Now, we aim to study the properties of the given function $y^0(\cdot)$. For $\forall\ 1\leq i\leq N$, solving ODE (8) directly, we have

$$
p_i(t)=H\bar{y}_i(0)e^{At}.
$$

Thus, the optimal control $\bar{u}_i(t)$ is given by

$$
\bar{u}_i(t)=-R^{-1}BH\bar{y}_i(0)e^{At}. \tag{11}
$$

Applying the decentralized control law (11) for the i^{th} agent $\mathcal{A}_i$, the closed-loop state in system (1) becomes

$$
\begin{cases}
-dy_i(t)=\Big[Ay_i(t)-B^2R^{-1}Hy_i(0)e^{At}+Cy^{(N)}(t)\Big]dt-z_i(t)dW_i(t)\\
\qquad\qquad-\displaystyle\sum_{j=1,j\neq i}^{N}z_{ij}(t)dW_j(t),\\
y_i(T)=\xi_i
\end{cases} \tag{12}
$$

where $y^{(N)}(t)=\frac{1}{N}\sum_{i=1}^{N}y_i(t)$. Summing the above N equations of (12) and dividing by N, we get

$$\begin{cases} -dy^{(N)}(t) = \Big[Ay^{(N)}(t) - B^2R^{-1}He^{At}y^{(N)}(0) + Cy^{(N)}(t)\Big]dt \\ \qquad\qquad - \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\left[z_i(t)dW_i(t) + \sum_{j=1,j\neq i}^{N} z_{ij}(t)dW_j(t)\right], \\ y^{(N)}(T) = \xi^{(N)}. \end{cases} \tag{13}$$

Letting $N\to+\infty$, replacing $y^{(N)}$ by y^0 and noting (4), we obtain the following limiting system

$$\begin{cases} -dy^0(t) = \Big[\big(A+C\big)y^0(t) - B^2R^{-1}He^{At}y^0(0)\Big]dt, \\ y^0(T) = \xi_0. \end{cases} \tag{14}$$

Comparing the coefficients with (3), we have

$$\begin{cases} \tilde{A}(t) \equiv A+C, \\ m(t) = -B^2R^{-1}He^{At}y^0(0). \end{cases} \tag{15}$$

Solving the ODE (3), we get

$$y^0(t) = \xi_0 e^{\int_t^T \tilde{A}(s)ds} + \int_t^T m(s)e^{\int_t^s \tilde{A}(u)du}ds.$$

Taking $t=0$ and noting (15), we have

$$y^0(0) = \xi_0 e^{(A+C)T} + \int_0^T m(s)e^{(A+C)s}ds.$$

Thus, $m(t)$ in (15) has the following expression:

$$m(t) = -B^2R^{-1}He^{At}\xi_0 e^{(A+C)T} - B^2R^{-1}He^{At}\int_0^T m(s)e^{(A+C)s}ds. \tag{16}$$

We have the following explicit representation of $m(t)$. As a sequel, $y^0(\cdot)$ in (3) can be determined.

Proposition 3.2: $m(\cdot)$ can be explicitly solved as

$$m(t) = \begin{cases} -\frac{B^2H(2A+C)\xi_0 e^{At+(A+C)T}}{R(2A+C)+B^2H\left(e^{(2A+C)T}-1\right)}, & \text{if } 2A+C\neq 0; \\ -\frac{B^2H[R+B^2H(T-1)]\xi_0 e^{-A(T-t)}}{R(R+B^2HT)}, & \text{if } 2A+C=0. \end{cases} \tag{17}$$

Proof: Denote $K := \int_0^T m(s)e^{(A+C)s}ds$, which is a constant depending on T. Then (16) can be rewritten as

$$m(t) = -B^2R^{-1}He^{At}\xi_0 e^{(A+C)T} - B^2R^{-1}He^{At}K.$$

Multiplying with $e^{(A+C)t}$ on both sides and taking integral from 0 to T w.r.t t, we have

$$\begin{aligned}K &= \int_0^T m(t)e^{(A+C)t}dt \\ &= -B^2R^{-1}H\xi_0 e^{(A+C)T}\int_0^T e^{(2A+C)t}dt - B^2R^{-1}HK\int_0^T e^{(2A+C)t}dt.\end{aligned}$$

Then we get

$$K = \begin{cases} -\frac{B^2H\xi_0 e^{(A+C)T}\left(e^{(2A+C)T}-1\right)}{R(2A+C)+B^2H\left(e^{(2A+C)T}-1\right)}, & \text{if } 2A+C\neq 0; \\ -\frac{B^2H\xi_0 e^{-AT}}{R+B^2HT}, & \text{if } 2A+C=0. \end{cases}$$

Thus, (17) is obtained. Noting (15), $y^0(\cdot)$ is also determined. □

Remark 3.1: (i) By Proposition 3.2, it follows that there exists a unique deterministic function y^0 in $C(0,T;\mathbb{R})$ to approximate the state average $y^{(N)}$. Applying the limiting function y^0, we get the optimal control for **(LFI)**, which plays an important role in obtaining the decentralized control and analyzing the properties of ϵ-Nash equilibrium.

(ii) Actually, in (17) if $2A+C>0(<0)$, $e^{(2A+C)T}-1>0(<0)$. Noting $R>0, H\geq 0$, we get $R(2A+C)+B^2H\left(e^{(2A+C)T}-1\right)>0(<0)$. Meanwhile, we have $R(R+B^2HT)>0$. Thus, the representation (17) is meaningful.

C. ϵ-Nash equilibrium for (FI)

In previous sections, we obtained the optimal control $\bar{u}_i(\cdot), 1\leq i\leq N$ of **(LFI)**. In this section, we analyze the asymptotic property of the decentralized control strategies and verify the ϵ-Nash equilibrium property for **(FI)**. To start, we first address the definition of ϵ-Nash equilibrium.

