

OPTIMALITY OF (s, S) POLICIES WITH NONLINEAR PROCESSES

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ABSTRACT. It is observed empirically that mean-reverting processes are more realistic in modeling the inventory level of a company. In a typical mean-reverting process, the inventory level is assumed to be linearly dependent on the deviation of the inventory level from the long-term mean. However, when the deviation is large, it is reasonable to assume that the company might want to increase the intensity of interference to the inventory level significantly rather than in a linear manner. In this paper, we attempt to model inventory replenishment as a nonlinear continuous feedback process. We study both infinite horizon discounted cost and the long-run average cost, and derive the corresponding optimal (s, S) policy.

1. Introduction. In inventory control problems, stochastic models account for the randomness in demands [3]. However, the problem becomes complicated with the addition of a number of parameters [1], such as fixed cost, variable cost, order latency and differentiate decision models [24] in the supply chains. Although numerical techniques can be easily applied for solving the inventory policies for both discrete [13, 15, 27] and continuous models [22, 23, 25], analytic solutions are still

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appealing as practitioners can gain more insights. It is possible to derive some closed form solution to the problem under certain conditions, such as under infinite time horizon. Indeed, under different demand models, closed form solutions have been derived. In the literature, the optimal policy has been derived for the diffusion process in [9]. An explicit solution to the problem was derived by Sulem [21] later. The policy was further extended in [16] for optimal band policy. For discrete demands, the optimal (s, S) policy was derived in [26]. If demands arrive at random epochs, it is traditionally modelled by the Poisson process [18, 19] or the renewal process [11, 20]. These early approaches are extended further by the use of the compound Poisson process [4, 17] and Markovian demands [6, 7].

As for the inventory control process, many types of inventory follow mean-reverting processes. This provides a feedback mechanism in adjusting the inventory level based on the deviation of the inventory level from the long-term mean. In addition, it can be used to model inventory dependent demand. By using the typical mean-reverting process and a quadratic cost function, a two-band replenishment policy has been derived in [8]. On the other hand, Benkherouf and Johnson [2] extended the standard mean-reverting process to a nonlinear one, where the intensity of adjustment to the inventory level depends nonlinearly on the current inventory level. With fixed and variable costs for every replenishment, the discounted cost problem is to be minimized. As a first attempt to model this nonlinearity, a Poisson jump demand model was assumed, and a new impulse control problem was formulated. The optimal (s, S) policy can be derived analytically. In addition to a Poisson jump demand model, further simplification was adopted in [2] in which the nonlinearity vanishes whenever there is a shortage in the inventory level.

In this paper, following [2, 8], we consider the control problem with a nonlinear inventory-dependent term incorporated into the diffusion demand model. Different from [2] which employed a jump process, we assume that the demand follows a Wiener process together with the nonlinearity introduced into the mean-reverting process. Furthermore, the nonlinearity is assumed for all inventory levels. Introducing fixed and variable costs, a new impulse control problem is proposed. It should be noted that only a linear mean-reverting process was considered in [3]. However, when a nonlinear term is introduced, the solution process becomes much more difficult, especially when we deal with the uniqueness of the policy in which some special properties of the Green function are needed. By extending the theory in [3] for this nonlinear feedback problem, we consider the optimal minimal discounted cost and tackle the problem under infinite time horizon. By the dynamic programming principle, the problem is reduced to solving a quasi-variational inequality (Q.V.I.). By constructing the Green function under the nonlinear process, the optimal (s, S) policy is developed. In addition, we also consider the long-run average cost function and derive the corresponding optimal (s, S) policy.

The main contributions this work are the following. First, we consider the nonlinear inventory-dependent term into inventory model, which captures the generalized phenomenon that traditional diffusion model cannot explain. Second, we don't require the nonlinear term to be zero when the inventory is out of order, which is more general than the one which is discussed in [2] with a jump model. Third, we also investigate the optimal replenishment problem with ergodic case, which is not considered in [2]. Finally, we show that an $(s; S)$ policy solves the Q.V.I. and solve the Q.V.I. analytically with a closed-form solution.

This paper is structured as follows. In Section 2 we formulate the stochastic inventory problem under a general nonlinear mean-reverting model. In section 3, we solve the discounted problem and obtain an explicit expression for the value function and hence the (s, S) policy. In Section 4 we consider the long-run average cost problem by considering the limit of the discount problem. Section 5 summarizes the results with some concluding remarks.

2. Problem formulation. We first formulate the stochastic inventory problem under consideration. Suppose that $W(t)$ is a Wiener process defined on a given probability space $(\Omega, \mathbb{A}, \mathbb{P})$. In the absence of intervention, the inventory in the interval $[0, t]$ can be described by the process

$$dy(t) = -\left((\gamma + f(y(t)))dt + \sigma dW(t)\right), \quad (1)$$

where $\gamma dt + \sigma dW(t)$ represents the external accumulated demand per unit time, $f(x)$ is part of the mean reversion process satisfying

$$\begin{aligned} 0 &\leq f'(x) \leq M, \\ \lim_{x \rightarrow +\infty} f(x) &= +\infty, \\ \lim_{x \rightarrow -\infty} f(x) &= -\infty. \end{aligned} \quad (2)$$

In the literature, $f(\cdot)$ usually takes a linear form. The mean reversion of the process can be explained by the assumption that if inventory recently has gone down because of a strong demand, one could expect the demand in the near future to be weaker, allowing the inventory to revert back toward its preferred target [3, 8]. Here we extend $f(x)$ to be a nondecreasing nonnegative function defined on \mathcal{R} . The term $f(\cdot)$ captures an inventory dependent demand, or a materialized deterioration of the inventory. If $f(\cdot)$ takes the mean reversion, it is reduced to the case discussed in [3]. If $f = 0$, it recovers a demand model when $\gamma t + \sigma W(t)$ represents the external accumulated demand on $(0, t)$. Denote

$$\mathcal{F}^t := \sigma(W(s), s \leq t).$$

An impulse control is a sequence

$$(\theta_n, v_n), n = 1, 2, \dots,$$

where θ_n is a stopping time with respect to \mathcal{F}^t and v_n is a \mathcal{F}^{θ_n} -measurable random variable. Here θ_n denotes the n th order and v_n denotes the amount at time θ_n . The cost for ordering an amount of $v_n > 0$ is given by

$$c(v_n) = K + cv_n, v_n > 0, \quad (3)$$

where $K > 0$ is the fixed set up cost of ordering and c denotes the unit cost for each item ordered. Let V denote an impulse control, the corresponding inventory level can be described by the formula

$$y_x(t, V) = x - \left(\gamma t + \int_0^t f(y_x(s, V)) ds + \sigma W(t) \right) + I(t; V), \quad (4)$$

where x is an initial inventory level, $\gamma > 0$ is a constant rate of demand, $\sigma > 0$ and

$$I(t) = I(t; V) = \sum_{\{n | \theta_n < t\}} v_n.$$

Here we first consider a discounted cost objective function. Let $\alpha > 0$ be a specified discount rate. For any given initial inventory level x and an ordering policy V , we define the discounted cost as

$$J_\alpha(x, V) = E \left[\sum_{n=0}^{\infty} (K + cv_n) \exp(-\alpha\theta_n) + \int_0^{\infty} g(y_x(t, V)) \exp(-\alpha t) dt \right], \quad (5)$$

where

$$g(x) = hx^+ + px^-$$

denotes the storage cost when $x > 0$ and the backlog cost when $x < 0$. Define the value function associated with (5) by

$$u_\alpha(x) = \inf_V J_\alpha(x, V). \quad (6)$$

In order that (5) is well defined, we let \mathcal{V} denote all V satisfying the following conditions

$$\begin{aligned} \int_0^{\infty} g(y_x(t, V)) \exp(-\alpha t) dt &< \infty, \\ E \sum_{n=0}^{\infty} \exp(-\alpha\theta_n) &< \infty, \\ E \sum_{n=0}^{\infty} \exp(-\alpha\theta_n) v_n &< \infty. \end{aligned} \quad (7)$$

The impulsive control V is said to be *admissible* if $V \in \mathcal{V}$. The problem is to find $V^* \in \mathcal{V}$ such that

$$u_\alpha(x) = J_\alpha(x, V^*).$$

3. The results with discounted problem. To solve the discounted problem, we first define the operator

$$\mathcal{A}u_\alpha(x) = -\frac{1}{2}\sigma^2 u_\alpha''(x) + (\gamma + f(x))u_\alpha'(x). \quad (8)$$

From the dynamic programming principle, the value function $u_\alpha(x)$ satisfies the following Q.V.I.:

$$\begin{cases} \mathcal{A}u_\alpha(x) + \alpha u_\alpha(x) \leq g(x), \\ u_\alpha(x) \leq M(u_\alpha)(x), \\ (\mathcal{A}u_\alpha(x) + \alpha u_\alpha(x) - g(x))(u_\alpha(x) - M(u_\alpha)(x)) = 0, \end{cases} \quad (9)$$

where

$$M(u_\alpha) = \inf_{v>0} [cv + K + u_\alpha(x + v)].$$

The derivation of Q.V.I. (9) follows the standard techniques described in Bensoussan and Lions [5] and Bensoussan [3]. We will sketch the proof in Appendix. We will study (9) in a continuous functional space and verify that it is equal to the solution to the value function (6) by a classical verification argument. We refer to [5] for the general theory of impulse control and Q.V.I.

