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On two novel types of three-way decisions in three-way decision spaces

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Abstract

In order to explore a unified theory of three-way decisions proposed by Yao, Hu introduced three-way decision spaces through an axiomatic method, established the corresponding three-way decisions, and proposed two open problems on the changes of decision parameters in definition of three-way decisions. For answering these two questions, this paper firstly discusses the parameter changes in the assumptions from $0 \le \beta < \alpha \le 1$ to $0 \le \beta \le \alpha \le 1$ and inequality $E(A)(x) \le \beta$ in the rejection region is replaced by $E(A)(x) < \beta$. Under the circumstance, this paper introduces new type of three-way decisions in three-way decision spaces and discusses properties of the three-way decisions, lower and upper approximations induced by three-way decisions, aggregation three-way decisions over multiple three-way decision spaces and dynamic three-way decisions on three-way decision spaces. Then this paper discusses another question on refusal decision region when the uncertain region is defined by using inequality $\beta < E(A)(x) < \alpha$ and gives one example to illustrate the similarity and difference among these three-way decisions based on three-way decision spaces.

Keywords: Partially ordered sets; Fuzzy sets; Interval-valued fuzzy sets; Rough sets; Three-way decision spaces; Three-way decisions.

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1. Introduction

The theory of three-way decisions (3WD) proposed by Yao is an extension of classic two-way decisions (2WD) [41-45], whose basic idea comes from Pawlak's rough sets [28] and probability rough sets proposed by Yao [38-46]. In recent years, researches on three-way decisions have been paid close attention by a growing number of scholars and are mainly reflected in the following three aspects [9, 10, 15, 47].

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The first one involves the background researches on three-way decisions, such as decision-theoretic rough sets (DTRS) [3, 38-40], game-theoretic rough sets (GTRS) [8], interval-valued fuzzy rough sets (IVFRS) [11, 13-14], interval-valued decision-theoretic rough sets [21], intuitionistic fuzzy decision-theoretic rough sets [22], triangular fuzzy decision-theoretic rough sets [23], dominance-based fuzzy rough sets [4-7], variable precision fuzzy rough sets [30, 34, 53], multi-granulation decision-theoretic rough sets [31], fuzzy covering-based rough sets [35, 36] and so on.

The second one refers to theoretical framework researches on three-way decisions, which mainly contain value domain of evaluation functions [44], construction and interpretation of evaluation functions [9, 10, 15, 16], the mode of three-way decisions [44, 45], three-way decisions based fuzzy probability [52, 54] and so on.

The third one is on application researches of three-way decisions, such as incomplete information system [18, 25], risk decision making [20], group decision making [24], prediction [17], cluster [12, 48], investment [26], multi-agent [37], recognition [19, 32], social networks [29] and cognitive networks [27], recommender systems [51], multi-granular mining [1] and so on.

Hu theorized three-way decisions and established three-way decision spaces (3WDS) [9, 10], such that existing three-way decisions are the special examples of three-way decision spaces, such as three-way decisions based on fuzzy sets, random sets and rough sets etc. At the same time, three-way decisions over multiple three-way decision spaces, dynamic two-way decisions and dynamic three-way decisions based on three-way decision spaces and three-way decisions with a pair of evaluation functions were also established in [9, 10, 15, 16]. In three-way decision spaces, three basic elements are unified, which are decision measurement, decision conditions and decision evaluation functions.

In a large number of existing literatures, the two decision parameters are generally considered to be not equal and we use the more general inequality $E(A)(x) \geq \alpha$ in the acceptance region and $E(A)(x) \leq \beta$ in the rejection region. In addition, there are other discussions on two decision parameters.

(1) Wei and Zhang suggested $0 < \beta \leq \alpha < 1$ in [33], in which inequality $E(A)(x) > \alpha$ is used in the acceptance region and $E(A)(x) < \beta$ in the rejection region.

(2) Yao and Wong assumed two decision parameters are equal and nonzero in [46], in which inequality $E(A)(x) > \alpha$ is used in the acceptance region and $E(A)(x) < \alpha$ in the rejection region.

In [9] two questions on the definition of three-way decisions were proposed which are not the same as existing methods. The two questions are listed as follows.

- The first question is what changes are there in three-way decisions when the condition $0 \leq \beta < \alpha \leq 1$ is changed to $0 \leq \beta \leq \alpha \leq 1$ and inequality $E(A)(x) \leq \beta$ in the rejection region is replaced by $E(A)(x) < \beta$.
- The second question is what changes are there in three-way decisions when the uncertain region is defined by using inequality $\beta < E(A)(x) < \alpha$.

This paper discusses the above two questions and gives the three-way decisions and dynamic three-way decisions under new assumption of two decision parameters.

The rest of this paper is organized as follows. In Section 2, the preliminary section, three-way decision spaces and three-way decisions are recalled for the further discussion in the following sections. Section 3 establishes one new type of three-way decisions in three-way decision spaces through considering parameter changes in assumptions, and discusses properties of the three-way decisions, lower and upper approximations induced by three-way decisions, aggregation three-way decisions over multiple three-way decision spaces and dynamic three-way decisions on three-way decision spaces. Section 4 establishes the second new type of three-way decisions on three-way decision spaces through the consideration of a refusal decision region and gives one example to illustrate the similarity and difference among these three-way decisions based on three-way decision spaces. Section 5 concludes.

2. Preliminary

This section recalls decision evaluation function, three-way decision space, three-way decisions, and the optimistic and pessimistic three-way decision.

2.1 Three-way decision space

In this paper $(P,\leq_P)$ is a partially ordered set with an involutive negator N_P, the minimum 0_P and maximum 1_P which is written as $(P,\leq_P,N_P,0_P,1_P)$ [10]. If this *P* constitutes a lattice, then it is written as $(P,\wedge_P,\vee_P,N_P,0_P,1_P)$. The following notations are used.

$$I^{(2)}=\left\{[a^-,a^+]\mid 0\leq a^-\leq a^+\leq 1\right\} \text{ and } \bar{a}=[a,a] \text{ for } a\in[0,1].$$

Namely $I^{(2)}$ is a family of closed interval numbers over [0, 1] and $\bar{a}=[a,a]$ is a singleton set of real number *a* over [0, 1].

Let *X* and *Y* be two universes. $Map(X,Y)$ is a set of all mappings from *X* to *Y*, i.e. $Map(X,Y)=\{f\mid f:X\rightarrow Y\}$. If $A\in Map(U,P)$, then *A* is called a *P*-fuzzy set of *U*. Especially, if $A\in Map(U,\{0,1\})$, then *A* is a subset of *U*, i.e. $Map(U,\{0,1\})$ is the power set of *U*. If $A\in Map(U,[0,1])$, then *A* is a fuzzy set of *U* [49], namely $Map(U,[0,1])$ is the fuzzy power set of *U*. If $A\in Map(U,I^{(2)})$, then *A* is an interval-valued fuzzy set of *U* [2, 50]. An interval-valued fuzzy set *A* is also denoted as $[A^-,A^+]$.

Let *U* be a universe and $A\in Map(U,P)$. Then the complement of *A* in *U* is defined by the following formula

$$N_P(A)(x)=N_P(A(x)).$$

If $A,B\in Map(U,P)$, then $A\subseteq_P B$ is defined by $A(x)\leq_P B(x)$, $\forall x\in U$. And $(0_P)_U(x)=0_P,\forall x\in U$ and $(1_P)_U(x)=1_P,\forall x\in U$. On lattice $(P,\wedge_P,\vee_P,N_P,0_P,1_P)$, we define for $A,B\in Map(U,P)$,

$$\left(A\bigcup\nolimits_P B\right)(x)=A(x)\vee_P B(x) \text{ and}$$

$$\left(A\bigcap\nolimits_P B\right)(x)=A(x)\wedge_P B(x).$$

Obviously $(Map(U,P),\subseteq_P)$ is a partially ordered set with an involutive negator N_P, the minimum $(0_P)_U$ and maximum $(1_P)_U$. Especially in [0,1], $\subseteq_P$, N_P, $(0_P)_U$ and $(1_P)_U$ are simply written as $\subseteq$, N, 0_U and 1_U respectively.

Let $(P_C,\leq_{P_C} N_{P_C},0_{P_C},1_{P_C})$ and $(P_D,\leq_{P_D},N_{P_D},0_{P_D},1_{P_D})$ be two partially ordered sets in the following. Let *U* be a nonempty universe for which a decision to make on it, called decision universe and *V* be a nonempty universe where a condition function is defined, named condition universe.

Definition 2.1. [10] Let *U* be a decision universe and *V* be a condition universe. Then a mapping $E: Map(V, P_C) \to Map(U, P_D)$ is called a *decision evaluation function* of *U*, if it satisfies the following axioms.

(E1) Minimum element axiom

$E((0_{P_C})_V) = (0_{P_D})_U$, i.e., $E((0_{P_C})_V)(x) = 0_{P_D}, \forall x \in U$

(E2) Monotonicity axiom

$A \subseteq_{P_C} B \Rightarrow E(A) \subseteq_{P_D} E(B), \forall A, B \in Map(V, P_C)$, i.e., $E(A)(x) \leq_{P_D} E(B)(x), \forall x \in U$ and

(E3) Complement axiom

$N_{P_D}(E(A)) = E(N_{P_C}(A)), \forall A \in Map(V, P_C)$, i.e., $N_{P_D}(E(A))(x) = E(N_{P_C}(A))(x), \forall x \in U$.

$E(A)(x)$ is called a decision evaluation function of *U* for $A \in Map(V, P_C)$.

As referred to [9, 10], background of these axioms comes from the common properties of the large amount of evaluation functions in probabilistic rough sets, such as $\frac{|A \cap [x]_R|}{|[x]_R|}$ for a crisp subset *A* and an equivalence relation *R* over a finite universe. In $\frac{|A \cap [x]_R|}{|[x]_R|}$, for Minimum element axiom, $\frac{|\varnothing \cap [x]_R|}{|[x]_R|} = 0$; for Monotonicity axiom, $A \subseteq B$ implies $\frac{|A \cap [x]_R|}{|[x]_R|} \leq \frac{|B \cap [x]_R|}{|[x]_R|}$; for Complement axiom, $\frac{|A^c \cap [x]_R|}{|[x]_R|} = 1 - \frac{|A \cap [x]_R|}{|[x]_R|}$. In order to understand further information on the three axioms, please see [6, 7, 12].

In [10], the following three-way decision space was introduced.

Given decision universe *U*, condition domain $Map(V, P_C)$, decision value domain P_D and evaluation function *E*, then $(U, Map(V, P_C), P_D, E)$ is called a *three-way decision space*.

2.2. Three-way decisions

For the convenience of expression, decision domain $(P_D, \leq_P, N_P, 0_P, 1_P)$ is denoted by $(P_D, \leq, N, 0, 1)$. In [10], three-way decisions and the corresponding lower and upper approximations were introduced in three-way decision spaces.

