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Stability and convergence of fully discrete Galerkin FEMs for the nonlinear thermistor equations in a nonconvex polygon

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Abstract In this paper, we establish the unconditional stability and optimal error estimates of a linearized backward Euler–Galerkin finite element method (FEM) for the time-dependent nonlinear thermistor equations in a two-dimensional nonconvex polygon. Due to the nonlinearity of the equations and the non-smoothness of the solution in a nonconvex polygon, the analysis is not straightforward, while most previous efforts for problems in nonconvex polygons mainly focused on linear models. Our theoretical analysis is based on an error splitting proposed in [30,31] together with rigorous regularity analysis of the nonlinear thermistor equations and the corresponding iterated (time-discrete) elliptic system in a nonconvex polygon. With the proved regularity, we establish the stability in $l^\infty(L^\infty)$ and the convergence in $l^\infty(L^2)$ for the fully discrete finite element solution without any restriction on the time-step size. The approach used in this paper may also be applied to other nonlinear parabolic systems in nonconvex polygons. Numerical results confirm our theoretical analysis and show clearly that no time-step condition is needed.

Keywords finite element method · nonconvex polygon · unconditional stability · optimal error estimate · thermistor problem

AMS subject classifications 65N12, 65N30, 35K61.

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1 Introduction

The solution of partial differential equations in nonconvex polygonal domains is involved in many physical applications. Due to the non-smoothness of the solution around the corner points, the analysis for such PDEs and for the corresponding numerical methods is challenging. Numerical methods and analysis for linear models defined in a nonconvex polygonal domain have been investigated extensively in the last several decades. A simple elliptic problem is

$$-\Delta u = f, \quad x \in \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega\,.$$

The regularity of the solution of the equation depends upon the interior angles ω_j of the corners. In a nonconvex polygon, we only have

$$u \in H^{1+s}, \quad 1/2 < s < \beta$$

for a smooth function f, where $\beta = \max_j \frac{\pi}{\omega_j} \in (1/2, 1)$ (see [16,20,23,25]). Under this regularity, the optimal L^2-error estimate of standard finite element methods for the Poisson equation in a nonconvex polygon is $O(h^{2s})$. Many analyses done for this equation can be found in the literature. For example, see [2,33,37] for estimates in the weighted Sobolev spaces and [9,10] for a singularity subtraction approach. A more precise analysis for the case $s = \beta$ was presented by Bacuta et al [6] in the framework of Besov spaces. Recently, error estimates of both semi-discrete and fully discrete finite element methods were investigated by Chatzipantelids et al [11] for the linear parabolic equation

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega. \tag{1.1}$$

The analysis was based on rigorous estimates of the corresponding analytic semigroup associated with the discrete Laplacian. The corresponding finite volume method was analyzed in [12]. Guo and Schwab [22] investigated the analytic regularity of the Stokes problem in a general polygonal domain and a weighted Sobolev space. Chrysafinos and Hou [13] studied the convergence of semi-discrete finite element approximations to the linear model (1.1) under a weaker regularity assumption, $u \in L^2((0,T); H_0^1(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$. The non-smoothness of the solution arises due to the low regularity of forcing term f, while the domain Ω was assumed to be convex in [13]. However, it is not clear whether these approaches can be extended to numerical analysis for time-dependent nonlinear problems under certain provable regularity assumptions, although the regularity analysis for the solution of some nonlinear models in a polygon was done by several authors, *e.g.* see [21,26,27] and references therein. The analysis of numerical methods for time-dependent nonlinear problems usually require a stronger regularity of the solution to handle the nonlinear terms. For fully discrete schemes, the stability or the time-step restriction of schemes due to the time discretization is another unknown issue.

The purpose of this paper is to study nonlinear parabolic equations (systems) in a nonconvex domain. For the illustration of our approach, we restrict our attention to a typical model, the time-dependent nonlinear thermistor problem,

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2, \tag{1.2}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0 \tag{1.3}$$

with the initial and boundary conditions:

$$\begin{aligned} &u(x,t)=0, \quad \phi(x,t)=g \quad \text{for} \quad x\in\partial\Omega, \quad t\in[0,T],\\ &u(x,0)=u_0 \qquad\qquad\qquad\;\; \text{for} \quad x\in\Omega. \end{aligned} \tag{1.4}$$

We shall focus on unconditional stability and optimal error estimates of fully

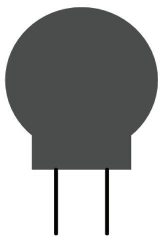

Fig. 1 Shape of a thermistor produced by the manufacturer EPCOS AG (No. B57235-S809-M).

discrete linearized Galerkin FEMs for the thermistor problem described above, and mention that the analysis presented in this paper may be extended to many other nonlinear parabolic equations.

The nonlinear system (1.2)-(1.4) describes the model of electric heating of a conducting body (see Figure 1), where u is the temperature, ϕ is the electric potential, and σ is the temperature-dependent electric conductivity. Following the previous works [17,46], we assume that $\sigma \in W^{1,\infty}(\mathbb{R})$ and

$$\kappa_1 \le \sigma(s) \le \kappa_2, \tag{1.5}$$

for some positive constants κ_1 and κ_2. Theoretical analysis for the thermistor system has been done by several authors [3,5,14,32,43,44,45]. In particular, existence and uniqueness of solutions of the thermistor problem were proved in [17,44,45] for uniformly elliptic electric conductivity $\sigma(u)$, and proved in [32] for non-uniformly elliptic electric conductivity. Further regularity of the solutions can be derived from these existing results with suitable assumptions on the initial and boundary data. In smooth or convex domains, numerical methods and analysis for the thermistor system have been studied in [4,17, 42,46,47]. For the system in the two-dimensional space, the optimal L^2-error estimate of a linearized semi-implicit scheme with Galerkin and mixed FEMs were obtained in [42] and [46] under a weak time step condition, respectively. Error analysis for the three-dimensional model was given in [17], in which the linearized semi-implicit Euler scheme with a linear Galerkin FEM was used. An optimal L^2-error estimate was presented under the time-step restriction $\tau = O(h^{1/2})$. The unconditional stability of the linearized schemes was proved in our recent work [30,29], where a new error splitting approach was introduced in terms of a time-discrete system. However, all these previous analyses relied on the assumption that the domain is either smooth or convex and the solution is smooth enough. Clearly, a corner singularity will make both stability analysis and error analysis difficult, since the solution of the nonlinear thermistor equations is only in H^{1+s} (see Lemma 1).

In this paper, we establish optimal error estimates and unconditional stability of a fully discrete linearized Galerkin FEM for the nonlinear thermistor equations in nonconvex polygonal domains. The key to our error analysis is the regularity of the solution of the nonlinear thermistor equations and the boundedness of numerical solution (U_h^n in L^∞ and Φ_h^n in $W^{1,4}$). Under certain assumptions on the initial and boundary data, we show that the unique solution of the nonlinear thermistor equations in a nonconvex polygon is also in H^{1+s}. To prove the boundedness of the numerical solution, we follow [30,31] to introduce an iterated sequence of elliptic PDEs (or time-discrete PDEs) so that the fully discrete finite element solution coincides with the finite element solution of the sequence of elliptic PDEs. Further analysis for the time-discrete equations in a nonconvex domain is presented to obtain the suitable regularity of the time-discrete solution U^n as required in the analysis of finite element approximations. With the proved regularity, the numerical solution U_h^n can be bounded via an inverse inequality, such as

$$\begin{aligned}
\|U_h^n\|_{L^\infty} &\le \|R_h U^n\|_{L^\infty} + \|U_h^n - R_h U^n\|_{L^\infty} \\
&\le C + Ch^{-1}\|U_h^n - R_h U^n\|_{L^2} \\
&\le C + Ch^{2s-1},
\end{aligned}$$

where R_h is the Ritz projection operator onto the finite element space. Then the optimal error estimate can be obtained in a traditional way.

The rest of the paper is organized as follows. In Section 2, we describe the linearized semi-implicit Euler–Galerkin finite element approximation and present our main results. After introducing an iterated sequence of time-discrete parabolic equations, we provide in Section 3 a priori estimates and optimal error estimates for the time-discrete solution, which imply the suitable regularity of the time-discrete solution. With the regularity obtained, we present optimal error estimates of the Galerkin finite element solution in the L^2 norm without any time-step restriction. In Section 4, we solve the nonlinear thermistor equations in a typical L-shape domain by the linearized Galerkin FEM with both uniform mesh and locally refined mesh. Numerical results confirm our theoretical analysis and also show that the method with the local refinement technique may give the optimal convergence rate.

2 Main results

Let Ω be a nonconvex polygon in $\mathbb{R}^2$ with the boundary $\partial\Omega = \cup_{j=1}^{m} \Gamma_j$, where Γ_j, $j = 1, \cdots, m$, are the edges of the polygon. Let π_h be a quasi-uniform triangulation of Ω into triangles T_j, $j = 1, \cdots, L$, with $h = \max_j\{\mathrm{diam}\, T_j\}$.