Definition 3.1: For $\epsilon\geq 0$, a set of controls $u_k\in\mathcal{U}_k,\ 1\leq k\leq N$, for N agents is called an ϵ-Nash equilibrium with respect to the costs $\mathcal{J}_k,\ 1\leq k\leq N$, if for any fixed $1\leq i\leq N$,

$$\mathcal{J}_i(u_i,u_{-i})\leq\mathcal{J}_i(u_i',u_{-i})+\epsilon \tag{18}$$

when any alternative control $u_i'\in\mathcal{U}_i$ is applied by $\mathcal{A}_i$.

Now, we state one main result of this paper and its proof will be given later.

Theorem 3.1: Let (H1) hold. Then $(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N)$ satisfies the ϵ-Nash equilibrium of **(FI)**, with ϵ is of order $1/\sqrt{N}$. Here, for $1\leq i\leq N$, $\bar{u}_i$ is given by

$$\bar{u}_i(t)=-R^{-1}BHy^0(0)e^{At}. \tag{19}$$

Before proving the theorem, some analysis is needed. Applying the optimal control (11) to (5), we have

$$\begin{cases} -d\bar{y}_i(t) = \big[A\bar{y}_i(t) - B^2R^{-1}H\bar{y}_i(0)e^{At} + Cy^0(t)\big]dt - \bar{z}_i(t)dW_i(t) - \sum_{j=1,j\neq i}^{N} \bar{z}_{ij}(t)dW_j(t), \\ \bar{y}_i(T) = \xi_i. \end{cases}$$

Taking expectation and solving the corresponding backward ODE, we get

$$\mathbb{E}\bar{y}_i(t) = \xi_0 e^{A(T-t)} - \int_t^T \Big[BR^{-1}BH\bar{y}_i(0)e^{As} - Cy^0(s)\Big]e^{A(s-t)}ds.$$

Taking $t = 0$ and noting $\bar{y}_i(0) = \mathbb{E}\bar{y}_i(0)$, we obtain

$$\bar{y}_i(0) = \Big[1 + \frac{B^2H}{2AR}\big(e^{2AT} - 1\big)\Big]^{-1}\Big[\xi_0 e^{AT} + C\int_0^T y^0(s)e^{As}ds\Big].$$

Thus, $\bar{y}_i(0)$ is a constant which can be determined by $y^0(\cdot)$ and ξ_0. Further, we have $\bar{y}_i(0) = y^0(0),\ i = 1, 2, \cdots, N$. For simplicity, we use the notation $y^0(0)$ in $\bar{u}_i(\cdot)$ instead of $\bar{y}_i(0)$ hereafter. Now, we formulate the dynamic systems as follows

$$\begin{cases} -dy_i(t) = \big[Ay_i(t) - B^2R^{-1}Hy^0(0)e^{At} + Cy^{(N)}(t)\big]dt - z_i(t)dW_i(t) \\ \qquad\qquad - \sum_{j=1,j\neq i}^{N} z_{ij}(t)dW_j(t), \\ y_i(T) = \xi_i \end{cases} \tag{20}$$

and

$$\begin{cases} -d\bar{y}_i(t) = \big[A\bar{y}_i(t) - B^2R^{-1}Hy^0(0)e^{At} + Cy^0(t)\big]dt - \bar{z}_i(t)dW_i(t) \\ \qquad\qquad - \sum_{j=1,j\neq i}^{N} \bar{z}_{ij}(t)dW_j(t), \\ \bar{y}_i(T) = \xi_i. \end{cases} \tag{21}$$

Then we have

Lemma 3.1:

$$\sup_{0\le t\le T} \mathbb{E}\Big|y^{(N)}(t) - y^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{22}$$

$$\sup_{1\le i\le N}\Big[\sup_{0\le t\le T} \mathbb{E}\Big|y_i(t) - \bar{y}_i(t)\Big|^2\Big] = O\Big(\frac{1}{N}\Big). \tag{23}$$

Proof: By (20) and (14), we have

$$\begin{cases} -d\big(y^{(N)}(t)-y^{0}(t)\big) = \big[(A+C)(y^{(N)}(t)-y^{0}(t))\big]dt \\ \qquad\qquad -\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\Big[z_i(t)dW_i(t)+\sum_{j=1,j\neq i}^{N} z_{ij}(t)dW_j(t)\Big], \\ y^{(N)}(T)-y^{0}(T) = \xi^{(N)}-\xi_0. \end{cases} \tag{24}$$

Introduce a 1-dimensional dual process $X(s,t)$ for (24), which satisfies

$$\begin{cases} dX(s,t) = (A+C)X(s,t)ds, \\ X(t,t) = 1,\ t\le s\le T. \end{cases}$$

$X(s,t)$ is deterministic and belongs to $L^2(0,T;\mathbb{R})$. Applying Itô's formula to $\langle y^{(N)}(s)-y^{0}(s), X(s,t)\rangle$, we get

$$y^{(N)}(t)-y^{0}(t) = X(T,t)\mathbb{E}\big(\xi^{(N)}-\xi_0\big|\mathcal{F}_t\big).$$

By (H1), we have

$$\mathbb{E}\Big|\xi^{(N)}-\xi_0\Big|^2 = \mathbb{E}\Big|\frac{1}{N}\sum_{i=1}^{N}\xi_i-\xi_0\Big|^2 = O\Big(\frac{1}{N}\Big).$$

Then (22) follows. Noting (20) and (21), applying the similar method, we can get (23). □

Lemma 3.2: For $\forall\ 1\le i\le N$,

$$\Big|\mathcal{J}_i(\bar u_i,\bar u_{-i})-J_i(\bar u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Proof: For $\forall\ 1\le i\le N$, by (21), we get $\sup\limits_{0\le t\le T}\mathbb{E}\big|\bar y_i(t)\big|^2<+\infty$. Applying Cauchy-Schwarz inequality and noting (23), we have

$$\begin{aligned} &\sup_{0\le t\le T}\mathbb{E}\Big||y_i(t)|^2-|\bar y_i(t)|^2\Big| \\ \le&\sup_{0\le t\le T}\mathbb{E}\Big|y_i(t)-\bar y_i(t)\Big|^2+2\Big(\sup_{0\le t\le T}\mathbb{E}|\bar y_i(t)|^2\Big)^{\frac12}\Big(\sup_{0\le t\le T}\mathbb{E}|y_i(t)-\bar y_i(t)|^2\Big)^{\frac12}, \\ =&O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned}$$

