MEAN-VARIANCE POLICY FOR DISCRETE-TIME CONE-CONSTRAINED MARKETS: TIME CONSISTENCY IN EFFICIENCY AND THE MINIMUM-VARIANCE SIGNED SUPERMARTINGALE MEASURE*

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The discrete-time mean-variance portfolio selection formulation, which is a representative of general dynamic mean-risk portfolio selection problems, typically does not satisfy time consistency in efficiency (TCIE), i.e., a truncated pre-committed efficient policy may become inefficient for the corresponding truncated problem. In this paper, we analytically investigate the effect of portfolio constraints on the TCIE of convex cone-constrained markets. More specifically, we derive semi-analytical expressions for the pre-committed efficient mean-variance policy and the minimum-variance signed supermartingale measure (VSSM) and examine their relationship. Our analysis shows that the pre-committed discrete-time efficient mean-variance policy satisfies TCIE if and only if the conditional expectation of the density of the VSSM (with respect to the original probability measure) is nonnegative, or once the conditional expectation becomes negative, it remains at the same negative value until the terminal time. Our finding indicates that the TCIE property depends only on the basic market setting, including portfolio constraints. This motivates us to establish a general procedure for constructing TCIE dynamic portfolio selection problems by introducing suitable portfolio constraints.

Key Words: cone-constrained market, discrete-time mean-variance policy, time consistency in efficiency, minimum-variance signed supermartingale measure

1 INTRODUCTION

In a dynamic decision problem, a decision maker may face a dilemma when the overall objective for the entire time horizon under consideration does not conform with a "local" objective for

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a tail part of the time horizon. In the language of dynamic programming, Bellman's principle of optimality is not applicable in such situations, as the global and local interests derived from their respective objectives are not consistent. This phenomenon has recently been investigated in the finance and financial engineering literature under the terminology of *time inconsistency*. In the language of portfolio selection, when a problem is not time consistent, the (global) optimal portfolio policy for the entire investment horizon determined at the initial time may not be optimal for a truncated investment problem at some intermediate time t and for a certain realized wealth level. Thus, investors have incentives to deviate from the global optimal policy and to seek the (local) optimal portfolio policy, instead, for the truncated time horizon.

As time consistency (or dynamic consistency) is a basic requirement for dynamic risk measures (see Rosazza Gianin (2006), Boda and Filar (2006), Artzner et al. (2007) and Jobert and Rogers (2008)), all of the appropriate dynamic risk measures should necessarily possess a certain functional structure such that Bellman's principle of optimality is satisfied. In particular, the time-0 risk of a future monetary position X_T, $\rho_0(X_T)$, can be obtained by first computing the time-t risk of X_T, $\rho_t(X_T)$, with $0 < t < T$, and then obtaining the time-0 risk of the monetary position $-\rho_t(X_T)$ at time t, i.e., a nested recursion $\rho_0(X_T) = \rho_0(-\rho_t(X_T))$ must hold true. Unfortunately, almost all of the static risk measures that investors have adopted in practice for decades, including the variance measure, VaR (Duffie and Pan (1997)) and CVaR (Uryasev (2000)), fail to be time consistent, when extended to dynamic situations (Boda and Filar (2006)). Researchers have proposed using nonlinear expectation ("g-expectation" (Peng (1997)) to construct time consistent dynamic risk measures.

As the time consistency requirement may result in a very conservative risk measure (see Roorda and Schumacher (2007)), there have been numerous proposals to relax it to weak time consistency (or sequential consistency), which represents the idea that a monetary position that is surely (un)acceptable at some future date should also be (un)acceptable now. We refer to Riedel (2004), Föllmer and Penner (2006, 2011), Tutsch (2006, 2008), and Roorda and Schumacher (2013, 2014) for detailed discussions of weak time consistency.

When a dynamic risk measure is time consistent, it not only justifies the mathematical formulation for risk management, but also facilitates the process of finding the optimal decision, as the corresponding dynamic mean-risk portfolio selection problem satisfies Bellman's principle of optimality, and is thus solvable by dynamic programming (e.g., see Cherny (2010)). When a dynamic risk measure is time inconsistent, the corresponding dynamic mean-risk portfolio selection problem is nonseparable in the sense of dynamic programming, and is thus intractable. Consider the dynamic mean-variance portfolio selection problem as an example, as it is the focus of this paper. As the nonseparable structure of the variance term leads to a challenging dynamic variance minimization problem, it took almost 50 years to figure out ways to extend the seminal static mean-variance formulation in Markowitz (1952) to its dynamic counterpart (see Li and Ng (2000) for the discrete-time (multi-period) mean-variance formulation and Zhou and Li (2000) for the continuous-time mean-variance formulation). The dynamic optimal investment policy derived in Li and Ng (2000) and Zhou and Li (2000) is called the *pre-committed* dynamic optimal investment policy by Basak and Chabakauri (2010), because the (adaptive) optimal policy is determined at time 0 to achieve an overall optimality for the entire investment horizon. As the original dynamic mean-variance formulation is not time consistent, the *pre-committed* dynamic optimal investment policy does not satisfy the principle of optimality and investors have incentives to deviate from it in certain circumstances, as revealed in Zhu et al. (2003) and Basak and Chabakauri (2010).

Two major research directions were pursued to alleviate the effects of the time inconsistency of the pre-committed optimal mean-variance policy. Basak and Chabakauri (2010) suggested building a time consistent policy by using backward induction to optimally choose the (time consistent) policy at any time t, on the premise that future time consistent policies have already been decided. Björk et al. (2014) extended the formulation in Basak and Chabakauri (2010) by introducing state dependent risk aversion, and used the backward time inconsistent control method (see Björk and Murgoci (2010)) to derive the corresponding time consistent policy. Czichowsky (2013) considered time consistent policies for both the discrete-time and continuous-time mean-variance models and revealed the connections between the two. Enforcing a time consistent policy in an inherently time inconsistent problem undoubtedly induces a cost, i.e., it results in a worse mean-variance efficient frontier compared with the one associated with the pre-committed mean-variance policy. This can be seen in numerical experiments reported in Wang and Forsyth (2011). On the other hand, Cui et al. (2012) relaxed the concept of time consistency to "*time consistency in efficiency*" (TCIE) based on a multi-objective version of the principle of optimality. This states that the principle of optimality holds if any tail part of an efficient policy is also efficient for any realizable state at any intermediate period (Li and Haimes (1987) and Li (1990)). Note that the essence of the groundbreaking work of Markowitz (1952) is to attain efficiency in portfolio selection by striking a balance between the two conflicting objectives of maximizing the expected return and minimizing the investment risk. In this sense, TCIE simply means requiring efficiency for any truncated mean-variance portfolio selection problem at every time instant during the investment horizon. Cui et al. (2012) showed that the dynamic mean-variance problem does not satisfy TCIE and developed a TCIE revised mean-variance policy by relaxing the self-financing restriction to allow the withdrawal of money from the market. While the revised policy achieves the same mean-variance pair for terminal wealth as the pre-committed dynamic optimal investment policy, it also enables investors to receive a free cash flow stream during the investment process. The revised policy proposed in Cui et al. (2012) thus strictly dominates the pre-committed dynamic optimal investment policy.

It is interesting to note that the current literature on time inconsistency has been mainly confined to investigations of time consistent risk measures. While portfolio constraints are an important part of the market setting, few studies have investigated the effects of portfolio constraints on the property of time consistency and TCIE. Let us consider an extreme situation in which only one admissible investment policy is available over the entire investment horizon. In such a situation, regardless of whether the adopted dynamic risk measure is time consistent, this policy is always optimal and time consistent, as it is the only choice available to investors. We can also learn from Wang and Forsyth (2011), who numerically compared the pre-committed optimal mean-variance policy and the time consistent mean-variance policy (proposed by Björk et al. (2014)) in a continuous-time market with no constraint, with no-bankruptcy constraint and with no-shorting constraint, respectively. They found that, with constraints, the efficient frontier generated by the time consistent mean-variance policy is closer to the efficient frontier generated by the pre-committed optimal mean-variance policy in the constrained market than in the unconstrained market. Thus, the presence of portfolio constraints may reduce the cost, when enforcing a time consistent policy in an inherent time inconsistent problem. Based on the above observation, in this paper we analytically investigate the effect of convex cone-type portfolio constraints on TCIE in a discrete-time market. Our analysis reveals an "if and only if" relationship between TCIE and the conditional expectation of the density of the minimum-variance signed supermartingale measure (with respect to the original probability measure). As our findings indicate that the property of TCIE depends only on the basic market setting, including portfolio constraints, we further establish a general procedure for constructing TCIE dynamic portfolio selection problems by introducing suitable portfolio constraints.

The main theme and the contribution of this paper is to address and answer the following question.

Given a financial market with known return statistics, what are the cone constraints on the portfolio policies which need to be introduced so that the derived optimal portfolio policy is time consistent in efficiency?

The paper thus establishes the following key points to achieve this overall goal. For a given convex cone-constrained market, we derive the pre-committed discrete-time efficient mean-variance policy by both duality theory and dynamic programming (Section 2). We then discuss the necessary and sufficient conditions for the pre-committed efficient policy to be time consistent in efficiency (Section 3). We define and derive the minimum-variance signed supermartingale measure (VSSM) for cone-constrained markets and reveal its close relationship with TCIE (Section 4). More specifically, we show that the pre-committed efficient mean-variance policy satisfies TCIE if and only if the conditional expectation of the density of the VSSM (with respect to the original probability measure) is nonnegative, or once the conditional expectation becomes negative, it keeps the same negative value until the terminal time. Finally we answer the question of how to completely eliminate time inconsistency in efficiency by adding cone constraints to the market (Section 5). We demonstrate this constructive process using illustrative examples. To make our presentation clear, we have placed all the proofs in the Appendix.

2 OPTIMAL MEAN-VARIANCE POLICY IN A DISCRETE-TIME CONE-CONSTRAINED MARKET

The capital market of T-time periods under consideration consists of n risky assets with random rates of return and one riskless asset with a deterministic rate of return. An investor with an initial wealth x_0 joins the market at time 0 and allocates wealth among these $(n+1)$ assets. Wealth can be reallocated among the $(n+1)$ assets at the beginning of each of the following $(T-1)$ consecutive time periods. The deterministic rate of return of the riskless asset at time period t is denoted by $s_t > 0$ and the rates of return of the risky assets at time period t are denoted by the vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$, where e_t^i is the random return of asset i at time period t and the notation $'$ denotes the transpose operation. It is assumed that the vectors $\mathbf{e}_t$, $t = 0, 1, \cdots, T-1$, are statistically independent with mean vector $\mathbb{E}[\mathbf{e}_t] = [\mathbb{E}[e_t^1], \cdots, \mathbb{E}[e_t^n]]'$ and a positive definite covariance matrix,

$$\mathrm{Cov}\left(\mathbf{e}_t\right) = \begin{bmatrix} \sigma_{t,11} & \cdots & \sigma_{t,1n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,1n} & \cdots & \sigma_{t,nn} \end{bmatrix} \succ 0.$$

Assume that all of the random vectors, $\mathbf{e}_t$, $t = 0, 1, \cdots, T-1$, are defined in a filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t = \sigma\left(\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_{t-1}\right)$ and $\mathcal{F}_0$ is the trivial σ-algebra over Ω. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is simply the unconditional expectation $\mathbb{E}[\cdot]$. Let x_t be the wealth of the investor at the beginning of the t-th time period, and u_t^i, $i = 1, 2, \cdots, n$, be the dollar amount invested in the ith risky asset at the beginning of the t-th time period. The dollar amount invested in the riskless asset at the beginning of the t-th time period is then equal to $x_t - \sum_{i=1}^n u_t^i$. It is assumed that the admissible investment strategy $\mathbf{u}_t = [u_t^1, u_t^2, \cdots, u_t^n]'$ is an $\mathcal{F}_t$-measurable Markov control, i.e., $\mathbf{u}_t \in \mathcal{F}_t$, and the realization of $\mathbf{u}_t$ is restricted to a deterministic and non-random convex cone $\mathcal{A}_t \subseteq \mathbb{R}^n$. Such cone type constraints are widely used to model regulatory restrictions; for example, the restrictions of no short selling and non-tradability. Cone constraints are also useful for representing portfolio restrictions, and can be generally expressed by $\mathcal{A}_t = \{\mathbf{u}_t \in \mathbb{R}^n | A\mathbf{u}_t \geq 0,\ A \in \mathbb{R}^{m\times n}\}$ (see Cuoco (1997) and Napp (2003) for more details).

An investor of mean-variance type seeks the best admissible investment strategy, $\{\mathbf{u}_t^*\}\ |_{t=0}^{T-1}$, such that the variance of the terminal wealth, $\mathrm{Var}(x_T)$, is minimized subject to the expected terminal wealth, $\mathbb{E}[x_T]$, being fixed at a preselected level d,

$$(P(d)) : \quad \begin{cases} \min & \mathrm{Var}(x_T) \equiv \mathbb{E}\left[(x_T - d)^2\right], \\ \text{s.t.} & \mathbb{E}[x_T] = d, \\ & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \\ & \mathbf{u}_t \in \mathcal{A}_t, \quad t = 0, 1, \ldots, T-1, \end{cases}$$

where

$$\mathbf{P}_t = \left[P_t^1, P_t^2, \cdots, P_t^n\right]' = \left[(e_t^1 - s_t), (e_t^2 - s_t), \cdots, (e_t^n - s_t)\right]'$$

is the vector of the excess rates of return. It is easy to see that $\mathbf{P}_t$ and $\mathbf{u}_t$ are independent, $\{x_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t)$.

Remark 2.1. *Varying the parameter d in $(P(d))$ from $-\infty$ to $+\infty$ yields the minimum variance set in the mean-variance space. Furthermore, as setting d equal to $\prod_{i=0}^{T-1} s_i x_0$ in $(P(d))$ gives rise to the minimum variance point, the upper branch of the minimum variance set corresponding to the range of d from $\prod_{i=0}^{T-1} s_i x_0$ to $+\infty$ characterizes the efficient frontier in the mean-variance space which enables investors to recognize the trade-off between the expected return and the risk, thus helping them specify their preferred expected terminal wealth.*

Note that condition $\mathrm{Cov}\left(\mathbf{e_t}\right) \succ 0$ implies the positive definiteness of the second moment of

$(s_t, \mathbf{e}_t')'$. The following is then true for $t = 0, 1, \ldots, T-1$:

$$\begin{bmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{P}_t'] \\ s_t\mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] \end{bmatrix} \begin{bmatrix} 1 & -1 & \ldots & -1 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \succ 0,$$

which further implies

$$\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] \succ 0, \quad \forall\, t = 0, 1, \ldots, T-1,$$
$$s_t^2(1 - \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]) > 0, \quad \forall\, t = 0, 1, \ldots, T-1.$$

Constrained dynamic mean-variance portfolio selection problems with various constraints have been attracting increasing attention in the last decade, e.g., Li et al. (2002), Zhu et al. (2004), Bielecki et al. (2005), Sun and Wang (2006), Labbé and Heunis (2007) and Czichowsky and Schweizer (2010). Recently, Czichowsky and Schweizer (2013) further considered cone-constrained continuous-time mean-variance portfolio selection with semimartingale price processes.