Let $W^{k,p}$ denote the usual Sobolev spaces [1] of functions defined in Ω and let $C(\overline{\Omega})$ be the space of continuous functions on $\overline{\Omega}$. For $s \in \mathbb{R}$, let H^s denote the Sobolev space defined on the domain Ω and let H_0^1 denote the space of functions of H^1 whose trace on the boundary $\partial\Omega$ is zero. For $\alpha \geq 0$ we define $H^\alpha(\partial\Omega)$ be the space of continuous functions defined on the boundary $\partial\Omega$ whose restriction to each edge Γ_j is in $H^\alpha(\Gamma_j)$. For a given triangulation π_h, we define the finite element spaces:

$$
\begin{aligned}
S_h &= \{v_h \in C(\overline{\Omega}) : v_h|_{T_j} \text{ is a linear polynomial } \},\\
V_h &= \{v_h \in C(\overline{\Omega}) : v_h|_{T_j} \text{ is a linear polynomial and } v_h = 0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

Let $\{t_n\}_{n=0}^N$ be a partition of the time interval $[0, T]$ with $t_n = n\tau$, $T = N\tau$ and set

$$u^n = u(x, t_n), \quad \phi^n = \phi(x, t_n)\,.$$

For any sequence of functions $\{f^n\}_{n=0}^N$ defined in Ω, we define

$$D_\tau f^{n+1} = (f^{n+1} - f^n)/\tau\,.$$

A linearized backward Euler–Galerkin finite element method is: to find $U_h^n \in V_h$ and $\Phi_h^n \in S_h$ for $n = 0, 1, \cdots, N$ such that for all $\xi_u,\ \xi_\phi \in V_h$

$$\big(D_\tau U_h^{n+1},\, \xi_u\big) + \big(\nabla U_h^{n+1},\, \nabla\xi_u\big) = \big(\sigma(U_h^n)|\nabla\Phi_h^n|^2,\, \xi_u\big), \tag{2.1}$$

$$\big(\sigma(U_h^n)\nabla\Phi_h^n,\, \nabla\xi_\phi\big) = 0, \tag{2.2}$$

with the boundary condition $\Phi_h^n = \Pi_h g^n$ on $\partial\Omega$ and the initial conditions $U_h^0 = \Pi_h u^0$, where $\Pi_h : C(\overline{\Omega}) \to S_h$ is the Lagrangian interpolation operator. Clearly, the numerical scheme above only requires the value of g on the boundary $\partial\Omega$. However, for the theoretical analysis and for the simplicity of notations, we assume that g is defined on the whole domain Ω and $g(x, 0) = g_0$ is a constant. It is easy to see that the above scheme is equivalent to find

U_h^n, $\Phi_h^n - \Pi_h g^n \in V_h$ for $n = 0, 1, \cdots, N$ such that (2.1)-(2.2) hold for all ξ_u, $\xi_\phi \in V_h$. This scheme was analyzed in [17,42,46] for problems in a convex domain under certain restrictions on the time-step size, and analyzed in [30, 31] for problems in a smooth domain without restriction on the time-step size.

The existence and uniqueness of a Hölder continuous solution to the initial/boundary value problem (1.2)-(1.4) was proved in [45]. Based on this result, we present further regularity of the solution in the following lemma. The proof will be given in Appendix. To simplify the notations, in the rest of this paper, we denote by C a generic positive constant which may be different at each occurrence. Similarly, C_s and C_ϵ denote generic positive constants depending on the parameter s and ϵ, respectively, which may be different at each occurrence.

Lemma 1 *Assume that $u_0, \Delta u_0 \in H_0^1$ and $g, g_t \in L^\infty((0,T); H^{2\beta}(\partial\Omega))$. Then the initial/boundary value problem (1.2)-(1.4) admits a unique solution satisfying*

$$\begin{aligned} &\|u\|_{C([0,T];H^{1+s})} + \|u_t\|_{C([0,T];L^2)} + \|u_t\|_{L^2((0,T);H^{1+s})} + \|u_{tt}\|_{L^2((0,T);L^2)} \\ &+ \|\phi\|_{C([0,T];H^{1+s})} + \|\phi_t\|_{L^\infty((0,T);W^{1,4})} \le C_s, \quad \textit{for any } s \in (1/2, \beta). \end{aligned} \tag{2.3}$$

Our main result is given in the following theorem. The proof will be presented in the next section.

Theorem 1 *Under the assumptions of Lemma 1, the finite element system (2.1)-(2.2) admits a unique solution (U_h^n, Φ_h^n), $n = 1, \cdots, N$, satisfying*

$$\max_{0\le n\le N} (\|U_h^n\|_{L^\infty} + \|\Phi_h^n\|_{L^\infty}) \le C, \tag{2.4}$$

$$\max_{1\le n\le N} \|U_h^n - u^n\|_{L^2} + \max_{1\le n\le N} \|\Phi_h^n - \phi^n\|_{L^2} \le C_s(\tau + h^{2s}), \tag{2.5}$$

for any given $s \in (1/2, \beta)$.

The following Sobolev embedding inequality will be frequently used in this paper:

$$\|u\|_{L^p} \le C\|u\|_{W^{\lambda,q}}, \quad \text{for} \quad \lambda \ge 2/q - 2/p, \;\; 1 < p, q < \infty. \tag{2.6}$$

3 Unconditional error analysis

For $U^0 = u_0$, we define U^n and Φ^n to be the solution of the following iterated time-discrete parabolic equations (or elliptic equations)

$$D_\tau U^{n+1} - \Delta U^{n+1} = \sigma(U^n)|\nabla\Phi^n|^2, \quad 0 \le n \le N-1, \tag{3.1}$$

$$-\nabla\cdot(\sigma(U^n)\nabla\Phi^n) = 0, \qquad 0 \le n \le N, \tag{3.2}$$

with the boundary conditions

$$U^{n+1}(x) = 0, \quad \Phi^n(x) = g(x, t_n) \quad \text{for} \quad x \in \partial\Omega\,. \tag{3.3}$$

Let $R_h : H_0^1(\Omega) \to V_h$ be the Ritz projection operator defined by

$$(\nabla(u - R_h u), \nabla v) = 0 \quad \text{for all } v \in V_h\,.$$

By classical finite element theory, see [11], we have

$$\|u - R_h u\|_{L^2} + h^s\|\nabla(u - R_h u)\|_{L^2} \le C\|u\|_{H^{1+s}} h^{2s}, \quad \text{for all } u \in H^{1+s} \cap H_0^1, \tag{3.4}$$

$$\|\nabla R_h u\|_{W^{1,p}} \le C\|u\|_{H^{1+s}}, \quad \text{for } p = 2/(1-s) \text{ and } u \in H^{1+s} \cap H_0^1. \tag{3.5}$$

Let $\Pi_h : C(\overline{\Omega}) \to S_h$ denote the Lagrangian interpolation operator. With the regularity in Lemma 2, we have the following estimates [11]:

$$\|v - \Pi_h v\|_{L^2} + h\|v - \Pi_h v\|_{H^1} \le Ch^{1+s}\|v\|_{H^{1+s}}, \quad \forall\, v \in H^{1+s} \cap H_0^1, \tag{3.6}$$

$$\|v - \Pi_h v\|_{H^{s'}} \le Ch^{1+s-s'}\|v\|_{H^{1+s}}, \quad \forall\, v \in H^{1+s} \cap H_0^1, \text{ for } 0 \le s' \le 1, \tag{3.7}$$

$$\|\Pi_h v\|_{W^{1,p}} \le C\|v\|_{H^{1+s}}, \quad \forall\, v \in H^{1+s} \cap H_0^1, \quad \text{for } p \le 2/(1-s), \tag{3.8}$$

for any $s \in (1/2, \beta)$.

We define

$$e^n = U^n - u^n, \quad e_h^n = U_h^n - R_h U^n,$$
$$\eta^n = \Phi^n - \phi^n, \quad \eta_h^n = \Phi_h^n - \Pi_h \Phi^n\,.$$

The error of the linearized Galerkin FEM can be splitted into

$$\|U_h^n - u^n\| \le \|e^n\| + \|e_h^n\| + \|U^n - R_h U^n\|$$
$$\|\Phi_h^n - \phi^n\| \le \|\eta^n\| + \|\eta_h^n\| + \|\Phi^n - R_h \Phi^n\|\,.$$

Then Theorem 1 is a consequence of the following two lemmas together with (3.4). The key to our theoretical analysis is the τ-independent boundedness of (e_h^n, η_h^n).

Lemma 2 *Under the assumption of Lemma 1, the time-discrete system (3.1)-(3.3) admits a unique solution (U^n, Φ^n) such that*

$$\max_{1\le n\le N} \|\Phi^n\|_{H^{1+s}} + \max_{1\le n\le N} \|U^n\|_{H^{1+s}} + \left(\sum_{n=1}^{N} \tau\|D_\tau U^n\|_{H^{1+s}}^2\right)^{1/2} \le C_s, \tag{3.9}$$

$$\max_{1\le n\le N} (\|e^n\|_{H^1} + \|\eta^n\|_{H^1}) \le C\tau. \tag{3.10}$$

for any $s \in (1/2, \beta)$.