Further,

$$\mathbb{E}\Big||y_i(0)|^2-|\bar y_i(0)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Then

$$\Big|\mathcal{J}_i(\bar u_i,\bar u_{-i})-J_i(\bar u_i)\Big|\le H\mathbb{E}\Big|y_i^2(0)-\bar y_i^2(0)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big),$$

which completes the proof. $\square$

Now, we have addressed some estimates of states and costs corresponding to control $\bar{u}_i$,$1 \leq i \leq N$. Our remaining analysis is to prove the control strategies set $(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N)$ is an ϵ-Nash equilibrium for **(FI)**. For any fixed i, $1 \leq i \leq N$, consider an admissible alternative control $u_i \in \mathcal{U}_i$ for $\mathcal{A}_i$ and introduce the dynamics

$$\begin{cases} -dx_i(t) = \big[Ax_i(t) + Bu_i(t) + Cx^{(N)}(t)\big]dt - q_i(t)dW_i(t) - \displaystyle\sum_{k=1,k\neq i}^{N} q_{ik}(t)dW_k(t), \\ x_i(T) = \xi_i \end{cases} \tag{25}$$

whereas other agents keep the control $\bar{u}_j, 1 \leq j \leq N, j \neq i$, i.e.,

$$\begin{cases} -dx_j(t) = \big[Ax_j(t) - B^2R^{-1}Hy^0(0)e^{At} + Cx^{(N)}(t)\big]dt - q_j(t)dW_j(t) \\ \qquad\qquad - \displaystyle\sum_{k=1,k\neq j}^{N} q_{jk}(t)dW_k(t), \\ x_j(T) = \xi_j \end{cases} \tag{26}$$

where $x^{(N)}(t) = \frac{1}{N}\sum\limits_{j=1}^{N} x_j(t)$.

If $\bar{u}_i,\ 1 \leq i \leq N$ is an ϵ-Nash equilibrium with respect to the cost $\mathcal{J}_i$, we have

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \geq \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i, \bar{u}_{-i}) \geq \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - \epsilon.$$

Then, when making the perturbation, we just need to consider $u_i \in \mathcal{U}_i$ such that $\mathcal{J}_i(u_i, \bar{u}_{-i}) \leq \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i})$, which implies

$$\mathbb{E}\int_0^T Ru_i^2(t)dt \leq \mathcal{J}_i(u_i, \bar{u}_{-i}) \leq \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) = J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big),$$

i.e.,

$$\mathbb{E}\int_0^T u_i^2(t)dt \leq C_0, \tag{27}$$

where C_0 is a positive constant which is independent of N.

Proposition 3.3: $\sup\limits_{1\leq i\leq N}\Big[\sup\limits_{0\leq t\leq T}\mathbb{E}|x_i(t)|^2\Big]$ is bounded.

Proof: By (25) and (26), it holds that

$$\begin{aligned} &\mathbb{E}\Big\{|x_i(t)|^2 + \int_t^T |q_i(s)|^2 ds + \int_t^T \sum_{k=1,k\neq i}^{N} |q_{ik}(s)|^2 ds\Big\} \\ \leq & C_1\mathbb{E}\Big\{|\xi_i|^2 + \int_t^T \Big[|x_i(s)|^2 + |u_i(s)|^2 + \frac{1}{N}\sum_{k=1}^{N}|x_k(s)|^2\Big]ds\Big\} \end{aligned}$$

and for $j \neq i$,

$$\mathbb{E}\left\{|x_j(t)|^2 + \int_t^T |q_j(s)|^2 ds + \int_t^T \sum_{k=1,k\neq j}^{N} |q_{jk}(s)|^2 ds\right\}$$
$$\leq C_1 \mathbb{E}\left\{|\xi_j|^2 + \int_t^T \left[|x_j(s)|^2 + |\bar{u}_j(s)|^2 + \frac{1}{N}\sum_{k=1}^{N}|x_k(s)|^2\right] ds\right\}$$

where C_1 is a positive constant. Thus,

$$\mathbb{E}\left[\sum_{k=1}^{N}|x_k(t)|^2\right] \leq C_1\left\{\mathbb{E}\left[\sum_{k=1}^{N}|\xi_k|^2\right] + \mathbb{E}\int_t^T \left[2\sum_{k=1}^{N}|x_k(s)|^2 + |u_i(s)|^2 + \sum_{k=1,k\neq i}^{N}|\bar{u}_k(s)|^2\right] ds\right\}.$$

By (27), we can see $u_i(t)$ is L^2-bounded. Besides, the optimal controls $\bar{u}_k(t), k \neq i$ are L^2-bounded. Then by Gronwall's inequality, we get

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\sum_{k=1}^{N}|x_k(t)|^2\right] = O(N),$$

and for any $1 \leq i \leq N$, $\sup\limits_{0\leq t\leq T} \mathbb{E}|x_i(t)|^2$ is bounded. □

For the i^{th} agent $\mathcal{A}_i$, consider the perturbation in **(LFI)** and introduce a new system

$$\begin{cases} -dx_i^0(t) = \left[Ax_i^0(t) + Bu_i(t) + Cy^0(t)\right]dt - q_i^0(t)dW_i(t) - \sum\limits_{k=1,k\neq i}^{N} q_{ik}^0(t)dW_k(t), \\ x_i^0(T) = \xi_i \end{cases} \tag{28}$$

and for the j^{th} agent $\mathcal{A}_j, j \neq i$,

$$\begin{cases} -d\bar{x}_j(t) = \left[A\bar{x}_j(t) - B^2R^{-1}Hy^0(0)e^{At} + Cy^0(t)\right]dt - \bar{q}_j(t)dW_j(t) \\ \qquad\qquad - \sum\limits_{k=1,k\neq j}^{N} \bar{q}_{jk}(t)dW_k(t), \\ \bar{x}_j(T) = \xi_j. \end{cases} \tag{29}$$