Definition 2.2. [10] Let $(U, Map(V, P_C), P_D, E)$ be a three-way decision space, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \leq \beta < \alpha \leq 1$. Then three-way decisions are defined as follows.

(1) Acceptance region: $ACP_{(\alpha,\beta)}(E, A) = \{x \in U \mid E(A)(x) \geq \alpha\}$.

(2) Rejection region: $REJ_{(\alpha,\beta)}(E, A) = \{x \in U \mid E(A)(x) \leq \beta\}$.

(3) Uncertain region: $UNC_{(\alpha,\beta)}(E, A) = \left(ACP_{(\alpha,\beta)}(E, A) \cup REJ_{(\alpha,\beta)}(E, A)\right)^c$.

If P_D is a linear order, then $UNC_{(\alpha,\beta)}(E, A) = \{x \in U \mid \beta < E(A)(x) < \alpha\}$.

Let $(U, Map(V, P_C), P_D, E)$ be a three-way decision space, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \leq \beta < \alpha \leq 1$. Then

$$\underline{apr}_{(\alpha,\beta)}(E, A) = ACP_{(\alpha,\beta)}(E, A)$$

and

$$\overline{apr}_{(\alpha,\beta)}(E, A) = \left(REJ_{(\alpha,\beta)}(E, A)\right)^c$$

are called *the lower approximation and upper approximation* of *A* respectively.

In [10], the optimistic and pessimistic three-way decisions and the corresponding lower and upper approximations were introduced over multiple three-way decision spaces.

Definition 2.3. [10]. Let $(U, Map(V,P_C), P_D, E_i)$ $(i=1,2,\cdots,n)$ be *n* three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\le\beta<\alpha\le 1$. Then the optimistic three-way decisions over *n* three-way decision spaces are defined as follows.

(1) Acceptance region: $ACP^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcup\limits_{i=1}^{n}ACP_{(\alpha,\beta)}(E_i,A)=\bigcup\limits_{i=1}^{n}\{x\in U\mid E_i(A)(x)\ge\alpha\}$,

(2) Rejection region: $REJ^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcap\limits_{i=1}^{n}REJ_{(\alpha,\beta)}(E_i,A)=\bigcap\limits_{i=1}^{n}\{x\in U\mid E_i(A)(x)\le\beta\}$,

(3) Uncertain region: $UNC^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(ACP^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$.

Then the pessimistic three-way decisions over *n* three-way decision spaces are defined as follows.

(1) Acceptance region: $ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcap\limits_{i=1}^{n}ACP_{(\alpha,\beta)}(E_i,A)=\bigcap\limits_{i=1}^{n}\{x\in U\mid E_i(A)(x)\ge\alpha\}$,

(2) Rejection region: $REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcup\limits_{i=1}^{n}REJ_{(\alpha,\beta)}(E_i,A)=\bigcup\limits_{i=1}^{n}\{x\in U\mid E_i(A)(x)\le\beta\}$,

(3) Uncertain region: $UNC^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$.

Defintion 2.4. [10]. If $A\in Map(V,P_C)$, then

$$\underline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP^{op}_{(\alpha,\beta)}(E_{1\sim n},A)$$

and

$$\overline{apr}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(REJ^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$$

are referred to as *the lower approximation and upper approximation* of *A* with regard to optimistic three-way decisions over *n* three-way decision spaces respectively. Similarly

$$\underline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)$$

and

$$\overline{apr}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$$

are referred to as the lower approximation and upper approximation of *A* with regard to pessimistic three-way decisions over *n* three-way decision spaces respectively.

3. Three-way decisions with a new parameter assumption in three-way decision space

3.1 Three-way decisions

In [9], the first question is what changes are there in three-way decisions when the condition $0\le\beta<\alpha\le 1$ is changed to $0\le\beta\le\alpha\le 1$ and inequality $E(A)(x)\le\beta$ in the rejection region is replaced by $E(A)(x)<\beta$ in Definition 2.2. In this section, the question is discussed.

Definition 3.1. Let $(U, Map(V,P_C), P_D, E)$ be a three-way decision space, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\le\beta\le\alpha\le 1$. Then the first new type of three-way decisions is defined as follows.

(1) Acceptance region： $ACP1_{(\alpha,\beta)}(E,A)=\{x\in U\mid E(A)(x)\geq\alpha\}$.

(2) Rejection region： $REJ1_{(\alpha,\beta)}(E,A)=\{x\in U\mid E(A)(x)<\beta\}$.

(3) Uncertain region： $UNC1_{(\alpha,\beta)}(E,A)=\left(ACP1_{(\alpha,\beta)}(E,A)\cup REJ1_{(\alpha,\beta)}(E,A)\right)^c$.

Note 3.1. (1) If P_D is a linear order, then $UNC1_{(\alpha,\beta)}(E,A)=\{x\in U\mid \beta\leq E(A)(x)<\alpha\}$.

(2) If P_D is a linear order, $\alpha,\beta\in P_D$ and $\alpha=\beta$, then $UNC1_{(\alpha,\beta)}(E,A)=\varnothing$.

(3) If $0\leq\beta<\alpha\leq 1$, then $REJ_{(\alpha,\beta)}(E,A)=REJ1_{(\alpha,\beta)}(E,A)\cup\{x\in U\mid E(A)(x)=\beta\}$.

It is worth noting that the condition $0\leq\beta<\alpha\leq 1$ in Definition 2.2 is replaced by $0\leq\beta\leq\alpha\leq 1$ in Definition 3.1 and inequality $E(A)(x)\leq\beta$ in the rejection region of Definition 2.2 is replaced by $E(A)(x)<\beta$. In order to distinguish Definitions 2.2, the notations of Acceptance region, Rejection region and Uncertain region are written as "*ACP*1", "*REJ*1" and "*UNC*1", respectively. There exist the following benefits for this definition:

(1) Two-way decisions can regard as special cases of three-way decisions when $\alpha=\beta$ and P_D is a linear order. Under the situation, acceptance region is $\{x\in U\mid E(A)(x)\geq\alpha\}$, rejection region is $\{x\in U\mid E(A)(x)<\alpha\}$ and there is no uncertain region.

(2) Acceptance with "$\geq$" and rejection with "$<$" are more in line with practical applications and semantic. Generally speaking, "$\geq$" means "you are eligible", otherwise "$<$" means "you are not eligible".

(3) Depending on the number of evaluation functions, Yao gave two modes of three-way decisions, which are the single evaluation function and dual evaluation functions [44]. The above three-way decisions are based on a single evaluation function. Three-way decisions based on dual evaluation functions are given below. Considering $(U,Map(V,P_C),P_D,E_a)$ and $(U,Map(V,P_C),P_D,E_b)$ are two three-way decisions spaces, $A,B\in Map(V,P_C)$ and $\alpha,\beta\in P_D$, then three-way decisions based on double evaluation functions are regarded as some operations of three-way decisions based on single evaluation function.

Acceptance region：

$$\begin{aligned}ACP_{(\alpha,\beta)}((E_a,E_b),(A,B))&=\{x\in U\mid E_a(A)(x)\geq\alpha\}\cap\{x\in U\mid E_b(B)(x)<\beta\}\\&=ACP_\alpha(E_a,A)\cap REJ_\beta(E_b,B)\ .\end{aligned}$$

Rejection region：

$$\begin{aligned}REJ_{(\alpha,\beta)}((E_a,E_b),(A,B))&=\{x\in U\mid F_a(A)(x)<\alpha\}\cap\{x\in U\mid E_b(B)(x)\geq\beta\}\\&=REJ_\alpha(E_a,A)\cap ACP_\beta(E_b,B)\ .\end{aligned}$$

Uncertain region：

$$UNC_{(\alpha,\beta)}((E_a,E_b),(A,B))=\left(ACP_{(\alpha,\beta)}(E_a,E_b,A,B)\cup REJ_{(\alpha,\beta)}(E_a,E_b,A,B)\right)^c .$$

That is to say, acceptance region $ACP_{(\alpha,\beta)}((E_a,E_b),(A,B))$ based on double evaluation functions E_a and E_b is an intersection of acceptance region $ACP_\alpha(E_a,A)$ for E_a and rejection region $REJ_\beta(E_b,B)$ for E_b in Definition 3.1. Rejection region

$REJ_{(\alpha,\beta)}((E_a, E_b),(A,B))$ based on double evaluation functions E_a and E_b is an intersection of rejection region $REJ_\alpha(E_a, A)$ for E_a and acceptance region $ACP_\beta(E_b, B)$ for E_b in Definition 3.1. So it unifies double evaluation function with a single evaluation function where three-way decisions based on double evaluation functions are classified as some operations of three-way decisions based on single evaluation function.

(4) When this definition is applied to probability rough set model there are some differences with popular three-way decisions, upper and lower approximations and so on. In the rejection region of Definition 3.1, it is clear that

$$\lim_{\beta\to 0+} REJ1_{(\alpha,\beta)}(E,A) = \lim_{\beta\to 0+}\{x\in U \mid E(A)(x) < \beta\} = \{x\in U \mid E(A)(x) = 0\}\ .$$

At this point Pawlak's rough sets can be considered to be probability rough sets while $\beta \to 0^+$ and $\alpha = 1$.

In this section, let us always assume $(U, Map(V, P_C), P_D, E)$ is a three-way decision space, $\alpha, \beta \in P_D$ and $0 \le \beta \le \alpha \le 1$.

Theorem 3.1. *Let* $A, B \in Map(V, P_C)$. *Then the following hold.*

(1) $REJ1_{(\alpha,\beta)}(E,A) = ACP1_{(N(\beta),N(\alpha))}(E, N_{P_C}(A)) \setminus \{x\in U \mid E(A)(x) = \beta\}$,

$ACP1_{(\alpha,\beta)}(E,A) = REJ1_{(N(\beta),N(\alpha))}(E, N_{P_C}(A)) \bigcup \{x\in U \mid E(A)(x) = \alpha\}$.

(2) $ACP1_{(\alpha,\beta)}(E,A) \bigcup UNC1_{(\alpha,\beta)}(E,A) = ACP1_{(\alpha,\beta)}(E,A) \bigcup (REJ1_{(\alpha,\beta)}(E,A))^c$.

Proof. We only prove the first formula of (1). The others can be proved in a similar way.

$$\begin{aligned} REJ1_{(\alpha,\beta)}(E,A) &= \{x\in U \mid E(A)(x) < \beta\} \\ &= \{x\in U \mid E(A)(x) \le \beta\} \setminus \{x\in U \mid E(A)(x) = \beta\} \\ &= \{x\in U \mid E(N_{P_c}(A), x) \ge N(\beta)\} \setminus \{x\in U \mid E(A)(x) = \beta\} \\ &= ACP1_{(N(\beta),N(\alpha))}(E, N_{P_C}(A)) \setminus \{x\in U \mid E(A)(x) = \beta\}\ . \qquad \square \end{aligned}$$

If we consider a lattice P_C and $A, B \in Map(V, P_C)$, then the following hold.