Lemma 3 *Under the assumption of Lemma 1, the finite element system (2.1)-(2.2) admits a unique solution (U_h^n, Φ_h^n) such that*

$$\max_{0\le n\le N} (\|U_h^n\|_{L^\infty} + \|\Phi_h^n\|_{L^\infty}) \le C \tag{3.11}$$

and

$$\max_{0\le n\le N} (\|e_h^n\|_{L^2} + \|\eta_h^n\|_{L^2}) \le C_s h^{2s} \tag{3.12}$$

for any $s \in (1/2, \beta)$.

We present the proof of Lemmas 2-3 in the subsections 3.1 and 3.2, respectively. The following inequality will be often used in our proof.

$$\|u\|_{W^{1,4}} \le C\|u\|_{H^{1+s}}, \quad s > 1/2\,. \tag{3.13}$$

3.1 Proof of Lemma 2

The following lemma is concerned with the regularity of the solution of an elliptic equation in a nonconvex polygon. The proof can be found in [16].

Lemma 4 *If u is the solution of the equation*

$$\begin{cases} -\Delta u = f_1 \ \textit{in } \Omega, \\ u = f_2 \qquad \textit{on } \partial\Omega, \end{cases}$$

then

$$\|u\|_{H^{s+1}} \le C_s(\|f_1\|_{H^{s-1}} + \|f_2\|_{H^{s+1/2}(\partial\Omega)}), \quad \textit{for} \ \ s \in (1/2, \beta).$$

We rewrite the system (1.2)-(1.3) by

$$D_\tau u^{n+1} - \Delta u^{n+1} = \sigma(u^n)|\nabla\phi^n|^2 + R_{\mathrm{tr}}^{n+1}, \tag{3.14}$$

$$-\nabla\cdot(\sigma(u^n)\nabla\phi^n) = 0, \tag{3.15}$$

where

$$\begin{aligned} R_{\mathrm{tr}}^{n+1} = D_\tau u^{n+1} - \frac{\partial u}{\partial t}\Big|_{t=t_{n+1}} &+ (\sigma(u^{n+1}) - \sigma(u^n))|\nabla\phi^{n+1}|^2 \\ &+\sigma(u^n)\nabla(\phi^{n+1}+\phi^n)\cdot\nabla(\phi^{n+1}-\phi^n) \end{aligned}$$

is the truncation error due to the time discretization. With the regularity (2.3), we have

$$\textstyle\sum_{n=0}^{N-1} \|R_{\mathrm{tr}}^{n+1}\|_{L^2}^2 \tau \le C\tau^2.$$

To prove Lemma 2, first we present the error estimate in an energy norm. By subtracting the equations (3.14)-(3.15) from the equations (3.1)-(3.2), respectively, we obtain

$$\begin{aligned} D_\tau e^{n+1} - \Delta e^{n+1} = {} & (\sigma(U^n) - \sigma(u^n))|\nabla\phi^n|^2 \\ & + \sigma(U^n)(\nabla\phi^n + \nabla\Phi^n)\cdot\nabla\eta^n + R_{\mathrm{tr}}^{n+1}, \end{aligned} \tag{3.16}$$

$$-\nabla\cdot(\sigma(U^n)\nabla\eta^n) = \nabla\cdot[(\sigma(u^n) - \sigma(U^n))\nabla\phi^n]\,. \tag{3.17}$$

Multiplying (3.17) by η^n gives

$$\|\nabla\eta^n\|_{L^2} \le C\|e^n\|_{L^4}\|\nabla\phi\|_{L^4} \le C\|e^n\|_{L^4}, \tag{3.18}$$

and multiplying (3.16) by e^{n+1} leads to

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}\|e^{n+1}\|_{L^2}^2\right)+\|\nabla e^{n+1}\|_{L^2}^2\\
&\le C\|e^n\|_{L^4}\|e^{n+1}\|_{L^4}\|\nabla\phi^n\|_{L^4}^2+|\big(\sigma(U^n)(\nabla\phi^n+\nabla\Phi^n)e^{n+1},\,\nabla\eta^n\big)|\\
&\quad+\|R_{\mathrm{tr}}^{n+1}\|_{L^2}\|e^{n+1}\|_{L^2}\\
&\le C\|e^n\|_{L^4}\|e^{n+1}\|_{L^4}+|\big(\sigma(U^n)(\nabla\phi^n+\nabla\Phi^n)e^{n+1},\,\nabla\eta^n\big)|+\|R_{\mathrm{tr}}^{n+1}\|_{L^2}\|e^{n+1}\|_{L^2}.
\end{aligned}
$$

With integration by parts and using (3.2), we have

$$
\begin{aligned}
&|\big(\sigma(U^n)(\nabla\phi^n+\nabla\Phi^n)e^{n+1},\,\nabla\eta^n\big)|\\
&=|\big(\sigma(U^n)e^{n+1}\nabla\phi^n,\,\nabla\eta^n\big)-\big(e^{n+1}\nabla\cdot(\sigma(U^n)\nabla\Phi^n)+\sigma(U^n)\nabla\Phi^n\cdot\nabla e^{n+1},\,\eta^n\big)|\\
&=|\big(\sigma(U^n)e^{n+1}\nabla\phi^n,\,\nabla\eta^n\big)-\sigma(U^n)\nabla(\phi^n+\eta^n)\cdot\nabla e^{n+1},\,\eta^n\big)|\\
&\le C\|e^{n+1}\|_{L^4}\|\nabla\phi^n\|_{L^4}\|\nabla\eta^n\|_{L^2}\\
&\quad+C(\|\nabla\phi^n\|_{L^4}\|\nabla e^{n+1}\|_{L^2}\|\eta^n\|_{L^4}+\|\nabla\eta^n\|_{L^2}\|\nabla e^{n+1}\|_{L^2}\|\eta^n\|_{L^\infty})\\
&\le C(\|e^{n+1}\|_{L^4}\|e^n\|_{L^4}+\|\nabla e^{n+1}\|_{L^2}\|e^n\|_{L^4}+\|e^n\|_{L^4}\|\nabla e^{n+1}\|_{L^2}\|\eta^n\|_{L^\infty}).
\end{aligned}
\tag{3.19}
$$

where (3.18) is used in the last step of the inequality above.

Applying the maximum principle to the elliptic equation (3.2) shows that $\|\Phi^n\|_{L^\infty}\le C$, which further implies $\|\eta^n\|_{L^\infty}\le C$. By using the Sobolev embedding inequality (2.6) we have

$$\|e^n\|_{L^4}\le C\|e^n\|_{H^{1/2}}\le C\epsilon^{-1}\|e^n\|_{L^2}+\epsilon\|\nabla e^n\|_{L^2},$$

where $\epsilon\in(0,1)$ is arbitrary, therefore

$$
\begin{aligned}
&D_\tau\left(\frac{1}{2}\|e^{n+1}\|_{L^2}^2\right)+\frac{1}{2}\|\nabla e^{n+1}\|_{L^2}^2\\
&\le\epsilon\|\nabla e^{n+1}\|_{L^2}^2+\epsilon\|\nabla e^n\|_{L^2}^2+\epsilon\|e^{n+1}\|_{L^2}^2+C_\epsilon\|e^n\|_{L^2}^2+C_\epsilon\|R_{\mathrm{tr}}^{n+1}\|_{L^2}^2.
\end{aligned}
\tag{3.20}
$$

By applying (explicit) Gronwall's inequality with $\epsilon\le 1/8$, we get

$$\max_{1\le n\le N}\|e^n\|_{L^2}^2+\sum_{n=1}^N\|e^n\|_{H^1}^2\tau\le C\tau^2,\tag{3.21}$$

which leads to

$$\|U^n\|_{H^1}\le C\quad\text{and}\quad\|D_\tau U^{n+1}\|_{L^2}\le C.\tag{3.22}$$

Secondly we prove the boundedness of $\|\Phi\|_{H^{1+s}}$ and $\|U^{n+1}\|_{H^{1+s}}$. We apply Lemma 4 with (3.22) to (3.1) to get

$$\|U^{n+1}\|_{H^{1+s}}\le C\|D_\tau U^{n+1}\|_{L^2}+C\|\nabla\Phi^n\|_{L^4}^2.\tag{3.23}$$