To simplify the second inequality of (9), we apply the transformation

$$G_\alpha(x) = u_\alpha(x) + cx.$$

Then solving (9) is reduced to find $G_\alpha(x)$ which satisfies

$$\begin{cases} \mathcal{A}G_\alpha(x) + \alpha G_\alpha(x) \leq \tilde{g}(x) + c\gamma, \\ G_\alpha(x) \leq K + \inf_{\eta \geq x} G_\alpha(\eta), \\ (\mathcal{A}G_\alpha(x) + \alpha G_\alpha(x) - \tilde{g}(x) - c\gamma)(G_\alpha(x) - K - \inf_{\eta \geq x} G_\alpha(\eta)) = 0, \end{cases} \quad (10)$$

where

$$\tilde{g}(x) := g(x) + c(\alpha x + f(x)).$$

We require $G_\alpha(\cdot)$ to be \mathcal{C}^1 with linear growth. Naturally, $u_\alpha(\cdot)$ has the same properties. For any fixed s , let $S_\alpha(s)$ denote the point where $G_{\alpha,s}(x)$ attains its smallest minimum, then

$$G'_{\alpha,s}(S_\alpha(s)) = 0.$$

Instead of (10), we will construct a pair $(s, G_{\alpha,s}(x))$, which is the solution of

$$\begin{cases} \mathcal{A}G_{\alpha,s}(x) + \alpha G_{\alpha,s}(x) = \tilde{g}(x) + c\gamma, x > s, \\ G_{\alpha,s}(x) = K + G_{\alpha,s}(S_\alpha(s)), x \leq s. \end{cases} \quad (11)$$

It follows from (10) that $G'_{\alpha,s}(s) = 0$ and $G'_{\alpha,s}(S_\alpha(s)) = 0$. Then the construction will be proceeded with the following three steps:

(a) First of all, for any fixed s , solve the first equation in (11) with the condition $G'_{\alpha,s}(s) = 0$ and obtain a \mathcal{C}^1 solution $G_{\alpha,s}(\cdot)$.

- (b) Show that $S_\alpha(s)$ exists, which satisfies the condition $G'_{\alpha,s}(S_\alpha(s)) = 0$.
(c) Use the second equation in (11) to determine a unique optimal s , which is denoted by s_α . This method leads to a unique function, which is \mathcal{C}^1 , and a unique pair (s_α, S_α) .

These steps will be described in details in the following subsection 3.1-subsection 3.3. Later we will verify that the function constructed in this way satisfies the original *Q.V.I.* (10) of the inventory problem, which will be given as Theorem 4.2.

3.1. Step 1: For any fixed s , construct $G_{\alpha,s}(x)$. Consider

$$\begin{cases} \mathcal{A}G_{\alpha,s}(x) + \alpha G_{\alpha,s}(x) = \tilde{g}(x) + c\gamma, x > s, \\ G_{\alpha,s}(x) = G_{\alpha,s}(s), x \leq s. \end{cases} \quad (12)$$

Because of the regularity, we necessarily have $G'_{\alpha,s}(s) = 0$.

Denote

$$H_{\alpha,s}(x) := G'_{\alpha,s}(x),$$

then it satisfies

$$\begin{cases} \mathcal{A}H_{\alpha,s}(x) + (\alpha + f'(x))H_{\alpha,s}(x) = \tilde{g}'(x), x > s, \\ H_{\alpha,s}(x) = 0, x \leq s. \end{cases} \quad (13)$$

Notice that $\tilde{g}'(x) = -p + c(\alpha + f'(x))$ if $x < 0$, we make the assumption

$$-p + c(\alpha + M) < 0, \quad (14)$$

so $\tilde{g}'(x) < -p + c(\alpha + f'(x)) < 0$ for $x < 0$.

To solve (13) we take advantage of the *Green function*, which is derived from

$$\begin{cases} -\frac{1}{2}\sigma^2\Phi''_\alpha(x) + (\gamma + f(x))\Phi'_\alpha(x) + (\alpha + f'(x))\Phi_\alpha(x) = 0, x \in R, \\ \Phi_\alpha(0) = 1, \Phi_\alpha(+\infty) = 0. \end{cases} \quad (15)$$

The following lemma shows the solution properties of (15).

Lemma 3.1. *There exists one and only one solution of (15), which has the properties $0 < \Phi_\alpha(x) < 1$ for $x > 0$, $\Phi_\alpha(x) > 1$ for $x < 0$, and*

$$\Phi'_\alpha(x) \leq 0,$$

for all $x \in (-\infty, +\infty)$.

Proof. First we claim that $\Phi_\alpha(x) > 0$ when $x \geq 0$. Otherwise, there exists a local minimum, where $\Phi_\alpha(x) \leq 0$, $\Phi'_\alpha(x) = 0$, $\Phi''_\alpha(x) > 0$. It contradicts (15). Moreover, we must have $\Phi'_\alpha(x) < 0$ when $x \geq 0$. Otherwise, there exists a positive local maximum on $[0, +\infty)$, which is again impossible due to (15). Similarly, we have $\Phi'_\alpha(x) < 0$ on $(-\infty, 0)$. Therefore, $\Phi_\alpha(x)$ is strictly decreasing on $(-\infty, \infty)$, namely, $0 < \Phi_\alpha(x) < 1$ for $x > 0$, $\Phi_\alpha(x) > 1$ for $x < 0$. \square

Before the construction, another function $\chi_\alpha(x)$ is needed. Denote

$$F(x) = \int_0^x f(y)dy, \vartheta(x) = \exp(-\frac{2}{\sigma^2}(\gamma x + F(x))).$$

and define

$$\chi_\alpha(x) = \vartheta(x)\Phi_\alpha(x), \quad x \in R, \quad (16)$$

Then $\chi_\alpha(x)$ satisfies the differential equation

$$\begin{cases} -\frac{\sigma^2}{2}\chi_\alpha''(x) - (\gamma + f(x))\chi_\alpha'(x) + \alpha\chi_\alpha(x) = 0, \\ \chi_\alpha(0) = 1, \chi_\alpha(+\infty) = 0. \end{cases} \quad (17)$$

The following lemma shows the properties of $\chi_\alpha(x)$, which will be used later.

Lemma 3.2. *There exists one and only one solution $\chi_\alpha(x)$ of (17). It has also the properties $0 < \chi_\alpha(x) < 1$ for $x > 0$, $\chi_\alpha(x) > 1$ for $x < 0$, and $\chi_\alpha'(x) < 0$ for all x . Moreover,*

$$\liminf_{x \rightarrow -\infty} \chi_\alpha''(x) \geq 0, \quad \lim_{x \rightarrow -\infty} \chi_\alpha(x) = +\infty. \quad (18)$$

Proof. Obviously $\chi_\alpha(x)$ defined by (16) is the solution of (17). By using a similar argument to $\Phi_\alpha(x)$, we can obtain $0 < \chi_\alpha(x) < 1$ for $x > 0$, $\chi_\alpha(x) > 1$ for $x < 0$, and $\chi_\alpha'(x) < 0$ for all x . Now let us prove the properties of $\chi_\alpha''(x)$ when x approaches to $-\infty$.

Denote $\Psi_\alpha(x) = \chi_\alpha'(x)$, then $\Psi_\alpha(x) < 0$ and

$$\begin{cases} -\frac{\sigma^2}{2}\Psi_\alpha''(x) - (\gamma + f(x))\Psi_\alpha'(x) + (\alpha - f'(x))\Psi_\alpha(x) = 0, \\ \Psi_\alpha(0) = \phi'_\alpha(0), \Psi'_\alpha(0) = \phi''_\alpha(0). \end{cases} \quad (19)$$

We prove $\liminf_{x \rightarrow -\infty} \chi_\alpha''(x) \geq 0$ under two separate cases:

Case 1: $\limsup_{x \rightarrow -\infty} f'(x) \leq \alpha$. There exists \bar{x} such that $f'(\bar{x}) < \alpha$.

(i) If $\Psi'_\alpha(\bar{x}) > 0$ and if there exists $x < \bar{x}$ satisfying $\Psi'_\alpha(x) \leq 0$, then a local minimum exists, where the first derivative is zero and the second derivative is positive, which contradicts (19). Thus, when $\Psi'_\alpha(\bar{x}) > 0$, we must have $\Psi'_\alpha(x) > 0$ for all $x \leq \bar{x}$, which means

$$\liminf_{x \rightarrow -\infty} \Psi'_\alpha(x) \geq 0,$$

namely,

$$\liminf_{x \rightarrow -\infty} \chi_\alpha''(x) \geq 0.$$

(ii) (a) If $\Psi'_\alpha(\bar{x}) \leq 0$ and if $\Psi'_\alpha(x) \leq 0$ for all $x \leq \bar{x}$, then we must have

$$\liminf_{x \rightarrow -\infty} \Psi'_\alpha(x) = 0.$$

Otherwise,

$$\liminf_{x \rightarrow -\infty} \Psi'_\alpha(x) < 0$$

leads to

$$\lim_{x \rightarrow -\infty} \Psi_\alpha(x) > 0,$$

which contradicts the fact that $\Psi_\alpha(x) \leq 0$.

(ii) (b) If $\Psi'_\alpha(\bar{x}) \leq 0$ and if there exists $\bar{\bar{x}} < \bar{x}$ such that $\Psi'_\alpha(\bar{\bar{x}}) > 0$, then it is reduced to the case in (i).

Case 2:¶¶ $\lim_{x \rightarrow -\infty} f'(x) > \alpha$. For some \tilde{x} , we have $f'(\tilde{x}) > \alpha$.

(i) If $\Psi'_\alpha(\tilde{x}) \leq 0$ and if there exists $x < \tilde{x}$ such that $\Psi'_\alpha(x) > 0$, then a local maximum exists, which is impossible from (19). Thus $\Psi'_\alpha(x) \leq 0$ for all $x \leq \tilde{x}$; similar to Case 1 (ii) (a), we have

$$\liminf_{x \rightarrow -\infty} \Psi'_\alpha(x) = 0.$$

(ii) If $\Psi'_\alpha(\tilde{x}) > 0$ and if there exists $\tilde{\tilde{x}} < \tilde{x}$ such that $\Phi'_\alpha(\tilde{\tilde{x}}) \leq 0$; then it is reduced to the case in (i). If $\Psi'_\alpha(x) > 0$ for all $x \leq \tilde{x}$, obviously, $\liminf_{x \rightarrow -\infty} \chi''_\alpha(x) \geq 0$.

¶ From **Case 1** and **Case 2**, we get the desired result.