(1) $ACP1_{(\alpha,\beta)}(E, A \bigcup_{P_C} B) \supseteq ACP1_{(\alpha,\beta)}(E,A) \bigcup ACP1_{(\alpha,\beta)}(E,B)$.

(2) $ACP1_{(\alpha,\beta)}(E, A \bigcap_{P_C} B) \subseteq ACP1_{(\alpha,\beta)}(E,A) \bigcap ACP1_{(\alpha,\beta)}(E,B)$.

(3) $REJ1_{(\alpha,\beta)}(E, A \bigcup_{P_C} B) \subseteq REJ1_{(\alpha,\beta)}(E,A) \bigcap REJ1_{(\alpha,\beta)}(E,B)$.

(4) $REJ1_{(\alpha,\beta)}(E, A \bigcap_{P_C} B) \supseteq REJ1_{(\alpha,\beta)}(E,A) \bigcup REJ1_{(\alpha,\beta)}(E,B)$.

Similarly we can define the concepts of lower and upper approximations from three-way decisions in Definition 3.1.

Definition 3.2. If $A \in Map(V, P_C)$, then

$$\underline{apr1}_{(\alpha,\beta)}(E,A) = ACP1_{(\alpha,\beta)}(E,A)$$

and

$$\overline{apr1}_{(\alpha,\beta)}(E,A) = \left(REJ1_{(\alpha,\beta)}(E,A)\right)^c$$

are called *the lower and upper approximations* of *A* respectively.

Note 3.2. (1) If P_D is a linear order, then

$\overline{apr1}_{(\alpha,\beta)}(E,A) = ACP1_{(\alpha,\beta)}(E,A) \bigcup UNC1_{(\alpha,\beta)}(E,A)$.

(2) If P_D is a linear order, then

$\underline{apr1}_{(\alpha,\beta)}(E,A) = \{x \in U \mid E(A)(x) \ge \alpha\}$,

$\overline{apr1}_{(\alpha,\beta)}(E,A) = \{x \in U \mid E(A)(x) \ge \beta\}$.

This is not the same as the definition suggested by Wei and Zhang [33] for $0 < \beta \le \alpha < 1$ in probabilistic rough set approximation:

$\underline{apr}_{(\alpha,\beta)} = \{x \in U \mid P(A \mid [x]) > \alpha\}$,

$\overline{apr}_{(\alpha,\beta)} = \{x \in U \mid P(A \mid [x]) \ge \beta\}$.

(3) If P_D is a linear order and $\alpha = \beta$, then

$\underline{apr1}_{(\alpha,\beta)}(E,A) = \overline{apr1}_{(\alpha,\beta)}(E,A) = \{x \in U \mid E(A)(x) \ge \alpha\}$.

This is also not the same as the definition of Yao and Wong [46] for $\alpha = \beta \ne 0$ in probabilistic rough set approximation:

$\underline{apr}_{\alpha} = \{x \in U \mid P(A \mid [x]) > \alpha\}$,

$\overline{apr}_{\alpha} = \{x \in U \mid P(A \mid [x]) \ge \alpha\}$.

The above definition of Yao and Wong [46] is an extension of the 0.5 probabilistic approximations.

In the following, we discuss the properties of Definition 3.2.

Theorem 3.2. *Let* $A, B \in Map(V, P_C)$ *. Then the following hold.*

(1) $\underline{apr1}_{(\alpha,\beta)}(E,A) \subseteq \overline{apr1}_{(\alpha,\beta)}(E,A)$.

Specially $\overline{apr1}_{(\alpha,\alpha)}(E,A) = \underline{apr1}_{(\alpha,\alpha)}(E,A) \bigcup \{x \in U \mid E(A)(x)$ *and* α *have no order relation about* $\le\}$

(2) $\underline{apr1}_{(\alpha,\beta)}(E,V) = U$, $\underline{apr1}_{(\alpha,\beta)}(E,\varnothing) = \begin{cases} \varnothing & \alpha > 0 \\ U & \alpha = 0 \end{cases}$,

$\overline{apr1}_{(\alpha,\beta)}(E,V) = U$, $\overline{apr1}_{(\alpha,\beta)}(E,\varnothing) = \begin{cases} \varnothing & \beta > 0 \\ U, & \beta = 0 \end{cases}$.

(3) $A \subseteq_{P_C} B \Rightarrow \underline{apr1}_{(\alpha,\beta)}(E,A) \subseteq \underline{apr1}_{(\alpha,\beta)}(E,B), \overline{apr1}_{(\alpha,\beta)}(E,A) \subseteq \overline{apr1}_{(\alpha,\beta)}(E,B)$.

(4) $\underline{apr1}_{(\alpha,\beta)}(E, N_{P_C}(A)) = \left(\overline{apr1}_{(N(\beta),N(\alpha))}(E,A)\right)^c \bigcup \{x \in U \mid E(A)(x) = N(\alpha)\}$,

$\overline{apr1}_{(\alpha,\beta)}(E, N_{P_C}(A)) = \left(\underline{apr1}_{(N(\beta),N(\alpha))}(E,A)\right)^c \bigcup \{x \in U \mid E(A)(x) = N(\beta)\}$.

Proof. (1) $\underline{apr1}_{(\alpha,\beta)}(E,A) = \{x \in U \mid E(A)(x) \ge \alpha\} \subseteq \{x \in U \mid E(A)(x) \ge \beta\}$

$\subseteq \left(\{x \in U \mid E(A)(x) < \beta\}\right)^c = \overline{apr1}_{(\alpha,\beta)}(E,A)$.

$\overline{apr1}_{(\alpha,\alpha)}(E,A) = \left(\{x \in U \mid E(A)(x) < \alpha\}\right)^c$

$= \{x \in U \mid E(A)(x) \ge \alpha\} \bigcup \{x \in U \mid E(A)(x)$ and α have no order relation about $\le\}$

$= \underline{apr1}_{(\alpha,\alpha)}(E,A) \bigcup \{x \in U \mid E(A)(x)$ and α have no order relation about $\le\}$.

(2) $\underline{apr1}_{(\alpha,\beta)}(E,\varnothing) = \{x \in U \mid E(\varnothing)(x) \ge \alpha\} = \begin{cases} \varnothing & \alpha > 0 \\ U & \alpha = 0 \end{cases}$

The others can be proved by a similar method.

(3) It is immediate from Definition 3.1.

(4) $\underline{apr1}_{(\alpha,\beta)}(E,N_{P_C}(A)) = ACP1_{(\alpha,\beta)}(E,N_{P_C}(A))$

$$= \{x \in U \mid E(N_{P_C}(A))(x) \geq \beta\}$$
$$= \{x \in U \mid E(A)(x) \leq N(\beta)\}$$
$$= \{x \in U \mid E(A)(x) < N(\beta)\} \bigcup \{x \in U \mid E(A)(x) = N(\beta)\}$$
$$= \left(\overline{apr1}_{(N(\beta),N(\alpha))}(E,A)\right)^c \bigcup \{x \in U \mid E(A)(x) = N(\beta)\} . \qquad \square$$

If we consider a lattice P_C and $A,B \in Map(V,P_C)$, then the following hold.

(1) $\underline{apr1}_{(\alpha,\beta)}(E,A \bigcap_{P_C} B) \subseteq \underline{apr1}_{(\alpha,\beta)}(E,A) \bigcap \underline{apr1}_{(\alpha,\beta)}(E,B)$.

(2) $\overline{apr1}_{(\alpha,\beta)}(E,A \bigcap_{P_C} B) \subseteq \overline{apr1}_{(\alpha,\beta)}(E,A) \bigcap \overline{apr1}_{(\alpha,\beta)}(E,B)$.

(3) $\underline{apr1}_{(\alpha,\beta)}(E,A \bigcup_{P_C} B) \supseteq \underline{apr1}_{(\alpha,\beta)}(E,A) \bigcup \underline{apr1}_{(\alpha,\beta)}(E,B)$.

(4) $\overline{apr1}_{(\alpha,\beta)}(E,A \bigcup_{P_C} B) \supseteq \overline{apr1}_{(\alpha,\beta)}(E,A) \bigcup \overline{apr1}_{(\alpha,\beta)}(E,B)$.

The following theorem can be proven.

Theorem 3.3. *Let* $A \in Map(V,P_C)$. *The following hold.*

(1) *If* $0 \leq \beta \leq \beta' \leq \alpha' \leq \alpha \leq 1$, *then*

$$\underline{apr1}_{(\alpha,\beta)}(E,A) \subseteq \underline{apr1}_{(\alpha',\beta')}(E,A), \overline{apr1}_{(\alpha,\beta)}(E,A) \supseteq \overline{apr1}_{(\alpha',\beta')}(E,A) .$$

(2) *If* P_D *is a lattice and* $\alpha,\beta \in P_D$, *then* $\forall t \in P_D, 0 \leq t \leq \alpha \wedge \beta$,

$$\underline{apr1}_{(\alpha\vee\beta,t)}(E,A) = \underline{apr1}_{(\alpha,t)}(E,A) \bigcap \underline{apr1}_{(\beta,t)}(E,A) .$$

(3) *If* P_D *is a lattice and* $\alpha,\beta \in P_D$, *then* $\forall t \in P_D, \alpha \vee \beta \leq t \leq 1$,

$$\overline{apr1}_{(t,\alpha\wedge\beta)}(E,A) \supseteq \overline{apr1}_{(t,\alpha)}(E,A) \bigcup \overline{apr1}_{(t,\beta)}(E,A)$$

and the equality holds while P_D *is linear.*

Proof. (1) It is straightforward from Definition 3.2.

(2) If $\alpha,\beta \in P_D$, then $\forall t \in P_D, 0 \leq t \leq \alpha \wedge \beta$,

$$\underline{apr1}_{(\alpha\vee\beta,t)}(E,A) = \{x \in U \mid E(A)(x) \geq \alpha \vee \beta\}$$
$$= \{x \in U \mid E(A)(x) \geq \alpha\} \bigcap \{x \in U \mid E(A)(x) \geq \beta\}$$
$$= \underline{apr1}_{(\alpha,t)}(E,A) \bigcap \underline{apr1}_{(\beta,t)}(E,A) .$$

(3) If $\alpha,\beta \in P_D$, then $\forall t \in P_D, \alpha \vee \beta \leq t \leq 1$,

$$\overline{apr}_{(t,\alpha\wedge\beta)}(E,A) = \left(REJ_{(t,\alpha\wedge\beta)}(E,A)\right)^c$$
$$= \left(x \in U \mid E(A)(x) < \alpha \wedge \beta\right)^c$$
$$\supseteq \left(\{x \in U \mid E(A)(x) < \alpha\} \bigcap \{x \in U \mid E(A)(x) < \beta\}\right)^c$$
$$= \overline{apr}_{(t,\alpha)}(E,A) \bigcup \overline{apr}_{(t,\beta)}(E,A) .$$

If P_D is linear, then the equality holds. $\square$

3.2 The optimistic three-way decisions over multiple three-way decision spaces

The above conclusions in Section 3.1 can be extended to multiple three-way decision spaces.