Remark 2.2. *In this section, we use duality theory and dynamic programming to analytically derive the discrete-time efficient mean-variance policy in convex cone-constrained markets. We demonstrate that the optimal mean-variance policy is a two-piece linear function of the current wealth level, which represents an extension of the result in Cui et al. (2014) for discrete-time markets under the no-shorting constraint (a special convex cone) and a discrete-time counterpart of the policy in Czichowsky and Schweizer (2013).*

We define the following two deterministic functions, $h_t^+(\mathbf{K}_t)$ and $h_t^-(\mathbf{K}_t)$, on $\mathbf{R}^n$ for $t = 0, 1, \ldots, T-1$,

$$h_t^{\pm}(\mathbf{K}_t) = \mathbb{E}\left[C_{t+1}^+\left(1 \mp \mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t \le \pm 1\}} + C_{t+1}^-\left(1 \mp \mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t > \pm 1\}}\right],$$

with terminal condition $C_T^+ = C_T^- = 1$, and denote their deterministic minimizers and optimal values, respectively, as

$$\mathbf{K}_t^{\pm} = \arg\min_{\mathbf{K}_t \in \mathcal{A}_t} \mathbb{E}\left[C_{t+1}^+\left(1 \mp \mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t \le \pm 1\}} + C_{t+1}^-\left(1 \mp \mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t > \pm 1\}}\right],$$

$$C_t^{\pm} = \mathbb{E}\left[C_{t+1}^+\left(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^-\left(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right]. \tag{1}$$

As we later show, parameters $\mathbf{K}_t^{\pm}$ and $C_t^{\pm}$ appear in the optimal policy for problem $(P(d))$. The following lemma is important for deriving our main results.

Lemma 2.1. *For $t = 0, 1, \ldots, T-1$, the following properties hold,*

$$C_t^{\pm} = \mathbb{E}\left[C_{t+1}^+\left(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\right) 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^-\left(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\right) 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right], \tag{2}$$
$$0 < C_t^{\pm} \le C_{t+1}^{\pm}.$$

Furthermore, $C_t^{\pm} = C_{t+1}^{\pm}$ if and only if $\mathbf{K}_t^{\pm} = \mathbf{0}$, where notation $\mathbf{0}$ denotes the n-dimensional zero vector.

Note that Lemma 2.1 reduces the piecewise quadratic form of $C_t^{\pm}$ in (1) to a piecewise linear one in (2). We now adopt Lagrangian duality and dynamic programming to solve problem $(P(d))$.

Theorem 2.1. *Define $\rho_t = \prod_{\ell=t}^{T-1} s_\ell$ (with $\prod_{i\in\emptyset} f_i$ being set to 1). When both $d > \rho_0 x_0$ and $C_0^+ = 1$ hold, or both $d < \rho_0 x_0$ and $C_0^- = 1$ hold, problem $(P(d))$ does not have any feasible solution.*

Under the assumption that problem $(P(d))$ is feasible, its optimal investment policy can be expressed by the following deterministic piecewise linear function of the wealth level x_t,

$$\mathbf{u}_t^\star(x_t) = s_t \mathbf{K}_t^+ \big((d-\mu^\star)\rho_t^{-1} - x_t\big) 1_{\{d-\mu^\star \geq \rho_t x_t\}} - s_t \mathbf{K}_t^- \big((d-\mu^\star)\rho_t^{-1} - x_t\big) 1_{\{d-\mu^\star < \rho_t x_t\}}, \quad t = 0, 1, \ldots, T-1, \tag{3}$$

where

$$\mu^\star = \frac{d-\rho_0 x_0}{1-(C_0^+)^{-1}} 1_{\{d \geq \rho_0 x_0\}} + \frac{d-\rho_0 x_0}{1-(C_0^-)^{-1}} 1_{\{d < \rho_0 x_0\}}. \tag{4}$$

Moreover, the minimum variance set is given as

$$\mathrm{Var}(x_T) = \frac{C_0^+ \big(\mathbb{E}[x_T] - \rho_0 x_0\big)^2}{1 - C_0^+} 1_{\{\mathbb{E}[x_T] \geq \rho_0 x_0\}} + \frac{C_0^- \big(\mathbb{E}[x_T] - \rho_0 x_0\big)^2}{1 - C_0^-} 1_{\{\mathbb{E}[x_T] < \rho_0 x_0\}},$$

and the mean-variance efficient frontier, which is the upper branch of the minimum variance set, is expressed as

$$\mathrm{Var}(x_T) = \frac{C_0^+ \big(\mathbb{E}[x_T] - \rho_0 x_0\big)^2}{1 - C_0^+}, \quad \textit{for} \quad \mathbb{E}[x_T] \geq \rho_0 x_0.$$

Note that every point on the lower branch of the minimum variance set corresponding to $d < \rho_0 x_0$ is dominated by the minimum variance point with $\mathbb{E}[x_T] = \rho_0 x_0$ and $\mathrm{Var}(x_T) = 0$. Although the cases with $d < \rho_0 x_0$ do not make sense from an economic point of view for the entire investment horizon, we do need this explicit expression for the lower branch of the minimum variance set for our later discussion. As we later demonstrate, the pre-committed investment policy is not time consistent in efficiency. Thus, applying the pre-committed mean-variance policy for a truncated time horizon could result in an inefficient mean-variance pair that falls onto the lower branch of the minimum variance set for the truncated time horizon. Time inconsistency in efficiency, which is difficult to justify from an economic point of view, hides behind this type of phenomenon. The purpose of this paper is to devise a solution to this problem.

Theorem 2.1 reveals that the optimal investment policy is a two-piece linear function with respect to the investor's current wealth level and this finding represents the discrete-time counterpart of the continuous-time result in Czichowsky and Schweizer (2013). In Section 5, we also demonstrate that Theorem 2.1 provides an extension of the result in Cui et al. (2014) for the multiperiod mean-variance formulation with no-shorting constraint.

When $d \geq \rho_0 x_0$ and $C_0^+ < 1$ hold, the optimal investment policy $\mathbf{u}_t^\star$, $t = 0, 1, \cdots, T-1$, in (3) is efficient. We call such a policy a pre-committed efficient mean-variance policy following Basak and Chabakauri (2010). While $d = \rho_0 x_0$, the optimal investment policy is achieved by $\mathbf{u}_t^\star = \mathbf{0}$, i.e., the investor invests all of his wealth in the riskless asset, which is exactly the minimum variance policy. When $d < \rho_0 x_0$ and $C_0^- < 1$ hold, the optimal investment policy of $(P(d))$, $\mathbf{u}_t^\star$, $t = 0, 1, \cdots, T-1$, in (3) is inefficient.

Remark 2.3. *By setting $\mathcal{A}_t = \mathbb{R}^n$, $t = 0, 1, \ldots, T-1$, the pre-committed discrete-time efficient mean-variance policy in (3) reduces to the one in the unconstrained market (Li and Ng (2000)),*

$$\mathbf{u}_t^\star(x_t) = s_t\big((d-\mu^\star)\rho_t^{-1} - x_t\big)\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right], \quad t = 0, 1, \ldots, T-1,$$

where

$$\mu^\star = \frac{d - \rho_0 x_0}{1 - \prod_{i=0}^{T-1}(1 - \mathbb{E}\left[\mathbf{P}_i'\right]\mathbb{E}^{-1}\left[\mathbf{P}_i\mathbf{P}_i'\right]\mathbb{E}\left[\mathbf{P}_i\right])^{-1}}.$$

There are three major differences between the pre-committed efficient mean-variance policies in a cone-constrained market and in an unconstrained market. First, in a cone-constrained market, problem ($P(d)$) may become infeasible, while feasibility is never an issue for the mean-variance portfolio selection in an unconstrained market. Second, when P_t, $t = 0, 1, \ldots, T-1$, are identically distributed, $\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]$, $t = 0, 1, \ldots, T-1$, take the same value, which implies that the investor holds a unique risky portfolio for any time period in an unconstrained market. This portfolio is also independent of wealth. In a cone-constrained market, however, the investor may hold two different risky portfolios, $\mathbf{K}_t^+$ and $\mathbf{K}_t^-$, while $\mathbf{K}_t^+$ and $\mathbf{K}_t^-$ are in general different for the same time t. A key observation, thus, is that the investor may switch his risky position according to his current wealth level. Third, in a cone-constrained market, although the excess rates of return of risky assets, $\mathbf{P}_t$, $t = 0, 1, \cdots, T-1$, are statistically independent, future $\mathbf{P}_\tau$, $\tau > t$, may influence the current risky portfolios, $\mathbf{K}_t^+$ and $\mathbf{K}_t^-$, through parameters C_t^+ and C_t^-, which implies that the independent structure of the optimal risky portfolio holdings (rooted in the independence assumption of the random rates of return) is destroyed by the presence of constraints. Thus, in general, $\mathbf{K}_t^\pm \neq \mathbf{K}_s^\pm$ when $t \neq s$, which further implies that the risky positions are no longer time-invariant. In summary, we can conclude that in a cone-constrained market, the risky positions are both state-dependent and time-dependent.

3 CONDITIONS FOR THE TIME CONSISTENCY IN EFFICIENCY OF THE PRE-COMMITTED EFFICIENT MEAN-VARIANCE POLICY

We now check the performance of the pre-committed optimal mean-variance policy $\{\mathbf{u}_t^\star\}\mid_{t=0}^{T-1}$, derived for the entire investment time horizon given in (3), in truncated time periods. More specifically, we examine the efficiency of $\{\mathbf{u}_t^\star\}\mid_{t=0}^{T-1}$ in shorter time periods and develop conditions under which $\{\mathbf{u}_t^\star\}\mid_{t=0}^{T-1}$ remains efficient at all times. Let us consider the following truncated mean-variance problem, for any realized wealth x_k in time period k,

$$(P_k(d_k) \mid x_k): \quad \begin{cases} \min & \mathrm{Var}(x_T) = \mathbb{E}\big[(x_T - d_k)^2\big] \\ \text{s.t.} & \mathbb{E}[x_T] = d_k, \\ & x_{t+1} = s_t x_t + \mathbf{P}_t'\mathbf{u}_t, \\ & \mathbf{u}_t \in \mathcal{A}_t, \quad t = k, k+1, \ldots, T-1, \\ & x_k \text{ is known,} \end{cases}$$

where d_k is a preselected level of the expected final wealth for the truncated mean-variance problem. As problem $(P_k(d_k) \mid x_k)$ has the same structure as problem ($P(d)$), based on Theorem

2.1, the corresponding optimal policy of $(P_k(d_k))$ is

(5)

$$\tilde{\mathbf{u}}_t^\star(x_t \mid d_k) = s_t\mathbf{K}_t^+\big((d_k - \mu_k^\star)\rho_t^{-1} - x_t\big)1_{\{d_k-\mu_k^\star \geq \rho_t x_t\}} - s_t\mathbf{K}_t^-\big((d_k - \mu_k^\star)\rho_t^{-1} - x_t\big)1_{\{d_k-\mu_k^\star < \rho_t x_t\}},$$
$$t = k, k+1, \dots, T-1,$$

where

$$\mu_k^\star = \frac{d_k - \rho_k x_k}{1-(C_k^+)^{-1}}1_{\{d_k \geq \rho_k x_k\}} + \frac{d_k - \rho_k x_k}{1-(C_k^-)^{-1}}1_{\{d_k < \rho_k x_k\}}.$$

From our discussion on $(P(d))$, the solution to $(P_k(d_k) \mid x_k)$ is mean-variance efficient at x_k if and only if $d_k \geq \rho_k x_k$.

Now we consider the following inverse optimization problem of $(P_k(d_k) \mid x_k)$: For any x_k, $k = 1, 2, \dots, T-1$, find an expected final wealth level d_k such that the truncated pre-committed optimal mean-variance policy $\mathbf{u}_t^\star(x_t)$ $(t = k, k+1, \dots, T-1)$, with $x_t = x_k$, specified in (3) solves $(P_k(d_k) \mid x_k)$. We call such a d_k an *induced expected final wealth level* by the pre-committed policy at x_k. It is now evident that, if for some x_k, $k = 1, 2, \dots, T-1$, the induced d_k is less than $\rho_k x_k$, then the truncated pre-committed optimal mean-variance policy $\mathbf{u}_t^\star(x_t)$ $(t = k, k+1, \dots, T-1)$, with $x_t = x_k$, is inefficient for the truncated mean-variance problem from stage k to T with given x_k.

Definition 3.1. *An efficient solution of $(P(d))$, $\{\mathbf{u}_t^\star(x_t)\}|_{t=0}^{T-1}$, is time consistent in efficiency if for all wealth x_k in time period k, k = 1, ..., $T-1$, the induced expected final wealth level d_k always satisfies $d_k \geq \rho_k x_k$, such that $\{\mathbf{u}_t^\star(x_t)\}|_{t=k}^{T-1}$ solves $(P_k(d_k) \mid x_k)$.*

In plain language, a globally mean-variance efficient solution is time consistent in efficiency if it is also locally mean-variance efficient for every intermediate stage and every possible realizable state (wealth level x_t).

Remark 3.1. *Note that the above definition of TCIE is conceptually the same as the one in Cui et al. (2012). However, the current one is defined in terms of the induced expected final wealth, whereas the one in Cui et al. (2012) is defined in terms of the induced trade off between the variance and the expectation of the terminal wealth.*

Remark 3.2. *Note also that insisting on time consistency of $\{\mathbf{u}_t^\star(x_t)\}|_{t=0}^{T-1}$ implies that $\{\mathbf{u}_t^\star(x_t)\}|_{t=k}^{T-1}$ solves $(P_k(d) \mid x_k)$ for any realized wealth x_k in every time period k, k = 1, ..., $T-1$.*

Remark 3.3. *Cui et al. (2012) showed that the discrete-time mean-variance portfolio selection problem is not time consistent in efficiency in unconstrained markets. When wealth exceeds a deterministic level determined by the market setting, the investor may become irrational and seek to minimize both the mean and the variance when continuing to apply the pre-committed efficient policy. In this section, we check whether the discrete-time mean-variance portfolio selection problem in cone-constrained markets also fails to be time consistent in efficiency.*

Note that the truncated minimum variance policy is always the minimum variance policy of the corresponding truncated mean-variance problem. Therefore, we only need to check whether the truncated pre-committed efficient policy (except for the minimum variance policy), $\mathbf{u}_t^\star(x_t)$, $t = k, k+1, \ldots, T-1$, is efficient with respect to the corresponding truncated mean-variance problem.