To prove $\lim_{x \rightarrow -\infty} \chi_\alpha(x) = +\infty$, we use the argument by reduction to absurdity. Indeed, because of $\chi_\alpha(x) > 0$ and $\chi'_\alpha(x) < 0$, if $\lim_{x \rightarrow -\infty} \chi_\alpha(x)$ is bounded, then $\lim_{x \rightarrow -\infty} \chi_\alpha(x)$ exists, we denote it by M_b , which is a constant. As a result,

$$\lim_{x \rightarrow -\infty} \chi'_\alpha(x) = 0, \quad \lim_{x \rightarrow -\infty} \chi''_\alpha(x) = 0.$$

It follows from (17) that

$$\frac{\chi'_\alpha(x)}{\chi_\alpha(x)} = \frac{\alpha}{\gamma + f(x)} - \frac{\sigma^2 \chi''_\alpha(x)}{2\chi_\alpha(x)(\gamma + f(x))}.$$

Therefore, there exists $\check{x} < 0$, when $x < \check{x}$,

$$\begin{aligned} \frac{\chi'_\alpha(x)}{\chi_\alpha(x)} &\leq \frac{\alpha}{\gamma + f(x)} \\ &\leq \frac{\alpha}{Mx}. \end{aligned} \tag{20}$$

Thus $\chi_\alpha(x)$ is larger than $\chi_\alpha(\check{x})(\frac{x}{\check{x}})^{\frac{\alpha}{M}}$, which contradicts the fact that $\lim_{x \rightarrow -\infty} \chi_\alpha(x)$ is bounded. This completes the proof. \square

Remark 3.1. When $f(x)$ takes the linear form as in [3], $\lim_{x \rightarrow -\infty} \chi_\alpha(x) = +\infty$ becomes more obvious. The proof in Lemma 3.2 is more involved to show that $\liminf_{x \rightarrow -\infty} \chi''_\alpha(x) \geq 0$, which is not only needed in the proof $\lim_{x \rightarrow -\infty} \chi_\alpha(x) = +\infty$, but also for later use in Lemma 3.8.

Now we are ready to construct a bounded solution of (13) by the following lemma.

Lemma 3.3. Denote

$$Q_\alpha(\xi) := \frac{2}{\sigma^2} \int_\xi^{+\infty} \tilde{g}'(\eta) \frac{\chi_\alpha(\eta)}{\chi_\alpha(\xi)} d\eta, \tag{21}$$

then

$$H_{\alpha,s}(x) = \int_s^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} Q_\alpha(\xi) d\xi, \tag{22}$$

is a bounded solution of (13).

Proof. Obviously, $H_{\alpha,s}(x)$ constructed by (22) is the solution of (13). The remaining work is to show that it is bounded. We first show that

$$\frac{2}{\sigma^2} \int_0^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^{+\infty} (\alpha + f'(\eta)) \frac{\chi_\alpha(\eta)}{\chi_\alpha(x)} d\eta d\xi = 1 - \Phi_\alpha(x), \quad (23)$$

and (23) is the bounded solution of

$$\begin{cases} -\frac{1}{2}\sigma^2 Z_\alpha''(x) + (\gamma + f(x))Z_\alpha'(x) + (\alpha + f'(x))Z_\alpha(x) = \alpha + f'(x), & x \in R^+, \\ Z_\alpha(0) = 0. \end{cases} \quad (24)$$

Denote

$$Z_\alpha(x) = 1 - \Phi_\alpha(x).$$

Obviously, $Z_\alpha(x)$ is the bounded solution of (24). To show (23) holds, we denote

$$Y_\alpha(x) = \frac{Z_\alpha(x)}{\Phi_\alpha(x)}, \quad (25)$$

and substitute $Z_\alpha(x) = \Phi_\alpha(x)Y_\alpha(x)$ into (24); it can be seen that

$$\begin{aligned} \frac{d}{dx}(Y_\alpha'(x)\Phi_\alpha^2(x)\vartheta(x)) &= -\frac{2}{\sigma^2}\Phi_\alpha(x)\vartheta(x)(\alpha + f'(\eta)) \\ &= -\frac{2}{\sigma^2}\chi_\alpha(x)(\alpha + f'(\eta)). \end{aligned} \quad (26)$$

On the other hand, the definition of $Y_\alpha(x)$ means that

$$Y_\alpha'(x)\Phi_\alpha^2(x)\vartheta(x) = -\Phi_\alpha'(x)\vartheta(x),$$

which implies $Y_\alpha'(x)\Phi_\alpha^2(x)\vartheta(x)$ is bounded and approaches to 0 when $x \rightarrow +\infty$. Thus, it follows from (26) that

$$Y_\alpha'(x)\Phi_\alpha^2(x)\vartheta(x) = \frac{2}{\sigma^2} \int_x^{+\infty} \chi_\alpha(\eta)(\alpha + f'(\eta))d\eta. \quad (27)$$

Substituting (25) into (27) leads to

$$Z_\alpha(x) = \frac{2}{\sigma^2} \int_0^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^{+\infty} (\alpha + f'(\eta)) \frac{\chi_\alpha(\eta)}{\chi_\alpha(\xi)} d\eta d\xi.$$

We can now go to the proof of (22). We prove it directly from the expression of $H_{\alpha,s}(x)$:

$$\begin{aligned} & H_{\alpha,s}(x) \\ &= \Phi_\alpha(x)H_{\alpha,s}(0) + \frac{2}{\sigma^2} \int_0^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^{+\infty} \tilde{g}'(\eta) \frac{\chi_\alpha(\eta)}{\chi_\alpha(x)} d\eta d\xi \\ &= \Phi_\alpha(x)H_{\alpha,s}(0) + \frac{2}{\sigma^2} \int_0^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^{+\infty} (\alpha + f'(\eta)) \left(c + \frac{g'(\eta)}{\alpha + f'(\eta)}\right) \frac{\chi_\alpha(\eta)}{\chi_\alpha(\xi)} d\eta d\xi \\ &= \Phi_\alpha(x)H_{\alpha,s}(0) + c(1 - \Phi_\alpha(x)) \\ &\quad + \int_0^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^{+\infty} \frac{h}{\alpha + f'(\eta)} (\alpha + f'(\eta)) \frac{\chi_\alpha(\eta)}{\chi_\alpha(\xi)} d\eta d\xi, \end{aligned}$$

where the third equation follows from (23). From the relation

$$\frac{h}{\alpha + M} < \frac{h}{\alpha + f'(\eta)} < \frac{h}{\alpha},$$

we have

$$\begin{aligned} \Phi_\alpha(x)H_{\alpha,s}(0) + c(1 - \Phi_\alpha(x)) + \frac{h}{\alpha + M}(1 - \Phi_\alpha(x)) &\leq H_{\alpha,s}(x) \\ &\leq \Phi_\alpha(x)H_{\alpha,s}(0) + c(1 - \Phi_\alpha(x)) + \frac{h}{\alpha}(1 - \Phi_\alpha(x)). \end{aligned} \quad (28)$$

Thus

$$c + \frac{h}{\alpha + M} \leq \lim_{x \rightarrow +\infty} H_{\alpha,s}(x) \leq c + \frac{h}{\alpha}. \quad (29)$$

which means that $H_{\alpha,s}(x)$ is bounded. \square

The following lemma guarantees that the bounded solution of (13) is unique.

Lemma 3.4. *The bounded solution of (13) is unique.*

Proof. It is sufficient to show that 0 is the unique solution of

$$\begin{cases} \mathcal{A}H_{\alpha,s}(x) + (\alpha + f'(x))H_{\alpha,s}(x) = 0, x > s, \\ H_{\alpha,s}(x) = 0, x \leq s. \end{cases} \quad (30)$$

In fact, if there exists $x_1 \geq s$ such that $H_{\alpha,s}(x_1) > 0$, then on $[x_1, +\infty)$, (a) there exists at least one local maximum or (b) $H'_{\alpha,s}(x) > 0$.

If (a) holds, then there exists x where $H_{\alpha,s}(x)$ attains its local maximum with

$$H_{\alpha,s}(x) > 0, H'_{\alpha,s}(x) = 0, H''_{\alpha,s}(x) < 0,$$

which contradicts (13). Thus (a) cannot hold.

If (b) holds, for large enough x , we have $\gamma + f(x) > 0$ if $H''_{\alpha,s}(x) \leq 0$, which is in contradiction with (13). If $H''_{\alpha,s}(x) > 0$, it implies that $H'_{\alpha,s}(x)$ increases with x , which contradicts the fact that $H_{\alpha,s}(x)$ is bounded. Thus (b) cannot hold.

From (a) and (b), we know that $H'_{\alpha,s}(x) > 0$ cannot be true and $H_{\alpha,s}(x)$ cannot be positive when $x > s$; therefore we must have

$$H_{\alpha,s}(x) \leq 0.$$

Similarly, $H_{\alpha,s}(x)$ cannot be negative. Therefore, the unique bounded solution of (30) is 0. Consequently the unique bounded solution of (13) is (22). \square

By Lemma 3.3 and Lemma 3.4, we can conclude that $H_{\alpha,s}(x)$ defined by (22) is the unique bounded solution of (13). Let's construct $G_{\alpha,s}(x)$ from $H_{\alpha,s}(x)$. For $x \geq s$, (12) can be rewritten as

$$-\frac{\sigma^2}{2}H'_{\alpha,s}(x) + (f(x) + \gamma)H_{\alpha,s}(x) + \alpha G_{\alpha,s}(s) = \tilde{g}(x) + c\gamma. \quad (31)$$

On the other hand, by integrating (13), we have

$$-\frac{\sigma^2}{2}H'_{\alpha,s}(x) + \frac{\sigma^2}{2}H'_{\alpha,s}(s) + (f(x) + \gamma)H_{\alpha,s}(x) = \tilde{g}(x) - \tilde{g}(s). \quad (32)$$

Comparing (31) and (32) on $[x, s]$, we have

$$\alpha G_{\alpha,s}(s) = \frac{\sigma^2}{2} H'_{\alpha,s}(s) + \tilde{g}(s) + c\gamma, \quad (33)$$

which decides the value of $G_{\alpha,s}(s)$. For any given s , define

$$G_{\alpha,s}(x) = \begin{cases} G_{\alpha,s}(s) + \int_s^x H_{\alpha,s}(\xi) d\xi, & x > s \\ G_{\alpha,s}(s), & x \leq s. \end{cases} \quad (34)$$

Obviously, $G_{\alpha,s}(x)$ constructed by (34) and (33) is in C^1 .