Definition 3.3. Let $(U, Map(V,P_C), P_D, E_i)$ ($i = 1,2,\cdots,n$) be n three-way decision spaces, $A \in Map(V,P_C)$, $\alpha,\beta \in P_D$ and $0 \leq \beta \leq \alpha \leq 1$. Then optimistic three-way decisions over n

three-way decision spaces are defined as follows.

(1) Acceptance region: $ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP^{op}_{(\alpha,\beta)}(E_{1\sim n},A)$.

(2) Rejection region: $REJ1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcap_{i=1}^{n}REJ1_{(\alpha,\beta)}(E_i,A)=\bigcap_{i=1}^{n}\{x\in U\mid E_i(A)(x)<\beta\}$.

(3) Uncertain region:

$$UNC1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c .$$

Note 3.3. (1) If P_D is a linear order, then

$$ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\vee}}E_i(A)(x)\geq\alpha\right\},$$

$$REJ1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\vee}}E_i(A)(x)<\beta\right\} \text{ and}$$

$$UNC1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \beta\leq\overset{n}{\underset{i=1}{\vee}}E_i(A)(x)<\alpha\right\}.$$

(2) If $0\leq\beta<\alpha\leq1$, $REJ^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(\bigcap_{i=1}^{n}REJ1_{(\alpha,\beta)}(E_i,A)\right)\cup\left(\bigcap_{i=1}^{n}\{x\in U\mid E_i(A)(x)=\beta\}\right)$.

Similarly, we can discuss the lower and upper approximations of optimistic three-way decisions over multiple three-way decision spaces.

Defintion 3.4. If $A\in Map(V,P_C)$, then

$$\underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)$$

and

$$\overline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(REJ1^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$$

are referred to as *the lower approximation and upper approximation* of *A* with regard to optimistic three-way decisions over *n* three-way decision spaces respectively.

Obviously, if P_D is a linear order, then

$$\underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\vee}}E_i(A)(x)\geq\alpha\right\}$$

and

$$\overline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\vee}}E_i(A)(x)\geq\beta\right\}.$$

Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be *n* three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\leq\beta\leq\alpha\leq1$. We can obtain similar results of Theorem 3.4 and 3.5 in [10].

We can prove the following theorem.

Theorem 3.4. *Let* $(U,Map(V,P_C),P_D,E_i)$ *(* $i=1,2,\cdots,n$ *) be n three-way decision spaces,* $A\in Map(V,P_C)$. *Then the following hold.*

(1) *If* $0\leq\beta\leq\beta'\leq\alpha'\leq\alpha\leq1$, *then* $\underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\subseteq\underline{apr1}^{op}_{(\alpha',\beta')}(E_{1\sim n},A)$ *and*

$$\overline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\supseteq\overline{apr1}^{op}_{(\alpha',\beta')}(E_{1\sim n},A).$$

(2) *If* P_D *is a lattice and* $\alpha,\beta\in P_D$, *then* $\forall t\in P_D, 0\leq t\leq\alpha\wedge\beta$,

$$\underline{apr1}^{op}_{(\alpha\vee\beta,t)}(E_{1\sim n},A)\subseteq\underline{apr1}^{op}_{(\alpha,t)}(E_{1\sim n},A)\cap\underline{apr1}^{op}_{(\beta,t)}(E_{1\sim n},A).$$

If P_D *is a lattice and* $\alpha,\beta\in P_D$, *then* $\forall t\in P_D, \alpha\vee\beta\le t\le 1$,

$$\overline{apr1}^{op}_{(t,\alpha\wedge\beta)}(E_{1\sim n},A)\supseteq \overline{apr1}^{op}_{(t,\alpha)}(E_{1\sim n},A)\cup \overline{apr1}^{op}_{(t,\beta)}(E_{1\sim n},A).$$

Proof. (1) It is straightforward from Definition 3.4.

(2) If P_D is a lattice and $\alpha,\beta\in P_D$, then $\forall t\in P_D, 0\le t\le \alpha\wedge\beta$,

$$\begin{aligned}\underline{apr1}^{op}_{(\alpha\vee\beta,t)}(E_{1\sim n},A)&=\bigcup_{i=1}^{n}\underline{apr1}_{(\alpha\vee\beta,t)}(E_i,A)\\&=\bigcup_{i=1}^{n}\left(\underline{apr1}_{(\alpha,t)}(E_i,A)\cap \underline{apr1}_{(\beta,t)}(E_i,A)\right)\\&\subseteq\left(\bigcup_{i=1}^{n}\underline{apr1}_{(\alpha,t)}(E_i,A)\right)\cap\left(\bigcup_{i=1}^{n}\underline{apr1}_{(\beta,t)}(E_i,A)\right)\\&=\underline{apr1}^{op}_{(\alpha,t)}(E_{1\sim n},A)\cap \underline{apr1}^{op}_{(\beta,t)}(E_{1\sim n},A).\end{aligned}$$

If P_D is a lattice and $\alpha,\beta\in P_D$, then $\forall t\in P_D, \alpha\vee\beta\le t\le 1$,

$$\begin{aligned}\overline{apr1}^{op}_{(t,\alpha\wedge\beta)}(E_{1\sim n},A)&=\bigcup_{i=1}^{n}\overline{apr1}_{(t,\alpha\wedge\beta)}(E_i,A)\\&\supseteq\bigcup_{i=1}^{n}\left(\overline{apr1}_{(t,\alpha)}(E_i,A)\cup \overline{apr1}_{(t,\beta)}(E_i,A)\right)\\&=\left(\bigcup_{i=1}^{n}\overline{apr1}_{(t,\alpha)}(E_i,A)\right)\cup\left(\bigcup_{i=1}^{n}\overline{apr1}_{(t,\beta)}(E_i,A)\right)\\&=\overline{apr1}^{op}_{(t,\alpha)}(E_{1\sim n},A)\cup \overline{apr1}^{op}_{(t,\beta)}(E_{1\sim n},A).\end{aligned}$$ □

3.3 The pessimistic three-way decisions over multiple three-way decision spaces

We can consider another type of three-way decisions over multiple three-way decision spaces.

Definition 3.5. Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be n three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\le\beta\le\alpha\le 1$. Then pessimistic three-way decisions over n three-way decision spaces are defined as

(1) Acceptance region：

$$ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n},A).$$

(2) Rejection region：

$$REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcup_{i=1}^{n}REJ1_{(\alpha,\beta)}(E_i,A)=\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)<\beta\}.$$

(3) Uncertain region：

$$UNC1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^{c}.$$

Note 3.4. (1) $ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\wedge}}E_i(A)(x)\ge\alpha\right\}$

(2) If P_D is a linear order, then

$$REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \overset{n}{\underset{i=1}{\wedge}}E_i(A)(x)<\beta\right\}$$ and

$$UNC1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \beta\le\overset{n}{\underset{i=1}{\wedge}}E_i(A)(x)<\alpha\right\}.$$

(3) If $0\le\beta<\alpha\le 1$, then

$$REJ^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(\bigcup_{i=1}^{n}REJ1_{(\alpha,\beta)}(E_i,A)\right)\cup\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)=\beta\}\right).$$

Similarly, we can discuss the lower and upper approximations of pessimistic three-way decisions over multiple three-way decision spaces.

Definition 3.6. If $A\in Map(V,P_C)$, then

$$\underline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)$$

and

$$\overline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$$

are referred to as *the lower approximation and upper approximation* of A with regard to pessimistic three-way decisions over n three-way decision spaces respectively.

Obviously, if P_D is a linear order, then

$$\overline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \bigwedge_{i=1}^{n}E_i(A)(x)\geq\beta\right\}.$$

Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be n three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\leq\beta\leq\alpha\leq1$. We can obtain similar results of Theorems 3.7 and 3.8 in [10].

We can prove the following theorem in a similar approach as Theorem 3.4.

Theorem 3.5. *Let $(U,Map(V,P_C),P_D,E_i\})$ ($i=1,2,\cdots,n$) be n three-way decision spaces and $A\in Map(V,P_C)$. Then we have the following statements.*

(1) *If $0\leq\beta\leq\beta'\leq\alpha'\leq\alpha\leq1$, then $\underline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\subseteq\underline{apr1}^{pe}_{(\alpha',\beta')}(E_{1\sim n},A)$ and*

$\overline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n},A)\supseteq\overline{apr1}^{pe}_{(\alpha',\beta')}(E_{1\sim n},B)$.

(2) *If P_D is a lattice and $\alpha,\beta\in P_D$, then $\forall t\in P_D, 0\leq t\leq\alpha\wedge\beta$,*

$\underline{apr1}^{pe}_{(\alpha\vee\beta,t)}(E_{1\sim n},A)=\underline{apr1}^{pe}_{(\alpha,t)}(E_{1\sim n},A)\cap\underline{apr1}^{pe}_{(\beta,t)}(E_{1\sim n},A)$.

If P_D is a lattice and $\alpha,\beta\in P_D$, then $\forall t\in P_D,\alpha\vee\beta\leq t\leq1$,

$\overline{apr1}^{pe}_{(t,\alpha\wedge\beta)}(E_{1\sim n},A)\supseteq\overline{apr1}^{pe}_{(t,\alpha)}(E_{1\sim n},A)\cup\overline{apr1}^{pe}_{(t,\beta)}(E_{1\sim n},A)$.

The following theorem shows the relationship between optimistic three-way decisions and pessimistic three-way decisions over n three-way decision spaces.

Theorem 3.6. *Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be n three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\leq\beta\leq\alpha\leq1$. Then the following hold.*

(1) $\underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},N_{P_C}(A))$

$$=\left(\overline{apr1}^{pe}_{(N(\beta),N(\alpha))}(E_{1\sim n},A)\right)^c\cup\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)=N(\alpha)\}\right).$$

(2) $\overline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},N_{P_C}(A))$

$$=\left(\underline{apr1}^{pe}_{(N(\beta),N(\alpha))}(E_{1\sim n},A)\right)^c\cup\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)=N(\beta)\}\right).$$

Proof. Only prove (1).

$$\underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n},N_{P_C}(A))=\bigcup_{i=1}^{n}\underline{apr1}_{(\alpha,\beta)}(E_i,N_{P_C}(A))$$
$$=\bigcup_{i=1}^{n}\left(\overline{apr1}_{(N(\beta),N(\alpha))}(E_i,A)\right)^c\cup\{x\in U\mid E_i(A)(x)=N(\alpha)\}$$
$$=\left(\bigcap_{i=1}^{n}\overline{apr1}_{(N(\beta),N(\alpha))}(E_i,A)\right)^c\cup\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)=N(\alpha)\}\right)$$
$$=\left(\overline{apr1}^{pe}_{(N(\beta),N(\alpha))}(E_{1\sim n},A)\right)^c\cup\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)=N(\alpha)\}\right).\qquad\square$$

3.4 The aggregation three-way decisions over multiple three-way decision spaces

Definition 3.7. ([15]) Let $(P,\leq_P,N_P,0_P,1_P)$ be a bounded partially ordered set. A mapping $f:P^n\to P$ is called an *n-ary complement-preserving aggregation function*, if it satisfies the following conditions:

(AF1) Regularity:

$f(x,x,\cdots,x)=x$, $\forall x_i\in P$;

(AF2) Nondecreasing property:

f is a nondecreasing function for each variable over P, i.e. $x_i^{(1)}\leq_P x_i^{(2)}$ $(i=1,2,\cdots,n)$ implies $f(x_1^{(1)},x_2^{(1)},\cdots,x_n^{(1)})\leq_P f(x_1^{(2)},x_2^{(2)},\cdots,x_n^{(2)})$, $\forall x_i^{(1)},x_i^{(2)}\in P$;

(AF3) Complement-preserving property:

$f\left(N_P(x_1),N_P(x_2),\cdots,N_P(x_n)\right)=N_P\left(f(x_1,x_2,\cdots,x_n)\right)$, $\forall x_i\in P$.