Theorem 3.1. *The truncated pre-committed efficient mean-variance policy (except for the minimum variance policy), $\mathbf{u}_t^\star(x_t), t = k, k+1, \ldots, T-1$, is an efficient policy of the truncated problem $(P(d_k) \mid x_k)$, if and only if*

$$\text{(i) } d - \mu^\star \geq \rho_k x_k, \quad \text{or} \quad \text{(ii) } d - \mu^\star < \rho_k x_k, \;\; C_k^- = 1.$$

Condition (i) in Theorem 3.1 for the efficiency of the truncated pre-committed efficient mean-variance policy at time k can be interpreted as a threshold condition for x_k,

$$x_k \leq \rho_k^{-1}(d - \mu^\star) = \rho_k^{-1} \frac{d - C_0^+ \rho_0 x_0}{1 - C_0^+},$$

which is similar to the result of Proposition 3.1 in Cui et al. (2012). However, note from the last statement in Lemma 2.1, if C_k^- becomes 1, then all C_t^- with $k < t \leq T-1$ will remain 1, implying $\mathbf{K}_t^- = 0$, $k \leq t \leq T-1$. Therefore, condition (ii) in Theorem 3.1 can be interpreted as follows: Once the wealth level at time k exceeds the deterministic level, $\rho_k^{-1}(d - \mu^\star)$, the investor switches to the minimum variance policy (investing all of wealth in the riskless asset). With the help of Eq. (3), under both conditions, the investor either holds portfolio $\mathbf{K}_k^+$ or only invests in the riskless asset. Thus, we call $\mathbf{K}_k^+$ an *efficient risky portfolio.* In contrast, when $d - \mu^\star < \rho_k x_k$, $C_k^- < 1$, the truncated pre-committed efficient mean-variance policy is inefficient and the corresponding portfolio $\mathbf{K}_k^-$ is thus termed an *inefficient risky portfolio.*

Based on Proposition 3.1 and the definition of TCIE, the following lemma for the TCIE of the pre-committed efficient mean-variance policy is apparent.

Lemma 3.1. *The pre-committed efficient mean-variance policy (except for the minimum variance policy) is time consistent in efficiency if and only if condition (i) or condition (ii) holds for all possible x_t achieved by the pre-committed efficient mean-variance policy and for all $t = 1, 2, \ldots, T-1$.*

The next proposition provides further insight about TCIE.

Proposition 3.1. *Adopting the pre-committed efficient mean-variance policy at time t yields the following conditional probabilities,*

$$\begin{aligned}
&Pr\left((d - \mu^\star) \geq \rho_{t+1}x_{t+1} \middle| (d - \mu^\star) > \rho_t x_t\right) = Pr\left(\mathbf{P}_t'\mathbf{K}_t^+ \leq 1\right), \\
&Pr\left((d - \mu^\star) < \rho_{t+1}x_{t+1} \middle| (d - \mu^\star) > \rho_t x_t\right) = Pr\left(\mathbf{P}_t'\mathbf{K}_t^+ > 1\right), \\
&Pr\left((d - \mu^\star) \geq \rho_{t+1}x_{t+1} \middle| (d - \mu^\star) < \rho_t x_t\right) = Pr\left(\mathbf{P}_t'\mathbf{K}_t^- \leq -1\right), \\
&Pr\left((d - \mu^\star) < \rho_{t+1}x_{t+1} \middle| (d - \mu^\star) < \rho_t x_t\right) = Pr\left(\mathbf{P}_t'\mathbf{K}_t^- > -1\right), \\
&Pr\left((d - \mu^\star) = \rho_{t+1}x_{t+1} \middle| (d - \mu^\star) = \rho_t x_t\right) = 1.
\end{aligned}$$

We can now conclude that for any pre-committed efficient mean-variance policy (except for the minimum variance policy), the probability that condition (i) or condition (ii) holds at time t

only depends on market parameters $\mathbf{P}_i$ and $\mathbf{K}_i^{\pm}$, $i = 0, 1, \ldots, t-1$, if we assume $(d - \mu^\star) > \rho_0 x_0$ (an equivalent form of $d > \rho_0 x_0$ (which implies $\mu^\star = \frac{d-\rho_0 x_0}{1-(C_0^+)^{-1}}$ from (4))). This finding suggests a link between TCIE and the minimum-variance signed supermartingale measure introduced in the next section.

4 THE MINIMUM-VARIANCE SIGNED SUPERMARTINGALE MEASURE

It is well known that the problems of mean-variance portfolio selection and mean-variance hedging are strongly connected (see Schweizer (2010)). Xia and Yan (2006) showed that in an unconstrained incomplete market, the optimal terminal wealth under an efficient dynamic mean-variance policy is related to the so-called variance-optimal signed martingale measure (VSMM) of the market, and the optimal terminal wealth has a nonnegative marginal utility if and only if the VSMM is nonnegative. Note that the VSMM is the particular signed measure with the minimum variance among all of the signed martingale measures, under which the discounted wealth process of any admissible policy is a martingale. In discrete-time unconstrained markets, the density of the VSMM with respect to the objective probability measure takes a product form (see Schweizer (1996) and Černý and Kallsen (2009)). In fact, the VSMM plays a central role in mean-variance hedging and is the pricing kernel for contingent claims in that context (see Schweizer (1995) and Schweizer (1996)).

Motivated by Xia and Yan (2006), we continue our analysis in this section, by deriving a similar "VSMM" in our constrained market. However, the situation is more complicated in a constrained market than in an unconstrained one. Pham and Touzi (1999) and Föllmer and Schied (2004) showed that in a constrained market, no arbitrage opportunity is equivalent to the existence of a supermartingale measure, under which the discounted wealth process of any admissible policy is a supermartingale (see Carassus et al. (2001) for a case with upper bounds on the fractions invested). Therefore, we define the particular measure with the minimum variance among all of the signed supermartingale measures as the minimum-variance signed supermartingale measure (VSSM) and derive its semi-analytical form for discrete-time cone-constrained markets. The newly defined VSSM can be considered as an extension of the VSMM in constrained markets and both take the product form. In this section, we also show that the VSSM is not only related to the optimal terminal wealth achieved by efficient mean-variance policies, but is also associated with the TCIE of efficient mean-variance policies. Our results explicitly assess the effect of portfolio constraints on TCIE.

We use $\mathcal{L}^2(\mathcal{F}_{t+1}, P)$ to denote the set of all $\mathcal{F}_{t+1}$-measurable square integrable random variables. According to Pham and Touzi (1999) and Chapter 9 of Föllmer and Schied (2004), a cone-constrained market does not have any arbitrage opportunity if and only if there exists an equivalent probability measure under which the discounted wealth process of any admissible policy is a supermartingale. Therefore, in this paper, we extend the definitions of the signed martingale measure and the variance-optimal signed martingale measure proposed in Schweizer (1996) to a signed supermartingale measure and the VSSM.

Definition 4.1. *A signed measure Q on $(\Omega, \mathcal{F}_T)$ is called a* signed supermartingale measure *if $Q[\Omega] = 1$, $Q \ll P$ with $dQ/dP \in \mathcal{L}^2(\mathcal{F}_T, P)$ and the discounted wealth process of any admissible* *policy is a supermartingale under Q, i.e., for $t = 0, 1, \ldots, T-1$,*

$$\mathbb{E}\left[\frac{dQ}{dP}\rho_t^{-1}x_T(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{T-1})\Big|\mathcal{F}_t\right] \leq x_t(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{t-1}), \quad \forall\ \mathbf{u}_t \in \mathcal{A}_t, \tag{6}$$

where $x_t(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{t-1})$ denotes the time-t wealth level achieved by applying $\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{t-1}\}$.

We denote by $\mathbb{P}_s$ the set of all signed supermartingale measures. It is easy to see that inequality (6) is equivalent to either one of the following two inequalities,

$$\mathbb{E}\left[\frac{dQ}{dP}\mathbf{P}_t'\mathbf{u}_t\Big|\mathcal{F}_t\right] \leq 0, \quad \forall\ \mathbf{u}_t \in \mathcal{A}_t,$$
$$\mathbb{E}\left[\frac{dQ}{dP}\mathbf{P}_t\Big|\mathcal{F}_t\right] \in \mathcal{A}_t^{\perp},$$

where $\mathcal{A}_t^{\perp}$ denotes the polar cone of $\mathcal{A}_t$, i.e.,

$$\mathcal{A}_t^{\perp} = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}'\mathbf{x} \leq 0,\ \mathbf{x} \in \mathcal{A}_t\}.$$

Definition 4.2. *A signed supermartingale measure $\tilde{P}$ is called a VSSM if $\tilde{P}$ minimizes*

$$\mathrm{Var}\left[\frac{dQ}{dP}\right] = \mathbb{E}\left[\left(\frac{dQ}{dP} - 1\right)^2\right] = \mathbb{E}\left[\left(\frac{dQ}{dP}\right)^2\right] - 1,$$

over all $Q \in \mathbb{P}_s$.

For $i = 0, 1, \ldots, T-1$, we define

$$m_i = \mathbb{E}\left[\frac{d\tilde{P}}{dP}\Big|\mathcal{F}_i\right] \Big/ \mathbb{E}\left[\frac{d\tilde{P}}{dP}\Big|\mathcal{F}_{i-1}\right].$$

Then we have

$$\frac{d\tilde{P}}{dP} = m_1 m_2 \cdots m_T.$$

If $m_i(\omega) = 0$, we can set $m_j(\omega)$, $j > i$, equal to any value. It is easy to check that $\mathbb{E}[m_i|\mathcal{F}_{i-1}] = 1$.

Next, we derive a semi-analytical form for the VSSM in the cone-constrained market. We first formulate the following pair of optimization problems for $t = 0, 1, \ldots, T-1$,

$$\begin{aligned}
(A^+(t)): \qquad \min \quad & \mathbb{E}\left[\left(\frac{1}{C_{t+1}^+}1_{\{m_{t+1}\geq 0\}} + \frac{1}{C_{t+1}^-}1_{\{m_{t+1}<0\}}\right)m_{t+1}^2\Big|\mathcal{F}_t\right]\\
\text{s.t.} \quad & \mathbb{E}\left[m_{t+1}|\mathcal{F}_t\right] = 1,\\
& \mathbb{E}\big[m_{t+1}\mathbf{P}_t\big|\mathcal{F}_t\big] \in \mathcal{A}_t^{\perp},\\
& m_{t+1} \in \mathcal{L}^2(\mathcal{F}_{t+1}, P),
\end{aligned}$$

and

$$\begin{aligned}
(A^-(t)): \qquad \min \quad & \mathbb{E}\left[\left(\frac{1}{C_{t+1}^+}1_{\{m_{t+1}\leq 0\}} + \frac{1}{C_{t+1}^-}1_{\{m_{t+1}>0\}}\right)m_{t+1}^2\Big|\mathcal{F}_t\right]\\
\text{s.t.} \quad & \mathbb{E}\left[m_{t+1}|\mathcal{F}_t\right] = 1,\\
& -\mathbb{E}\big[m_{t+1}\mathbf{P}_t\big|\mathcal{F}_t\big] \in \mathcal{A}_t^{\perp},\\
& m_{t+1} \in \mathcal{L}^2(\mathcal{F}_{t+1}, P).
\end{aligned}$$

Lemma 4.1. *The solutions of $(A^+(t))$ and $(A^-(t))$ are respectively given by*

$$m_{t+1}^+ = \frac{1}{C_t^+}\left[C_{t+1}^+(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{m_{t+1}^+\geq 0\}} + C_{t+1}^-(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{m_{t+1}^+<0\}}\right],$$
$$m_{t+1}^- = \frac{1}{C_t^-}\left[C_{t+1}^+(1+\mathbf{P}_t'\mathbf{K}_t^-)1_{\{m_{t+1}^-\leq 0\}} + C_{t+1}^-(1+\mathbf{P}_t'\mathbf{K}_t^-)1_{\{m_{t+1}^->0\}}\right],$$

and the optimal objective values of $(A^+(t))$ and $(A^-(t))$ are $\frac{1}{C_t^+}$ and $\frac{1}{C_t^-}$ respectively.

Theorem 4.1. *The density of the VSSM $\tilde{P}$ (with respect to the objective probability measure P) is given by*

$$\frac{d\tilde{P}}{dP} = (C_0^+)^{-1}\prod_{i=0}^{T-1} B_i,$$

where

$$B_0 = 1-\mathbf{P}_0'\mathbf{K}_0^+,$$
$$B_i = (1-\mathbf{P}_i'\mathbf{K}_i^+)1_{\{\prod_{j=0}^{i-1}B_j\geq 0\}} + (1+\mathbf{P}_i'\mathbf{K}_i^-)1_{\{\prod_{j=0}^{i-1}B_j<0\}},\ i=1,2,\cdots,T-1.$$

Furthermore,

$$\mathbb{E}\left[\frac{d\tilde{P}}{dP}\Big|\mathcal{F}_t\right] = (C_0^+)^{-1}\prod_{i=0}^{t-1}B_i\left(C_t^+1_{\{\prod_{j=0}^{t-1}B_j\geq 0\}} + C_t^-1_{\{\prod_{j=0}^{t-1}B_j<0\}}\right), \tag{7}$$

$$\mathbb{E}\left[\left(\frac{d\tilde{P}}{dP}\right)^2\right] = \frac{1}{C_0^+}. \tag{8}$$

There is a strong connection between the VSSM and the optimal terminal wealth achieved by the pre-committed efficient mean-variance policy. Substituting the pre-committed efficient mean-variance policy in (3) into the wealth dynamic equation yields

$$x_{t+1}^\star = \begin{cases} s_t x_t^\star + s_t\mathbf{P}_t'\mathbf{K}_t^+\left((d-\mu^\star)\rho_t^{-1}-x_t^\star\right), & \text{if } d-\mu^\star\geq \rho_t x_t^\star,\\ s_t x_t^\star - s_t\mathbf{P}_t'\mathbf{K}_t^-\left((d-\mu^\star)\rho_t^{-1}-x_t^\star\right), & \text{if } d-\mu^\star< \rho_t x_t^\star,\end{cases} \tag{9}$$

with $x_0^\star = x_0$. Set $y_t^\star \triangleq x_t^\star - (d-\mu^\star)\rho_t^{-1}$. From the wealth equation in (9), which $x_t^\star$ satisfies, we deduce

$$\begin{cases} y_{t+1}^\star = s_t y_t^\star - s_t\mathbf{P}_t'\mathbf{K}_t^+ y_t^\star 1_{\{y_t^\star\leq 0\}} + s_t\mathbf{P}_t'\mathbf{K}_t^- y_t^\star 1_{\{y_t^\star>0\}},\\ y_0^\star = x_0-(d-\mu^\star)\rho_0^{-1}.\end{cases}$$

Note that $y_0^\star = x_0-(d-\mu^\star)\rho_0^{-1} = \frac{d\rho_0^{-1}-x_0}{C_0^+-1}\leq 0$ by virtue of the fact that $d\geq x_0\rho_0$ and $C_0^+<1$. We can show

$$y_t^\star = y_0^\star\prod_{i=0}^{t-1}s_i\prod_{i=0}^{t-1}B_i,\quad t=1,2,\ldots,T.$$

For $t = 1$, it is trivial. Assume that the statement holds true for t, we now show that the statement also holds true for $t+1$, as

$$\begin{aligned} y^\star_{t+1} &= s_t y^\star_t - s_t \mathbf{P}'_t \mathbf{K}^+_t y^\star_t 1_{\{y^\star_t \le 0\}} + s_t \mathbf{P}'_t \mathbf{K}^-_t y^\star_t 1_{\{y^\star_t > 0\}} \\ &= y^\star_0 \prod_{i=0}^{t} s_i \prod_{i=0}^{t-1} B_i \left[(1 - \mathbf{P}'_t \mathbf{K}^+_t) 1_{\{\prod_{j=0}^{t-1} B_j \ge 0\}} + (1 + \mathbf{P}'_t \mathbf{K}^-_t) 1_{\{\prod_{j=0}^{t-1} B_j < 0\}} \right]. \end{aligned}$$

Thus, the time t optimal wealth achieved by the pre-committed efficient mean-variance policy is given by

$$x^\star_t = (d - \mu^\star)\rho_t^{-1} - [(d - \mu^\star) - x_0 \rho_0] \rho_t^{-1} \prod_{i=0}^{t-1} B_i, \tag{10}$$

which leads to the following theorem.