3.2. Step 2: The existence of $S_\alpha(s)$. Let $x_{0,\alpha}$ be the root of $Q_\alpha(x)$ defined by (21). Then such $x_{0,\alpha}$ exists and is unique. In fact, from $\chi_\alpha(x) = \vartheta(x)\Phi_\alpha(x)$, $\chi_\alpha(x) > 0$ and $\lim_{x \rightarrow -\infty} \chi_\alpha(x) = \infty$, we obtain the existence of $x_{0,\alpha}$ and the property that $x_{0,\alpha} < 0$. Because the first derivative of $Q_\alpha(s)\chi_\alpha(x)$ is negative, then $x_{0,\alpha}$ is unique.

Lemma 3.5. *If $s \geq x_{\alpha,0}$, then*

$$S_\alpha(s) = s,$$

otherwise, $G_{\alpha,s}(x)$ attains its minimum at

$$S_\alpha(s) \geq x_{\alpha,0}.$$

Proof. The definition of $x_{\alpha,0}$ and $Q_\alpha(x)$ implies that $Q_\alpha(\xi) > 0$ when $\xi > x_{\alpha,0}$.

Thus, if $x > s \geq x_{\alpha,0}$, then

$$H_{\alpha,s}(x) = \int_s^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} Q_\alpha(\xi) d\xi > 0. \quad (35)$$

In this case, we have $S_\alpha(s) = s$.

If $s < x_{\alpha,0}$, $Q_\alpha(x) < 0$ on $[s, x_0]$, then $H_{\alpha,s}(x_{\alpha,0}) < 0$. Combined with (29), we conclude that there exists $S_\alpha(s) \geq s$ such that $H_{\alpha,s}(S_\alpha(s)) = 0$. \square

3.3. Step 3: Decide the value of s . Notice that the second equation of (11) is to decide the value of s satisfying

$$K + \int_s^{S_\alpha(s)} H_{\alpha,s}(x) dx = 0. \quad (36)$$

Denote $\Gamma_\alpha(s) := \int_s^{S_\alpha(s)} H_{\alpha,s}(y) dy$. We have the following lemma about $\Gamma_\alpha(s)$.

Lemma 3.6. *There exists an*

$$s < x_{\alpha,0}$$

such that

$$\Gamma_\alpha(s) = -K.$$

Proof. When $s \geq x_{\alpha,0}$, according to Lemma 3.5,

$$S_\alpha(s) = s,$$

which means $\Gamma_\alpha(s) = 0$. Therefore, we only need discuss the case that $s < x_{\alpha,0}$.

When $s < x_{\alpha,0}$, we investigate the properties of $\Gamma_\alpha(s)$. As $H_\alpha(s) = H_\alpha(S_\alpha(s)) = 0$, we have

$$\Gamma'_\alpha(s) = \int_s^{S_\alpha(s)} H'_{\alpha,s}(y) dy = -Q_\alpha(s) \int_s^{S_\alpha(s)} \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy. \quad (37)$$

Obviously, when $s < x_{\alpha,0}$ $\Gamma'_\alpha(s) > 0$ due to $Q_\alpha(s) < 0$. We will show

$$\lim_{s \rightarrow -\infty} \Gamma'_\alpha(s) \geq \frac{p - c(\alpha + M)}{2\alpha}. \quad (38)$$

It follows from (37) that

$$\begin{aligned} \Gamma'_\alpha(s) &\geq -Q_\alpha(s) \int_s^{x_{\alpha,0}} \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy \\ &= -\frac{Q_\alpha(s)}{\Phi_\alpha(s)} \int_s^0 \Phi_\alpha(\eta) d\eta \left(1 - \frac{\int_{x_{\alpha,0}}^0 \Phi_\alpha(\eta) d\eta}{\int_s^0 \Phi_\alpha(\eta) d\eta} \right) \\ &= -Q_\alpha(s) \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy \\ &\geq \frac{2}{\sigma^2} (p - c(\alpha + M)) \int_s^{+\infty} \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy. \\ &\geq \frac{2}{\sigma^2} (p - c(\alpha + M)) \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy. \end{aligned} \quad (39)$$

In the next lemma, we will show that

$$\lim_{s \rightarrow -\infty} \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy \geq \frac{\sigma^2}{4\alpha}. \quad (40)$$

From (39) and (40), we have

$$\lim_{s \rightarrow -\infty} \Gamma'_\alpha(s) \geq \frac{p - c(\alpha + M)}{2\alpha}.$$

Thus we have proved (38), which implies $\lim_{s \rightarrow -\infty} \Gamma_\alpha(s) = -\infty$. Combined with $\Gamma_\alpha(x_{\alpha,0}) = 0$, we conclude that there exists $s < x_{\alpha,0}$ such that

$$\Gamma_\alpha(s) = -K.$$

□

Next we construct the proof of (40) by the following lemma. One way is to approximate $\Phi'_\alpha(x)$ as suggested in the previous work [3]. However, it requires more algebra in the process. Here a shorter proof is provided as follows.

Lemma 3.7.

$$\lim_{s \rightarrow -\infty} \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy \geq \frac{\sigma^2}{4\alpha}. \quad (41)$$

Proof. Rewrite $\lim_{s \rightarrow -\infty} \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy$ as $\lim_{s \rightarrow -\infty} \frac{\int_s^0 \chi_\alpha(y) dy}{\chi_\alpha(s)} \frac{\int_s^0 \Phi_\alpha(y) dy}{\Phi_\alpha(s)}$. Since

$$\lim_{x \rightarrow -\infty} \chi_\alpha(x) = +\infty$$

and

$$\lim_{x \rightarrow -\infty} \vartheta(x) = +\infty,$$

we have

$$\lim_{x \rightarrow -\infty} \Phi_\alpha(x) = +\infty.$$

Thus, it follows from the l'Hospital's rule that

$$\begin{aligned} & \lim_{s \rightarrow -\infty} \frac{\int_s^0 \chi_\alpha(y) dy}{\chi_\alpha(s)} \frac{\int_s^0 \Phi_\alpha(y) dy}{\Phi_\alpha(s)} \\ &= \frac{\chi_\alpha(s)}{\chi'_\alpha(s)} \frac{\Phi_\alpha(s)}{\Phi'_\alpha(s)}. \end{aligned} \quad (42)$$

Then we will consider $\frac{\chi_\alpha(s)}{\chi'_\alpha(s)}$ and $\frac{\Phi_\alpha(s)}{\Phi'_\alpha(s)}$, respectively.

As $\liminf_{x \rightarrow -\infty} \chi''_\alpha(x) \geq 0$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$, therefore, there exists $\hat{x} < 0$ such that

$$\chi''_\alpha(x) > -\frac{2}{\sigma^2}$$

and

$$\gamma + f(x) < -K_1$$

for all $x \leq \hat{x}$, here K_1 is any positive constant.

When $x \leq \hat{x} < 0$, it follows from (17) that

$$(\gamma + f(x))\chi'_\alpha(x) \leq \alpha\chi_\alpha(x) + 1 \leq 2\alpha\chi_\alpha(x). \quad (43)$$

Then

$$0 \geq \frac{\chi'_\alpha(x)}{\chi_\alpha(x)} \geq \frac{2\alpha}{\gamma + f(x)}, \quad (44)$$

or equivalently

$$\frac{\chi_\alpha(x)}{\chi'_\alpha(x)} \leq \frac{\gamma + f(x)}{2\alpha}. \quad (45)$$

On the other hand,

$$\begin{aligned} \frac{\Phi'_\alpha(x)}{\Phi_\alpha(x)} &= \frac{\chi'_\alpha(x)}{\chi_\alpha(x)} + \frac{\vartheta'(x)}{\vartheta(x)} \\ &\geq \frac{2\alpha}{\gamma + f(x)} + \frac{2}{\sigma^2}(\gamma + f(x)). \end{aligned} \quad (46)$$

Equivalently

$$\frac{\Phi_\alpha(x)}{\Phi'_\alpha(x)} \leq 1 / \left(\frac{2\alpha}{\gamma + f(x)} + \frac{2}{\sigma^2}(\gamma + f(x)) \right). \quad (47)$$

Combined (45) and (47), we have

$$\begin{aligned}
& \lim_{s \rightarrow -\infty} \frac{\chi_\alpha(s) \Phi_\alpha(s)}{\chi'_\alpha(s) \Phi'_\alpha(s)} \\
& \geq \lim_{s \rightarrow -\infty} \frac{\gamma + f(s)}{2\alpha} / \left(\frac{2\alpha}{\gamma + f(s)} + \frac{2}{\sigma^2} (\gamma + f(s)) \right) \\
& \geq \frac{\sigma^2}{4\alpha}.
\end{aligned} \tag{48}$$

□

Finally, it remains to prove the uniqueness of s .

Lemma 3.8. *The value of s satisfying the property $\Gamma_\alpha(s) = -K$ is unique.*

Proof. From Lemma 3.6, the value s satisfying $\Gamma_\alpha(s) = -k$ is on $(-\infty, x_{\alpha,0})$, and

$$\Gamma'_\alpha(s) = -\frac{Q_\alpha(s)}{\Phi_\alpha(s)} \int_s^{S_\alpha(s)} \Phi(\eta) d\eta. \tag{49}$$

As $Q_\alpha(s) < 0$ in $(-\infty, x_{\alpha,0})$ and $\Phi_\alpha(s) > 0$, therefore, $\Gamma'_\alpha(s) > 0$, i.e., $\Gamma_\alpha(s)$ is strictly increasing with s . Then the value s satisfying the property $\Gamma_\alpha(s) = -K$ is unique. □

We denote the pair $(s, S_\alpha(s))$, which satisfies $\gamma_\alpha(s) = -K$, by (s_α, S_α) . By summarizing the lemmas above, we come to the main result of this work.