Let $K=\max_{0\le n\le N}\|\nabla\phi^n\|_{L^4}$ and now, we prove

$$\|\nabla\Phi^n\|_{L^4}\le K+1\tag{3.24}$$

by mathematical induction. It is easy to see that the above inequality holds for $n = 0$. If we assume that (3.24) holds for $0 \le n \le k$ for some nonnegative integer k, then (3.23) implies that

$$\|U^{n+1}\|_{H^{1+s}} \le C \tag{3.25}$$

for $0 \le n \le k$. Since $H^{1+s}(\Omega) \hookrightarrow C^{1/2}(\overline{\Omega})$ for $s > 1/2$, it follows that $\|\sigma(U^n)\|_{C^{1/2}(\overline{\Omega})} \le C$. By applying the $W^{1,4}$ estimate of [24] to (3.17) we obtain (under the regularity $\|\sigma(U^n)\|_{C^{1/2}(\overline{\Omega})} \le C$, the $W^{1,4}$ estimate of [24] can be extended to elliptic equations with variable coefficients by the standard perturbation argument)

$$\begin{aligned}
\|\nabla\eta^{n+1}\|_{L^4} &\le \|(\sigma(u^{n+1}) - \sigma(U^{n+1}))\nabla\phi^{n+1}\|_{L^4} \\
&\le C\|e^{n+1}\|_{L^\infty}\|\nabla\phi^{n+1}\|_{L^4} \\
&\le C\|e^{n+1}\|_{L^2}^{s/(1+s)}\|e^{n+1}\|_{H^{1+s}}^{1/(1+s)} \\
&\le C\tau^{s/(1+s)},
\end{aligned} \tag{3.26}$$

where we have used (3.21) and (3.25). Then there exists $\tau_0 > 0$ such that $\|\nabla\eta^{n+1}\|_{L^4} \le 1$ when $\tau < \tau_0$, which further implies that

$$\|\nabla\Phi^{n+1}\|_{L^4} \le \|\nabla\phi^{n+1}\|_{L^4} + \|\nabla\eta^{n+1}\|_{L^4} \le K + 1.$$

The induction on (3.24) is closed, which implies that (3.24)-(3.25) hold for $0 \le n \le N$.

Moreover, we rewrite (3.2) as

$$-\Delta\Phi^n = \frac{1}{\sigma(U^n)}\sigma'(U^n)\nabla U^n \cdot \nabla\Phi^n\,. \tag{3.27}$$

By Lemma 4, we get

$$\begin{aligned}
\|\Phi^n\|_{H^{1+s}} &\le C\|\nabla U^n \cdot \nabla\Phi^n\|_{L^2} + C\|g^n\|_{H^{1/2+s}(\partial\Omega)} \\
&\le C\|\nabla U^n\|_{L^4}\|\nabla\Phi^n\|_{L^4} + C \le C
\end{aligned} \tag{3.28}$$

for $1 \le n \le N$. We multiply (3.16) by $-\Delta e^{n+1}$ to derive that

$$\begin{aligned}
&D_\tau\left(\frac{1}{2}\|e^{n+1}\|_{H^1}^2\right) + \|\Delta e^{n+1}\|_{L^2}^2 \\
&\le \|(\sigma(U^n) - \sigma(u^n))|\nabla\phi^n|^2\|_{L^2}^2 + \|\sigma(U^n)(\nabla\phi^n + \nabla\Phi^n)\cdot\nabla\eta^n\|_{L^2}^2 + C\|R_{\rm tr}^{n+1}\|_{L^2}^2 \\
&\le C\|e^n\|_{L^\infty}^2 + C\|\nabla\eta^n\|_{L^4}^2 + C\|R_{\rm tr}^{n+1}\|_{L^2}^2 \\
&\le C\|e^n\|_{L^\infty}^2 + C\|R_{\rm tr}^{n+1}\|_{L^2}^2 \\
&\le \epsilon\|\Delta e^n\|_{L^2}^2 + C_\epsilon\|e^n\|_{L^2}^2 + C\|R_{\rm tr}^{n+1}\|_{L^2}^2 \\
&\le C_\epsilon\tau^2 + \epsilon\|\Delta e^n\|_{L^2}^2 + C\|R_{\rm tr}^{n+1}\|_{L^2}^2,
\end{aligned}$$

where we have used (3.26) in the third step. With Gronwall's inequality, the last inequality implies that

$$\max_{1\le n\le N} \|e^n\|_{H^1}^2 + \sum_{n=1}^{N} \tau \|\Delta e^n\|_{L^2}^2 \le C\tau^2, \tag{3.29}$$

which leads to $\sum_{n=1}^{N} \tau \|D_\tau \Delta e^n\|_{L^2}^2 \le C$. By using Lemma 4 again, we see that

$$\sum_{n=1}^{N} \tau \|D_\tau e^n\|_{H^{1+s}}^2 \le \sum_{n=1}^{N} \tau \|D_\tau \Delta e^n\|_{L^2}^2 \le C\,.$$

It follows that

$$\sum_{n=1}^{N} \tau \|D_\tau U^n\|_{H^{1+s}}^2 \le C. \tag{3.30}$$

From (3.18) and (3.29) we also see that

$$\|\eta^{n+1}\|_{H^1} \le C\|e^{n+1}\|_{L^4} \le C\|e^{n+1}\|_{H^1} \le C\tau. \tag{3.31}$$

So far we have proved that (3.9)-(3.10) hold when $\tau < \tau_0$. Now we consider the case $\tau_0 \le \tau \le T$. In this case, (3.22) and (3.23) still hold. The equation (3.2) implies

$$\max_{1\le n\le N} \|\Phi^n\|_{H^1} \le C\|g^n\|_{H^{1/2}(\partial\Omega)}.$$

By (3.23), we have

$$\|U^{n+1}\|_{H^{1+s}} \le C + C\|\Phi^n\|_{H^{1+s}}^2. \tag{3.32}$$

Similarly, applying Lemma 4 to (3.27) gives

$$\begin{aligned}
\|\Phi^{n+1}\|_{H^{1+s}} &\le C\|\nabla U^{n+1}\|_{L^4}\|\nabla \Phi^{n+1}\|_{L^4} + C\|g^n\|_{H^{1/2+s}(\partial\Omega)} \\
&\le C\|U^{n+1}\|_{H^{1+s}}\|\Phi^{n+1}\|_{H^1}^{(2s-1)/(2s)}\|\Phi^{n+1}\|_{H^{s+1}}^{1/(2s)} + C \\
&\le \frac{1}{2}\|\Phi^{n+1}\|_{H^{1+s}} + C\|U^{n+1}\|_{H^{1+s}}^{2s/(2s-1)} + C\,,
\end{aligned}$$

which in turn produces

$$\|\Phi^{n+1}\|_{H^{1+s}} \le C\|U^{n+1}\|_{H^{1+s}}^{2s/(2s-1)} + C \le (C + C\|\Phi^n\|_{H^{1+s}}^2)^{2s/(2s-1)} + C.$$

Let $f(z) = (C + Cz^2)^{2s/(2s-1)} + C$ and define $f^{(n)}(s) = f(f^{(n-1)}(s))$ for $n = 1, \cdots, N$. The last inequality implies that

$$\max_{0\le n\le N} \|\Phi^n\|_{H^{1+s}} \le f^{([T/\tau_0]+1)}(\|\phi(\cdot,0)\|_{H^{1+s}}) \le C, \tag{3.33}$$

and from (3.32) we see that

$$\max_{0\le n\le N} \|U^n\|_{H^{1+s}} \le C. \tag{3.34}$$

Since $\tau \geq \tau_0$, we further derive that

$$\begin{aligned}
&\max_{1\leq n\leq N}(\|D_\tau U^n\|_{H^{1+s}} + \|D_\tau \Phi^n\|_{H^{1+s}}) \\
&\leq 2\tau_0^{-1} \max_{0\leq n\leq N}(\|U^n\|_{H^{1+s}} + \|\Phi^n\|_{H^{1+s}}) \leq C. \qquad (3.35)
\end{aligned}$$

Combining the two cases, $\tau < \tau_0$ and $\tau \geq \tau_0$, we complete the proof of Lemma 2. ∎

3.2 Proof of Lemma 3

At each time step of the linearized Galerkin scheme, one only needs to solve the two discrete linear systems (2.2) and (2.1). It is easy to see that coefficient matrices in both systems are symmetric and positive definite. The existence and uniqueness of the Galerkin finite element solution follow immediately.

To prove (3.11)-(3.12), we write the weak form of the time-discrete system (3.1)-(3.3) by

$$\big(D_\tau U^{n+1},\, \xi_u\big) + \big(\nabla U^{n+1},\, \nabla \xi_u\big) = \big(\sigma(U^n)|\nabla \Phi^n|^2,\, \xi_u\big), \qquad (3.36)$$

$$\big(\sigma(U^n)\nabla \Phi^n,\, \nabla \xi_\phi\big) = 0, \qquad (3.37)$$

for any $\xi_u,\ \xi_\phi \in V_h$. From the above equations and the finite element system (2.1)-(2.2), we find that the error function (e_h^n, η_h^n) satisfies

$$\begin{aligned}
&\big(D_\tau e_h^{n+1},\, \xi_u\big) + \big(\nabla e_h^{n+1},\, \nabla \xi_u\big) \\
&= \big(D_\tau(U^{n+1} - R_h U^{n+1}),\, \xi_u\big) + \big((\sigma(U_h^n) - \sigma(U^n))|\nabla \Phi^n|^2,\, \xi_u\big) \\
&\quad + 2\big((\sigma(U_h^n) - \sigma(U^n))\nabla \Phi^n \cdot \nabla(\Phi_h^n - \Phi^n),\, \xi_u\big) \\
&\quad + \big(\sigma(U_h^n)|\nabla(\Phi_h^n - \Phi^n)|^2,\, \xi_u\big) + 2\big(\sigma(U^n)\nabla \Phi^n \cdot \nabla(\Phi_h^n - \Phi^n),\, \xi_u\big) \\
&:= \sum_{i=1}^{5} J_i^{n+1} \qquad (3.38)
\end{aligned}$$

and

$$\begin{aligned}
\big(\sigma(U_h^n)\nabla \eta_h^n,\, \nabla \xi_\phi\big) &= \big((\sigma(U_h^n) - \sigma(U^n))\nabla \Phi^n,\, \nabla \xi_\phi\big) \\
&\quad + \big(\sigma(U_h^n)\nabla(\Phi^n - \Pi_h \Phi^n),\, \nabla \xi_\phi\big) \qquad (3.39)
\end{aligned}$$

for all $\xi_u, \xi_\phi \in V_h$.