Theorem 4.2. *The optimal terminal wealth achieved by the pre-committed efficient mean-variance policy $x^\star_T$ and the VSSM $\tilde{P}$ have the following duality relationship:*

$$x^\star_T = (d - \mu^\star) - \frac{(d - \mu^\star) - x_0 \rho_0}{\mathbb{E}\left[\left(\frac{d\tilde{P}}{dP} \right)^2 \right]} \cdot \frac{d\tilde{P}}{dP}.$$

Remark 4.1. *Xia and Yan (2006) considered the mean-variance portfolio selection problem in an incomplete, albeit unconstrained, market and established the relationship between the mean-variance efficient portfolio and the VSMM by analyzing the geometric structure of the problem. In fact, Theorem 4.2 above is an extension of Theorem 3.1 in Xia and Yan (2006) for the discrete-time cone-constrained market. When the convex cone constraint is chosen as the whole space, our theorem reduces to the result in Xia and Yan (2006). However, different from Xia and Yan (2006), we prove the theorem by solving both the optimal terminal wealth and the VSSM directly.*

Most importantly, we now demonstrate that the VSSM is also related to the TCIE of the pre-committed efficient mean-variance policy.

Theorem 4.3. *The pre-committed efficient mean-variance policy (except for the minimum variance policy) in a cone-constrained market is time consistent in efficiency if and only if the VSSM of this market satisfies:*

$$\mathbb{E}\left[\frac{d\tilde{P}}{dP} \Big| \mathcal{F}_t \right](\omega) \ge 0, \ \forall\, 0 < t < T, \ \forall\, \omega \in \Omega; \tag{11}$$

or

$$\mathbb{E}\left[\frac{d\tilde{P}}{dP} \Big| \mathcal{F}_k \right](\omega) = \mathbb{E}\left[\frac{d\tilde{P}}{dP} \Big| \mathcal{F}_\tau \right](\omega) < 0, \ \forall\, \tau \le k \le T, \ \forall\, \omega \in \Omega, \tag{12}$$

where the stopping time τ is defined as

$$\tau = \inf \left\{ t \,\Big|\, \mathbb{E}\left[\frac{d\tilde{P}}{dP} \Big| \mathcal{F}_t \right] < 0, \quad t = 1, 2, \ldots, T \right\}.$$

We can conclude from Theorem 4.3, that the pre-committed efficient mean-variance policy (except for the minimum variance policy) satisfies TCIE if and only if the conditional expectation of the density of the VSSM (with respect to the original probability measure) is nonnegative, or once the conditional expectation takes a negative value, it keeps the same value until the terminal time.

It is also easy to see that condition (11) implies that $\forall\ \omega \in \Omega,\ (1 - \mathbf{P}'_t(\omega)\mathbf{K}^+_t) \geq 0$ and

$$\mathbf{u}^\star_t = s_t \mathbf{K}^+_t \big((d - \mu^\star)\rho_t^{-1} - x_t\big).$$

In such a case, every mean-variance investor holds a long position in the efficient risky portfolio $\mathbf{K}^+_t$, whose excess rate of return does not exceed 100% ($\mathbf{P}'_t\mathbf{K}^+_t \leq 1$), and achieves efficiency during the entire investment horizon.

The stopping time τ can also be expressed as

$$\tau = \inf \left\{ \ t \ \middle| \ (1 - \mathbf{P}'_{t-1}\mathbf{K}^+_{t-1}) < 0, \quad t = 1, 2, \ldots, T \right\}.$$

Then, condition (12) implies that, for $t < \tau$,

$$\mathbf{u}^\star_t = s_t \mathbf{K}^+_t \big((d - \mu^\star)\rho_t^{-1} - x_t\big),$$

and for $k \geq \tau$,

$$\mathbf{K}^-_k = \mathbf{0}, \quad \mathbf{u}^\star_k = s_k \mathbf{K}^-_k \big((d - \mu^\star)\rho_k^{-1} - x_k\big) = \mathbf{0}.$$

In this situation, every mean-variance investor starts with a long position in the efficient risky portfolio $\mathbf{K}^+_t$ and switches all wealth into the riskless asset once the excess rate of return of $\mathbf{K}^+_t$ exceeds 100%, i.e., $\mathbf{P}'_t\mathbf{K}^+_t > 1$.

Theorem 4.3 shows that whether the pre-committed efficient mean-variance policy (except for the minimum variance policy) is time consistent in efficiency only depends on the basic market setting (the distribution of the excess rate of return $\mathbf{P}_t$ and the portfolio constraint set $\mathcal{A}_t$) and does not depend on the initial wealth level, x_0, and the objective level that the investor aspires to achieve, d. This property suggests that market constraints can be added to eliminate time inconsistency in efficiency.

5 ELIMINATION OF TIME INCONSISTENCY IN EFFICIENCY WITH PORTFOLIO CONSTRAINTS

From previous sections, it is clear that portfolio constraints have an effect on TCIE. Suppose that a given discrete-time mean-variance problem fails to be TCIE. Can the time inconsistency in efficiency be eliminated by introducing suitable portfolio constraints into the market? This section provides a positive answer to this question.

Remark 5.1. *We continue our investigation starting from an unconstrained market, then a market without shorting, before dealing with a general cone-constrained market.*

i) Case of unconstrained markets:

If the market is constraint free, i.e., $\mathcal{A}_t = \mathbb{R}^n$, we have

$$\mathbf{K}_t^{\pm} = \pm\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right],$$

$$C_t^{\pm} = \prod_{i=t}^{T-1}(1-\mathbb{E}\left[\mathbf{P}_i'\right]\mathbb{E}^{-1}\left[\mathbf{P}_i\mathbf{P}_i'\right]\mathbb{E}\left[\mathbf{P}_i\right]).$$

Therefore, the optimal mean-variance policy of $(P(d))$ is

$$\mathbf{u}_t^{\star} = s_t\big((d-\mu^{\star})\rho_t^{-1} - x_t\big)\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right], \quad t = 0, 1, \ldots, T-1,$$

where

$$\mu^{\star} = \frac{d-\rho_0 x_0}{1-\prod_{i=0}^{T-1}(1-\mathbb{E}\left[\mathbf{P}_i'\right]\mathbb{E}^{-1}\left[\mathbf{P}_i\mathbf{P}_i'\right]\mathbb{E}\left[\mathbf{P}_i\right])^{-1}}, \tag{13}$$

which is exactly the result in Li and Ng (2000). We can assume here that $\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right] \neq \mathbf{0}$. Otherwise, all efficient policies reduce to the one investing only in the riskless asset.

Furthermore, the VSSM in the unconstrained market is

$$\frac{d\tilde{P}}{dP} = \prod_{i=0}^{T-1}\frac{1-\mathbf{P}_i'\mathbb{E}^{-1}\left[\mathbf{P}_i\mathbf{P}_i'\right]\mathbb{E}\left[\mathbf{P}_i\right]}{1-\mathbb{E}\left[\mathbf{P}_i'\right]\mathbb{E}^{-1}\left[\mathbf{P}_i\mathbf{P}_i'\right]\mathbb{E}\left[\mathbf{P}_i\right]},$$

which is the VSMM obtained in Schweizer (1995), Schweizer (1996) and Černý and Kallsen (2009).

Theorem 4.3 shows that the pre-committed efficient mean-variance policy (except for the minimum variance policy) in the unconstrained market satisfies TCIE if and only if the VSMM is a nonnegative measure for any $\mathcal{F}_t$, i.e.,

$$\mathbf{P}_t'\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right] \leq 1, \quad a.s. \tag{14}$$

Actually, Cui et al. (2012) proved that condition (14) does not hold only if the market is an incomplete market and proposed a TCIE revised mean-variance policy which i) achieves the same mean-variance pair as the pre-committed efficient policy and ii) receives an additional positive free cash flow during the investment horizon.

ii) Case of markets without shorting:

Assume that the shorting of risky assets is not allowed in the market, i.e., $\mathcal{A}_t = \mathbb{R}_+^n$, and the expected excess rates of return of risky assets are nonnegative, i.e., $\mathbb{E}[\mathbf{P}_t] \geq \mathbf{0}$, t = 0, 1, …, $T-1$. In this situation,

$$\mathbf{K}_t^{+} = \arg\min_{\mathbf{K}_t\in\mathbb{R}_+^n}\mathbb{E}\left[C_{t+1}^{+}\Big(1-\mathbf{P}_t'\mathbf{K}_t\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t\leq 1\}} + C_{t+1}^{-}\Big(1-\mathbf{P}_t'\mathbf{K}_t\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t> 1\}}\right],$$

$$\mathbf{K}_t^{-} = \arg\min_{\mathbf{K}_t\in\mathbb{R}_+^n}\mathbb{E}\left[C_{t+1}^{+}\Big(1+\mathbf{P}_t'\mathbf{K}_t\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t\leq -1\}} + C_{t+1}^{-}\Big(1+\mathbf{P}_t'\mathbf{K}_t\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t> -1\}}\right] = \mathbf{0}.$$

In addition, we also have

$$\left(\nabla_{\mathbf{K}_t} h_t^-(\mathbf{0})\right)' (\mathbf{K}_t - \mathbf{0}) = 2C_{t+1}^- \mathbb{E}[\mathbf{P}_t'] \mathbf{K}_t \geq 0, \ \forall \ \mathbf{K}_t \in \mathbb{R}_+^n.$$

Therefore, the optimal policy of $(P(d))$ *is*

$$\mathbf{u}_t^\star = s_t\big((d-\mu^\star)\rho_t^{-1} - x_t\big)\mathbf{K}_t^+ 1_{\{d-\mu^\star \geq \rho_t x_t\}}, \quad t = 0, 1, \ldots, T-1,$$

where

$$\mu^\star = \frac{d - \rho_0 x_0}{1 - (C_0^+)^{-1}},$$

which is the result derived in Cui et al. (2014).

Furthermore, the VSSM in such a market setting is given by

$$\frac{d\tilde{P}}{dP} = (C_0^+)^{-1} \prod_{i=0}^{(T-1)\wedge(\tau-1)} (1 - \mathbf{P}_i' \mathbf{K}_i^+),$$

where

$$\tau = \inf \left\{ \ t \ \middle| \ (1 - \mathbf{P}_{t-1}' \mathbf{K}_{t-1}^+) < 0, \quad t = 1, 2, \ldots, T \right\}.$$

We can see that $C_t^- = 1$, $t = 0, 1, \cdots, T-1$. *Therefore, according to Theorem 4.3, all pre-committed efficient policies are time consistent in efficiency in a market with no shorting and with nonnegative expected excess rates of return.*

We now discuss a general cone-constrained market setting.

Theorem 5.1. *If a convex cone* $\mathcal{A}_t$ *is chosen to restrict portfolios such that the expected excess rate of return vector* $\mathbb{E}[\mathbf{P}_t]$ *lies in the dual cone of* $\mathcal{A}_t$, *i.e.,*

$$\mathbb{E}[\mathbf{P}_t] \in \mathcal{A}_t^*,$$

where $\mathcal{A}_t^* = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}'\mathbf{x} \geq 0, \ \mathbf{x} \in \mathcal{A}_t\} = -\mathcal{A}_t^\perp$, *then the corresponding optimal discrete-time pre-committed efficient mean-variance policy is time consistent in efficiency.*

Figure 5 graphically illustrates the above proposition. Essentially, this is the inverse process of finding the convex cone $\mathcal{A}_t$. For a given market, $\mathbb{E}[\mathbf{P}_t]$ is known. We first identify a cone $\mathcal{A}_t^*$ such that $\mathbb{E}[\mathbf{P}_t] \in \mathcal{A}_t^*$. We then find another cone $\mathcal{A}_t$ such that the selected $\mathcal{A}_t^*$ becomes its dual cone. The condition in Theorem 5.1 aims to enforce the inefficient risky portfolio $\mathbf{K}_t^-$ equal to zero to achieve condition (12). Note that condition (11) is much harder to satisfy, as it is related to the distribution of the excess rate of return, which is uncontrollable in general.

Example 5.1. We now consider, as an example, the construction of a three-year pension fund consisting of the S&P 500 (SP), the index of Emerging Markets (EM), and Small Stocks (MS) of the U.S market and a bank account. The annual rates of return of these three indices have the expected values, standard deviations and correlations given in Table 1, based on the data provided in Elton et al. (2007).