Theorem 3.1. *The function $G_{\alpha,s}(x)$ defined by (11) is equal to the solution $G_\alpha(x)$ of Q.V.I. (10). The strategy (s_α, S_α) is optimal, which can be derived from the relation*

$$\begin{cases} H_{\alpha,s_\alpha}(S_\alpha) = 0, \\ \int_{s_\alpha}^{S_\alpha} H_{\alpha,s_\alpha}(x) dx + K = 0, \end{cases} \tag{50}$$

where

$$H_{\alpha,s_\alpha}(x) = \frac{2}{\sigma^2} \int_{s_\alpha}^x \frac{\Phi_\alpha(x)}{\Phi_\alpha(\xi)} \int_\xi^\infty \tilde{g}'(\eta) \frac{\chi_\alpha(\eta)}{\chi_\alpha(\xi)} d\eta d\xi, \tag{51}$$

$$\Phi_\alpha(x) = e^{\frac{2}{\sigma^2}(\gamma x + F(x))} \chi_\alpha(x), \tag{52}$$

and $\chi_\alpha(x)$ is the solution of (17).

Proof. The proof is given in Appendix. □

4. The long run average cost. One important problem is to study the behavior when the discount factor approaches to 1. Although the objective function tends to infinity, an average cost function can be employed instead. This is referred as the ergodic control problem. Define

$$J(x, V) = \limsup_T \frac{1}{T} E \left[\sum_{n=0}^{N(T)} (K + cv_n) + \int_0^T g(y_x(t, V)) dt \right] \tag{53}$$

where

$$N(T) = \max\{i, \theta_i \leq T\}.$$

The problem is thus to investigate the inventory control problem with an ergodic cost criterion of minimizing (53):

$$\rho_0 = \inf_V J(x, V).$$

Denote $\tilde{u}_\alpha(x) := u_\alpha(x) - u_\alpha(s_\alpha)$. By the “vanishing discount method”, we will show $(\tilde{u}_\alpha(x), \alpha u_\alpha(s_\alpha))$ converges to a pair $(u(x), \rho)$, which is the solution of Q.V.I.:

$$\begin{aligned} \mathcal{A}u(x) + \rho &\leq g(x), \\ u(x) &\leq K + \inf_{\eta \geq x} u(\eta), \\ (\mathcal{A}u(x) + \rho - g(x))(u(x) - K - \inf_{\eta \geq x} u(\eta)) &= 0, \end{aligned} \quad (54)$$

followed by showing that ρ is equal to the value of ρ_0 . To simplify the algebra, denote

$$\tilde{G}_\alpha(x) = G_\alpha(x) - G_\alpha(s_\alpha), \quad \rho_\alpha = \alpha G_{s_\alpha}(s_\alpha).$$

Obviously,

$$\lim_{\alpha \rightarrow 0} \rho_\alpha = \rho.$$

In the following sections, instead of proving the convergence of $(\tilde{u}_\alpha(x), \alpha u_\alpha(s_\alpha))$ and solving (54), we prove that $(\tilde{G}_\alpha(x), \alpha G_\alpha(s_\alpha))$ converges to the solution $(G(x), \rho)$ of the Q.V.I.:

$$\begin{aligned} \mathcal{A}G(x) + \rho &\leq \tilde{g}(x) + c\gamma, \\ G(x) &\leq K + \inf_{\eta \geq x} G(\eta), \\ (\mathcal{A}G(x) + \rho - \tilde{g}(x) - c\gamma)(G(x) - K - \inf_{\eta \geq x} G(\eta)) &= 0, \end{aligned} \quad (55)$$

and solve ρ from the transformed problem (55).

4.1. The ergodic control. In the previous section, we construct $G_{\alpha,s}(x)$ through its derivative $H_{\alpha,s}(x)$, which in turn can be derived by employing the *Green* function $\Phi_\alpha(x)$ and $\chi_\alpha(x)$. Let $\alpha = 0$ in $\chi_\alpha(x)$ of (17), $\Phi_\alpha(x)$ of (15), $Q_{\alpha,s}(x)$ of (21) and $H_{\alpha,s}(x)$ of (22), and denote them by $\chi(x)$, $\Phi(x)$, $Q(x)$ and $H_s(x)$, respectively. We first consider the solution of

$$\begin{cases} \mathcal{A}G_s(x) + \alpha G_s(x) = \tilde{g}(x) + c\gamma, x > s, \\ G_s(x) = K + G_s(S_s), x \leq s. \end{cases} \quad (56)$$

Theorem 4.1. *The policy (s_α, S_α) are uniformly bounded. Moreover, $(\tilde{G}_\alpha(x), \alpha \tilde{G}_\alpha(s_\alpha))$ converges to the solution $(G(x), \rho)$ of (56).*

Proof. We first prove that $H_\alpha(x)$ uniformly converges to $H(x)$ on any compact set of \mathbb{R} , then $\tilde{G}_\alpha(x)$ converges to $G(x)$. The idea is to prove that $\Phi_\alpha(x)$ and $\chi_\alpha(x)$ uniformly converge to $\Phi(x)$ and $\chi(x)$. The details for the proof are given in Appendix. \square

Lemma 4.1. $H_s(x)$ is bounded.

Proof. The proof is similar to Lemma 3.3, which is omitted here. \square

Based on Theorem 4.1 and Lemma 4.1, we obtain the following important results.

Theorem 4.2. *There exists a unique s such that*

$$\int_s^S H_s(x) dx + K = 0. \quad (57)$$

The pair (s, S) can be derived from the relation $H_s(S) = 0$ and (57), where

$$H_s(x) = \frac{2}{\sigma^2} \int_s^x \frac{\Phi(x)}{\Phi(\xi)} \int_\xi^\infty \tilde{g}'(\eta) \frac{\chi(\eta)}{\chi(\xi)} d\eta d\xi, \quad (58)$$

$$\begin{aligned} \chi(x) &= \frac{\int_x^{+\infty} e^{-\frac{2}{\sigma^2}(\gamma y + F(y))} dy}{\int_0^\infty e^{-\frac{2}{\sigma^2}(\gamma y + F(y))} dy}, \\ \Phi(x) &= e^{\frac{2}{\sigma^2}(\gamma x + F(x))} \chi(x). \end{aligned} \quad (59)$$

Let

$$G_s(x) = \begin{cases} \int_s^x H_s(\xi) d\xi, & x > s, \\ 0, & x \leq s. \end{cases} \quad (60)$$

and

$$\rho = \frac{\sigma^2}{2} H'_s(s) + \tilde{g}(s) + c\gamma,$$

then $(G_s(x), \rho)$ is the solution of the Q.V.I. (55).

Proof. The proof can be achieved by repeating a similar procedure to that in the discounted case in Theorem 3.1. The procedure is to show firstly that $(G_s(x), \rho)$ is the solution of (56), then it is the solution of Q.V.I. (55). \square

Obviously, from the relation $G_\alpha(x) = u_\alpha(x) + cx$, we have $G(x) = u(x) + cx$. Moreover, it is the same ρ and the same optimal strategy for (54) and (55). The following verification theorem says the long run average cost ρ_0 can be obtained by the value ρ from Q.V.I. (54), and (s, S) policy is the optimal control.

Theorem 4.3. *Let $(\rho, u(x))$ be the solution of Q.V.I. (55) and (s, S) be given by Theorem 4.2, then*

$$\rho_0 = \rho = \frac{\sigma^2}{2} \int_s^\infty \tilde{g}'(\eta) \frac{\chi(\eta)}{\chi(\xi)} d\eta + \tilde{g}(s) + c\gamma$$

and

$$\limsup_{T \rightarrow \infty} \frac{EG(y_x^{s,S}(T))}{T} = 0.$$

The proof is given in Appendix.

5. Conclusion. In this paper, we have considered a new stochastic inventory control problem under a nonlinear process which varies depending on the current inventory level. With the formulated model, we have derived and proved the optimality of the (s, S) strategy, and shown that the strategy is unique. The strategy is reduced to the case in [3] when $f(x)$ takes a linear form. Furthermore, we have considered the limiting case with ergodic control when the discount factor vanishes under the nonlinear process using the long run average cost function. Again we have derived the (s, S) strategy and proved the optimality. We hope this work can shed light on the nonlinear inventory control process and can be used for practical problems.

6. Appendix.

6.1. The derivation of (9). From the dynamic programming theory (See Bensoussan [3]; Bensoussan and Lions [5]; Fleming and Soner[12]). It can be shown that u_α satisfies the dynamic programming principle,

$$u_\alpha(x) = \inf_V E \left[\sum_{n=0}^t (K + cv_n) \exp(-\alpha\theta_n) + \int_0^t g(y_x(t, V)) \exp(-\alpha t) dt + \exp(-\alpha t) u_\alpha(y_x(t, V)) \right]. \quad (\text{A.1})$$

If an order with quantity v is made at the initial time, then $K + cv$ should be paid and the inventory level becomes $x + v$. If we proceed optimally from now on, the best we can obtain is

$$cv + K + u_\alpha(x + v).$$

Therefore,

$$u_\alpha(x) \leq cv + K + \inf_{v>0} u_\alpha(x + v). \quad (\text{A.2})$$

If no order is made at the initial time, we assume that the first order is made after time δ . Then

$$\sum_{n=0}^{\delta} (K + cv_n) \exp(-\alpha\theta_n) = 0.$$

Substituting δ with t in (5) results in

$$u_\alpha(x) \leq \inf_V E \left[\int_0^\delta g(y_x(t, V)) \exp(-\alpha t) dt + \exp(-\alpha\delta) u_\alpha(y_x(\delta, V)) \right]. \quad (\text{A.3})$$

Applying Ito lemma to $\exp(-\alpha\delta) u_\alpha(y_x(\delta, V))$ leads to

$$0 \leq \inf_V E \int_0^\delta \left[g(y_x(s, V)) - \alpha u_\alpha(y_x(s, V)) - \mathcal{A}u_\alpha(y_x(s, V)) \exp(-\alpha t) \right] ds. \quad (\text{A.4})$$

Let $\delta \rightarrow 0$ in (A.4), we have

$$\mathcal{A}u_\alpha(x) + \alpha u_\alpha(x) \leq g(x). \quad (\text{A.5})$$

For any initial inventory x , either (A.2) or (A.5) is tight. That is, the third equation of (9) holds.