In order to obtain necessary estimates for **(FI)** and **(LFI)**, we need introduce some intermediate states as follows

$$\begin{cases} -d\check{x}_i(t) = \left[A\check{x}_i(t) + Bu_i(t) + \dfrac{N-1}{N}C\check{x}^{(N-1)}(t)\right]dt - \check{q}_i(t)dW_i(t) \\ \qquad\qquad - \sum\limits_{k=1,k\neq i}^{N} \check{q}_{ik}(t)dW_k(t), \\ \check{x}_i(T) = \xi_i \end{cases} \tag{30}$$

and for $j \neq i$,

$$\begin{cases} -d\check{x}_j(t) = & \Big[A\check{x}_j(t) - B^2R^{-1}Hy^0(0)e^{At} + \frac{N-1}{N}C\check{x}^{(N-1)}(t)\Big]dt - \check{q}_j(t)dW_j(t) \\ & - \sum_{k=1,k\neq j}^{N} \check{q}_{jk}(t)dW_k(t), \\ \check{x}_j(T) = & \xi_j \end{cases} \tag{31}$$

where $\check{x}^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} \check{x}_j(t)$.

Denote $x^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_j(t)$, by (26) and (31), we get

$$\begin{cases} -dx^{(N-1)}(t) = & \Big[\Big(A + \frac{N-1}{N}C\Big)x^{(N-1)}(t) - B^2R^{-1}Hy^0(0)e^{At} + \frac{C}{N}x_i(t)\Big]dt \\ & - \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\Big[q_j(t)dW_j(t) + \sum_{k=1,k\neq j}^{N} q_{jk}(t)dW_k(t)\Big], \\ x^{(N-1)}(T) = & \xi^{(N-1)} \end{cases} \tag{32}$$

and

$$\begin{cases} -d\check{x}^{(N-1)}(t) = & \Big[\Big(A + \frac{N-1}{N}C\Big)\check{x}^{(N-1)}(t) - B^2R^{-1}Hy^0(0)e^{At}\Big]dt \\ & - \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\Big[\check{q}_j(t)dW_j(t) + \sum_{k=1,k\neq j}^{N} \check{q}_{jk}(t)dW_k(t)\Big], \\ \check{x}^{(N-1)}(T) = & \xi^{(N-1)} \end{cases} \tag{33}$$

where $\xi^{(N-1)} = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} \xi_j$.

We have the following estimates.

Proposition 3.4:

$$\sup_{0\leq t\leq T} \mathbb{E}\Big|x^{(N-1)}(t) - \check{x}^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N^2}\Big), \tag{34}$$

$$\sup_{0\leq t\leq T} \mathbb{E}\Big|x^{(N)}(t) - x^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{35}$$

$$\sup_{0\leq t\leq T} \mathbb{E}\Big|\check{x}^{(N-1)}(t) - y^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{36}$$

Proof: By (32) and (33), we have

$$\begin{cases} -d\Big(x^{(N-1)}(t)-\check{x}^{(N-1)}(t)\Big)=\Bigg[\Big(A+\dfrac{N-1}{N}C\Big)\big(x^{(N-1)}(t)-\check{x}^{(N-1)}(t)\big)+\dfrac{C}{N}x_i(t)\Bigg]dt \\ \qquad\qquad -\dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\Bigg[\Big(q_j(t)-\check{q}_j(t)\Big)dW_j(t)+\sum_{k=1,k\neq j}^{N}\Big(q_{jk}(t)-\check{q}_{jk}(t)\Big)dW_k(t)\Bigg], \\ x^{(N-1)}(T)-\check{x}^{(N-1)}(T)=0. \end{cases}$$

By the estimates of BSDE, Proposition 3.3, and Gronwall's inequality, the assertion (34) holds. (35) follows from assumption (H1) and the L^2-boundness of controls $u_i(\cdot)$ and $\tilde{u}_j(\cdot), j\neq i$. By (14) and (33), making similar analysis, we get (36). □

In addition, based on Proposition 3.4, we obtain more direct estimates to prove Theorem 3.1.

Lemma 3.3:

$$\sup_{0\leq t\leq T}\mathbb{E}\Big||x_i(t)|^2-|x_i^0(t)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big), \tag{37}$$

$$\Big|\mathcal{J}_i(u_i,\bar{u}_{-i})-J_i(u_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big). \tag{38}$$

Proof: By Proposition 3.4, we get

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|x^{(N)}(t)-y^0(t)\Big|^2=O\Big(\frac{1}{N}\Big).$$

Besides, by (25) and (28), we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|x_i(t)-x_i^0(t)\Big|^2=O\Big(\frac{1}{N}\Big).$$

Noting $\sup\limits_{0\leq t\leq T}\mathbb{E}|x_i^0(t)|^2<+\infty$, applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sup_{0\leq t\leq T}\mathbb{E}\Big||x_i(t)|^2-|x_i^0(t)|^2\Big|\\ \leq&\sup_{0\leq t\leq T}\mathbb{E}|x_i(t)-x_i^0(t)|^2+2\sup_{0\leq t\leq T}\mathbb{E}|x_i^0(t)(x_i(t)-x_i^0(t))|\\ \leq&\sup_{0\leq t\leq T}\mathbb{E}|x_i(t)-x_i^0(t)|^2+2\Big(\sup_{0\leq t\leq T}\mathbb{E}|x_i^0(t)|^2\Big)^{\frac{1}{2}}\Big(\sup_{0\leq t\leq T}\mathbb{E}|x_i(t)-x_i^0(t)|^2\Big)^{\frac{1}{2}}\\ =&O\Big(\frac{1}{\sqrt{N}}\Big)\end{aligned}$$

which is (37). Further, we get

$$\mathbb{E}\Big||x_i(0)|^2-|x_i^0(0)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).$$

Thus, (38) is obtained. □

Proof of Theorem 3.1: Now, we consider the ϵ-Nash equilibrium of $\mathcal{A}_i$ for **(FI)**. Combining Lemma 3.2 and 3.3, we have

$$\begin{aligned}\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i}) =&J_i(\bar{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ \leq& J_i(u_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ =&\mathcal{J}_i(u_i,\bar{u}_{-i})+O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 3.1 follows by taking $\epsilon=O\Big(\frac{1}{\sqrt{N}}\Big)$.