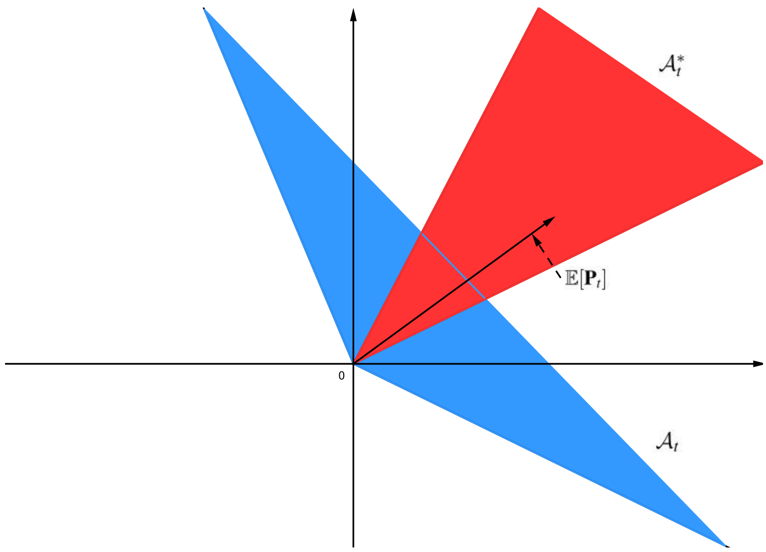


Figure 1: Construction of a Suitable Cone Constraint

	SP	EM	MS
Expected Return	14%	16%	17%
Standard Deviation	18.5%	30%	24%
Correlation			
SP	1	0.64	0.79
EM		1	0.75
MS			1

Table 1: Data for Example 5.1

We further assume that all annual rates of return are statistically independent and follow i) the identical multivariate normal distribution (with the statistics described above) or ii) the identical multivariate t distribution with freedom 5 (and with the statistics described above) for all 3 years, and that the annual risk free rate is 5%, i.e., $s_t = 1.05$, $t = 0, 1, 2$. We first compute $\mathbb{E}[\mathbf{P}_t]$, $\mathrm{Cov}(\mathbf{P}_t)$ and $\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']$ as follows, for $t = 0, 1, 2$,

$$\mathbb{E}[\mathbf{P}_t] = \begin{bmatrix} 0.09 \\ 0.11 \\ 0.12 \end{bmatrix}, \ \mathrm{Cov}(\mathbf{P}_t) = \begin{bmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{bmatrix}, \ \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] = \begin{bmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{bmatrix}.$$

To examine violation of TCIE (by observing the number of events where wealth exceeds the threshold $(d - \mu^\star)\rho_t^{-1}$), we simulate 2×10^6 sample paths for each distribution assumption, with initial wealth equal to $x_0 = 1$ and the target expected return equal to $d = 1.35$.

Case 1: When the market is unconstrained, the optimal mean-variance policy of $(P(d))$ is

$$\mathbf{u}_t^\star = s_t\big((d - \mu^\star)\rho_t^{-1} - x_t\big)\mathbb{E}^{-1}\left[\mathbf{P}_t\mathbf{P}_t'\right]\mathbb{E}\left[\mathbf{P}_t\right] = 1.05\left((1.35 + 0.1808)1.05^{t-3} - x_t\right)\begin{bmatrix} 1.0580 \\ -0.1207 \\ 1.1052 \end{bmatrix}, \quad t = 0, 1, 2,$$

with $\mu^\star = -0.1808$ (based on (13)) for both distributional assumptions. Under both the *unbounded* multivariate normal distribution and the multivariate t distribution, equation (14) does not hold, which implies that TCIE may fail. More specifically, recalling Theorem 3.1 and Lemma 3.1 and noticing $C_t^+ = C_t^- < 1$ with $t < T$, the pre-committed efficient mean-variance policy does not satisfy TCIE if and only if the optimal wealth level $x_t^\star$ exceeds the threshold

$$(d - \mu^\star)\rho_t^{-1} = (1.35 + 0.1808)1.05^{t-3}.$$

The simulation results show that the probabilities that $x_t^\star$ exceeds the threshold $(d - \mu^\star)\rho_t^{-1}$ are 0.055 for the multivariate normal distribution and 0.0558 for the multivariate t distribution. This simulation outcome indicates that a distribution with a heavier tail tends to generate a higher degree of time inconsistency in efficiency in an unconstrained market.

Case 2: To eliminate the time inconsistency in efficiency, we first consider adding the following cone constraint to the market,

$$\mathcal{A}_t = \{\mathbf{u}_t \in \mathbb{R}^n \mid \mathbb{E}[\mathbf{P}_t']\mathbf{u}_t \geq 0\},$$

which is a half-space with boundary $\mathbb{E}[\mathbf{P}_t']\mathbf{u}_t = 0$ that is a hyperplane orthogonal to $\mathbb{E}[\mathbf{P}_t]$. The dual cone of $\mathcal{A}_t$ is

$$\mathcal{A}_t^* = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \lambda\mathbb{E}[\mathbf{P}_t],\ \lambda \geq 0\},$$

which is exactly the ray along $\mathbb{E}[\mathbf{P}_t]$ (see Proposition 3.2.1 of Bertsekas (2003)). Notice that the constraint cone, $\mathcal{A}_t$, defined above is the largest cone (thus the loosest constraint) that we can identify to eliminate the time inconsistency in efficiency in this example.

Based on the proof of Theorem 5.1, we have $\mathbf{K}_0^- = \mathbf{K}_1^- = \mathbf{K}_2^- = \mathbf{0}$ for both distributions. By Lemma 2.1, we can numerically compute $\mathbf{K}_t^+$ using the penalty function method (see Appendix A of Cui et al. (2014)) with the initial point $[1.06, -0.12, 1.11]'$ as

$$\text{i)}\ \ \mathbf{K}_0^+ = \begin{bmatrix} 1.0589 \\ -0.1212 \\ 1.1086 \end{bmatrix},\ \mathbf{K}_1^+ = \begin{bmatrix} 1.0600 \\ -0.1200 \\ 1.1100 \end{bmatrix},\ \mathbf{K}_2^+ = \begin{bmatrix} 1.0600 \\ -0.1200 \\ 1.1100 \end{bmatrix}$$

for the multivariate normal distribution and

$$\text{ii)}\ \ \mathbf{K}_0^+ = \begin{bmatrix} 1.0461 \\ -0.1335 \\ 1.0929 \end{bmatrix},\ \mathbf{K}_1^+ = \begin{bmatrix} 1.0548 \\ -0.1263 \\ 1.1034 \end{bmatrix},\ \mathbf{K}_2^+ = \begin{bmatrix} 1.0600 \\ -0.1200 \\ 1.1100 \end{bmatrix}$$

for the multivariate t distribution. The optimal investment policies are

$$\text{i)}\ \ \mathbf{u}_t^\star(x_t) = 1.05\big((1.35 + 0.1810)1.05^{t-3} - x_t\big)\mathbf{K}_t^+ 1_{\{x_t < 1.05^{(t-3)}(1.5310)\}}$$

for the multivariate normal distribution and

$$\text{ii)}\ \ \mathbf{u}_t^\star(x_t) = 1.05\big((1.35 + 0.1831)1.05^{t-3} - x_t\big)\mathbf{K}_t^+ 1_{\{x_t < 1.05^{(t-3)}(1.5331)\}}$$

for the multivariate t distribution. The simulation shows that the probabilities that $x_t^\star$ exceeds the threshold $(d - \mu^\star)\rho_t^{-1}$ are 0.0559 for the multivariate t distribution and 0.0533 for the multivariate t distribution. Once $x_t^\star$ exceeds the threshold $(d-\mu^\star)\rho_t^{-1}$, the investor puts wealth entirely into the riskless asset, which eliminates the time inconsistency in efficiency in this example.

Case 3: In this case, we introduce a more realistic convex cone constraint into the market,

$$\mathcal{A}_t = \{\mathbf{u}_t \in \mathbb{R}^n \mid u_t^2 \geq 0,\ u_t^3 \geq 0,\ u_t^1 + u_t^2 + u_t^3 \geq 0\},$$

which implies that short selling is not allowed for the index of Emerging Markets and the Small Stocks of the U.S market, and that the negative position on the S&P 500 cannot be too large. The dual cone of $\mathcal{A}_t$ in this case is

$$\mathcal{A}_t^* = \left\{ \mathbf{y} \in \mathbb{R}^n \;\middle|\; \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \lambda,\ \lambda \geq \mathbf{0} \right\}.$$

Note that specifying λ at $[0.09, 0.02, 0.03]' \in \mathcal{A}_t^*$ yields the ray along $\mathbb{E}[\mathbf{P}_t]$.

Based on the proof in Theorem 5.1, we have $\mathbf{K}_0^- = \mathbf{K}_1^- = \mathbf{K}_2^- = \mathbf{0}$ for both distributions. By Lemma 2.1, we can compute $\mathbf{K}_t^+$ numerically using the penalty function method (see Appendix A of Cui et al. (2014)) with the initial point $[1.06, 0.05, 1.11]'$ as

$$\text{i)}\ \ \mathbf{K}_0^+ = \begin{bmatrix} 1.0076 \\ 0.0044 \\ 1.0324 \end{bmatrix},\ \mathbf{K}_1^+ = \begin{bmatrix} 1.0133 \\ 0.0037 \\ 1.0373 \end{bmatrix},\ \mathbf{K}_2^+ = \begin{bmatrix} 1.0147 \\ 0.0031 \\ 1.0401 \end{bmatrix}$$

for the multivariate normal distribution and

$$\text{ii)}\ \ \mathbf{K}_0^+ = \begin{bmatrix} 1.0112 \\ 0.0030 \\ 1.0413 \end{bmatrix},\ \mathbf{K}_1^+ = \begin{bmatrix} 1.0201 \\ 0.0026 \\ 1.0522 \end{bmatrix},\ \mathbf{K}_2^+ = \begin{bmatrix} 1.0216 \\ 0.0039 \\ 1.0501 \end{bmatrix}$$

for the multivariate t distribution. The optimal investment policies are

$$\text{i)}\ \ \mathbf{u}_t^\star(x_t) = 1.05\big((1.35 + 0.1818)1.05^{t-3} - x_t\big)\mathbf{K}_t^+ 1_{\{x_t < 1.05^{(t-3)}(1.5318)\}}$$

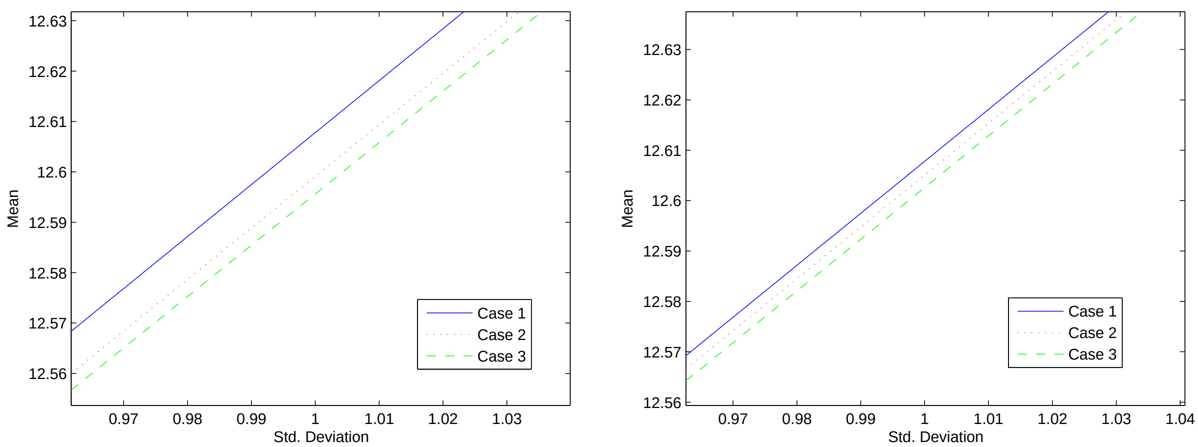
for the multivariate normal distribution and

$$\text{ii)}\ \ \mathbf{u}_t^\star(x_t) = 1.05\big((1.35 + 0.1843)1.05^{t-3} - x_t\big)\mathbf{K}_t^+ 1_{\{x_t < 1.05^{(t-3)}(1.5343)\}}$$

for the multivariate t distribution. The simulation shows that the probabilities that $x_t^\star$ exceeds the threshold $(d-\mu^\star)\rho_t^{-1}$ are 0.0569 for the multivariate normal distribution and 0.0588 for the multivariate t distribution. Although, compared to the unconstrained case, both probabilities increase, the investor puts wealth entirely into the riskless asset immediately after $x_t^\star$ exceeds the threshold $(d-\mu^\star)\rho_t^{-1}$.

For the unconstrained market in Case 1, the expression for the efficient frontier achieved by the pre-committed policy is given in (76) in Li and Ng (2000). For the cone-constrained markets in Cases 2 and 3, the efficient frontiers achieved by the pre-committed policy are given in Theorem 2.1 of this paper. For problem $(P(d))$, Appendix A9 of this paper derives the efficient frontier achieved by the time consistent policy proposed by Basak and Chabakauri (2010) and Björk et al. (2014), with its characterization given in (20).

Figure 2 depicts the efficient frontiers in the mean-standard deviation space for Cases 1, 2 and 3, and demonstrates a clear dominance relationship among the three. Furthermore, Figure 3 illustrates a clear dominance relationship between Case 3 and the efficient frontier achieved by the time consistent policy. As both the TCIE policies and the time consistent policy aim to align the inherently inconsistent global and local interests, they all sacrifice certain degrees of global performance, and are thus all dominated by the pre-committed policy. Case 2 dominates Case 3 because it is associated with a looser constraint, while Case 3 is associated with a tighter constraint. It is interesting to note that both TCIE policies significantly dominate the time consistent policy, which indicates that insisting on time consistency for an inherently time inconsistent problem may result in a significant loss in global performance. Expression (20) reveals that the time consistent policy achieves a good efficient frontier globally only if B_t is large. In conclusion, by introducing appropriate constraints into the model, we can not only eliminate time inconsistency in efficiency, but also strike a good balance between the global and local mean-variance efficiency. In fact, relaxing the time consistency requirement to TCIE offers us the flexibility to decide which level of good global performance to maintain by introducing suitable portfolio constraints and deriving the corresponding pre-committed TCIE policy.



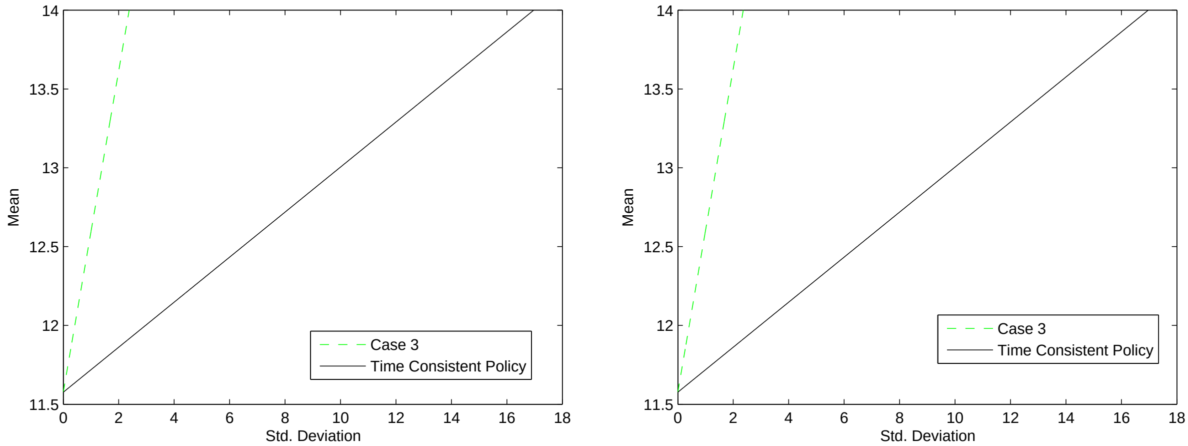
(a) Multivariate t distribution

(b) Multivariate normal distribution

Figure 2: Comparison of efficient frontiers for Cases 1, 2 and 3 of Example 5.1

6 CONCLUSION

This paper provides a complete answer to the following question: Given a financial market with known return statistics, what are the cone constraints required to ensure that the derived optimal portfolio policy is time consistent in efficiency. Our paper makes three main contributions to the literature by developing i) an analytical solution to the mean-variance formulation for discrete-time cone-constrained markets; ii) a complete characterization of TCIE and its relationship to



(a) Multivariate t distribution

(b) Multivariate normal distribution

Figure 3: Comparison of efficient frontiers between Case 3 and the time consistent policy of Example 5.1

the VSSM; and iii) a systematic framework for guaranteeing TCIE by enforcing suitable cone constraints on portfolios.