6.2. The proof of Theorem 3.1.

Proof. From the construction, we need to check the complementary slackness condition of (9). When $x \leq s_\alpha$,

$$\begin{aligned} G_{\alpha, s_\alpha}(x) &= G_{\alpha, s_\alpha}(s_\alpha) \\ &= \inf_{\eta > s_\alpha} G_{\alpha, s_\alpha}(\eta) + K \\ &= \inf_{\eta > x} G_{\alpha, s_\alpha}(\eta) + K. \end{aligned} \tag{A.6}$$

The third equation above follows from

$$G_{\alpha, s_\alpha}(x') = G_{\alpha, s_\alpha}(s_\alpha) > \inf_{\eta > s_\alpha} G_{\alpha, s_\alpha}(\eta)$$

when $x \leq x' \leq s_\alpha$. Then

$$G_{\alpha, s_\alpha}(x) = \inf_{\eta > x} G_{\alpha, s_\alpha}(\eta) + K.$$

We also need to show that

$$\begin{aligned} \mathcal{A}G_{\alpha, s_\alpha}(x) + \alpha G_{\alpha, s_\alpha}(x) &\leq \tilde{g}(x) + c\gamma, x < s_\alpha, \\ G_{\alpha, s_\alpha}(x) &\leq M(u)(x), x \geq s_\alpha. \end{aligned} \tag{A.7}$$

When $x < s_\alpha$,

$$\mathcal{A}G_{\alpha, s_\alpha}(x) + \alpha G_{\alpha, s_\alpha}(x) = \alpha G_{\alpha, s_\alpha}(s_\alpha).$$

As

$$\alpha G_{\alpha, s_\alpha}(s_\alpha) = \frac{\sigma^2}{2} H'_{\alpha, s_\alpha}(s_\alpha) + \tilde{g}(s_\alpha) + c\gamma \leq \tilde{g}(x) + c\gamma,$$

it follows readily that

$$\mathcal{A}G_{\alpha, s_\alpha}(x) + \alpha G_{\alpha, s_\alpha}(s_\alpha) = \alpha G_{\alpha, s_\alpha}(s_\alpha) \leq \tilde{g}(x) + c\gamma, x < s_\alpha.$$

From $H_{\alpha, s_\alpha}(x)$ defined by (22), it is easily seen that

$$H_{\alpha, s_\alpha}(x) < 0 \text{ on } x \leq S_\alpha$$

and

$$H_{\alpha, s_\alpha}(x) > 0 \text{ on } x > S_\alpha,$$

which means that $G_{\alpha, s_\alpha}(x)$ is strictly increasing (decreasing) on $x > S_\alpha$ ($x < S_\alpha$).

Thus,

$$G_{\alpha, s_\alpha}(x) \leq \inf_{\eta > x} G_{\alpha, s_\alpha}(\eta) < M(u)(x), x \geq S_\alpha.$$

It remains to show the case $s_\alpha < x < S_\alpha$. In fact,

$$G_{\alpha, s_\alpha}(x) < G_{\alpha, s_\alpha}(s_\alpha) \leq K + G_{\alpha, s_\alpha}(S_\alpha) = K + \inf_{\eta > x} G_{\alpha, s_\alpha}(\eta)$$

on $s_\alpha < x < S_\alpha$, namely,

$$G_{\alpha, s_\alpha}(x) \leq M(u)(x)$$

on $s_\alpha < x < S_\alpha$. Thus we have

$$G_{\alpha, s_\alpha}(x) \leq M(u)(x)$$

on $x > s_\alpha$.

Let u be the solution of (9). By Ito's lemma¹, for any policy $V \in \mathcal{V}$, we apply the Ito differential rule to $u_\alpha(x)e^{-\alpha t}$:

$$\begin{aligned}
& E[e^{-\alpha t}u_\alpha(y_x(t, V)) - u_\alpha(x)] \\
&= E\left[\int_0^t -e^{-\alpha s}(\mathcal{A}u_\alpha(y_x(s, V))ds + \alpha u_\alpha(x_s))ds\right. \\
&\quad \left. + \sum_{n=0}^{N(t)} e^{-\alpha\theta_n}(u_\alpha(y_x(\theta_n, V)) - u_\alpha(y_x(\theta_n^-, V)))\right] \\
&\geq E\int_0^t e^{-\alpha s}(-g(y_x(s, V)))ds - \sum_{n=0}^{N(t)} e^{-\alpha\theta_n}(cv_n + K). \tag{A.8}
\end{aligned}$$

Let $t \rightarrow \infty$ for $\forall V \in \mathcal{V}$, we have

$$\int_0^\infty g(y_x(t, V)) \exp(-\alpha t) dt < \infty,$$

which implies

$$\lim_{t \rightarrow \infty} E[e^{-\alpha t}|y_x(t, V)] = 0. \tag{A.9}$$

Because $u'_\alpha(\cdot)$ are bounded, therefore,

$$\lim_{t \rightarrow \infty} E[e^{-\alpha t}u(y_x(t, V))] = 0. \tag{A.10}$$

Then, we have

$$u_\alpha(x) \leq \lim_{t \rightarrow \infty} \int_0^\infty e^{-\alpha t}(g(y_x(t, V)))dt + \sum_{n=0}^\infty e^{-\alpha v_n}(cv_n + K). \tag{A.11}$$

That is, for any admissible V , we have

$$u_\alpha(x) \leq J(x, V),$$

and therefore,

$$u_\alpha(x) \leq \inf_{V \in \mathcal{V}} J(x, V). \tag{A.12}$$

Let V^{s^*, S^*} denote the (s^*, S^*) policy. It is defined as follows. Let

$$\begin{aligned}
\theta_1 &= \{t, t \geq 0, y_x(t, V^{s^*, S^*}) \leq s^*\}, v_1 = S^* - \min(x, s^*), \\
\theta_n &= \inf\{t, t > \theta_{n-1}, y_x(t, V^{s^*, S^*}) \leq s^*\}, v_n = S^* - s^*, n = 2, \dots.
\end{aligned}$$

If inventory $y_x(t, V^{s^*, S^*}) > s^*$, then

$$\mathcal{A}u_\alpha(x) + \alpha u_\alpha(x) = g(x).$$

Otherwise, there is a replenishment, and

$$u(y_x(\theta_n, V^{s^*, S^*})) = u(y_x(\theta_n^-, V^{s^*, S^*})) + K + cv_n, n = 1, \dots.$$

¹It can be seen from Dellacherie and Meyer 1980 [10], Thm. VIII.27, Ito's lemma still holds for the function of $u_\alpha(x) \in \mathcal{C}^2$ except for s^* . We can also refer to the Remark 3.2 of Liu *et al.* [14] to see how Ito's lemma can be applied here.

Therefore, with (s^*, S^*) policy, the inequality of (A.8) becomes an equality.

In the next proposition we show $E \int_0^{+\infty} e^{-\alpha t} y_x^2(t, V^{s^*, S^*}) dt < \infty$, which implies

$$\int_0^\infty g(y_x(t, V^{s^*, S^*})) \exp(-\alpha t) dt < \infty.$$

Then

$$\lim_{t \rightarrow \infty} E(e^{-\alpha t} |y_x(t, V^{s^*, S^*})|) = 0,$$

and the inequality of (A.11) becomes an equality. That is,

$$u_\alpha(x) = J(x, V^{s^*, S^*})$$

and the policy (s^*, S^*) is optimal. \square

Proposition 6.1. $E \int_0^{+\infty} e^{-\alpha t} y_x^2(t, V^{s^*, S^*}) dt < \infty$.

Proof. When $0 < t < \theta_1$, the dynamics $y(t)$

$$dy(t) = -(\gamma + f(y))dt - \sigma dW(t), y(0) = x. \quad (\text{A.13})$$

When $n \geq 1$, $\theta_n < t < \theta_{n+1}$, the dynamics $y(t)$

$$dy(t) = -(\gamma + f(y))dt - \sigma dW(t), y(\theta_n) = S. \quad (\text{A.14})$$

Applying Ito Lemma to $y^2(t)e^{-\alpha t}$ leads to

$$\begin{aligned} dy^2(t)e^{-\alpha t} &= -\alpha e^{-\alpha t} y^2(t)dt + 2e^{-\alpha t} y(t)(\gamma + f(y(t)))dt - 2\sigma e^{-\alpha t} y(t)dW(t) \\ &\leq -\alpha e^{-\alpha t} y^2(t)dt + 2e^{-\alpha t} y(t)(\gamma + f(0))dt - 2\sigma e^{-\alpha t} y(t)dW(t), \end{aligned}$$

where the inequality follows from that

$$f(y) = f(0) + y \int_0^1 f'(\lambda y) d\lambda \quad (\text{A.15})$$

and

$$yf(y) = yf(0) + y^2 \int_0^1 f'(\lambda y) d\lambda, \quad (\text{A.16})$$

which yields to

$$yf(y) \geq yf(0).$$

By integrating $y^2(t)e^{-\alpha t}$ from θ_n to θ_{n+1}^- and taking expectation, we have

$$\begin{aligned} &E(y^2(\theta_{n+1}^-)e^{-\alpha \theta_{n+1}}) - E(y^2(\theta_n)e^{-\alpha \theta_n}) \\ &= s^2 E(e^{-\alpha \theta_{n+1}}) - S^2 E(e^{-\alpha \theta_n}) \\ &\leq E \int_{\theta_n}^{\theta_{n+1}} e^{-\alpha t} (-\alpha y^2(t) - 2y(t)(\gamma + f(0))) dt. \\ &\leq E \int_{\theta_n}^{\theta_{n+1}} e^{-\alpha t} (-\alpha y^2(t) + \frac{\alpha y^2(t)}{2} + \frac{8}{\alpha} (\gamma + f(0))^2) dt. \end{aligned} \quad (\text{A.17})$$