IV. Problem (PI): Partial Information and Coupling in Cost

Now, we turn to study the backward mean-field LQG games with partial information **(PI)**. Similar to full information structure, we need also introduce and study the associated mean-field LQG games via limiting state average. We also assume $V_i=\mathbb{R}$ for $i=1,2,\cdots,N$.

A. The limiting control of (LPI)

Considering the large population system with partial information structure, suppose the feedback control for $\mathcal{A}_i$ takes the following feedback form on the state filters

$$u_i(t)=-a(t)\mathbb{E}\big(y_i(t)|\mathcal{F}_t^{w_i}\big)+\sum_{j=1,j\neq i}^{N}\tilde{a}(t)\mathbb{E}\big(y_j(t)|\mathcal{F}_t^{w_i}\big)+b(t) \tag{39}$$

where the regulator coefficients $a(\cdot),\tilde{a}(\cdot),b(\cdot)\in L^2(0,T;\mathbb{R})$ and $\tilde{a}(\cdot)=O(\frac{1}{N})$. Inserting (39) into the state equation in (2), we have

$$\begin{aligned}-dy_i(t)=&\Big[Ay_i(t)-Ba(t)\mathbb{E}\big(y_i(t)|\mathcal{F}_t^{w_i}\big)+B\tilde{a}(t)\sum_{j=1,j\neq i}^{N}\mathbb{E}\big(y_j(t)|\mathcal{F}_t^{w_i}\big)+Bb(t)\Big]dt\\ &-z_i(t)dW_i(t)-\tilde{z}_i(t)dW(t),\ 1\leq i\leq N.\end{aligned} \tag{40}$$

Then consider the state average, we get

$$\begin{aligned}-d\Bigg(\frac{1}{N}\sum_{i=1}^{N}y_i(t)\Bigg)=&\Bigg[A\frac{1}{N}\sum_{i=1}^{N}y_i(t)-Ba(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(y_i(t)|\mathcal{F}_t^{w_i})\\ &+B\tilde{a}(t)\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\mathbb{E}(y_j(t)|\mathcal{F}_t^{w_i})+Bb(t)\Bigg]dt\\ &-\frac{1}{N}\sum_{i=1}^{N}z_i(t)dW_i(t)-\frac{1}{N}\sum_{i=1}^{N}\tilde{z}_i(t)dW(t).\end{aligned}$$

Thus, we assume there exists a limiting process $(y^*(t), z^*(t))$, which satisfies the following backward stochastic differential equation

$$\begin{cases} -dy^*(t) = \big[Ay^*(t) + \tilde{B}(t)\mathbb{E}y^*(t) + r(t)\big]dt - z^*(t)dW(t), \\ \quad y^*(T) = \eta \end{cases} \tag{41}$$

where $\eta \in \mathcal{F}_T^w$ is obtained by (H2), $\tilde{B}(\cdot)$ and $r(\cdot) \in L^2(0,T;\mathbb{R})$ are to be determined.

Now, we introduce the limiting partial-information system

$$\begin{cases} -dy_i(t) =[Ay_i(t) + Bu_i(t)]dt - z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t), \\ \quad y_i(T) =\eta_i \end{cases} \tag{42}$$

with the cost functional

$$J_i(u_i) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + 2y_i(0)\big(\alpha - \beta y^*(0)\big)\right] \tag{43}$$

where $y^*(\cdot)$ is given by (41).

Now, we formulate the limiting partial information LQG games.

Problem (LPI). For the i^{th} agent, $i = 1, 2, \cdots, N$, find $\hat{u}_i \in \mathcal{V}_i$ satisfying

$$J_i(\hat{u}_i) = \inf_{u_i \in \mathcal{V}_i} J_i(u_i).$$

Then $\hat{u}_i$ is called an optimal control of problem **(LPI)**. Further we have

Proposition 4.1: Let (H2) hold. Then the optimal control of **(LPI)** is

$$\hat{u}_i(t) = -R^{-1}Bh_i(t)$$

where $h_i(t) \in L^2(0,T;\mathbb{R})$ satisfies the following ODE:

$$\begin{cases} dh_i(t) = Ah_i(t)dt, \\ \quad h_i(0) = \alpha - \beta y^*(0), \ \ i = 1, 2, \cdots, N. \end{cases} \tag{44}$$

Proof: Similar to the proof of Proposition 3.1, applying the standard variational method, the result is obtained. We omit it. □

B. The fixed point principle with partial information

For $\forall\ 1 \leq i \leq N$, solving ODE (44) directly, we have

$$h_i(t) = \big(\alpha - \beta y^*(0)\big)e^{At}.$$

Thus, the optimal control $\hat{u}_i(t)$ is given by

$$\hat{u}_i(t) = -R^{-1}B\big(\alpha - \beta y^*(0)\big)e^{At}. \tag{45}$$

Applying the control law (45) for i^{th} agent $\mathcal{A}_i$, the closed-loop system (2) becomes

$$\begin{cases} -dy_i(t) =& \Big[Ay_i(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t), \\ y_i(T) =& \eta_i. \end{cases} \tag{46}$$