For any given $s \in (1/2, \beta)$, we choose $s_1 = (s+\beta)/2$. Taking $\xi_\phi = \eta_h^n$ in (3.39), we get

$$\begin{aligned}
\|\nabla \eta_h^n\|_{L^2} &\leq C\|(\sigma(U_h^n) - \sigma(U^n))\nabla \Phi^n\|_{L^2} + C\|\nabla(\Phi^n - \Pi_h \Phi^n)\|_{L^2} \\
&\leq C\|U_h^n - U^n\|_{L^4}\|\nabla \Phi^n\|_{L^4} + Ch^{s_1} \\
&\leq C(\|e_h^n\|_{L^4} + \|U^n - R_h U^n\|_{L^4}) + Ch^{s_1} \\
&\leq \epsilon h^\alpha \|\nabla e_h^n\|_{L^2} + C\epsilon^{-1}h^{-\alpha}\|e_h^n\|_{L^2} + Ch^{s_1}, \quad \forall\ \alpha > 0. \qquad (3.40)
\end{aligned}$$

Let ψ be the solution to the equation

$$\begin{cases} -\nabla\cdot(\sigma(U^n)\nabla\psi) = \varphi, \text{ in } \Omega, \\ \psi = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

From (3.9) and (3.13) we see that $\|U^n\|_{W^{1,4}} + \|U^n\|_{C^{s_1}} \le C\|U^n\|_{H^{1+s_1}} \le C$. Therefore, we can rewrite the equation above as

$$-\Delta\psi = \frac{1}{\sigma(U^n)}\varphi + \frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\psi \tag{3.41}$$

and apply Lemma 4. Then we obtain

$$\begin{aligned}
\|\psi\|_{H^{1+s_1}} &\le C\left\|\frac{1}{\sigma(U^n)}\varphi + \frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\psi\right\|_{H^{s_1-1}} \\
&\le C\left\|\frac{1}{\sigma(U^n)}\varphi\right\|_{H^{s_1-1}} + C\left\|\frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\psi\right\|_{L^2} \\
&\le C\|\varphi\|_{H^{s_1-1}} + C\|\nabla\psi\|_{L^4} \\
&\le C\|\varphi\|_{H^{s_1-1}} + \epsilon\|\psi\|_{H^{1+s_1}} + C_\epsilon\|\psi\|_{H^1}
\end{aligned}$$

where we have used the Sobolev embedding inequality (2.6), i.e.

$$\|\nabla\psi\|_{L^4} \le C\|\psi\|_{H^{3/2}} \le \epsilon\|\psi\|_{H^{1+s_1}} + C_\epsilon\|\psi\|_{H^1}.$$

It follows that

$$\|\psi\|_{H^{1+s_1}} \le C\|\varphi\|_{H^{s_1-1}}\,.$$

Moreover, by the trace theorem, we have

$$\|\partial_\nu\psi\|_{L^2(\partial\Omega)} \le C\|\psi\|_{H^{1+s_1}} \le C\|\varphi\|_{H^{s_1-1}}\,.$$

By (3.6)-(3.8) and (3.9) we have further

$$\begin{aligned}
&\|\nabla(\psi - \Pi_h\psi)\|_{L^2} \le Ch^{s_1}\|\psi\|_{H^{1+s_1}} \le Ch^{s_1}\|\varphi\|_{H^{s_1-1}}, \\
&\|\Pi_h\psi\|_{W^{1,4}} \le C\|\psi\|_{H^{1+s_1}} \le C\|\varphi\|_{H^{s_1-1}}\,, \\
&\|\Pi_h\Phi\|_{W^{1,4}} \le C\|\Phi\|_{H^{1+s_1}} \le C\,.
\end{aligned}$$

Therefore, using (3.39) we obtain

$$\begin{aligned}
(\eta_h^n,\varphi) &= \big(\sigma(U^n)\nabla\eta_h^n,\, \nabla(\psi-\Pi_h\psi)\big) + \big((\sigma(U^n)-\sigma(U_h^n))\nabla\eta_h^n,\, \nabla\Pi_h\psi\big) \\
&\quad + \big(\sigma(U_h^n)\nabla\eta_h^n,\, \nabla\Pi_h\psi\big) \\
&= \big(\sigma(U^n)\nabla\eta_h^n,\, \nabla(\psi-\Pi_h\psi)\big) + \big((\sigma(U^n)-\sigma(U_h^n))\nabla\eta_h^n,\, \nabla\Pi_h\psi\big) \\
&\quad + \big((\sigma(U_h^n)-\sigma(U^n))\nabla\Phi^n,\, \nabla\Pi_h\psi\big) \\
&\quad + \big(\sigma(U_h^n)\nabla(\Phi^n-\Pi_h\Phi^n),\, \nabla(\Pi_h\psi-\psi)\big) \\
&\quad + \big((\sigma(U_h^n)-\sigma(U^n))\nabla(\Phi^n-\Pi_h\Phi^n),\, \nabla\psi\big) \\
&\quad + \big(\sigma(U^n)\nabla(\Phi^n-\Pi_h\Phi^n),\, \nabla\psi\big)
\end{aligned}$$

$$:= \sum_{j=1}^{6} L_j^n,$$

where

$$\begin{aligned}
L_1^n &\le C\|\nabla \eta_h^n\|_{L^2}\|\nabla(\psi - \Pi_h\psi)\|_{L^2} \\
&\le Ch^{s_1}\|\nabla \eta_h^n\|_{L^2}\|\varphi\|_{H^{s_1-1}}, &(3.42)\\
L_2^n &\le C(\|U^n - R_h u^n\|_{L^2} + \|e_h^n\|_{L^2})\|\nabla \eta_h^n\|_{L^4}\|\varphi\|_{H^{s_1-1}} \\
&\le C(h^{2s_1} + \|e_h^n\|_{L^2})\|\nabla \eta_h^n\|_{L^4}\|\varphi\|_{H^{s_1-1}}, &(3.43)\\
L_3^n &\le C(\|U^n - R_h u^n\|_{L^2} + \|e_h^n\|_{L^2})\|\nabla \Phi^n\|_{L^4}\|\varphi\|_{H^{s_1-1}} \\
&\le C(h^{2s_1} + \|e_h^n\|_{L^2})\|\varphi\|_{H^{s_1-1}}, &(3.44)\\
L_4^n &\le Ch^{2s_1}\|\Phi^n\|_{H^{1+s_1}}\|\psi\|_{H^{1+s_1}} \\
&\le Ch^{2s_1}\|\varphi\|_{H^{s_1-1}}, &(3.45)\\
L_5^n &\le C(\|U^n - R_h u^n\|_{L^2} + \|e_h^n\|_{L^2})\|\Phi^n\|_{H^{1+s_1}}\|\psi\|_{H^{1+s_1}} \\
&\le C(h^{2s_1} + \|e_h^n\|_{L^2})\|\varphi\|_{H^{s_1-1}}, &(3.46)\\
L_6^n &= (\Phi^n - \Pi_h\Phi^n, -\nabla\cdot(\sigma(U^n)\nabla\psi)) + (\Phi^n - \Pi_h\Phi^n, \sigma(U^n)\partial_\nu\psi)_{\partial\Omega} \\
&= (\Phi^n - \Pi_h\Phi^n, \varphi) + (\Phi^n - \Pi_h\Phi^n, \sigma(U^n)\partial_\nu\psi)_{\partial\Omega} \\
&\le C\|\Phi^n - \Pi_h\Phi^n\|_{H^{1-s_1}}\|\varphi\|_{H^{s_1-1}} + C\|\Phi^n - \Pi_h\Phi^n\|_{L^2(\partial\Omega)}\|\partial_\nu\psi\|_{L^2(\partial\Omega)} \\
&\le C(\|\Phi^n - \Pi_h\Phi^n\|_{H^{1-s_1}} + \|g^n - \Pi_h g^n\|_{L^2(\partial\Omega)})\|\varphi\|_{H^{s_1-1}} \\
&\le Ch^{2s_1}\|\varphi\|_{H^{s_1-1}}. &(3.47)
\end{aligned}$$