Similarly, we have

$$\begin{aligned}
& E(y^2(\theta_1^-)e^{-\alpha\theta_1}) - E(y^2(0)) \\
&= s^2 E(e^{-\alpha\theta_{n+1}}) - y_0^2 E(e^{-\alpha\theta_n}) \\
&\leq E \int_0^{\theta_1} e^{-\alpha t} (-\alpha y^2(t) + \frac{\alpha y^2(t)}{2} + \frac{8}{\alpha} (\gamma + f(0))^2) dt. \tag{A.18}
\end{aligned}$$

It follows from (A.17) and (A.18) that

$$\begin{aligned}
& s^2 E e^{-\alpha\theta_{n+1}} - y_0^2 + (S^2 - s^2) \sum_{i=1}^n E(e^{-\alpha\theta_n}) \\
&\leq -\frac{\alpha}{2} E \int_0^{+\infty} e^{-\alpha t} y^2(t) dt + \frac{8}{\alpha} (\gamma + f(0))^2 E \int_0^{\infty} e^{-\alpha t} dt. \tag{A.19}
\end{aligned}$$

That is,

$$\begin{aligned}
& \frac{\alpha}{2} E \int_0^{+\infty} e^{-\alpha t} y^2(t) dt \\
&\leq \frac{8}{\alpha} (\gamma + f(0))^2 + s^2 E e^{-\alpha\theta_{n+1}} - y_0^2 + (S^2 - s^2) \sum_{i=1}^n E(e^{-\alpha\theta_n}). \tag{A.20}
\end{aligned}$$

If $s^2 E e^{-\alpha\theta_{n+1}} + (S^2 - s^2) \sum_{i=1}^n E(e^{-\alpha\theta_n}) < \infty$ when $n \rightarrow \infty$, we obtain the desired result. In the remainder we show this property.

Let $\tau = \inf\{t | y_S(t) = s\}$ and $E e^{-\alpha\tau} = \eta$. When $n \geq 2$, we have $\theta_n = \theta_{n-1} + \tau$ and

$$E e^{-\alpha(\theta_n)} = E e^{-\alpha\theta_1} \eta^{n-1}.$$

It is easy to show that $E e^{-\alpha\tau} = u(S) < 1$. Therefore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n E(e^{-\alpha\theta_n}) = \lim_{n \rightarrow \infty} E e^{-\alpha\theta_1} \sum_{i=1}^n \eta^{n-1} < \infty$$

and

$$s^2 E e^{-\alpha\theta_{n+1}} = 0,$$

which finishes the proof of (A.20). \square

6.3. The proof of Theorem 4.1. The following Lemma 6.1 and Lemma 6.2 will be needed for the proof of Theorem 6.1 below, which in turn is the main result for proving Theorem 4.1.

Lemma 6.1. *For $\alpha_1 < \alpha_2$, Suppose that $\chi_{\alpha_1}(x)$ and $\chi_{\alpha_2}(x)$ are the solutions of the following equations, respectively,*

$$\begin{cases} -\frac{\sigma^2}{2} \chi_{\alpha_1}''(x) - (f(x) + \gamma) \chi_{\alpha_1}'(x) + \alpha_1 \chi_{\alpha_1}(x) = 0, \\ -\frac{\sigma^2}{2} \chi_{\alpha_2}''(x) - (f(x) + \gamma) \chi_{\alpha_2}'(x) + \alpha_2 \chi_{\alpha_2}(x) = 0, \end{cases} \tag{A.21}$$

with boundary condition

$$\chi_{\alpha_i}(0) = 1, \chi_{\alpha_i}(+\infty) = 0, \quad i = 1, 2.$$

Then $\chi_{\alpha_1}(x) > \chi_{\alpha_2}(x)$ for $x > 0$, and $\chi_{\alpha_1}(x) < \chi_{\alpha_2}(x)$ for $x < 0$.

Proof. We first consider the case that $x \geq 0$. From the fact $\chi_{\alpha_2}(x) > 0$ and the assumption that $\alpha_1 < \alpha_2$, we know

$$-\frac{\sigma^2}{2}\chi''_{\alpha_1}(x) - (f(x) + \gamma)\chi'_{\alpha_1}(x) + \alpha_2\chi_{\alpha_1}(x) > 0. \quad (\text{A.22})$$

Denote

$$L(x) = \chi_{\alpha_1}(x) - \chi_{\alpha_2}(x).$$

Then

$$\lim_{x \rightarrow \infty} L(x) = 0.$$

From (A.21) and (A.22), we have

$$-\frac{\sigma^2}{2}L''(x) - (f(x) + \gamma)L'(x) + (\alpha_2 - \alpha_1)L(x) > 0. \quad (\text{A.23})$$

If there exists x such that $L(x) < 0$, then there must exist x^* such that

$$L(x^*) = \min_{x \geq 0} L(x).$$

At the point x^* , we have

$$L''(x^*) > 0, L'(x^*) = 0, L(x^*) < 0.$$

However, these properties contradict with (A.22). Thus, $L(x) > 0$ holds. Namely, $\chi_{\alpha_1}(x) > \chi_{\alpha_2}(x)$ for $x > 0$.

As $\chi_{\alpha_1}(x) > \chi_{\alpha_2}(x)$ for $x > 0$ and $\chi_{\alpha_1}(0) = \chi_{\alpha_2}(0)$, we have

$$\chi'_{\alpha_1}(0) > \chi'_{\alpha_2}(0).$$

We claim that $\chi_{\alpha_1}(x) < \chi_{\alpha_2}(x)$ for all $x < 0$. Otherwise, if there exists x such that $\chi_{\alpha_1}(x) \geq \chi_{\alpha_2}(x)$ when $x < 0$, let

$$x_- = \max\{x, x < 0, \chi_{\alpha_1}(x) = \chi_{\alpha_2}(x)\}.$$

Due to $\chi'_{\alpha_1}(0) > \chi'_{\alpha_2}(0)$, we have

$$x_- < 0.$$

Then

$$\chi_{\alpha_1}(x) < \chi_{\alpha_2}(x), \quad x \in [x_-, 0].$$

On interval $[x_-, 0]$, there is a contradiction with (A.22) by repeating the proof of case $x > 0$. So we have

$$\chi_{\alpha_1}(x) < \chi_{\alpha_2}(x)$$

for all $x < 0$. □

Lemma 6.2. *Let $\chi(x)$ be the solution of*

$$-\frac{\sigma^2}{2}\chi''(x) - (f(x) + \gamma)\chi'(x) = 0, \quad (\text{A.24})$$

with boundary condition

$$\chi(0) = 1, \chi(+\infty) = 0.$$

Then, when $\alpha \rightarrow 0$, $\chi_\alpha(x)$ increasingly(decreasingly) converges to $\chi(x)$ for $x > 0(x < 0)$.

Proof. Similar to the proof of Lemma 6.1, we just prove the case $x > 0$. It can be easily checked that $\chi_\alpha(x) \leq \chi(x)$ for $x > 0$ and $\chi_\alpha(x)$ increases with α . Moreover, from Lemma 6.1 we know $\chi_\alpha(x)$ is increasing/decreasing with α on $[-\infty, 0)/[0, +\infty)$, and bounded by $\chi(x)$. Thus, $\lim_{\alpha \rightarrow 0} \chi_\alpha(x)$ exists. We denote it by $\chi_{lim}(x)$. Also, it can be proven that

$$\lim_{\alpha \rightarrow 0} \chi'_\alpha(x) = \chi'_{lim}(x), \lim_{\alpha \rightarrow 0} \chi''_\alpha(x) = \chi''_{lim}(x).$$

Thus, we have

$$-\frac{\sigma^2}{2} \chi''_{lim}(x) - (f(x) + \gamma) \chi'_{lim}(x) = 0, \quad (\text{A.25})$$

with $\chi_{lim}(0) = 1$. Suppose that $\chi''_{lim}(+\infty) = a$, obviously, $a \geq 0$. As $\chi_\alpha(x) < \chi(x)$, we have

$$\lim_{x \rightarrow +\infty} \chi_{lim}(x) \leq \lim_{x \rightarrow \infty} \chi(x),$$

which implies $a \leq 0$. Thus we have

$$a = 0.$$

So $\chi_{lim}(x)$ is the solution of (A.25).

Theorem 6.1. $H_{\alpha,s}(x) \rightarrow H_s(x)$ uniformly on any compact set of \mathbb{R} , with

$$H_s(x) = \frac{2}{\sigma^2} \int_s^x \frac{\Phi(x)}{\Phi(\xi)} Q(\xi) d\xi,$$

Proof. Without lose of generality, it suffices to prove that for any closed interval $[s, b]$, $H_{\alpha,s}(x) \rightarrow H_s(x)$ uniformly. Namely, it suffices to prove that for any ϵ , there exists α_0 such that

$$|H_s(x) - H_{\alpha,s}(x)| \leq \frac{\epsilon}{2}, x \in [s, b] \quad (\text{A.26})$$

when $\alpha < \alpha_0$. In the following we will prove (A.26).

Denote

$$\begin{aligned} M_\alpha(\xi) &= \frac{1}{\Phi_\alpha(\xi) \chi_\alpha(\xi)}, \\ N_\alpha(\xi) &= \int_\xi^\infty \tilde{g}'_\alpha(\eta) \chi_\alpha(\eta) d\eta, \\ M(\xi) &= \frac{1}{\Phi(\xi) \chi(\xi)}, \\ N(\xi) &= \int_\xi^\infty \tilde{g}'(\eta) \chi(\eta) d\eta. \end{aligned} \quad (\text{A.27})$$

From Lemma 6.2, it can be easily verified that $M_\alpha(\xi)$ uniformly converges and is uniformly bounded on any compact set of \mathbb{R} and $N_\alpha(\xi)$ converges uniformly on any

compact set of \mathbb{R} .