Summing the above N equations of (46) and dividing by N, we get

$$\begin{cases} -dy^{(N)}(t) =& \Big[Ay^{(N)}(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} z_i(t)dW_i(t) \\ & - \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \tilde{z}_i(t)dW(t), \\ y^{(N)}(T) =& \eta^{(N)} \end{cases} \tag{47}$$

where $\eta^{(N)} = \frac{1}{N}\sum\limits_{i=1}^{N}\eta_i$. Taking $N \to +\infty$ and noting (41), we have $\tilde{B}(t) \equiv 0$ and

$$r(t) = -B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}. \tag{48}$$

Then we rewrite (41) as

$$\begin{cases} -dy^*(t) = \big[Ay^*(t) + r(t)\big]dt - z^*(t)dW(t), \\ y^*(T) = \eta. \end{cases} \tag{49}$$

Taking expectation and solving the corresponding backward ODE, we get

$$\mathbb{E}y^*(t) = \eta_0 e^{A(T-t)} + \int_t^T r(s)e^{A(s-t)}ds$$

where $\eta_0 := \mathbb{E}\eta$. Thus,

$$y^*(0) = \mathbb{E}y^*(0) = \eta_0 e^{AT} + \int_0^T r(s)e^{As}ds.$$

Further we have

$$r(t) = -B^2R^{-1}e^{At}\bigg\{\alpha - \beta\Big[\eta_0 e^{AT} + \int_0^T r(s)e^{As}ds\Big]\bigg\}.$$

Then we have the following proposition.

Proposition 4.2: $r(\cdot)$ can be explicitly solved as

$$r(t)=\begin{cases}-\frac{2AB^2e^{At}\left(\alpha-\beta\eta_0e^{AT}\right)}{2AR-B^2\beta\left(e^{2AT}-1\right)}, & \text{if } A\neq 0,\ 2AR-B^2\beta\big(e^{2AT}-1\big)\neq 0;\\ -\frac{B^2(\alpha-\beta\eta_0)}{R-B^2\beta T}, & \text{if } A=0,\ R-B^2\beta T\neq 0.\end{cases} \tag{50}$$

Moreover, $y^*(\cdot)$ in (49) can be determined based on $r(\cdot)$.

Proof: The proof is similar to that of Proposition 3.2 and omitted. □

Remark 4.1: By Proposition 4.2 it follows that there exists a unique bounded continuous function $r(\cdot)$. Then (49) admits a unique solution $\big(y^*(\cdot), z^*(\cdot)\big)$, in which $y^*(\cdot)$ is approximated by the state average $y^{(N)}$. Applying $y^*(\cdot)$, we get the optimal control for **(LPI)**, which is important to analyze the properties of ϵ-Nash equilibrium.

C. ϵ-Nash equilibrium for (PI)

In this section, we analyze the asymptotic property of the decentralized control strategies and verify the ϵ-Nash equilibrium property for **(PI)**. To begin with, we state the main result.

Theorem 4.1: Let (H2) hold. Then the strategy set $(\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)$ satisfies the ϵ-Nash equilibrium of **(PI)**, with ϵ is of order $1/\sqrt{N}$.

Let y_i denote the state process corresponding to $\hat{u}_i$ for **(PI)**, $\hat{y}_i$ denote the state process corresponding to $\hat{u}_i$ for **(LPI)**. Note that in partial information structure, state average is coupled in cost only therefore applying $\hat{u}_i$, y_i is same to $\hat{y}_i,\ i=1,2,\cdots,N$.

Lemma 4.1:

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|y^{(N)}(t)-y^*(t)\Big|^2=O\Big(\frac{1}{N}\Big), \tag{51}$$

$$\Big|\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})-J_i(\hat{u}_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big),\ \forall\ 1\leq i\leq N. \tag{52}$$

Proof: By (47) and (49), we have

$$\begin{cases}-d\big(y^{(N)}(t)-y^*(t)\big)=A\big[y^{(N)}(t)-y^*(t)\big]dt-\dfrac{1}{N}\displaystyle\sum_{i=1}^N z_i(t)dW_i(t)\\ \qquad\qquad\qquad\qquad+\left[z^*(t)-\dfrac{1}{N}\displaystyle\sum_{i=1}^N\tilde{z}_i(t)\right]dW(t),\\ y^{(N)}(T)-y^*(T)=\eta^{(N)}-\eta.\end{cases} \tag{53}$$

Similar to the proof in Lemma 3.1, introducing a 1-dimensional dual process $X(s,t)$, which satisfies

$$\begin{cases} dX(s,t) =& AX(s,t)ds, \\ X(t,t) =& 1,\ t \leq s \leq T, \end{cases}$$

and applying Itô's formula, we get

$$y^{(N)}(t) - y^*(t) = X(T,t)\mathbb{E}\big(\eta^{(N)} - \eta \big| \mathcal{G}_t\big).$$

It is easy to obtain that

$$\mathbb{E}\Big|\eta^{(N)} - \eta\Big|^2 = \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\Big|\eta_i - \eta\Big|^2 + \frac{2}{N^2}\sum_{i<j}\mathbb{E}\big(\eta_i - \eta\big)\big(\eta_j - \eta\big).$$

Since $\mathbb{E}\Big|\eta_i - \eta\Big|^2 < +\infty$, we have $\frac{1}{N^2}\sum\limits_{i=1}^{N}\mathbb{E}\Big|\eta_i - \eta\Big|^2 = O\Big(\frac{1}{N}\Big)$. Besides, it follows that

$$\mathbb{E}\big(\eta_i - \eta\big)\big(\eta_j - \eta\big) = \mathbb{E}\Big[\mathbb{E}\big[\big(\eta_i - \eta\big)\big(\eta_j - \eta\big)\big|\mathcal{F}_T^w\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[\eta_i\eta_j\big|\mathcal{F}_T^w\big] - \eta^2\Big].$$

Under (H2), applying the results of [25] or [36], we can derive that

$$\mathbb{E}\big[\eta_i\eta_j\big|\mathcal{F}_T^w\big] = \mathbb{E}\big[\eta_i\big|\mathcal{F}_T^w\big]\mathbb{E}\big[\eta_j\big|\mathcal{F}_T^w\big] = \eta^2.$$