The family of all *n*-ary complement-preserving aggregation functions over *P* is denoted by $AF_n(P)$. Sometimes we have to use the following condition:

(AF4) $f(x_1,x_2\cdots,x_n)\leq_P f(y_1,y_2,\cdots,y_n)$ implies that there is an *i* such that $x_i\leq_P y_i$, $\forall x_i,y_i\in P$.

Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be *n* three-way decision spaces, $f\in AF_n(P_D)$ and $E^f(A)(x)=f\left(E_1(A)(x),E_2(A)(x),\cdots,E_n(A)(x)\right)$ for $A\in Map(V,P_C)$ and $x\in U$. Then $(U,Map(V,P_C),P_D,E^f)$ is a three-way decision space and the aggregation three-way decisions over *n* three-way decision spaces for $\alpha,\beta\in P_D$ and $0\leq\beta\leq\alpha\leq 1$ are defined as follows.

(1) Acceptance region：

$ACP1^f_{(\alpha,\beta)}(E_{1\sim n},A)=ACP_{(\alpha,\beta)}(E^f,A)$,

(2) Rejection region：

$REJ1^f_{(\alpha,\beta)}(E_{1\sim n},A)=REJ1_{(\alpha,\beta)}(E^f,A)=\left\{x\in U\mid E^f(A)(x)<\beta\right\}$,

(3) Uncertain region：

$UNC1^f_{(\alpha,\beta)}(E_{1\sim n},A)=\left(ACP1^f_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ1^f_{(\alpha,\beta)}(E_{1\sim n},A)\right)^c$.

If P_D is a linear order, then $UNC1^f_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid\beta\leq E^f(A)(x)<\alpha\right\}$.

The lower and upper approximations of the aggregation three-way decisions over *n* three-way decision spaces are $\underline{apr}^f_{(\alpha,\beta)}(E_{1\sim n},A)=\underline{apr}_{(\alpha,\beta)}(E^f,A)$ and $\overline{apr}^f_{(\alpha,\beta)}(E_{1\sim n},A)=\overline{apr}_{(\alpha,\beta)}(E^f,A)$ respectively**.**

In the following, we discuss the properties of the aggregation three-way decisions and the lower and upper approximations, and relationships among the aggregation three-way decisions, the optimistic three-way decisions and the pessimistic three-way decisions.

Theorem 3.7. *Let* $(U, Map(V,P_C), P_D, E_i)$ *(* $i=1,2,\cdots,n$ *) be n three-way decision spaces,* $f \in AF_n(P_D)$ *and satisfy (AF4),* $A \in Map(V,P_C)$ *and* $0 \le \beta < \alpha \le 1$ *. Then the following statements hold.*

(1) $ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP1^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$,

(2) $REJ1^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ1^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

Proof. Let $x \in ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} ACP1_{(\alpha,\beta)}(E_i, A)$. Then $\forall i$, $x \in ACP1_{(\alpha,\beta)}(E_i, A)$, i.e. $E_i(A)(x) \ge \alpha$. So

$$E^f(A)(x) = f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \\ \ge f(\alpha, \alpha, \cdots, \alpha) = \alpha,$$

i.e. $x \in ACP1^{f}_{(\alpha,\beta)}(E_{1\sim n}, A)$. If $x \in ACP1^{f}_{(\alpha,\beta)}(E_{1\sim n}, A)$, then

$$E^f(A)(x) = f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \ge \alpha.$$

Hence there is an i such that $E_i(A)(x) \ge \alpha$ due to the condition (AF4) of *f*, i.e. $x \in \bigcup_{i=1}^{n} \underline{apr1}_{(\alpha,\beta)}(E_i, A) = ACP1^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

The second relation of inclusion can be proved in a similar way. □

Theorem 3.8. *Let* $(U, Map(V,P_C), P_D, E_i)$ *(* $i=1,2,\cdots,n$ *) be n three-way decision spaces,* $f \in AF_n(P_D)$, $A, B \in Map(V,P_C)$ *and* $0 \le \beta < \alpha \le 1$. *Then*

$$\underline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} \underline{apr1}_{(\alpha,\beta)}(E_i, A) \subseteq \underline{apr1}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \bigcup_{i=1}^{n} \underline{apr1}_{(\alpha,\beta)}(E_i, A) = \underline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$$

and

$$\overline{apr1}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} \overline{apr1}_{(\alpha,\beta)}(E_i, A) \subseteq \overline{apr1}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \bigcup_{i=1}^{n} \overline{apr1}_{(\alpha,\beta)}(E_i, A) = \overline{apr1}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions and Theorem 3.7. □

Theorem 3.9. *Let* $(U, Map(V,P_C), P_D, E_i)$ *(* $i=1,2,\cdots,n$ *) be n three-way decision spaces,* $f \in AF_n(P_D)$, $A \in Map(V,P_C)$. *If* $0 \le \beta \le \beta' < \alpha' \le \alpha \le 1$, *then*

$$\underline{apr1}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \underline{apr1}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A) \quad \textit{and}$$

$$\overline{apr1}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \supseteq \overline{apr1}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions. □

3.5 Dynamic three-way decisions

The following decisions are considered to be made up of more than one three-way decisions.

Definition 3.7. Let $(U, Map(V,P_C), P_D, E_i)$ be i^{th} three-way decision space, $A_i \in Map(V,P_C)$, $\alpha_i, \beta_i \in P_D$ and $0 \le \beta_i \le \alpha_i \le 1$ ($i=1,2,\cdots,n$). Then the definition of dynamic three-way decisions is given below.

(Decision 1) Acceptance region 1： $ACP1^{(1)}_{(\alpha_1,\beta_1)}(E_1, A_1) = \{x \in U \mid E_1(A_1)(x) \ge \alpha_1\}$.

Rejection region 1： $REJ1^{(1)}_{(\alpha_1,\beta_1)}(E_1, A_1) = \{x \in U \mid E_1(A_1)(x) < \beta_1\}$.

Uncertain region 1： $UNC1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=\left(ACP1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup REJ1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\right)^c$.

(Decision 2) If $UNC1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\neq\varnothing$, then

Acceptance region 2：

$$ACP1^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=ACP1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup\{x\in UNC1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\mid E_2(A_2)(x)\geq\alpha_2\}.$$

Rejection region 2：

$$REJ1^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=REJ1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup\{x\in UNC1^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\mid E_2(A_2)(x)<\beta_2\}.$$

Uncertain region 2： $UNC1^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=\left(ACP1^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)\cup REJ1^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)\right)^c$.

……

(Decision n) If $UNC1^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\neq\varnothing$, then

Acceptance region n： $ACP^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=$

$$ACP1^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\cup\{x\in UNC1^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\mid E_n(A_n)(x)\geq\alpha_n\}.$$

Rejection region n： $REJ^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=$

$$REJ1^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\cup\{x\in UNC1^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\mid E_n(A_n)(x)<\beta_n\}.$$

Uncertain region n： $UNC1^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=\left(ACP1^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)\cup REJ1^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)\right)^c$.

4. On the refusal decision region in three-way decision space

4.1 Three-way decisions

In [9], the second question is what changes are there in three-way decisions when the uncertain region is defined by using inequality $\beta<E(A)(x)<\alpha$ in Definition 2.3. In this section, the question is discussed.

Definition 4.1. Let $(U,Map(V,P_C),P_D,E)$ be a three-way decision space, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\leq\beta<\alpha\leq1$. Then the second new type of three-way decisions over the three-way decision space is defined as follows.

(1) Acceptance region： $ACP2_{(\alpha,\beta)}(E,A)=\{x\in U\mid E(A)(x)\geq\alpha\}$.

(2) Rejection region： $REJ2_{(\alpha,\beta)}(E,A)=\{x\in U\mid E(A)(x)\leq\beta\}$.

(3) Uncertain region： $UNC2_{(\alpha,\beta)}(E,A)=\{x\in U\mid\beta<E(A)(x)<\alpha\}$.

In order to distinguish Definitions 2.2 and Definition 3.1, the notations of Acceptance region, Rejection region and Uncertain region are written as "*ACP*2", "*REJ*2" and "*UNC*2", respectively.

If P_D is not a linear order, then $ACP2_{(\alpha,\beta)}(E,A)\cup REJ2_{(\alpha,\beta)}(E,A)\cup UNC2_{(\alpha,\beta)}(E,A)$ is not necessarily *U*. At this point $U-\left(ACP2_{(\alpha,\beta)}(E,A)\cup REJ2_{(\alpha,\beta)}(E,A)\cup UNC2_{(\alpha,\beta)}(E,A)\right)$ is referred to as a refusal decision region, written as $REF_{(\alpha,\beta)}(E,A)$. There are large numbers of the nonlinear order sets, e.g., $I^2=[0,1]\times[0,1]$, I^2_s (truth value set of intuitionistic fuzzy sets, Example 2.1(4) in [9]), $I^{(2)}$ (truth value set of interval-valued fuzzy sets, Example 2.1(5) in [9]), $I(2^U)$ (interval sets, Example 2.1(6) in [6]), $2^{[0,1]}-\varnothing$ (truth value set of hesitant fuzzy sets, Theorem 4.2 in [10]), $2^{I^{(2)}}-\varnothing$ (truth value set of interval-valued hesitant fuzzy sets, Theorem 5.4 in [10]).

The uncertain region may be seen to objects of the delayed decision. The existence of a refusal decision region is due to noise data or the method is not applicable to the data. As in fuzzy pattern recognition, if the identified object does not reach a predefined threshold, then it is refused to make decision or it cannot be identified through the method.

The acceptance region and the rejection region in Definition 4.1 are the same as the ones in Definition 2.2. Therefore we can obtain similar results with [10] on $ACP2_{(\alpha,\beta)}(E,A)$ and $REJ2_{(\alpha,\beta)}(E,A)$. We can also define $\underline{apr2}_{(\alpha,\beta)}(E,A)=ACP2_{(\alpha,\beta)}(E,A)$ and $\overline{apr2}_{(\alpha,\beta)}(E,A)=\left(REJ2_{(\alpha,\beta)}(E,A)\right)^c$ and can obtain similar results with [7].