More specifically, we have investigated the discrete-time mean-variance portfolio selection problem in a convex cone-constrained market, and have given a condition under which there exists an admissible policy. We have also analytically derived the pre-committed efficient mean-variance policy and identified the explicit conditions under which the pre-committed efficient mean-variance policy satisfies TCIE. The optimal policy derived has a two-piece linear form, which indicates that in a cone-constrained market, mean-variance investors may switch between one efficient risky portfolio $\mathbf{K}_t^+$ and one inefficient risky portfolio $\mathbf{K}_t^-$ depending on the current wealth level. Another prominent finding that may also require special attention is that market constraints induce a dependence between the current risky portfolios and the future market conditions, even when the rates of return among different time periods are independent.

Furthermore, we have extended the definition of the variance-optimal signed martingale measure (VSMM) in unconstrained markets to the minimum-variance signed supermartingale measure (VSSM) in constrained markets, and have derived a semi-analytical expression for the density of the VSSM (with respect to the original probability measure), which only depends on the basic market setting (including the distribution of the excess rate of return, $\mathbf{P}_t$, and the set of portfolio constraints, $\mathcal{A}_t$). Our major finding demonstrates that the property of TCIE and the VSSM are closely related, i.e., the pre-committed discrete-time efficient mean-variance policy (except for the minimum variance policy) satisfies TCIE if and only if the conditional expectation of the density of the VSSM is nonnegative, or once the conditional expectation becomes negative, it keeps the same value until the terminal time. This interesting finding is the first analytical result to explicitly assess the effect of constraints on the property of time consistency in dynamic decision problems. It motivated us to develop a general procedure for constructing TCIE dynamic portfolio selection models based on the introduction of suitable portfolio constraints.

The semi-analytical expression for the density of the VSSM may also benefit the research on mean-variance hedging in constrained markets.

The extension of our result to continuous-time cone-constrained markets is straightforward, at least conceptually. However, if the rates of return among different periods are correlated, the problem becomes more complicated and the idea of opportunity-neutral measure change to handle stochastic opportunity sets presented in Černý and Kallsen (2009) may be helpful. The real challenge emerges when considering general markets with convex portfolio constraints (which may not be a cone type). In such markets, the pre-committed efficient mean-variance policy may depend on more than two risky portfolios, which complicates the analysis.

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APPENDIX:

A1: The proof of Lemma 2.1

Proof: From the definition in (1), it is easy to see that $C_t^{\pm} > 0$ for all $t = 0, 1, \ldots, T-1$.
The first-order and second-order derivatives of $h_t^{\pm}(\mathbf{K}_t)$ with respect to $\mathbf{K}_t$ are

$$\begin{aligned}
&\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t) = 2\mathbb{E}\left[C_{t+1}^{+}\Big(\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t \mp \mathbf{P}_t\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t \le \pm 1\}} + C_{t+1}^{-}\Big(\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t \mp \mathbf{P}_t\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t > \pm 1\}}\right],\\
&\triangledown^2_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t) = 2\mathbb{E}\left[\Big(C_{t+1}^{+}1_{\{\mathbf{P}_t'\mathbf{K}_t \le \pm 1\}} + C_{t+1}^{-}1_{\{\mathbf{P}_t'\mathbf{K}_t > \pm 1\}}\Big)\mathbf{P}_t\mathbf{P}_t'\right] \succeq 2\min(C_{t+1}^{+}, C_{t+1}^{-})\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] \succ 0.
\end{aligned}$$

Therefore, $h_t^{\pm}(\mathbf{K}_t)$ are strictly convex with respect to $\mathbf{K}_t$, which implies that $\mathbf{K}_t^{\pm}$ are uniquely determined. Furthermore, $\mathbf{K}_t^{\pm}$ are optimal if and only if $\left(\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t^{\pm})\right)'(\mathbf{K}_t - \mathbf{K}_t^{\pm}) \ge 0$, $\forall\ \mathbf{K}_t \in \mathcal{A}_t$ (see Theorem 27.4 in Rockafellar (1970)), which implies

$$\left(\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t^{\pm})\right)'(\alpha\mathbf{K}_t^{\pm} - \mathbf{K}_t^{\pm}) \ge 0, \ \ \forall\ \alpha > 0, \quad \Leftrightarrow \quad \left(\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t^{\pm})\right)'\mathbf{K}_t^{\pm} = 0,$$

due to the assumption that $\mathcal{A}_t$ is a cone.
Then,

$$\begin{aligned}
&\mathbb{E}\left[C_{t+1}^{+}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right]\\
=&\mathbb{E}\left[C_{t+1}^{+}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right] + \frac{1}{2}\left(\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t^{\pm})\right)'\mathbf{K}_t^{\pm}\\
=&\mathbb{E}\left[C_{t+1}^{+}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[C_{t+1}^{+}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 \mp \mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right]\\
=&\mathbb{E}\left[C_{t+1}^{+}\Big(1 - (\mathbf{K}_t^{\pm})'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 - (\mathbf{K}_t^{\pm})'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right]\\
&+\left(\triangledown_{\mathbf{K}_t} h_t^{\pm}(\mathbf{K}_t^{\pm})\right)'\mathbf{K}_t^{\pm}\\
=&\mathbb{E}\left[C_{t+1}^{+}\Big(1 - (\mathbf{K}_t^{\pm})'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} \le \pm 1\}} + C_{t+1}^{-}\Big(1 - (\mathbf{K}_t^{\pm})'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t^{\pm}\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^{\pm} > \pm 1\}}\right].
\end{aligned}$$

Therefore,

$$C_t^+ \le \mathbb{E}\left[C_{t+1}^+\Big(1-(\mathbf{K}_t^+)'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t^+\Big)1_{\{\mathbf{P}_t'\mathbf{K}_t^+\le 1\}}\right] \le C_{t+1}^+.$$

The equality holds in the above inequality if and only if $\mathbf{K}_t^+ = \mathbf{0}$. The situation for C_t^- can be proved similarly. □

A2: The proof of Theorem 2.1

Proof: Consider an auxiliary problem of $(P(d))$ by introducing the Lagrangian multiplier 2μ,

$$(15)\qquad \begin{aligned} \min\quad & \mathbb{E}\big[(x_T-d)^2+2\mu(x_T-d)\big],\\ \text{s.t.}\quad & x_{t+1}=s_tx_t+\mathbf{P}_t'\mathbf{u}_t,\\ & \mathbf{u}_t\in\mathcal{A}_t,\quad t=0,1,\ldots,T-1, \end{aligned}$$

which is equivalent to

$$(L(\mu)):\qquad \begin{aligned} \min\quad & \mathbb{E}\left[\frac{1}{2}y_T^2\right],\\ \text{s.t.}\quad & y_{t+1}=s_ty_t+\mathbf{P}_t'\mathbf{u}_t,\\ & \mathbf{u}_t\in\mathcal{A}_t,\quad t=0,1,\ldots,T-1, \end{aligned}$$

where $y_t \triangleq x_t-(d-\mu)\rho_t^{-1}$, $t=0,\,1,\,\ldots,\,T$.
We now prove that the value function of $(L(\mu))$ at time t is

$$(16)\qquad J_t(y_t)=\min_{\mathbf{u}_t\in\mathcal{A}_t,\cdots,\mathbf{u}_{T-1}\in\mathcal{A}_{T-1}}\mathbb{E}\left[\frac{1}{2}y_T^2\Big|\mathcal{F}_t\right]=\frac{1}{2}\rho_t^2\left[C_t^+y_t^21_{\{y_t\le 0\}}+C_t^-y_t^21_{\{y_t>0\}}\right],$$

where C_t^+ and C_t^- are given in Lemma 2.1.
At time T, we have

$$J_T(y_T)=\frac{1}{2}y_T^2=\frac{1}{2}\rho_T^2\left[C_T^+y_T^21_{\{y_T\le 0\}}+C_T^-y_T^21_{\{y_T>0\}}\right].$$

Thus, statement (16) holds true for time T. Assume that statement (16) holds true for time $t+1$. We now prove that the statement also remains true for time t. Applying the recursive relationship between J_{t+1} and J_t yields
(17)

$$\begin{aligned} J_t(y_t) &= \min_{\mathbf{u}_t\in\mathcal{A}_t}\mathbb{E}[J_{t+1}(y_{t+1})|\mathcal{F}_t]\\ &= \min_{\mathbf{u}_t\in\mathcal{A}_t}\frac{1}{2}\rho_{t+1}^2\mathbb{E}\Big[C_{t+1}^+y_{t+1}^21_{\{y_{t+1}\le 0\}}+C_{t+1}^-y_{t+1}^21_{\{y_{t+1}>0\}}\big|\mathcal{F}_t\Big]\\ &= \min_{\mathbf{u}_t\in\mathcal{A}_t}\frac{1}{2}\rho_{t+1}^2\mathbb{E}\Big[C_{t+1}^+(s_ty_t+\mathbf{P}_t'\mathbf{u}_t)^21_{\{\mathbf{P}_t'\mathbf{u}_t\le -s_ty_t\}}+C_{t+1}^-(s_ty_t+\mathbf{P}_t'\mathbf{u}_t)^21_{\{\mathbf{P}_t'\mathbf{u}_t>-s_ty_t\}}\big|\mathcal{F}_t\Big]. \end{aligned}$$

When $y_t<0$, identifying the optimal $\mathbf{u}_t$ within the convex cone $\mathbf{u}_t\in\mathcal{A}_t$ is equivalent to identifying the optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t\in\mathcal{A}_t$ when we set $\mathbf{u}_t=-s_t\mathbf{K}_ty_t$. Thus,

$$J_t(y_t)=\min_{\mathbf{K}_t\in\mathcal{A}_t}\frac{1}{2}\rho_t^2y_t^2\mathbb{E}\left[C_{t+1}^+\Big(1-\mathbf{P}_t'\mathbf{K}_t\Big)^21_{\{\mathbf{P}_t'\mathbf{K}_t\le 1\}}+C_{t+1}^-\Big(1-\mathbf{P}_t'\mathbf{K}_t\Big)^21_{\{\mathbf{P}_t'\mathbf{K}_t>1\}}\right].$$

From Lemma 2.1, the optimal control takes the form $\mathbf{u}_t^\star = -s_t\mathbf{K}_t^+ y_t$. Substituting $\mathbf{u}_t^\star$ back to the value function (17) leads to

$$J_t(y_t) = \frac{1}{2}\rho_t^2 y_t^2 \mathbb{E}\left[C_{t+1}^+\left(1-\mathbf{P}_t'\mathbf{K}_t^+\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^+\le 1\}} + C_{t+1}^-\left(1-\mathbf{P}_t'\mathbf{K}_t^+\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^+>1\}}\right] = \frac{1}{2}C_t^+\rho_t^2 y_t^2.$$

When $y_t > 0$, identifying the optimal $\mathbf{u}_t$ within the convex cone $\mathbf{u}_t \in \mathcal{A}_t$ is equivalent to identifying the optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t \in \mathcal{A}_t$ when we set $\mathbf{u}_t = s_t\mathbf{K}_t y_t$. Thus,

$$J_t(y_t) = \min_{\mathbf{K}_t\in\mathcal{A}_t} \frac{1}{2}\rho_t^2 y_t^2 \mathbb{E}\left[C_{t+1}^+\left(1+\mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t\le -1\}} + C_{t+1}^-\left(1+\mathbf{P}_t'\mathbf{K}_t\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t>-1\}}\right].$$

From Lemma 2.1, the optimal control takes the form $\mathbf{u}_t^\star = s_t\mathbf{K}_t^- y_t$. Substituting $\mathbf{u}_t^\star$ back to the value function (17) leads to

$$J_t(y_t) = \frac{1}{2}\rho_t^2 y_t^2 \mathbb{E}\left[C_{t+1}^+\left(1+\mathbf{P}_t'\mathbf{K}_t^-\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^-\le -1\}} + C_{t+1}^-\left(1+\mathbf{P}_t'\mathbf{K}_t^-\right)^2 1_{\{\mathbf{P}_t'\mathbf{K}_t^->-1\}}\right] = \frac{1}{2}C_t^-\rho_t^2 y_t^2.$$

When $y_t = 0$, we can easily verify that $\mathbf{u}_t^\star = \mathbf{0}$ is the minimizer. We can thus set $J_t(y_t) = \frac{1}{2}C_t^+\rho_t^2 y_t^2$.