Thus, $\mathbb{E}\Big|\eta^{(N)} - \eta\Big|^2 = O\Big(\frac{1}{N}\Big)$ and (51) follows. In addition, note that the state equation of **(PI)** coincides with its limiting equation (42), since the state equation in (2) does not contain the state-average term $y^{(N)}$. Therefore, after applying the optimal control $\hat{u}_i$ in (45), we get that $y_i = \hat{y}_i$ P-$a.s.$. Thus, we have

$$\begin{aligned} \Big|\mathcal{J}_i(\hat{u}_i, \hat{u}_{-i}) - J_i(\hat{u}_i)\Big| \leq& 2\beta\mathbb{E}\Big[\big|\hat{y}_i(0)\big|\big|y^{(N)}(0) - y^*(0)\big|\Big] \\ =& O\Big(\frac{1}{\sqrt{N}}\Big) \end{aligned}$$

where the last equality follows by Hölder's inequality and (51). □

For any fixed i, $1 \leq i \leq N$, consider an admissible alternative control $u_i \in \mathcal{V}_i$ for the i^{th} agent $\mathcal{A}_i$ and denote the corresponding state as

$$\begin{cases} -dk_i(t) = \big[Ak_i(t) + Bu_i(t)\big]dt - n_i(t)dW_i(t) - \tilde{n}_i(t)dW(t), \\ k_i(T) = \eta_i \end{cases} \tag{54}$$

while all other agents keep the control $\hat{u}_j, 1 \leq j \leq N, j \neq i$, i.e.,

$$\begin{cases} -dk_j(t) =& \Big[Ak_j(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - n_j(t)dW_i(t) - \tilde{n}_j(t)dW(t), \\ k_j(T) =& \eta_j \end{cases} \tag{55}$$

with the cost functional

$$\mathcal{J}_i\big(u_i(\cdot),\hat{u}_{-i}(\cdot)\big)=\mathbb{E}\left[\int_0^T Ru_i^2(t)dt+2k_i(0)\big(\alpha-\beta k^{(N)}(0)\big)\right] \tag{56}$$

where $k^{(N)}(t)=\frac{1}{N}\sum\limits_{j=1}^{N}k_j(t)$, $k^{(N)}(0)=\frac{1}{N}\sum\limits_{j=1}^{N}k_j(0)$ is its initial value.

If $\hat{u}_i,\ 1\leq i\leq N$ is an ϵ-Nash equilibrium with respect to the cost $\mathcal{J}_i$, we have

$$\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})\geq\inf_{u_i\in\mathcal{V}_i}\mathcal{J}_i(u_i,\hat{u}_{-i})\geq\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})-\epsilon.$$

Then, when making the perturbation, we just need to consider $u_i\in\mathcal{V}_i$ such that $\mathcal{J}_i(u_i,\hat{u}_{-i})\leq\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})$. Besides, by (54) (55) and applying the estimates of BSDE, we obtain the L^2 boundness of $k_j, j\neq i$ and the following inequality

$$\sup_{0\leq t\leq T}\mathbb{E}|k_i(t)|^2\leq C_4\Big[1+\mathbb{E}\int_0^T|u_i(s)|^2ds\Big]$$

where $C_4=C_4(A,B)$ is a positive constant which is independent of N but depends on A,B. Then we have

$$\begin{aligned}\mathcal{J}_i\big(u_i,\hat{u}_{-i}\big)\geq&\mathbb{E}\int_0^T Ru_i^2(t)dt-2\mathbb{E}\Big[\big|k_i(0)\big|\big|\alpha-\beta k^{(N)}(0)\big|\Big]\\ \geq&\big(R-C_5\big)\mathbb{E}\int_0^T|u_i(s)|^2ds-C_6\end{aligned}$$

where $C_5=C_5(A,B,\alpha,\beta),C_6=C_6(A,B,\alpha,\beta)$ are positive constants which are independent of N. For simplicity, we introduce the following assumption:

(H3) $R>C_5$.

Then (H3) implies

$$\big(R-C_5\big)\mathbb{E}\int_0^T u_i^2(t)dt\leq\mathcal{J}_i(u_i,\hat{u}_{-i})+C_6\leq\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})+C_6=J_i(\hat{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big)+C_6,$$

i.e.,

$$\mathbb{E}\int_0^T u_i^2(t)dt\leq C_7 \tag{57}$$

where C_7 is a positive constant which is independent of N. Further, we can get the boundness of $\sup\limits_{0\leq t\leq T}\mathbb{E}|k_i(t)|^2$.

Remark 4.2: Note that $C_5=C_5(A,B,\alpha,\beta)$ is independent of N. Actually, C_5 contains the item $\frac{1}{N^2}$ which is brought in by $k^{(N)}$. However, obviously it vanishes as N is large enough.

For the i^{th} agent $\mathcal{A}_i$, consider the perturbation in **(LPI)** and introduce some auxiliary system

$$\begin{cases} -dk_i^0(t) = \big[Ak_i^0(t) + Bu_i(t)\big]dt - n_i^0(t)dW_i(t) - \tilde{n}_i^0(t)dW(t), \\ \quad k_i^0(T) = \eta_i \end{cases} \tag{58}$$

and for $j \neq i$,

$$\begin{cases} -d\hat{k}_j(t) = \Big[A\hat{k}_j(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - \hat{n}_j(t)dW_i(t) - \hat{\tilde{n}}_j(t)dW(t), \\ \quad \hat{k}_j(T) = \eta_j \end{cases} \tag{59}$$

with the cost functional

$$J_i\big(u_i(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + 2k_i^0(0)\big(\alpha - \beta y^*(0)\big)\right]. \tag{60}$$

Noting (54) and (58), we can see that $(k_i, n_i, \tilde{n}_i)$ is same to $(k_i^0, n_i^0, \tilde{n}_i^0)$. Besides, by (54) and (55), we have

$$\begin{cases} -dk^{(N)}(t) = \left[Ak^{(N)}(t) + \frac{B}{N}\Big(u_i(t) + \sum_{j=1, j\neq i}^{N} \hat{u}_j(t)\Big)\right]dt - \frac{1}{N}\sum_{j=1}^{N} n_j(t)dW_j(t) \\ \qquad\qquad - \frac{1}{N}\sum_{j=1}^{N} \tilde{n}_j(t)dW(t), \\ \quad k^{(N)}(T) = \eta^{(N)} \end{cases} \tag{61}$$

where $\hat{u}_j(t) = -BR^{-1}\big(\alpha - \beta y^*(0)\big)e^{At},\ j \neq i$. Then we have the following lemma.