The following example illustrates the similarity and difference among different types of three-way decisions. In the following discussion, for fuzzy set $A\in Map(U,I)$ and fuzzy relation $R\in Map(U\times U,I)$, we define $\left(A\cdot[x]_R\right)(y)=A(y)R(x,y)$.

Example 4.1. Let $P_C=P_D=I^{(2)}$, $U=\{x_1,x_2,x_3,x_4,x_5\}$,
$A=\frac{[0.4,0.7]}{x_1}+\frac{[0.3,0.4]}{x_2}+\frac{[0.1,0.4]}{x_3}+\frac{[0.8,0.9]}{x_4}+\frac{[0.6,0.8]}{x_5}=[A^-,A^+]\in Map(U,I^{(2)})$, and

$$R=\begin{bmatrix}1 & 0.8 & 0.6 & 0.4 & 0.6\\ 0.8 & 1 & 0.6 & 0.4 & 0.6\\ 0.6 & 0.6 & 1 & 0.4 & 0.9\\ 0.4 & 0.4 & 0.4 & 1 & 0.4\\ 0.6 & 0.6 & 0.9 & 0.4 & 1\end{bmatrix}$$

be a fuzzy equivalence relation on *U*. Then

$$E(A)(x)=\left[\frac{|A^-\cdot[x]_R|}{|[x]_R|},\frac{|A^+\cdot[x]_R|}{|[x]_R|}\right]$$

is a decision evaluation function of *U*.

By computing we obtain $E(A)(x_1)=\left[0.406,0.618\right]$, $E(A)(x_2)=\left[0.4,0.6\right]$, $E(A)(x_3)=\left[0.394,0.611\right]$, $E(A)(x_4)=[0.523,0.7]$ and $E(A)(x_5)=\left[0.409,0.623\right]$.

Consider $\alpha=[0.5,0.7]$ and $\beta=[0.4,0.6]$, three types of three-way decisions based on three-way decision spaces are shown in Table 4.1.

Table 4.1. Three types of three-way decisions based on three-way decision spaces

Acceptance region			Rejection region			Uncertain region		
ACP	*ACP*1	*ACP*2	*REJ*	*REJ*1	*REJ*2	*UNC*	*UNC*1	*UNC*2
$\{x_4\}$	$\{x_4\}$	$\{x_4\}$	$\{x_2\}$	$\varnothing$	$\{x_2\}$	$\{x_1,x_3,x_5\}$	$\{x_1,x_2,x_3,x_5\}$	$\{x_1,x_5\}$

It follows from Table 4.1 that

$ACP_{(\alpha,\beta)}(E,A)\cup REJ_{(\alpha,\beta)}(E,A)\cup UNC_{(\alpha,\beta)}(E,A)=U$ and
$ACP1_{(\alpha,\beta)}(E,A)\cup REJ1_{(\alpha,\beta)}(E,A)\cup UNC1_{(\alpha,\beta)}(E,A)=U$, but
$ACP2_{(\alpha,\beta)}(E,A)\cup REJ2_{(\alpha,\beta)}(E,A)\cup UNC2_{(\alpha,\beta)}(E,A)=\{x_1,x_2,x_4,x_5\}$ and
$REF_{(\alpha,\beta)}(E,A)=\{x_3\}$.

In the first new type (Definition 3.1), the rejection region is empty, i.e., $REJ1=\varnothing$. In Definition 2.2, however, it is not reasonable that $E(A)(x_2)=\left[0.4,0.6\right]$ is rejected in accordance with the standard $\beta=[0.4,0.6]$, i.e., $REJ=\{x_2\}$. And x_2 is added to uncertain region, since it does not meet the standard $\alpha=[0.5,0.7]$.

In the second new type, x_4 is accepted, x_2 is rejected, x_1 and x_5 are uncertain, and x_3 is refused to make decision. In accordance with Definition 4.1, it is reasonable that x_3 is added to refusal decision region for the standards $\alpha=[0.5,0.7]$ and $\beta=[0.4,0.6]$, since the width of $E(A)(x_3)=[0.394,0.611]$ is too large. □

When $\{E(A)(x),\alpha,\beta\}$ is a linear order subset of P_D, there is one inequality to be true in $E(A)(x)\geq\alpha$, $E(A)(x)\leq\beta$ and $\beta<E(A)(x)<\alpha$. Thus it follows the following properties on refusal decision region.

Theorem 4.1. *Let* $(U,Map(V,P_C),P_D,E)$ *be a three-way decision space,* $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ *and* $0\leq\beta<\alpha\leq1$. *Then* $REF_{(\alpha,\beta)}(E,A)=\varnothing$ *if and only if* $\{E(A)(x),\alpha,\beta\}$ *is a linear order subset of* P_D *for any* $x\in U$.

Corollary 4.1. *Let* $(U,Map(V,P_C),P_D,E)$ *be a three-way decision space,* $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ *and* $0\leq\beta<\alpha\leq1$. *Then the following hold.*

(1) $REF_{(\alpha,\beta)}(E,\varnothing)=REF_{(\alpha,\beta)}(E,V)=\varnothing$.

(2) *If* P_D *is linear, then* $REF_{(\alpha,\beta)}(E,A)=\varnothing$.

4.2 The optimistic three-way decisions over multiple three-way decision spaces

The above conclusions in Section 4.1 can be extended to multiple three-way decision spaces.

Definition 4.2. Let $(U,Map(V,P_C),P_D,E_i)$ ($i=1,2,\cdots,n$) be n three-way decision spaces, $A\in Map(V,P_C)$, $\alpha,\beta\in P_D$ and $0\leq\beta<\alpha\leq1$. Then optimistic three-way decisions over n three-way decision spaces are defined as follows.

(1) Acceptance region： $ACP2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=ACP^{op}_{(\alpha,\beta)}(E_{1\sim n},A)$.

(2) Rejection region： $REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\bigcap_{i=1}^{n}REJ2_{(\alpha,\beta)}(E_i,A)=\bigcap_{i=1}^{n}\{x\in U\mid E_i(A)(x)\leq\beta\}$.

(3) Uncertain region：

$$UNC2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left(\bigcup_{i=1}^{n}\{x\in U\mid E_i(A)(x)>\beta\}\right)\cap\left(\bigcap_{i=1}^{n}\{x\in U\mid E_i(A)(x)<\alpha\}\right).$$

Note 4.1. If P_D is a linear order, then

$$ACP2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \bigvee_{i=1}^{n}E_i(A)(x)\geq\alpha\right\},$$

$$REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \bigvee_{i=1}^{n}E_i(A)(x)\leq\beta\right\}\text{ and}$$

$$UNC2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)=\left\{x\in U\mid \beta<\bigvee_{i=1}^{n}E_i(A)(x)<\alpha\right\}.$$

$U-\left(ACP2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\cup REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\cup UNC2^{op}_{(\alpha,\beta)}(E_{1\sim n},A)\right)$ is referred to as a refusal decision region, written as $REF^{op}_{(\alpha,\beta)}(E_{1\sim n},A)$.

4.3 The pessimistic three-way decisions over multiple three-way decision spaces

We can consider another type of three-way decisions over multiple three-way decision spaces.

Definition 4.3. Let $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, n$) be n three-way decision spaces, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \le \beta < \alpha \le 1$. Then pessimistic three-way decisions over n three-way decision spaces are defined as

(1) Acceptance region:

$$ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A).$$

(2) Rejection region:

$$REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcup_{i=1}^{n} REJ2_{(\alpha,\beta)}(E_i, A) = \bigcup_{i=1}^{n} \{x \in U \mid E_i(A)(x) \le \beta\}.$$

(3) Uncertain region:

$$UNC2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left(\bigcap_{i=1}^{n} \{x \in U \mid E_i(A)(x) > \beta\}\right) \cap \left(\bigcup_{i=1}^{n} \{x \in U \mid E_i(A)(x) < \alpha\}\right).$$

Note 4.2. (1) $ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left\{x \in U \mid \overset{n}{\underset{i=1}{\wedge}} E_i(A)(x) \ge \alpha\right\}$

(2) If P_D is a linear order, then

$$REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = \left\{x \in U \mid \overset{n}{\underset{i=1}{\wedge}} E_i(A)(x) \le \beta\right\}.$$

$U - \left(ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \cup REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \cup UNC2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)\right)$ is referred to as a refusal decision region, written as $REF^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

Here, we use an example to illustrate some notions of the optimistic and pessimistic three-way decisions over multiple three-way decision spaces.

Example 4.2. Let $P_C = P_D = I^{(2)}$, $U = \{x_1, x_2, x_3, x_4, x_5\}$,

$A = \frac{[0.4, 0.7]}{x_1} + \frac{[0.2, 0.3]}{x_2} + \frac{[0.1, 0.38]}{x_3} + \frac{[0.8, 0.9]}{x_4} + \frac{[0.6, 0.8]}{x_5} = [A^-, A^+] \in Map(U, I^{(2)})$, and

$$R = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & 0.6 \\ 0.8 & 1 & 0.6 & 0.4 & 0.6 \\ 0.6 & 0.6 & 1 & 0.4 & 0.9 \\ 0.4 & 0.4 & 0.4 & 1 & 0.4 \\ 0.6 & 0.6 & 0.9 & 0.4 & 1 \end{bmatrix}$$

be a fuzzy equivalence relation on U.

$E_1(A) = A$ and

$$E_2(A)(x) = \left[\frac{|A^- \cap [x]_R|}{|[x]_R|}, \frac{|A^+ \cap [x]_R|}{|[x]_R|}\right]$$

both are evaluation functions of U.

By computing we obtain

$$E_2(A) = \frac{[0.5, 0.7]}{x_1} + \frac{[0.5, 0.7]}{x_2} + \frac{[0.486, 0.709]}{x_3} + \frac{[0.731, 0.915]}{x_4} + \frac{[0.486, 0.709]}{x_5}.$$

In two three-way decision spaces $(U, Map(U, \{0,1\}), [0,1], E_1)$ and $(U, Map(U, \{0,1\}), [0,1], E_2)$, if we consider three groups of different parameters α, β, then acceptance regions, rejection regions, uncertain regions and refusal decision regions of optimistic and pessimistic three-way decisions of A over two three-way decision spaces are listed in Table

4.2.

Table 4.2

Acceptance regions, rejection regions, uncertain regions and refusal decision regions of the optimistic and pessimistic three-way decisions of *A* over two three-way decision spaces for different α,β.