To conclude, we have

$$(\eta_h^n, \varphi) \le C(h^{s_1}\|\nabla \eta_h^n\|_{L^2} + (h^{2s_1} + \|e_h^n\|_{L^2})\|\nabla \eta_h^n\|_{L^4} + \|e_h^n\|_{L^2} + h^{2s_1})\|\varphi\|_{H^{s_1-1}}, \tag{3.48}$$

and by duality we derive

$$\|\eta_h^n\|_{H^{1-s_1}} \le C(h^{s_1}\|\nabla \eta_h^n\|_{L^2} + (h^{2s_1} + \|e_h^n\|_{L^2})\|\nabla \eta_h^n\|_{L^4} + \|e_h^n\|_{L^2} + h^{2s_1}). \tag{3.49}$$

Taking $\xi_u = e_h^{n+1}$ in (3.38), we have

$$\begin{aligned}
J_1^{n+1} &\le \epsilon\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|D_\tau U^{n+1} - R_h D_\tau U^{n+1}\|_{L^2}^2 \\
&\le \epsilon\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|D_\tau U^{n+1}\|_{H^{1+s_1}}^2 h^{4s_1}, &(3.50)\\
J_2^{n+1} &\le C\|U_h^n - U^n\|_{L^2}\|\nabla\Phi^n\|_{L^{2/(1-s_1)}}^2\|e_h^{n+1}\|_{L^{2/(2s_1-1)}} \\
&\le C(\|e_h^n\|_{L^2} + \|U^n - R_h U^n\|_{L^2})\|\Phi^n\|_{H^{1+s_1}}^2\|\nabla e_h^{n+1}\|_{L^2} \\
&\le C(\|e_h^n\|_{L^2} + h^{2s_1})\|\nabla e_h^{n+1}\|_{L^2} \\
&\le \epsilon\|\nabla e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\|e_h^n\|_{L^2}^2 + h^{4s_1}), &(3.51)
\end{aligned}$$

where we have used the Sobolev embedding inequalities (special cases of (2.6))

$$
\begin{aligned}
&\|e_h^{n+1}\|_{L^{2/(2s_1-1)}} \le C\|\nabla e_h^{n+1}\|_{L^2},\\
&\|\nabla \Phi^n\|_{L^{2/(1-s_1)}} \le C\|\Phi^n\|_{H^{1+s_1}} \le C,
\end{aligned}
$$

$$
\begin{aligned}
J_3^{n+1} &\le C\|U_h^n - U^n\|_{L^2}\|\nabla\Phi^n\|_{L^{2/(1-s_1)}}\|\nabla(\Phi_h^n-\Phi^n)\|_{L^4}\|e_h^{n+1}\|_{L^{4/(2s_1-1)}}\\
&\le C(\|e_h^n\|_{L^2} + \|U^n - R_hU^n\|_{L^2})\|\Phi^n\|_{H^{1+s_1}}(\|\nabla\Phi_h^n\|_{L^4} + \|\nabla\Phi^n\|_{L^4})\|\nabla e_h^{n+1}\|_{L^2}\\
&\le C(\|e_h^n\|_{L^2} + h^{2s_1})(C + \|\nabla\Phi_h^n\|_{L^4})\|\nabla e_h^{n+1}\|_{L^2}\\
&\le \epsilon\|\nabla e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(1+\|\nabla\Phi_h^n\|_{L^4}^2)(\|e_h^n\|_{L^2}^2 + h^{4s_1}), \qquad (3.52)\\
J_4^{n+1} &\le C\|e_h^{n+1}\|_{L^\infty}(\|\nabla\eta_h^n\|_{L^2}^2 + \|\nabla(\Phi^n - \Pi_h\Phi^n)\|_{L^2}^2)\\
&\le C|\ln h|\|\nabla e_h^{n+1}\|_{L^2}(h^s\|\nabla e_h^n\|_{L^2}^2 + h^{-s}\|e_h^n\|_{L^2}^2 + h^{2s_1})\\
&\le (\epsilon + C_3h^s|\ln h|)\|\nabla e_h^{n+1}\|_{L^2}^2 + C_3h^s|\ln h|\|\nabla e_h^n\|_{L^2}^4\\
&\quad + C\epsilon^{-1}h^{-2s}|\ln h|^2\|e_h^n\|_{L^2}^4 + C\epsilon^{-1}h^{4s_1}|\ln h|^2\,, \qquad (3.53)
\end{aligned}
$$

where we have used (3.40) with $\alpha = s_1$ and the discrete Sobolev embedding inequality

$$
\|e_h^{n+1}\|_{L^\infty} \le C|\ln h|\|\nabla e_h^{n+1}\|_{L^2}, \qquad (3.54)
$$

and

$$
\begin{aligned}
&J_5^{n+1}\\
&= -2\big(\sigma(U^n)\nabla\Phi^n(\Phi_h^n - \Phi^n),\, \nabla e_h^{n+1}\big)\\
&\le C\|\nabla\Phi^n\|_{L^{2/(1-s_1)}}\|\Phi_h^n - \Phi^n\|_{L^{2/s_1}}\|\nabla e_h^{n+1}\|_{L^2}\\
&\le C\|\Phi^n\|_{H^{1+s_1}}(\|\eta_h^n\|_{L^{2/s_1}} + \|\Phi^n - \Pi_h\Phi^n\|_{L^{2/s_1}})\|\nabla e_h^{n+1}\|_{L^2}\\
&\le C(\|\eta_h^n\|_{H^{1-s_1}} + \|\Phi^n - \Pi_h\Phi^n\|_{H^{1-s_1}})\|\nabla e_h^{n+1}\|_{L^2}\\
&\le C(h^{s_1}\|\nabla\eta_h^n\|_{L^2} + (h^{2s_1} + \|e_h^n\|_{L^2})\|\nabla\eta_h^n\|_{L^4} + \|e_h^n\|_{L^2} + h^{2s_1})\|\nabla e_h^{n+1}\|_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3.7) and (3.49))}\\
&\le C(h^{s_1}\|\nabla e_h^n\|_{L^2} + (h^{2s_1} + \|e_h^n\|_{L^2})\|\nabla\eta_h^n\|_{L^4} + \|e_h^n\|_{L^2} + h^{2s_1})\|\nabla e_h^{n+1}\|_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3.40) with } \alpha = 0)\\
&\le (\epsilon + Ch^{2s_1})\|\nabla e_h^{n+1}\|_{L^2}^2 + Ch^{2s_1}\|\nabla e_h^n\|_{L^2}^2\\
&\quad + C\epsilon^{-1}(\|\nabla\Phi_h^n\|_{L^4}^2 + 1)(h^{4s_1} + \|e_h^n\|_{L^2}^2), \qquad (3.55)
\end{aligned}
$$

where we have used the Sobolev embedding inequality (a special case of (2.6))

$$
\|\eta_h^n\|_{L^{2/s_1}} \le C\|\eta_h^n\|_{H^{1-s_1}}.
$$

Let $\widehat{K} = \max_{1\le n\le N}\|\nabla\Pi_h\Phi^n\|_{L^4}$. Now we prove

$$
\|\nabla\Phi_h^n\|_{L^4} \le \widehat{K} + 1 \quad \text{and} \quad \|e_h^n\|_{L^2} \le h \qquad (3.56)
$$

by mathematical induction. Since (3.56) holds for $n = 0$, we assume that it holds for $0 \le n \le k$. Then there exists a positive constant h_1 such that when $h < \min(h_1, 1/2)$, (3.38) together with (3.50)-(3.56) reduces to

$$\begin{aligned} & D_\tau \left(\|e_h^{n+1}\|_{L^2}^2\right) + \|\nabla e_h^{n+1}\|_{L^2}^2 \\ & \le h^s |\ln h| \|\nabla e_h^n\|_{L^2}^2 + C\|e_h^n\|_{L^2}^2 + Ch^{4s} + C\|D_\tau U^{n+1}\|_{H^{1+s}}^2 h^{4s}, \end{aligned}$$

for $0 \le n \le k$, which in turn produces

$$\|e_h^{n+1}\|_{L^2}^2 + \sum_{m=0}^{n} \tau \|\nabla e_h^{m+1}\|_{L^2}^2 \le C \sum_{m=0}^{n} \tau \|e_h^m\|_{L^2}^2 + Ch^{4s},$$

for $0 \le n \le k$. By applying Gronwall's inequality, we derive that

$$\|e_h^{n+1}\|_{L^2} \le Ch^{2s}. \tag{3.57}$$

By an inverse inequality, we have $\|\nabla e_h^{n+1}\|_{L^2} \le Ch^{-1}\|e_h^{n+1}\|_{L^2} \le Ch^{2s-1} \le C$, and from (3.40) we see that (by choosing $\alpha = s$)

$$\|\nabla \eta_h^{n+1}\|_{L^2} \le Ch^s \tag{3.58}$$

for $0 \le n \le k$. When $h < h_2$ for some $h_2 > 0$, we can apply inverse inequalities to (3.57)-(3.58) and get

$$\|\nabla \eta_h^{k+1}\|_{L^4} \le Ch^{-1/2}\|\nabla \eta_h^{k+1}\|_{L^2} \le Ch^{s-1/2} \le 1, \tag{3.59}$$

$$\|e_h^{k+1}\|_{L^2} \le Ch^{2s} \le h\,. \tag{3.60}$$

The induction on (3.56) is closed and (3.57)-(3.58) hold for all $0 \le n \le N-1$.