In summary, the optimal value for problem (15) is

$$g(\mu) = \left[C_0^+(d-\rho_0x_0-\mu)^2-\mu^2\right]1_{\{\mu\le d-\rho_0x_0\}} + \left[C_0^-(d-\rho_0x_0-\mu)^2-\mu^2\right]1_{\{\mu> d-\rho_0x_0\}}, \tag{18}$$

which is a first-order continuously differentiable concave function. To obtain the optimal value and optimal strategy for problem $(P(d))$, we maximize (18) over $\mu\in\mathbb{R}$ according to the Lagrangian duality theorem. We derive our results for three different value ranges of d.

i) $d = \rho_0x_0$. The optimal Lagrangian multiplier takes value zero, i.e., $\mu^\star = 0$. The optimal investment policy is thus $\mathbf{u}_t^\star = \mathbf{0}$, $t = 0, 1, \ldots, T-1$.

ii) $d > \rho_0x_0$. When $C_0^+ = 1$, i.e., $K_t^+ = \mathbf{0}$, $t = 0, 1, \ldots, T-1$, we can take $\mu^\star = -\infty$ resulting in $g(\mu^\star) = +\infty$. This means that $P(d)$ does not have a feasible solution. When $C_0^+ < 1$ and $C_0^- = 1$, $C_0^+(d-\rho_0x_0-\mu)^2-\mu^2$ is a strictly concave function and $C_0^-(d-\rho_0x_0-\mu)^2-\mu^2$ is a decreasing linear function. The optimal Lagrangian multiplier satisfies $\mu^\star = \frac{d-\rho_0x_0}{1-(C_0^+)^{-1}} < (d-\rho_0x_0)$. When $C_0^+ < 1$ and $C_0^- < 1$, $C_0^\pm(d-\rho_0x_0-\mu)^2-\mu^2$ are both strictly concave. The optimal Lagrangian multiplier satisfies $\mu^\star = \frac{d-\rho_0x_0}{1-(C_0^+)^{-1}} < (d-\rho_0x_0)$. Therefore, the optimal mean-variance pair is presented by

$$\left(\mathbb{E}[x_T],\ \mathrm{Var}(x_T)\right) = (d,\ g(\mu^\star)) = \left(d,\ \frac{C_0^+(d-\rho_0x_0)^2}{1-C_0^+}\right).$$

iii) $d < \rho_0x_0$. Similarly, when $C_0^- = 1$, $P(d)$ does not have a feasible solution. When $C_0^- < 1$, the optimal Lagrangian multiplier satisfies $\mu^\star = \frac{d-\rho_0x_0}{1-(C_0^-)^{-1}} > (d-\rho_0x_0)$. Then, the optimal mean-variance pair is presented by

$$\left(\mathbb{E}[x_T],\ \mathrm{Var}(x_T)\right) = (d,\ g(\mu^\star)) = \left(d,\ \frac{C_0^-(d-\rho_0x_0)^2}{1-C_0^-}\right).$$

Therefore, $g(\mu)$ attains its maximum value at $\mu^\star$ as expressed in (4). Moreover, the optimal mean-variance pair of problem $(P(d))$ is presented by

$$\big(\mathbb{E}[x_T],\ \mathrm{Var}(x_T)\big) = \left(d,\ \frac{C_0^+\left(d-\rho_0 x_0\right)^2}{1-C_0^+}1_{\{d\geq\rho_0 x_0\}} + \frac{C_0^-\left(d-\rho_0 x_0\right)^2}{1-C_0^-}1_{\{d<\rho_0 x_0\}}\right).$$

Finally, the efficient frontier follows naturally from our above discussion. □

A3: The proof of Theorem 3.1

Proof: Comparing Eq. (3) with Eq. (5), we can conclude that at time k, the truncated pre-committed efficient mean-variance policy, $\mathbf{u}_t^\star, t = k, k+1, \ldots, T-1$, also solves $(P(d_k) \mid x_k)$ when d_k satisfies $d - \mu^\star = d_k - \mu_k^\star$. Note from the discussion following Theorem 2.1 that the solution to $(P(d_k) \mid x_k)$ is inefficient if and only if $d_k < \rho_k x_k$ and $C_k^- < 1$ (or equivalently, the solution to $(P(d_k) \mid x_k)$ is efficient if i) $d_k \geq \rho_k x_k$, or ii) $d_k < \rho_k x_k$ and $C_k^- = 1$). When $0 < C_k^+ < 1$, we have

$$\begin{aligned}
d_k \geq \rho_k x_k &\Leftrightarrow (d_k - \rho_k x_k)\frac{1}{1-C_k^+} \geq 0 \\
&\Leftrightarrow d_k - \frac{d_k - \rho_k x_k}{1-(C_k^+)^{-1}} \geq \rho_k x_k \\
&\Leftrightarrow d_k - \mu_k^\star \geq \rho_k x_k,\ \text{if } d_k \geq \rho_k x_k \\
&\Leftrightarrow d_k - \mu_k^\star \geq \rho_k x_k \\
&\Leftrightarrow d - \mu^\star \geq \rho_k x_k.
\end{aligned}$$

Therefore, when both $d - \mu^\star \geq \rho_k x_k$ and $C_k^+ < 1$ hold, the truncated pre-committed efficient mean-variance policy remains efficient for the truncated problem $(P(d_k) \mid x_k)$.

Similarly, when $0 < C_k^- < 1$, we have

$$\begin{aligned}
d_k < \rho_k x_k &\Leftrightarrow (d_k - \rho_k x_k)\frac{1}{1-C_k^-} < 0 \\
&\Leftrightarrow d_k - \frac{d_k - \rho_k x_k}{1-(C_k^-)^{-1}} < \rho_k x_k \\
&\Leftrightarrow d_k - \mu_k^\star < \rho_k x_k,\ \text{if } d_k < \rho_k x_k \\
&\Leftrightarrow d_k - \mu_k^\star < \rho_k x_k \\
&\Leftrightarrow d - \mu^\star < \rho_k x_k,
\end{aligned}$$

which implies that when both $d - \mu^\star < \rho_k x_k$ and $C_k^- < 1$ hold, the truncated pre-committed efficient mean-variance policy becomes inefficient for the truncated problem $(P(d_k) \mid x_k)$.

When $d - \mu^\star \geq \rho_k x_k$, $C_k^+ = 1$ or $d - \mu^\star < \rho_k x_k$, $C_k^- = 1$ hold, we have $\mathbf{u}_t^\star = \mathbf{0}$, $t = k, k+1, \ldots, T-1$, i.e., the truncated pre-committed efficient mean-variance policy becomes the minimum variance policy for the truncated problem $(P(d_k) \mid x_k)$.

The proposition follows after combining the results for all of the situations discussed above. □

A4: The proof of Proposition 3.1

Proof: We only need to prove the first, third and fifth equalities.

Condition $(d-\mu^\star) > \rho_t x_t$ dictates that the optimal policy at time t is $\mathbf{u}_t^\star = s_t \mathbf{K}_t^+ \big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)$. The wealth level at time $t+1$ follows $x_{t+1} = s_t x_t + s_t \mathbf{P}_t' \mathbf{K}_t^+ \big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)$, which implies

$$\begin{aligned}(d-\mu^\star) \geq \rho_{t+1}x_{t+1} &\Leftrightarrow (d-\mu^\star) \geq \rho_t x_t + \rho_t \mathbf{P}_t'\mathbf{K}_t^+\big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)\\ &\Leftrightarrow\ [(d-\mu^\star)-\rho_t x_t]\,(1-\mathbf{P}_t'\mathbf{K}_t^+) \geq 0\\ &\Leftrightarrow \mathbf{P}_t'\mathbf{K}_t^+ \leq 1.\end{aligned}$$

Thus the first statement holds.

Condition $(d-\mu^\star) < \rho_t x_t$ dictates that the optimal policy at time t is $\mathbf{u}_t^\star = -s_t \mathbf{K}_t^- \big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)$. The wealth level at time $t+1$ is $x_{t+1} = s_t x_t - s_t \mathbf{P}_t' \mathbf{K}_t^- \big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)$, which implies

$$\begin{aligned}(d-\mu^\star) \geq \rho_{t+1}x_{t+1} &\Leftrightarrow (d-\mu^\star) \geq \rho_t x_t - \rho_t \mathbf{P}_t'\mathbf{K}_t^-\big((d_k-\mu_k^\star)\rho_t^{-1} - x_t\big)\\ &\Leftrightarrow\ [(d-\mu^\star)-\rho_t x_t]\,(1+\mathbf{P}_t'\mathbf{K}_t^-) \geq 0\\ &\Leftrightarrow \mathbf{P}_t'\mathbf{K}_t^- \leq -1.\end{aligned}$$

Thus the third statement holds.

Condition $(d-\mu^\star) = \rho_t x_t$ dictates that the optimal policy at time t is $\mathbf{u}_t^\star = \mathbf{0}$. The wealth level at time $t+1$ is $x_{t+1} = s_t x_t$, which implies $(d-\mu^\star) =\rho_{t+1}x_{t+1} = \rho_t x_t$. Thus the fifth statement holds. □

A5: The proof of Lemma 4.1

Proof: We solve both problems by duality theory. The dual problem of $(A^+(t))$ is

$$\max_{\nu_t\in\mathbb{R}}\ \max_{-\lambda_t\in\mathcal{A}_t} \mathbb{E}\left[\min_{m_{t+1}} L_t(m_{t+1},\nu_t,\lambda_t)\Big|\mathcal{F}_t\right],$$

where the Lagrangian function is defined as

$$L_t(m_{t+1},\nu_t,\lambda_t) \triangleq \left(\frac{1}{C_{t+1}^+}1_{\{m_{t+1}\geq 0\}} + \frac{1}{C_{t+1}^-}1_{\{m_{t+1}<0\}}\right) m_{t+1}^2 - \nu_t m_{t+1} + \nu_t - \lambda_t'\mathbf{P}_t m_{t+1}.$$

We also define

$$\mathcal{D}(\nu_t,\lambda_t) \triangleq \mathbb{E}\left[\min_{m_{t+1}} L_t(m_{t+1},\nu_t,\lambda_t)\Big|\mathcal{F}_t\right].$$

The first order condition of $L_t(m_{t+1},\nu_t,\lambda_t)$ with respect to m_{t+1} gives rise to

$$m_{t+1} = \frac{C_{t+1}^+}{2}(\nu_t+\lambda_t'\mathbf{P}_t)1_{\{\nu_t+\lambda_t'\mathbf{P}_t\geq 0\}} + \frac{C_{t+1}^-}{2}(\nu_t+\lambda_t'\mathbf{P}_t)1_{\{\nu_t+\lambda_t'\mathbf{P}_t< 0\}}. \tag{19}$$

Note that $m_{t+1}\geq 0$ if and only if $\nu_t+\lambda_t'\mathbf{P}_t\geq 0$.

Then we have

$$\mathcal{D}(\nu_t, \lambda_t) = \mathbb{E}\left[-\frac{1}{4}(\nu_t + \lambda_t'\mathbf{P}_t)^2 \left(C_{t+1}^{+} 1_{\{\nu_t+\lambda_t'\mathbf{P}_t \geq 0\}} + C_{t+1}^{-} 1_{\{\nu_t+\lambda_t'\mathbf{P}_t<0\}}\right) + \nu_t\right].$$

If $\nu_t > 0$, identifying the optimal λ_t within the convex cone $-\lambda_t \in \mathcal{A}_t$ is equivalent to identifying the optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t \in \mathcal{A}_t$ when we set $\lambda_t = -\nu_t \mathbf{K}_t$. Then,

$$\begin{aligned}
&\max_{\nu_t>0} \max_{-\lambda_t\in\mathcal{A}_t} \mathcal{D}(\lambda_t, \nu_t) \\
=& \max_{\nu_t>0} \left\{-\frac{1}{4}\nu_t^2 \left\{\min_{\mathbf{K}_t\in\mathcal{A}_t} \mathbb{E}\left[(1-\mathbf{K}_t'\mathbf{P}_t)^2 \left(C_{t+1}^{+} 1_{\{\mathbf{K}_t'\mathbf{P}_t\leq 1\}} + C_{t+1}^{-} 1_{\{\mathbf{K}_t'\mathbf{P}_t>1\}}\right)\right]\right\} + \nu_t\right\}.
\end{aligned}$$

Therefore, $\mathcal{D}(\lambda_t, \nu_t)$ attains its maximum $\frac{1}{C_t^+}$ at $\lambda_t^+ = -\nu_t \mathbf{K}_t^+$ and $\nu_t^+ = \frac{2}{C_t^+}$.
If $\nu_t < 0$, identifying the optimal λ_t within the convex cone $-\lambda_t \in \mathcal{A}_t$ is equivalent to identifying the optimal $\mathbf{K}_t$ within the convex cone $\mathbf{K}_t \in \mathcal{A}_t$ when we set $\lambda_t = \nu_t \mathbf{K}_t$. Then,

$$\begin{aligned}
&\max_{\nu_t<0} \max_{-\lambda_t\in\mathcal{A}_t} \mathcal{D}(\lambda_t, \nu_t) \\
=& \max_{\nu_t<0} \left\{-\frac{1}{4}\nu_t^2 \left\{\min_{\mathbf{K}_t\in\mathcal{A}_t} \mathbb{E}\left[(1-\mathbf{K}_t'\mathbf{P}_t)^2 \left(C_{t+1}^{+} 1_{\{\mathbf{K}_t'\mathbf{P}_t\leq -1\}} + C_{t+1}^{-} 1_{\{\mathbf{K}_t'\mathbf{P}_t>-1\}}\right)\right]\right\} + \nu_t\right\}.
\end{aligned}$$

Now, $\mathcal{D}(\lambda_t, \nu_t)$ attains its maximum 0 when $\nu_t \uparrow 0$.
Substituting both λ_t^+ and ν_t^+ into (19) yields the expression of m_{t+1}^+,

$$\begin{aligned}
m_{t+1}^+ &= \frac{1}{C_t^+}\left[C_{t+1}^+(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{\mathbf{P}_t'\mathbf{K}_t^+\leq 1\}} + C_{t+1}^-(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{\mathbf{P}_t'\mathbf{K}_t^+>1\}}\right], \\
&= \frac{1}{C_t^+}\left[C_{t+1}^+(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{m_{t+1}^+\geq 0\}} + C_{t+1}^-(1-\mathbf{P}_t'\mathbf{K}_t^+)1_{\{m_{t+1}^+<0\}}\right],
\end{aligned}$$

and the optimal objective value of $(A^+(t))$,

$$\mathbb{E}\left[\left(\frac{1}{C_{t+1}^+}1_{\{m_{t+1}^+\geq 0\}} + \frac{1}{C_{t+1}^-}1_{\{m_{t+1}^+<0\}}\right)(m_{t+1}^+)^2\Big|\mathcal{F}_t\right] = \frac{1}{C_t^+}.$$

Notice that $m_{t+1}^+ \geq 0$ if and only if $\mathbf{P}_t'\mathbf{K}_t^+ \leq 1$.
Applying a similar approach to problem $(A^-(t))$ gives rise to the expression for m_{t+1}^- and the corresponding optimal value $\frac{1}{C_t^-}$. Notice that $m_{t+1}^- \leq 0$ if and only if $\mathbf{P}_t'\mathbf{K}_t^- \leq -1$. □

A6: The proof of Theorem 4.1

Proof: The problem of finding the density of the VSSM is formulated as

$$\begin{aligned}
(P_{VSSM}): \qquad & \min \quad \mathbb{E}\left[m_1^2 m_2^2 \cdots m_T^2\right] \\
& \text{s.t.} \quad \mathbb{E}\left[m_{t+1}\big|\mathcal{F}_t\right] = 1,\ t = 0,\ldots,T-1, \\
& \qquad \mathbb{E}\big[m_1 m_2 \cdots m_T \mathbf{P}_t\big|\mathcal{F}_t\big] \in \mathcal{A}_t^{\perp},\ t = 0,\ldots,T-1, \\
& \qquad m_{t+1} \in \mathcal{L}^2(\mathcal{F}_{t+1}, P),\ t = 0,\ldots,T-1.
\end{aligned}$$

We prove by induction that the cost-to-go function of (P_{VSSM}) at time t is given by

$$J(m_1 m_2 \dots m_t) = \frac{1}{C_t^+} m_1^2 m_2^2 \cdots m_t^2 1_{\{m_1 m_2 \cdots m_t \geq 0\}} + \frac{1}{C_t^-} m_1^2 m_2^2 \cdots m_t^2 1_{\{m_1 m_2 \cdots m_t < 0\}},$$

which implies (8).

At time T, the statement holds true by recognizing that $C_T^{\pm} = 1$. Assume that the statement holds true for time $t+1$. We now prove that the statement also remains true for time t.