Then

$$\begin{aligned}
& |H_s(x) - H_{\alpha,s}(x)| \\
& \leq \frac{2\Phi(x)}{\sigma^2} \int_s^x (M_\alpha(\xi)N_\alpha(\xi) - M(\xi)N(\xi))d\xi \\
& = \frac{2\Phi(x)}{\sigma^2} \int_s^x (M_\alpha(\xi)(N_\alpha(\xi) - N(\xi)) + N(\xi)(M_\alpha(\xi) - M(\xi)))d\xi \\
& \leq \frac{2\Phi(x)}{\sigma^2} (B_1 \int_s^b |N_\alpha(\xi) - N(\xi)|d\xi + B_2 \int_s^b (M_\alpha(\xi) - M(\xi))d\xi), \quad (\text{A.28})
\end{aligned}$$

where

$$\begin{aligned}
B_1 &:= \sup\{M_\alpha(\xi), \xi \in [a, b]\}, \\
B_2 &:= \sup\{N(\xi), \xi \in [a, b]\}.
\end{aligned}$$

Denote

$$B = \max(B_1\Phi(s), B_2\Phi(s)).$$

Then there exists $\alpha_1 > 0$, when $\alpha < \alpha_1$,

$$|M_\alpha(\xi) - M(\xi)| < \frac{\epsilon}{B}, |N_\alpha(\xi) - N(\xi)| < \frac{\epsilon}{B}, \forall \xi \in [a, b].$$

Thus

$$|(H_s(a) - H_s(x)) - (H_{\alpha,s}(a) - H_{\alpha,s}(x))| < \epsilon.$$

□

Theorem 6.2. *There exists M such that $S_\alpha \leq M$ for small enough α .*

Proof. As $H_{\alpha,s}(x)$ converges to $H_s(x)$ uniformly on any compact set of \mathbb{R} , then there exist $\alpha_{h_1}, M_{h_1} > 0$, when $\alpha < \alpha_2$, we have

$$H_{\alpha,s}(0) < M_{h_1}.$$

For $x > 0$, from (28), we have

$$H_{\alpha,s}(x) \geq \frac{h}{\alpha + M} + c + \Phi_\alpha(x)(H_{\alpha,s}(0) - (\frac{h}{\alpha + M} + c)). \quad (\text{A.29})$$

Since $\Phi_\alpha(x)$ converges to $\Phi(x)$ uniformly on any compact set of \mathbb{R} , then there exist M_{h_2} and α_{h_2} such that

$$\Phi_\alpha(x) \leq \frac{\epsilon}{M_{h_1} + \frac{h}{k} + c}$$

for any $\alpha < \alpha_{h_2}$ and $x \geq M_{h_2}$.

Denote $M_h := \max(M_{h_i}, i = 1, 2)$, $\alpha_h = \min(1, \alpha_{h_i}, i = 1, 2)$, then

$$H_{\alpha,s}(x) \geq \frac{h}{\alpha + M} + c - \epsilon \geq c + \frac{h}{1 + M} - \epsilon \quad (\text{A.30})$$

for any $x \geq M_h$ and $\alpha \leq \alpha_h$.

Let x_0 be the root of $Q(x) = 0$. If $s \geq x_0$, it follows from (A.26) that $H_{\alpha,s}(x) > 0$ on $(s, +\infty)$, thus $S_\alpha(s) = s$. Since $S_\alpha(s)$ satisfies $H_{\alpha,s}(S_\alpha(s)) = 0$, it follows from (A.30) that $S_\alpha(s) < M_h$, specially, $S_\alpha < M_h$, when $\alpha \leq \alpha_h$. \square

Theorem 6.3. s_α is bounded below.

Proof. Let s_0 be a fixed value with $s_0 < x_0 < 0$. We can assume that for sufficiently small α , $s_0 < x_{\alpha,0}$. If $s_\alpha \geq s_0$, then it is bounded below by s_0 .

Now let us check the case that $s_\alpha < s_0$. Similar to Lemma 3.6, there exists M_γ such that

$$\begin{aligned} \Gamma'(s) &\geq -\frac{Q(s)}{\Phi(s)} \int_s^0 \Phi(\eta) d\eta \left(1 - \frac{\int_{x_0}^0 \Phi(\eta) d\eta}{\int_s^0 \Phi(\eta) d\eta} \right) \\ &\geq \lim_{s \rightarrow -\infty} \frac{2}{\sigma^2} (p - cM) \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy. \end{aligned} \quad (\text{A.31})$$

Let $\alpha = 0$ in (17), we have

$$(\gamma + f(x))\chi'(x) \leq \chi(x) + 1. \quad (\text{A.32})$$

Then

$$\frac{\chi(x)}{\chi'(x)} \leq \gamma + f(x), \quad (\text{A.33})$$

and

$$\frac{\Phi'(x)}{\Phi(x)} \geq \frac{1}{\gamma + f(x)} + \frac{2}{\sigma^2} (\gamma + f(x)). \quad (\text{A.34})$$

Therefore,

$$\lim_{s \rightarrow -\infty} \int_s^0 \frac{\chi_\alpha(y)}{\chi_\alpha(s)} dy \int_s^0 \frac{\Phi_\alpha(y)}{\Phi_\alpha(s)} dy \geq \lim_{s \rightarrow -\infty} \frac{\chi(s)}{\chi'(s)} \frac{\Phi'(s)}{\Phi(s)} \geq \frac{\sigma^2}{2}, \quad (\text{A.35})$$

which implies

$$\Gamma'(s) \geq P - cM. \quad (\text{A.36})$$

Thus, for sufficiently small α ,

$$-\frac{Q_\alpha(s)}{\Phi_\alpha(s)} \int_s^0 \Phi_\alpha(\eta) d\eta \left(1 - \frac{\int_{x_{\alpha,0}}^0 \Phi_\alpha(\eta) d\eta}{\int_s^0 \Phi_\alpha(\eta) d\eta} \right) \geq \frac{p - cM}{2}, s \leq -M_\gamma. \quad (\text{A.37})$$

Denote

$$B_\gamma = \frac{p - cM}{2}.$$

If $s < \min(-M_\gamma, s_0)$, then

$$\Gamma'_\alpha(s) \geq B_\gamma,$$

and

$$\Gamma_\alpha(-M_\gamma) - \Gamma_\alpha(s) \geq (-M_\gamma - s)B_\gamma. \quad (\text{A.38})$$

As $\Gamma_\alpha(x_{\alpha,0}) = 0$ and $\Gamma'_\alpha(x) > 0$ for $x < x_{\alpha,0}$, therefore, we have $\Gamma_\alpha(-M_\gamma) < 0$. Thus, (A.38) becomes

$$-\Gamma_\alpha(s) \geq (-M_\gamma - s)B_\gamma. \quad (\text{A.39})$$

Notice that $\Gamma_\alpha(s_\alpha) = -K$, substituting s with s_α in (A.38) yields to

$$K \geq (-M_\gamma - s_\alpha)B_\gamma,$$

that is, $s_\alpha \geq -M_\gamma - \frac{K}{B_\gamma}$. So s_α is bounded below by $-M_\gamma - \frac{K}{B_\gamma}$ if α is small enough.

As $-M_\gamma - \frac{K}{B_\gamma} \leq s_\alpha \leq S_{s_\alpha} \leq M$ and $\gamma_\alpha(s_\alpha) = -K$, thus we can extract a subsequence such that $s_\alpha \rightarrow s$, $S_{s_\alpha} \rightarrow S$ and $\gamma(s) = -K$. Similar to Lemma 3.8, the policy (s, S) is unique. \square

6.4. The proof of Theorem 4.3.

Proof. Suppose that $(u(x), \rho)$ is the solution of (55), then for any admissible control V , by Ito lemma, we have

$$\begin{aligned} & E[u(y_x(T, V)) - u(y_x(0, V))] \\ &= E\left[\int_0^T -\mathcal{A}u(y_x(s, V))ds + \sum_{n=0}^{N(T)} (u(y_x(\tau_n, V)) - u(y_x(\tau_n^-, V)))\right] \\ &\geq E\int_0^T (\rho - g(y_x(s, V)))ds - \sum_{n=0}^{N(T)} (c\gamma_n + K), \end{aligned} \quad (\text{A.40})$$

thus,

$$\begin{aligned} & \frac{1}{T} E(u(y_x(T, V)) - u(y_x(0, V))) \\ &\geq \frac{1}{T} E\left[\int_0^T (\rho - g(y_x(s, V)))ds - \sum_{n=0}^{N(T)} (c\gamma_n + K)\right]. \end{aligned} \quad (\text{A.41})$$

Since $u_\alpha(x) \geq 0$, we have $u(x) \geq 0$. Then

$$\begin{aligned} \rho &\leq \limsup_T \frac{1}{T} [E\int_0^T g(y_x(s, V))ds - Eu(y_x(T, V)) + \sum_{n=0}^{N(T)} (c\gamma_n + K)] \\ &\leq J(x, y_x(0, V)), \end{aligned} \quad (\text{A.42})$$

which means

$$\rho \leq \rho^0. \quad (\text{A.43})$$

On the other hand, as $(\rho, u(x))$ is the solution with (s, S) , it satisfies

$$\begin{aligned} & \mathcal{A}u(x) + \rho = g(x) + c(\nu - k\gamma), x > s \\ & u(x) = K + \inf_{\eta \geq x} u(\eta) = K + u(S), x \leq s. \end{aligned} \quad (\text{A.44})$$

Then by this (s, S) strategy, the inequality of (A.40) becomes equality. Thus,

$$\rho = J(x, y_x(0, V^*),$$

which implies that

$$\rho \geq \rho_0. \quad (\text{A.45})$$

From (A.43) and (A.45), we show that $\rho = \rho_0$ and $\limsup_{T \rightarrow \infty} \frac{Eu(y_x^{s,S}(T))}{T} = 0$. It follows from (33) that

$$\rho = \frac{\sigma^2}{2} \int_s^\infty \tilde{g}'(\eta) \frac{\chi(\eta)}{\chi(\xi)} d\eta + \tilde{g}(s) + c(\nu + \gamma).$$

Thus we get

$$\rho_0 = \frac{\sigma^2}{2} \int_s^\infty \tilde{g}'(\eta) \frac{\chi(\eta)}{\chi(\xi)} d\eta + \tilde{g}(s) + c(\nu + \gamma).$$

□

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