From (3.49) we derive that

$$\|\eta_h^n\|_{L^2} \le Ch^{2s}. \tag{3.61}$$

By applying an inverse inequality to (3.57) and (3.61), we obtain (2.4).

So far we have proved (3.11)-(3.12) for $h < h_0 := \min(h_1, h_2, 1/2)$. If $h \ge h_0$, we set $\xi_\phi = \Phi_h^n$ and $\xi_u = U_h^{n+1}$ in (2.2) and (2.1), respectively, to get

$$\max_{0 \le n \le N} \|\nabla \Phi_h^n\|_{L^2} \le C \le Ch_0^{-2s} h^{2s} \tag{3.62}$$

and

$$\begin{aligned} & D_\tau \left(\frac{1}{2}\|U_h^{n+1}\|_{L^2}^2\right) + \|\nabla U_h^{n+1}\|_{L^2}^2 \\ & \le C\|\nabla \Phi_h^n\|_{L^2}^2 \|U_h^{n+1}\|_{L^\infty} \\ & \le C\|U_h^{n+1}\|_{L^\infty} \\ & \le Ch_0^{-1}\|U_h^{n+1}\|_{L^2} && \text{(by the inverse inequality)} \\ & \le Ch_0^{-1}\|\nabla U_h^{n+1}\|_{L^2} && \text{(Sobolev embedding inequality)} \\ & \le \epsilon\|\nabla U_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}h_0^{-2}, \end{aligned}$$

which implies that

$$\max_{0\le n\le N} \|U_h^n\|_{L^2} \le Ch_0^{-2} \le Ch_0^{-2-2s}h^{2s}. \tag{3.63}$$

By using the inverse inequality

$$\|U_h^n\|_{L^\infty} \le Ch_0^{-1}\|U_h^n\|_{L^2}, \qquad \|\Phi_h^n\|_{L^\infty} \le Ch_0^{-1}\|\Phi_h^n\|_{L^2} \le Ch_0^{-1}\|\nabla\Phi_h^n\|_{L^2}$$

from (3.62)-(3.63) we derive (3.11)-(3.12) for $h \ge h_0$.

Combining the two cases, $h < h_0$ and $h \ge h_0$, we complete the proof of Lemma 3. ∎

4 Numerical results

In this section, we present some numerical results to confirm our theoretical analysis. All the computations are performed with the software FEniCS.

We rewrite the system (1.2)-(1.3) by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2 + f_1, \tag{4.1}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = f_2, \tag{4.2}$$

where $\Omega = (-1,1)\times(-1,1)\backslash[0,1)\times[0,1)$ is an L shape domain and

$$\sigma(u) = \frac{1}{1+u^2} + 1.$$

The functions f_1, f_2, and the Dirichlet boundary conditions are chosen corresponding to the exact solution

$$u(x,y,t) = r^{2/3}\sin(2\theta/3)\exp(2t)\,, \quad \phi(x,y,t) = r^{2/3}\sin(2\theta/3)\cos(4t) \tag{4.3}$$

in polar coordinates. In this case, $u \in H^{1+s}$ for $s < 2/3$. All numerical results are obtained by using the linearized Galerkin FEMs defined in (2.1)-(2.2) with $T = 1.0$ in our tests.

To test the convergence rate, a uniform triangulation is made on the L-shape domain Ω, see Figure 2 for a sample mesh, where $M+1$ nodal points locate in the interval $[0,1]$. To confirm our error estimates, we choose $\tau = 1/M^2$ for the linear FE method. From our theoretical analysis, the L^2-norm errors are in the order $O(h^2 + h^{4/3}) \sim O(h^{4/3})$ and the H^1-norm errors are in the order $O(h^2+h^{2/3}) \sim O(h^{2/3})$. We present the L^2 and H^1-norm errors in Table 1. One can see clearly that the L^2-norm errors of u and ϕ are proportional to $h^{4/3}$ and the H^1-norm errors are proportional to $h^{2/3}$, which is in good agreement with our theoretical analysis.

To verify the first order convergence of the scheme in the temporal direction, we test the scheme on a very fine uniform mesh ($M = 128$) with $\tau = 0.1$, 0.05, 0.025. We present the L^2-norm errors in Table 2. One can also observe

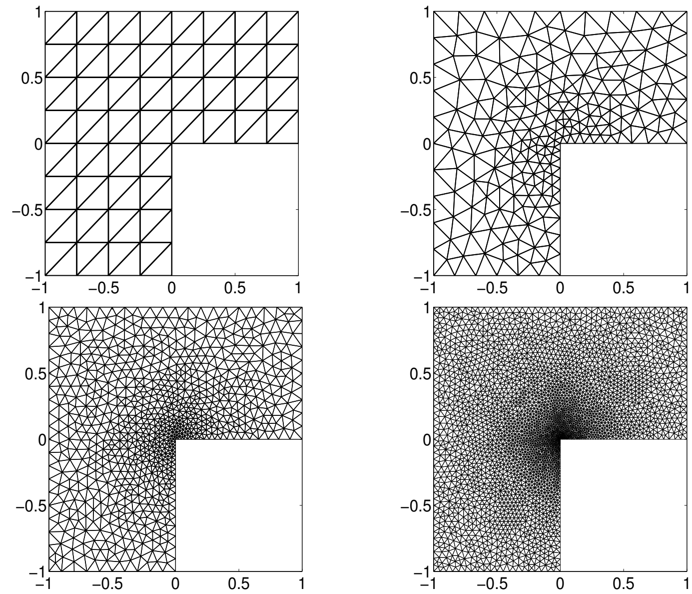

Fig. 2 Top Left: the uniform mesh with $M = 4$. Top Right: the first mesh with 227 nodes and 400 elements. Bottom Left: the second mesh with 851 nodes and 1600 elements. Bottom Right: the third mesh with 3402 nodes and 6579 elements.

Table 1 L^2 and H^1 errors of the linear FEM on uniform meshes.

	$\|U_h^N - u(\cdot,1)\|_{L^2}$	$\|U_h^N - u(\cdot,1)\|_{H^1}$	$\|\Phi_h^N - \phi(\cdot,1)\|_{L^2}$	$\|\Phi_h^N - \phi(\cdot,1)\|_{H^1}$
M=4	9.4486e-02	1.5123e+00	1.6199e-02	1.3495e-01
M=8	3.7414e-02	9.5685e-01	6.5429e-03	8.5561e-02
M=16	1.5634e-02	6.0845e-01	2.5948e-03	5.4180e-02
M=32	6.5769e-03	3.8615e-01	1.0360e-03	3.4271e-02
order	1.28	0.66	1.32	0.66

that the L^2-norm errors for both u and ϕ are proportional to τ, which confirm our theoretical analysis again.

To show the unconditional convergence of the scheme, we use the linear FEM to solve (4.1)-(4.2) with three different time steps $\tau = 0.10,\ 0.05,\ 0.01$ on gradually refined meshes with $M = 4, 8, 16, 32, 64, 128$. The log-log plots of the L^2-norm errors against M are given in Figure 3. Based on our theoretical analysis, in this case,

$$\|U_h^N - u(\cdot, t_N)\|_{L^2} + \|\Phi_h^N - \psi(\cdot, t_N)\|_{L^2} = O(\tau + h^{4/3})\,.$$

Table 2 L^2 errors on the uniform mesh $M = 128$.

	$\|U_h^N - u(\cdot,1)\|_{L^2}$	$\|\Phi_h^N - \phi(\cdot,1)\|_{L^2}$
$\tau = 0.1$	1.1524e-01	3.9441e-03
$\tau = 0.05$	6.0485e-02	1.9597e-03
$\tau = 0.025$	3.0792e-02	1.0316e-03
order	0.95	0.96

We can observe from Figure 3 that for a fixed τ, the L^2-norm errors asymptotically converge to a constant as $h \to 0$, *i.e.* the temporal error of the order $O(\tau)$. This shows clearly that the proposed scheme is unconditionally stable and no time-step condition is needed.

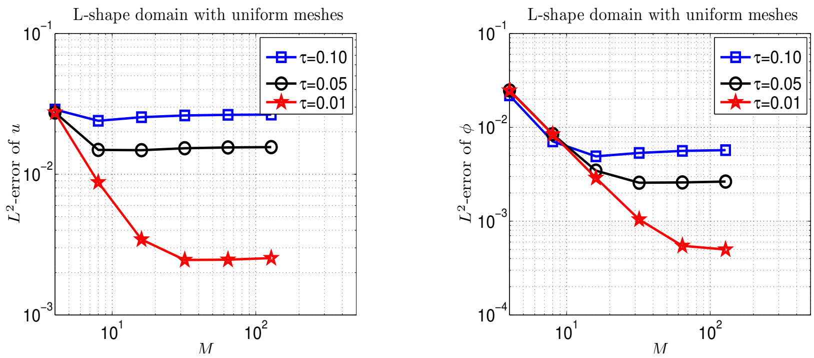

Fig. 3 L^2-norm errors of the linear FEM on uniform meshes.