Lemma 4.2:

$$\sup_{0\le t\le T} \mathbb{E}\Big|k^{(N)}(t) - y^*(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{62}$$

$$\Big|\mathcal{J}_i(u_i, \hat{u}_{-i}) - J_i(u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{63}$$

Proof: By (49) and (61), we have

$$\begin{cases} -d\big(k^{(N)}(t) - y^*(t)\big) = \left[A\big(k^{(N)}(t) - y^*(t)\big) + \frac{1}{N}\big(Bu_i(t) - r(t)\big)\right] - \frac{1}{N}\sum_{j=1}^{N} n_j(t)dW_j(t) \\ \qquad\qquad + \left[z^*(t) - \frac{1}{N}\sum_{j=1}^{N}\tilde{n}_j(t)\right]dW(t), \\ \quad k^{(N)}(T) - y^*(T) = \eta^{(N)} - \eta. \end{cases}$$

Noting (57) (48) and applying the estimates of BSDE, we obtain (62). Thus, we have

$$\begin{aligned}\Big|\mathcal{J}_i(u_i,\hat{u}_{-i})-J_i(u_i)\Big| \leq& 2\beta\mathbb{E}\Big[\big|k_i(0)\big|\big|k^{(N)}(0)-y^*(0)\big|\Big]\\ =&O\Big(\frac{1}{\sqrt{N}}\Big)\end{aligned}$$

which completes the proof. □

Proof of Theorem 4.1: Consider the ϵ-Nash equilibrium of $\mathcal{A}_i$ for **(PI)**. Combining Lemma 4.1 and 4.2, we have

$$\begin{aligned}\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i}) =& J_i(\hat{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ \leq& J_i(u_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ =&\mathcal{J}_i(u_i,\hat{u}_{-i})+O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 4.1 follows by taking $\epsilon=O\Big(\frac{1}{\sqrt{N}}\Big)$.

D. Extensions

Now, we present some possible extensions based on our previous analysis. The first extension is to consider the following cost functional

$$\mathcal{J}_i^1\big(u_i,u_{-i}\big)=\mathbb{E}\left[\int_0^T Ru_i^2(t)dt+2y_i(0)\Big(\alpha+\frac{\beta}{y^{(N)}(0)}\Big)\right] \tag{64}$$

where α,β are nonnegative constants. Such cost functional characterizes the so-called benchmark performance criteria in investment. To be more precise, suppose there has a large population system which consists of considerable small investors who aim to achieve (or, hedge) some terminal targets η_i by portfolio selection. The term $y^{(N)}(0)$ denotes the average hedging cost for all investors while $\frac{y_i(0)}{y^{(N)}(0)}$ denotes the relative hedging costs for i^{th} investor, and β denotes its weight. In case $\beta=0$, it is reduced to the classical individual own performance. In case $\beta>0$, the investor should get some balance between its own individual performance and the average population performance. In other words, the investor aims to minimize its initial hedging cost by taking account of the average cost of the whole market participants. In this case, we aim to minimize the weighted cost functional $\mathcal{J}_i^1$.

Another extension is to consider the so-called convex portfolio selection. In this case, the given individual investor will take into account their relative performance by comparison to their peers in convex combination. In accordance with [12], in which the security writers aim to maximize the utility function of terminal wealth. Here, we aim to minimize the following initial hedging cost

$$\mathcal{J}_i^2\big(u_i,u_{-i}\big)=\mathbb{E}\left[\frac{1}{2}\int_0^T Ru_i^2(t)dt+(1-\lambda)y_i(0)+\lambda\Big(y_i(0)-y^{(N)}(0)\Big)\right] \tag{65}$$

where $\lambda\in[0,1]$ is the parameter of relative interest.

For above two extensions, following the similar arguments to our previous analysis, we can get the corresponding optimal decentralized controls as

$$\begin{cases}\bar{u}_i^1(t)=-\,R^{-1}B\Big(\alpha+\dfrac{\beta}{y^1(0)}\Big)e^{At},\\ \bar{u}_i^2(t)=-\,R^{-1}Be^{At}\end{cases} \tag{66}$$

where $y^1(0)$ is the initial value of the limiting process of state average. Besides, the fixed points principle and the ϵ-Nash equilibrium properties for $\mathcal{J}_i^1,\mathcal{J}_i^2$ are obtained respectively. Since there are some other financial models in the form of large population with partial information structure, our theoretical results may have potential applications in finance and economics.

V. Conclusion

In this paper, we introduce the backward mean-field LQG games. Different to the well-studied *forward* mean-field LQG games, the terminal conditions of individual players are specified here as a priori and as a result, the decentralized control and consistency condition are determined in *backward* manner. Both the full and partial information cases are addressed and the ϵ-Nash equilibrium are verified using the estimates of backward stochastic differential equation and its limiting equation. Our work suggests some future research directions. One is to include the first solution component $y_i(t)$ and its average $y^{(N)}(t)$ into the running cost to be minimized. This brings additional technical difficulty as the decoupling method via Riccati equation is not workable for *backward* setup. Another one is to introduce the second component $z_i(t)$ into the state or cost functional. We plan to discuss them in our future work.

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