	$\beta=[0.5,0.7]$ $\alpha=[0.6,0.8]$	$\beta=[0.5,0.6]$ $\alpha=[0.6,0.7]$	$\beta=[0.6,0.8]$ $\alpha=[0.8,0.9]$
$ACP^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4,x_5\}$	$\{x_4,x_5\}$	$\{x_4\}$
$REJ^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2\}$	$\varnothing$	$\{x_1,x_2,x_3,x_5\}$
$UNC^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_3\}$	$\{x_1,x_2,x_3\}$	$\varnothing$
$ACP^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4\}$	$\{x_4\}$	$\varnothing$
$REJ^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3,x_5\}$
$UNC^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_5\}$	$\{x_5\}$	$\{x_4\}$
$ACP1^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4,x_5\}$	$\{x_4,x_5\}$	$\{x_4\}$
$REJ1^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\varnothing$	$\varnothing$	$\{x_1,x_2,x_3\}$
$UNC1^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}$	$\{x_5\}$
$ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4\}$	$\{x_4\}$	$\varnothing$
$REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3,x_5\}$
$UNC1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_5\}$	$\{x_5\}$	$\{x_4\}$
$ACP2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4,x_5\}$	$\{x_4,x_5\}$	$\{x_4\}$
$REJ2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2\}$	$\varnothing$	$\{x_1,x_2,x_3,x_5\}$
$UNC2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\varnothing$	$\{x_1,x_2\}$	$\varnothing$
$REF^{op}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_3\}$	$\{x_3\}$	$\varnothing$
$ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_4\}$	$\{x_4\}$	$\varnothing$
$REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3\}$	$\{x_1,x_2,x_3,x_5\}$
$UNC2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\varnothing$	$\varnothing$	$\varnothing$
$REF^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)$	$\{x_5\}$	$\{x_5\}$	$\{x_4\}$

It follows from Table 4.2 that

$$ACP^{op}_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ^{op}_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC^{op}_{(\alpha,\beta)}(E_{1\sim2},A)=U \quad \text{and}$$

$$ACP1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC1^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)=U,$$

for three groups of different parameters α,β.

But

$$ACP2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC2^{op}_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_1,x_2,x_4,x_5\}\neq U \quad \text{and}$$

$$REF^{op}_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_3\},$$

for $(\alpha,\beta)=([0.6,0.8],[0.5,0.7])$ or $(\alpha,\beta)=([0.6,0.7],[0.5,0.6])$.

$$ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC2^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_1,x_2,x_3,x_4\}\neq U \quad \text{and}$$

$$REF^{pe}_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_5\},$$

for three groups of different parameters α,β.

In the first new type (Definition 3.3), the rejection region of optimistic three-way decisions over multiple three-way decision spaces is empty, i.e., $REJ1^{op}_{([0.6,0.8],[0.5,0.7])}(E_{1\sim2},A)=\varnothing$. By Definition 4.2, however, $REJ2^{op}_{([0.6,0.8],[0.5,0.7])}(E_{1\sim2},A)=\{x_1,x_2\}$. This is mainly because $E_1(A)(x_2)=E_2(A)(x_2)=[0.5,0.7]$ and different conditions for *REJ*1 and *REJ*2 in accordance with the standard $\beta = [0.5, 0.7]$.

On the other hand, by Definition 3.5 and Definition 4.3,

$$REJ1^{pe}_{([0.6,0.8],[0.5,0.7])}(E_{1\sim 2}, A) = REJ2^{pe}_{([0.6,0.8],[0.5,0.7])}(E_{1\sim 2}, A) = \{x_1, x_1, x_3\}$$

Since $E_1(A)(x_3) = [0.1, 0.38]$ and $E_2(A)(x_3) = [0.486, 0.709]$, x_3 is rejected from a pessimistic point of view, but x_3 is not rejected from an optimistic point of view, here x_3 is in uncertain and in refusal decision region according to Definition 3.5 (*UNC*1) and Definition 4.3 (*REF*), respectively. □

4.4 The aggregation three-way decisions over multiple three-way decision spaces

Let $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, n$) be n three-way decision spaces, $f \in AF_n(P_D)$, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \le \beta < \alpha \le 1$. Then the aggregation three-way decisions over n three-way decision spaces are defined as follows.

(1) Acceptance region:

$$ACP2^f_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP_{(\alpha,\beta)}(E^f, A),$$

(2) Rejection region:

$$REJ2^f_{(\alpha,\beta)}(E_{1\sim n}, A) = REJ2_{(\alpha,\beta)}(E^f, A) = \{x \in U \mid E^f(A)(x) \le \beta\},$$

(3) Uncertain region:

$$UNC2^f_{(\alpha,\beta)}(E_{1\sim n}, A) = U - ACP2^f_{(\alpha,\beta)}(E_{1\sim n}, A) \cup REJ2^f_{(\alpha,\beta)}(E_{1\sim n}, A).$$

If P_D is a linear order, then $UNC2^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \{x \in U \mid \beta < E^f(A)(x) < \alpha\}$.

The lower and upper approximations of the aggregation three-way decisions over n three-way decision spaces are $\underline{apr2}^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \underline{apr2}_{(\alpha,\beta)}(E^f, A)$ and $\overline{apr2}^f_{(\alpha,\beta)}(E_{1\sim n}, A) = \overline{apr2}_{(\alpha,\beta)}(E^f, A)$ respectively.

In the following, we discuss properties of the aggregation three-way decisions and the lower and upper approximations, and relationships among the aggregation three-way decisions, the optimistic three-way decisions and the pessimistic three-way decisions.

Theorem 4.2. *Let* $(U, Map(V, P_C), P_D, E_i)$ *(* $i = 1, 2, \cdots, n$ *) be* n *three-way decision spaces,* $f \in AF_n(P_D)$ *and satisfy (AF4),* $A \in Map(V, P_C)$ *and* $0 \le \beta < \alpha \le 1$. *Then the following statements hold.*

(1) $ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP2^f_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq ACP2^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$,

(2) $REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ2^f_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq REJ2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A)$.

Proof. (1) It takes notice of $ACP2^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A$, $ACP2^f_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^f_{(\alpha,\beta)}(E_{1\sim n}, A)$ and $ACP2^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = ACP^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$. So it holds from Theorem 3.7 in [12].

(2) Let $x \in REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n}, A) = \bigcap_{i=1}^{n} REJ2_{(\alpha,\beta)}(E_i, A)$. Then $\forall i$, $x \in REJ2_{(\alpha,\beta)}(E_i, A)$, i.e. $E_i(A)(x) \le \beta$. So

$$\begin{aligned} E^f(A)(x) &= f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \\ &\le f(\beta, \beta, \cdots, \beta) = \beta, \end{aligned}$$

i.e. $x \in REJ2^f_{(\alpha,\beta)}(E_{1\sim n}, A)$. If $x \in REJ2^f_{(\alpha,\beta)}(E_{1\sim n}, A)$, then

$$E^f(A)(x) = f(E_1(A)(x), E_2(A)(x), \cdots, E_n(A)(x)) \le \beta.$$

Hence there is an i such that $E_i(A)(x) \le \beta$ due to the condition (AF4) of *f*, i.e. $x \in \bigcup_{i=1}^{n} \underline{apr2}_{(\alpha,\beta)}(E_i, A) = REJ2^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$. □

Theorem 4.3. *Let* $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, n$) *be* n *three-way decision spaces,* $f \in AF_n(P_D)$ *and satisfy (AF4),* $A, B \in Map(V, P_C)$ *and* $0 \le \beta < \alpha \le 1$. *Then*

$$\underline{apr2}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \underline{apr2}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \underline{apr2}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A)$$

and

$$\overline{apr2}^{pe}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \overline{apr2}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \overline{apr2}^{op}_{(\alpha,\beta)}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions and Theorem 4.2. □

Theorem 4.4. *Let* $(U, Map(V, P_C), P_D, E_i)$ ($i = 1, 2, \cdots, n$) *be* n *three-way decision spaces,* $f \in AF_n(P_D)$, $A \in Map(V, P_C)$. *If* $0 \le \beta \le \beta' < \alpha' \le \alpha \le 1$, *then*

$$\underline{apr2}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \subseteq \underline{apr2}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A) \quad \text{and}$$

$$\overline{apr2}^{f}_{(\alpha,\beta)}(E_{1\sim n}, A) \supseteq \overline{apr2}^{f}_{(\alpha',\beta')}(E_{1\sim n}, A).$$

Proof. It is straightforward from the definition of the lower and upper approximations of the aggregation three-way decisions. □

In the following, we use an example to illustrate some notions of the aggregation three-way decisions over multiple three-way decision spaces.

Example 4.3. Consider two three-way decision spaces $(U, Map(U, \{0,1\}), [0,1], E_1)$ and $(U, Map(U, \{0,1\}), [0,1], E_2)$ in Example 4.2 again, where

$$E_1(A) = \frac{[0.4, 0.7]}{x_1} + \frac{[0.2, 0.3]}{x_2} + \frac{[0.1, 0.38]}{x_3} + \frac{[0.8, 0.9]}{x_4} + \frac{[0.6, 0.8]}{x_5},$$

$$E_2(A) = \frac{[0.5, 0.7]}{x_1} + \frac{[0.5, 0.7]}{x_2} + \frac{[0.486, 0.709]}{x_3} + \frac{[0.731, 0.915]}{x_4} + \frac{[0.486, 0.709]}{x_5}.$$

If we consider 2-ary complement-preserving aggregation function

$$f([x_1^-, x_2^-], [x_2^-, x_2^+]) = [\min\{x_1^-, x_2^-\}, \max\{x_2^-, x_2^+\}],$$

then

$$E^f(A) = \frac{[0.4, 0.7]}{x_1} + \frac{[0.2, 0.7]}{x_2} + \frac{[0.1, 0.709]}{x_3} + \frac{[0.731, 0.915]}{x_4} + \frac{[0.486, 0.8]}{x_5}$$

Consider $\alpha = [0.6, 0.8]$ and $\beta = [0.4, 0.7]$, then acceptance regions, rejection regions, uncertain regions and refusal decision regions of aggregation three-way decisions of *A* over two three-way decision spaces are listed in Table 4.3.

Table 4.3. Three types of aggregation three-way decisions based on three-way decision spaces

Acceptance region			Rejection region			Uncertain region		
ACP	*ACP*1	*ACP*2	*REJ*	*REJ*1	*REJ*2	*UNC*	*UNC*1	*UNC*2
$\{x_4\}$	$\{x_4\}$	$\{x_4\}$	$\{x_1,x_2\}$	$\{x_2\}$	$\{x_1,x_2\}$	$\{x_3,x_5\}$	$\{x_1,x_3,x_5\}$	$\{x_5\}$

It follows from Table 4.3 that

$ACP^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC^f_{(\alpha,\beta)}(E_{1\sim2},A)=U$ and

$ACP1^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ1^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC1^f_{(\alpha,\beta)}(E_{1\sim2},A)=U$, but

$ACP2^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup REJ2^f_{(\alpha,\beta)}(E_{1\sim2},A)\cup UNC2^f_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_1,x_2,x_4,x_5\}$ and

$REF^f_{(\alpha,\beta)}(E_{1\sim2},A)=\{x_3\}$.