At time t, when $m_1 m_2 \cdots m_t > 0$, (P_{VSSM}) reduces to

$$\begin{aligned} \min \quad & m_1^2 m_2^2 \cdots m_t^2 \mathbb{E}\left[\left(\frac{1}{C_{t+1}^+} 1_{\{m_{t+1} \geq 0\}} + \frac{1}{C_{t+1}^-} 1_{\{m_{t+1} < 0\}} \right) m_{t+1}^2 \middle| \mathcal{F}_t \right] \\ \text{s.t.} \quad & \mathbb{E}[m_{t+1} | \mathcal{F}_t] = 1, \\ & \mathbb{E}\big[m_{t+1} \mathbf{P}_t \big| \mathcal{F}_t\big] \in \mathcal{A}_t^{\perp}, \\ & m_{t+1} \in \mathcal{L}^2(\mathcal{F}_{t+1}, P). \end{aligned}$$

On the other hand, when $m_1 m_2 \cdots m_t < 0$, (P_{VSSM}) reduces to

$$\begin{aligned} \min \quad & m_1^2 m_2^2 \cdots m_t^2 \mathbb{E}\left[\left(\frac{1}{C_{t+1}^+} 1_{\{m_{t+1} \leq 0\}} + \frac{1}{C_{t+1}^-} 1_{\{m_{t+1} > 0\}} \right) m_{t+1}^2 \middle| \mathcal{F}_t \right] \\ \text{s.t.} \quad & \mathbb{E}[m_{t+1} | \mathcal{F}_t] = 1, \\ & -\mathbb{E}\big[m_{t+1} \mathbf{P}_t \big| \mathcal{F}_t\big] \in \mathcal{A}_t^{\perp}, \\ & m_{t+1} \in \mathcal{L}^2(\mathcal{F}_{t+1}, P). \end{aligned}$$

With the help of Lemma 4.1, the optimal solution is

$$m_{t+1}^* = m_{t+1}^+ 1_{\{m_1 m_2 \cdots m_t \geq 0\}} + m_{t+1}^- 1_{\{m_1 m_2 \cdots m_t < 0\}}.$$

(When $m_1 m_2 \cdots m_t = 0$, we can set $m_{t+1}^* = m_{t+1}^+$.)

Then, the cost-to-go function becomes

$$J(m_1 m_2 \dots m_t) = \frac{1}{C_t^+} m_1^2 m_2^2 \cdots m_t^2 1_{\{m_1 m_2 \cdots m_t \geq 0\}} + \frac{1}{C_t^-} m_1^2 m_2^2 \cdots m_t^2 1_{\{m_1 m_2 \cdots m_t < 0\}}.$$

Now, it remains to prove that $m_1^* m_2^* \cdots m_T^* = (C_0^+)^{-1} \prod_{i=0}^{T-1} B_i$. We prove that

$$m_1^* m_2^* \cdots m_t^* = (C_0^+)^{-1} \prod_{i=0}^{t-1} B_i \left(C_t^+ 1_{\{m_1^* m_2^* \cdots m_t^* \geq 0\}} + C_t^- 1_{\{m_1^* m_2^* \cdots m_t^* < 0\}} \right),$$

which implies the conditional expectation in (7).

When $t = 1$, the following is obvious,

$$m_1^* = m_1^+ = (C_0^+)^{-1} C_1^+ (1 - \mathbf{P}_0' \mathbf{K}_0^+) = (C_0^+)^{-1} \prod_{i=0}^{0} B_i \left(C_1^+ \right).$$

Assume that at time t our statement holds true, we now prove that the statement also holds for time $t+1$, as

$$
\begin{aligned}
& m_1^* m_2^* \cdots m_{t+1}^* \\
= & (C_0^+)^{-1} \prod_{i=0}^{t-1} B_i \left(C_t^+ 1_{\{m_1^* \cdots m_t^* \geq 0\}} + C_t^- 1_{\{m_1^* \cdots m_t^* < 0\}} \right) \left(m_{t+1}^+ 1_{\{m_1^* \cdots m_t^* \geq 0\}} + m_{t+1}^- 1_{\{m_1^* \cdots m_t^* < 0\}} \right) \\
= & (C_0^+)^{-1} \prod_{i=0}^{t-1} B_i \big[C_{t+1}^+ (1 - \mathbf{P}_t' \mathbf{K}_t^+) 1_{\{m_1^* \cdots m_t^* \geq 0\}} 1_{\{m_{t+1}^+ \geq 0\}} + C_{t+1}^- (1 - \mathbf{P}_t' \mathbf{K}_t^+) 1_{\{m_1^* \cdots m_t^* \geq 0\}} 1_{\{m_{t+1}^+ < 0\}} \\
& + C_{t+1}^+ (1 + \mathbf{P}_t' \mathbf{K}_t^-) 1_{\{m_1^* \cdots m_t^* < 0\}} 1_{\{m_{t+1}^- \leq 0\}} + C_{t+1}^- (1 + \mathbf{P}_t' \mathbf{K}_t^-) 1_{\{m_1^* \cdots m_t^* < 0\}} 1_{\{m_{t+1}^- > 0\}} \big] \\
= & (C_0^+)^{-1} \prod_{i=0}^{t-1} B_i \left[(1 - \mathbf{P}_t' \mathbf{K}_t^+) 1_{\{m_1^* \cdots m_t^* \geq 0\}} + (1 + \mathbf{P}_t' \mathbf{K}_t^-) 1_{\{m_1^* \cdots m_t^* < 0\}} \right] \\
& \cdot \left(C_{t+1}^+ 1_{\{m_1^* \cdots m_{t+1}^* \geq 0\}} + C_{t+1}^- 1_{\{m_1^* \cdots m_{t+1}^* < 0\}} \right) \\
= & (C_0^+)^{-1} \prod_{i=0}^{t} B_i \left(C_{t+1}^+ 1_{\{m_1^* \cdots m_{t+1}^* \geq 0\}} + C_{t+1}^- 1_{\{m_1^* \cdots m_{t+1}^* < 0\}} \right).
\end{aligned}
$$

Therefore,

$$
m_1^* m_2^* \cdots m_T^* = (C_0^+)^{-1} \prod_{i=0}^{T-1} B_i \left(C_T^+ 1_{\{m_1^* m_2^* \cdots m_T^* \geq 0\}} + C_T^- 1_{\{m_1^* m_2^* \cdots m_T^* < 0\}} \right) = (C_0^+)^{-1} \prod_{i=0}^{T-1} B_i.
$$

□

A7: The proof of Theorem 4.3

Proof: Lemma 3.1 states the necessary and sufficient condition under which the pre-committed efficient policy (except for the minimum variance policy) satisfies TCIE, which can be summarized as follows:

For $t = 1, 2, \cdots, T-1$, $\forall\ x_t^\star,\ d - \mu^\star \geq \rho_t x_t^\star$ holds or $\exists\ x_t^\star,\ d - \mu^\star < \rho_t x_t^\star,\ C_t^- = 1$ holds.

If at time t, $\exists\ x_t^\star,\ d - \mu^\star < \rho_t x_t^\star,\ C_t^- = 1$, then

$$
x_{t+1}^\star = s_t x_t^\star, \quad d - \mu^\star < \rho_t x_t^\star = \rho_{t+1} x_{t+1}^\star,\ C_{t+1}^- = 1,
$$

if and only if $\mathbf{K}_i^- = \mathbf{0}$, $(\mathbf{u}_i^\star = \mathbf{0})$, $i = t, t+1, \cdots, T-1$.

Therefore, the necessary and sufficient condition can be reexpressed as

$$
\begin{aligned}
& \forall\ T > t > 0,\ \forall\ x_t^\star, \quad d - \mu^\star \geq \rho_t x_t^\star; \\
\text{or } & C_\tau^- = 1, \text{ where } \tau = \inf \left\{ t \;\middle|\; d - \mu^\star < \rho_t x_t^\star, \quad t = 1, 2, \cdots, T \right\}.
\end{aligned}
$$

Based on the expression of $x_t^\star$ in (10), we have

$$
d - \mu^\star \geq (<)\ \rho_t x_t^\star \quad \Leftrightarrow \quad [(d - \mu^\star) - x_0 \rho_0] \prod_{i=0}^{t-1} B_i \geq (<)\ 0 \quad \Leftrightarrow \quad \prod_{i=0}^{t-1} B_i \geq (<)\ 0.
$$

Here, we use the property of a pre-committed efficient policy, i.e., $d - \mu^\star > \rho_0 x_0$. Furthermore, for $T \geq k \geq \tau$ and $T > j \geq \tau$,

$$
\begin{aligned}
& C_\tau^- = 1, \quad \prod_{i=0}^{\tau-1} B_i < 0, \\
\Leftrightarrow\ & \mathbf{K}_j^- = \mathbf{0}, \quad C_k^- = 1, \quad \prod_{i=0}^{\tau-1} B_i < 0, \\
\Leftrightarrow\ & \prod_{i=0}^{k-1} B_i \left(C_k^+ 1_{\{\prod_{i=0}^{k-1} B_i \geq 0\}} + C_k^- 1_{\{\prod_{i=0}^{k-1} B_i < 0\}} \right) = \prod_{i=0}^{\tau-1} B_i \left(C_\tau^+ 1_{\{\prod_{i=0}^{\tau-1} B_i \geq 0\}} + C_\tau^- 1_{\{\prod_{i=0}^{\tau-1} B_i < 0\}} \right) < 0.
\end{aligned}
$$

The expression for $\mathbb{E}\left[\frac{d\tilde{P}}{dP}\middle|\mathcal{F}_t\right]$ in (7) finally concludes our proof. $\square$

A8: The proof of Theorem 5.1

Proof: Under the condition in the proposition, we have

$$
\left(\bigtriangledown_{\mathbf{K}_t} h_t^-(\mathbf{0}) \right)' (\mathbf{K}_t - \mathbf{0}) = 2C_{t+1}^- \mathbb{E}[\mathbf{P}_t'] \mathbf{K}_t \geq 0, \ \forall\ \mathbf{K}_t \in \mathcal{A}_t,
$$

which implies $\mathbf{K}_t^- = \mathbf{0}$ and $C_t^- = 1$ for all t. $\square$

A9: The time consistent policy of $(P(d))$

In the solution framework proposed by Basak and Chabakauri (2010) and Björk et al. (2014), the so-called time consistent policy at time t is derived by backward induction, taking into account that the optimal investment decisions have already been made in the future. Thus, the time consistent policy is the collection of equilibrium strategies adopted by fictitious investors at different times in a sequential game. More specifically, the time t investor considers the following problem,

$$
(P_t(d)) : \quad \left\{ \begin{array}{rl}
\min\limits_{\mathbf{u}_t \in \mathbb{R}^n} & \mathrm{Var}(x_T|x_t) \equiv \mathbb{E}\big[(x_T - d)^2 | x_t\big], \\
\text{s.t.} & \mathbb{E}[x_T|x_t] = d, \\
& x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \\
& x_{j+1} = s_j x_j + \mathbf{P}_j' \bar{\mathbf{u}}_j, \quad j = t+1, \ldots, T-1, \\
& \bar{\mathbf{u}}_j \ \text{ solves Problem } (P_j(d)), \quad j = t+1, \ldots, T-1.
\end{array} \right.
$$

We prove by induction that the time consistent policy is given by

$$
\bar{\mathbf{u}}_t = -\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] \frac{x_t \rho_t - d}{B_t \rho_{t+1}},
$$

and the corresponding conditional mean and conditional variance of terminal wealth are expressed, respectively, as $\mathbb{E}[x_T|x_t]_{\{\bar{\mathbf{u}}\}} = d$ and $\mathrm{Var}(x_T|x_t)_{\{\bar{\mathbf{u}}\}} = (d - x_t\rho_t)^2 D_t$ with $B_t = \mathbb{E}[\mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]$ and $D_t = \prod_{j=t}^{T-1} \frac{1 - B_j}{B_j} > 0$.

We start our proof from time $T-1$ when the investor faces the following optimization problem,

$$(P_{T-1}(d)): \quad \begin{cases} \min\limits_{\mathbf{u}_{T-1}\in\mathbb{R}^n} & \mathbb{E}\big[(s_{T-1}x_{T-1}+\mathbf{P}'_{T-1}\mathbf{u}_{T-1}-d)^2|x_{T-1}\big], \\ \text{s.t.} & \mathbb{E}[s_{T-1}x_{T-1}+\mathbf{P}'_{T-1}\mathbf{u}_{T-1}|x_{T-1}]=d, \end{cases}$$

which can be solved by the Lagrangian method with its solution given as

$$\bar{\mathbf{u}}_{T-1}=-\mathbb{E}^{-1}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\mathbb{E}[\mathbf{P}_{T-1}]\frac{x_{T-1}\rho_{T-1}-d}{B_{T-1}}.$$

We can also show that $\mathbb{E}[x_T|x_{T-1}]_{\{\bar{\mathbf{u}}\}}=d$ and $\mathrm{Var}(x_T|x_{T-1})_{\{\bar{\mathbf{u}}\}}=(d-x_{T-1}\rho_{T-1})^2D_{T-1}$.
Assume that the statements hold true at time $t+1$. Then, at time t, the investor faces the following optimization problem,

$$(P_t(d)): \quad \begin{cases} \min\limits_{\mathbf{u}_t\in\mathbb{R}^n} & \mathrm{Var}(x_T|x_t)\equiv\mathbb{E}[\mathrm{Var}(x_T|x_{t+1})_{\{\bar{\mathbf{u}}\}}|x_t\big]+\mathrm{Var}\big(\mathbb{E}[x_T|x_{t+1}]_{\{\bar{\mathbf{u}}\}}|x_t\big), \\ \text{s.t.} & \mathbb{E}[\mathbb{E}[x_T|x_{t+1}]_{\{\bar{\mathbf{u}}\}}|x_t]=d, \\ & x_{t+1}=s_tx_t+\mathbf{P}'_t\mathbf{u}_t, \end{cases}$$

which is equivalent to

$$\min_{\mathbf{u}_t\in\mathbb{R}^n} \quad D_{t+1}\mathbb{E}[(d-(s_tx_t+\mathbf{P}'_t\mathbf{u}_t)\rho_{t+1})^2|x_t\big].$$

It is not difficult to verify the following optimal solution for $(P_t(d))$,

$$\bar{\mathbf{u}}_t=-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}'_t]\mathbb{E}[\mathbf{P}_t]\frac{x_t\rho_t-d}{B_t\rho_{t+1}},$$

with $\mathbb{E}[x_T|x_t]_{\{\bar{\mathbf{u}}\}}=d$ and $\mathrm{Var}(x_T|x_t)_{\{\bar{\mathbf{u}}\}}=(d-x_t\rho_t)^2D_t$.
Therefore, the efficient frontier of the time consistent policy is given as

$$\mathrm{Var}(x_T)_{\{\bar{\mathbf{u}}\}}=(\mathbb{E}[x_T]_{\{\bar{\mathbf{u}}\}}-x_0\rho_0)^2D_0, \quad \mathbb{E}[x_T]_{\{\bar{\mathbf{u}}\}}\geq x_0\rho_0. \tag{20}$$