In the first new type, the rejection region is $REJ1_{([0.6,0.8],[0.4,0.7])}(E^f,A)=\{x_2\}$. In Definition 2.2, however, it is not reasonable that $E^f(A)(x_1)=[0.4,0.7]$ is rejected in accordance with the standard $\beta=[0.4,0.7]$, i.e., *REJ*= $\{x_2\}$. And x_1 is added to uncertain region, since it does not meet the standard $\alpha=[0.6,0.8]$.

In the second new type, x_4 is accepted, x_1 and x_2 are rejected, x_5 is uncertain, and x_3 is refused to make decision. In accordance with definition of aggregation three-way decisions, it is reasonable that x_3 is added to refusal decision region for the standards $\alpha=[0.6,0.8]$ and $\beta=[0.4,0.7]$, since the width of $E^f(A)(x_3)=[0.1,0.709]$ is too large. □

4.5 Dynamic three-way decisions

The following decisions are considered to be made up of more than one three-way decisions.

Definition 4.4. Let $(U,Map(V,P_C),P_D,E_i)$ be i^{th} three-way decision space, $A_i\in Map(V,P_C)$, $\alpha_i,\beta_i\in P_D$ and $0\le\beta_i<\alpha_i\le1$ ($i=1,2,\cdots,n$). Then strict definition of dynamic three-way decisions is given below.

(Decision 1)

Acceptance region 1： $ACP2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=\{x\in U\mid E_1(A_1)(x)\ge\alpha_1\}$.

Rejection region 1： $REJ2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=\{x\in U\mid E_1(A_1)(x)\le\beta_1\}$.

Uncertain region 1： $UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=\{x\in U\mid \beta_1<E_1(A_1)(x)<\alpha_1\}$.

Refusal decision region 1：

$$REF^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=U-\left(ACP2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup REJ2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\right)$$

If $UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)=\varnothing$, then it is end of three-way decisions; or continue to the next step.

(Decision 2)

Acceptance region 2：

$$ACP2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=ACP2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup\left\{x\in UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\mid E_2(A_2)(x)\ge\alpha_2\right\}.$$

Rejection region 2：

$$REJ2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=REJ2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\cup\left\{x\in UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\mid E_2(A_2)(x)\le\beta_2\right\}.$$

Uncertain region 2：

$$UNC2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=\left\{x\in UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A_1)\mid \beta_2<E_2(A_2)(x)<\alpha_2\right\}.$$

Refusal decision region 2：

$$REF^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)=U-\left(ACP2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)\cup REJ2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)\cup UNC2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A_2)\right).$$

……

If $UNC2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})=\varnothing$, then it is end of three-way decisions; or continue to the next step.

(Decision n)

Acceptance region n:

$$ACP2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=ACP2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\cup$$
$$\left\{x\in UNC2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\mid E_n(A_n)(x)\geq\alpha_n\right\}.$$

Rejection region n:

$$REJ2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=REJ2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\cup$$
$$\left\{x\in UNC2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\mid E_n(A_n)(x)\leq\beta_n\right\}.$$

Uncertain region n:

$$UNC2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=\left\{x\in UNC2^{(n-1)}_{(\alpha_{n-1},\beta_{n-1})}(E_{n-1},A_{n-1})\mid \beta_n<E_n(A_n)(x)<\alpha_n\right\}.$$

Refusal decision region n:

$$REF^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)=U-\left(ACP2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)\cup REJ2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)\cup UNC2^{(n)}_{(\alpha_n,\beta_n)}(E_n,A_n)\right).$$

Dynamic three-way decisions are shown as Fig. 4.1.

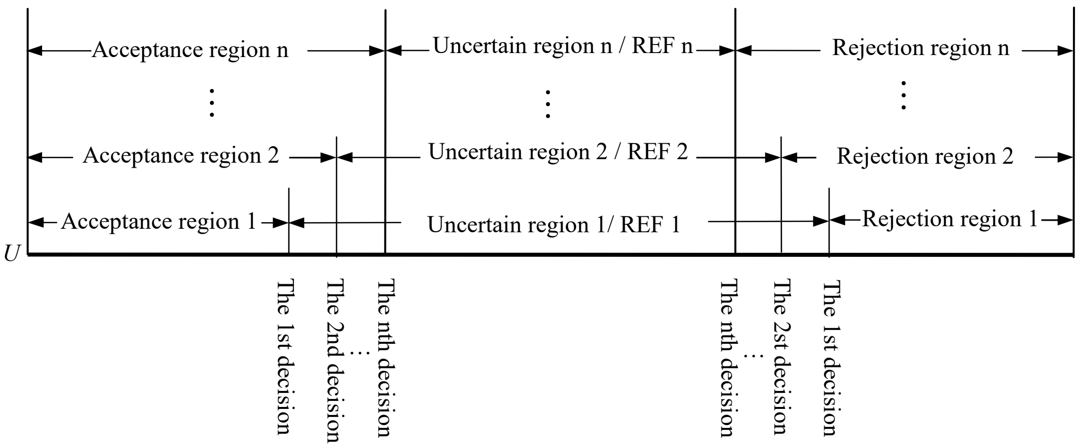

Fig. 4.1 Dynamic three-way decisions

In the following, we use an example to illustrate some notions of the dynamic three-way decisions over multiple three-way decision spaces.

Example 4.4. Consider two three-way decision spaces $(U,Map(U,\{0,1\}),[0,1],E_1)$ and $(U,Map(U,\{0,1\}),[0,1],E_2)$ in Example 4.2 again, where

$$E_1(A)=\frac{[0.4,0.7]}{x_1}+\frac{[0.2,0.3]}{x_2}+\frac{[0.1,0.38]}{x_3}+\frac{[0.8,0.9]}{x_4}+\frac{[0.6,0.8]}{x_5},$$

$$E_2(A)=\frac{[0.5,0.7]}{x_1}+\frac{[0.5,0.7]}{x_2}+\frac{[0.486,0.709]}{x_3}+\frac{[0.731,0.915]}{x_4}+\frac{[0.486,0.709]}{x_5}.$$

Consider $\alpha_1=[0.7,0.8]$ and $\beta_1=[0.5,0.6]$, $\alpha_2=[0.5,0.7]$ and $\beta_2=[0.4,0.5]$, then acceptance regions, rejection regions, uncertain regions and refusal decision regions of dynamic three-way decisions of A over two three-way decision spaces are listed in Table 4.4.

Table 4.4. Three types of dynamic three-way decisions based on three-way decision spaces

	Decision 1	Decision 2
Acceptance region	$ACP2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A)=\{x_4\}$	$ACP2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A)=\{x_1,x_4\}$
Rejection region	$REJ2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A)=\{x_2,x_3\}$	$REJ2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A)=\{x_2,x_3\}$
Uncertain region	$UNC2^{(1)}_{(\alpha_1,\beta_1)}(E_1,A)=\{x_5\}$	$UNC2^{(2)}_{(\alpha_2,\beta_2)}(E_2,A)=\varnothing$
Refusal decision region	$REF^{(1)}_{(\alpha_1,\beta_1)}(E_1,A)=\{x_1\}$	$REF^{(2)}_{(\alpha_2,\beta_2)}(E_2,A)=\{x_5\}$

It follows from Table 4.4 that

$ACP^{f}_{(\alpha,\beta)}(E,A)\bigcup REJ^{f}_{(\alpha,\beta)}(E,A)\bigcup UNC^{f}_{(\alpha,\beta)}(E,A)=U$ and

$ACP1^{f}_{(\alpha,\beta)}(E,A)\bigcup REJ1^{f}_{(\alpha,\beta)}(E,A)\bigcup UNC1^{f}_{(\alpha,\beta)}(E,A)=U$, but

$ACP2^{f}_{(\alpha,\beta)}(E,A)\bigcup REJ2^{f}_{(\alpha,\beta)}(E,A)\bigcup UNC2^{f}_{(\alpha,\beta)}(E,A)=\{x_1,x_2,x_4,x_5\}$ and

$REF^{f}_{(\alpha,\beta)}(E,A)=\{x_3\}$.

In the first new type (Definition 3.1), the rejection region is $REJ1=\{x_2\}$. In Definition 2.2, however, it is not reasonable that $E^f(A)(x_1)=[0.4,0.7]$ is rejected in accordance with the standard $\beta=[0.4,0.7]$, i.e., $REJ=\{x_2\}$. And x_1 is added to uncertain region, since it does not meet the standard $\alpha=[0.6,0.8]$.

In the second new type, x_4 is accepted, x_1 and x_2 are rejected, x_5 is uncertain, and x_3 is refused to make decision. In accordance with Definition 4.1, it is reasonable that x_3 is added to refusal decision region for the standards $\alpha=[0.6,0.8]$ and $\beta=[0.4,0.7]$, since the width of $E^f(A)(x_3)=[0.1,0.709]$ is too large. □

5. Conclusions

This paper discusses two new types of three-way decisions in three-way decision spaces through answering two questions in [9]. The first type relates to parameter changes in assumptions and another type involves refusal decision region. For the sake of clarity, we summarize the main conclusions on parameter assumption, acceptance region, rejection region and uncertain region under three types in Table 5.1.

Table 5.1.

Parameter assumption, acceptance region, rejection region and uncertain region under three types.

	First type	Second type	Existing type
Parameter assumption	$0\le\beta\le\alpha\le1$	$0\le\beta<\alpha\le1$	$0\le\beta<\alpha\le1$
Acceptance region	$\{x\in U\mid E(A)\ge\alpha\}$	$\{x\in U\mid E(A)\ge\alpha\}$	$\{x\in U\mid E(A)\ge\alpha\}$
Rejection region	$x\in U\mid E(A)<\beta\}$	$\{x\in U\mid E(A)\le\beta\}$	$\{x\in U\mid E(A)\le\beta\}$
Uncertain region	$(\{x\in U\mid E(A)\ge\alpha\}\bigcup\{x\in U\mid E(A)<\beta\})^c$	$\{x\in U\mid\beta<E(A)<\alpha\}$	$(\{x\in U\mid E(A)\ge\alpha\}\bigcup\{x\in U\mid E(A)<\beta\})^c$

From Table 5.1, we can see the following conclusions..

(1) Acceptance regions of either new types or existing type are same.

(2) If we consider linear order, second type and existing type are same. Only while we consider nonlinear order, uncertain regions of second type and existing type are different.

There exist the following benefits in the first new type:

(1) Two-way decisions can regard as special cases of three-way decisions when two decision parameters are equal and the order of decision measurement is a linear order.

(2) Acceptance with "large than or equal to" and rejection with "less than" are more in line with practical applications and semantic.

(3) It unifies double evaluation function with a single evaluation function where three-way decisions based on double evaluation functions are classified as some operations of three-way decisions based on single evaluation function.

There exist the following benefits in the second new type:

There are refusal decision region in pattern recognition and decision making. This paper introduces refusal decision region in three-way decision. The type may be applied to pattern recognition, decision making, cluster analysis, etc.

We will discuss inference rules and potential applications of these new types of three-way decisions in three-way decision spaces in future work.

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