Clearly the regularity of the exact solution near the L-shape corner is not enough to get a second order convergence for the linear FEM in a uniform mesh. It has been noted that a local refinement may improve further the convergence rate. Here we test our scheme with locally refined meshes, although our analysis was given only for a quasi-uniform mesh. We present three non-uniform meshes in Figure 2 with a finer mesh distribution around the L-shape corner. These meshes are generated by the software Gmsh [19] with a specified element-size parameter at the corners of the polygon and a local refinement (smoothing) technique in FeniCS. In order to test the convergence rate in the spatial direction we set $\tau = 1/M^2$ with $M = 8, 16, 32$, respectively and we present the L^2 and H^1-norm errors in Table 3 and also plot in Figure 4 the errors against the number of total nodal points N in the log-log form. One can observe that the L^2-norm errors of u and ϕ are in the second order (proportional to N) and the H^1-norm errors are in the first order (proportional to $\sqrt{N}$).

Table 3 L^2 and H^1 errors of the linear FEM on adaptive meshes.

node number	$\|U_h^N - u(\cdot,1)\|_{L^2}$	$\|U_h^N - u(\cdot,1)\|_{H^1}$	$\|\Phi_h^N - \phi(\cdot,1)\|_{L^2}$	$\|\Phi_h^N - \phi(\cdot,1)\|_{H^1}$
227	6.7106e-02	6.4475e-01	3.9879e-03	5.3611e-02
851	1.7649e-02	3.2107e-01	1.0739e-03	2.7922e-02
3402	4.4707e-03	1.5756e-01	2.7398e-04	1.3878e-02
order	1.9539	1.0164	1.9317	0.9749

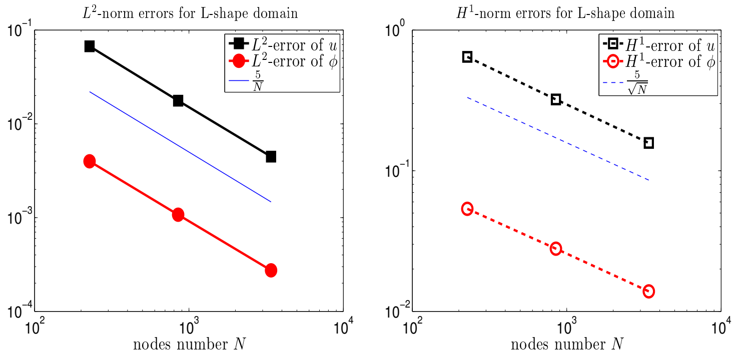

Fig. 4 L^2 and H^1-norm errors of the linear FEM on three locally refined meshes.

5 Conclusions

Clearly, analyses for nonlinear parabolic equations in a nonconvex polygon are very limited and previous efforts for the thermistor equations were restricted only on smooth domains, which produces a smooth solution. In this paper, we have established unconditional stability and optimal error estimates of a fully discrete Galerkin FEM for the time-dependent nonlinear thermistor equations in a nonconvex polygon. Our work is based on the error splitting technique proposed in [30,31], together with rigorous analysis for the nonlinear thermistor equations and the corresponding iterated time-discrete parabolic system. In these analyses, we have proved that the numerical solutions U_h^n in L^∞ and Φ_h^n in $W^{1,4}$ are bounded uniformly, with which the optimal error estimate can be established in a traditional way. Numerical results confirm our analysis and show the unconditional stability of the numerical method.

It is noted that for a uniform mesh (or quasi-uniform mesh), the singularity at the corner pollutes the global solution of finite element methods. The optimal L^2-error bound with a uniform mesh is only proportional to h^{2s}. For a fixed corner point, the accuracy may be improved by some local refinement techniques, see [2,11,33] for some linear models. Our numerical results show that the linear FE approximation with certain locally refined meshes may give the convergence rate $O(h^2)$ for the nonlinear thermistor equations. It is possible to extend our analysis to the problem with local refinement around the nonconvex corner. However, the analysis of linearized schemes with such locally refined meshes is more difficult.

The optimal error estimates in Theorem 1 are established for the thermistor problem based on the natural regularity of the solution in a two-dimensional nonconvex polygon. The corresponding analysis for the thermistor problem in a three-dimensional nonconvex polyhedron remains open.

Appendix — Proof of Lemma 1

The following lemmas are consequences of [24] (which can be extended to Hölder continuous coefficients via a basic perturbation argument) and [41].

Lemma A.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^2$. Suppose that $\sigma(u)$ is Hölder continuous and satisfies (1.5). Then there exists a positive constant $p_1 > 4$ (depending on the domain Ω) such that the solution v of the equation*

$$\begin{cases} -\nabla\cdot(\sigma(u)\nabla v) = \nabla\cdot\mathbf{b} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies that

$$\|v\|_{W^{1,q}} \le C_q\|\mathbf{b}\|_{L^q} \quad \text{for } p_1/(p_1-1) \le q \le p_1.$$

Lemma A.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^2$. Then the solution of the inhomogeneous heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \nabla\cdot f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) & \text{for } x\in\Omega, \end{cases}$$

satisfies that

$$\|u\|_{L^2((0,T);W^{1,4})} \le C\sum_{j=1}^{2}\|f_j\|_{L^2((0,T);L^4)} + C\|u_0\|_{H^{1/2}}.$$

From [7] we know that the regularity of g given in Lemma 1 implies that g can be extended to the interior of the domain Ω with $g\in L^\infty((0,T);H^{1+\beta})$ and $g_t\in L^\infty((0,T);W^{1,4})$.

Based on Yuan and Liu's results [44,45], the solution of (1.2)-(1.4) satisfies that

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C. \tag{A.1}$$

By Lemma A.1, the equation (1.3) with the Hölder continuity of u implies that

$$\|\phi\|_{L^\infty((0,T);W^{1,p_1})} \le C\|g\|_{L^\infty((0,T);W^{1,p_1})} \le C, \quad \text{for some } p_1 > 4. \tag{A.2}$$

Then (1.2) implies that

$$\|u_t\|_{L^2((0,T);L^2)} \le C\|\sigma(u)|\nabla\phi|^2\|_{L^2((0,T);L^2)} + C\|u_0\|_{H^1} \le C. \tag{A.3}$$

Let $w = u_t$. Differentiating (1.2)-(1.3) with respect to t, we obtain

$$\partial_t w - \Delta w = \nabla\cdot(\sigma'(u)w\phi\nabla\phi) + \nabla\cdot(\sigma(u)\phi_t\nabla\phi) + \nabla\cdot(\sigma(u)\phi\nabla\phi_t), \tag{A.4}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi_t) = \nabla\cdot(\sigma'(u)w\nabla\phi). \tag{A.5}$$

with the initial condition $w(x,0) = w_0(x)$, where $w_0 = \Delta u_0 + \sigma(u_0)|\nabla\phi_0|^2$ and ϕ_0 is the solution of the elliptic PDE

$$\begin{cases} -\nabla\cdot(\sigma(u_0)\nabla\phi_0) = 0, \text{ in } \ \Omega \\ \phi_0(x) = g_0 \qquad\qquad \text{for } \ x\in\partial\Omega. \end{cases}$$

It follows that $w_0 \in H_0^1$. Since $w \in L^2((0,T);L^2)$ and $\nabla\phi \in L^\infty((0,T);L^4)$, applying Lemma A.1 to (A.5) gives

$$\begin{aligned}\|\phi_t\|_{L^2((0,T);W^{1,4/3})} &\le C\|w\nabla\phi\|_{L^2((0,T);L^{4/3})} + C\|g_t\|_{L^2((0,T);W^{1,4/3})} \\ &\le C\|w\|_{L^2((0,T);L^2)}\|\nabla\phi\|_{L^2((0,T);L^4)} + C\|g_t\|_{L^2((0,T);W^{1,4/3})} \\ &\le C. \end{aligned} \tag{A.6}$$

and by the Sobolev embedding theorem, we obtain,

$$\|\phi_t\|_{L^2((0,T);L^4)} \le C\|\phi_t\|_{L^2((0,T);W^{1,4/3})} \le C. \tag{A.7}$$

Again applying Lemma A.2 to (A.4) shows

$$\begin{aligned}\|w\|_{L^2((0,T);W^{1,4/3})} &\le C\|w\nabla\phi\|_{L^2((0,T);L^{4/3})} + C\|\phi_t\nabla\phi\|_{L^2((0,T);L^{4/3})} \\ &\quad + C\|\nabla\phi_t\|_{L^2((0,T);L^{4/3})} + \|w_0\|_{H^{1/2}} \\ &\le C \end{aligned} \tag{A.8}$$

and with the Sobolev embedding theorem, we have

$$\|w\|_{L^2((0,T);L^4)} \le C\|w\|_{L^2((0,T);W^{1,4/3})} \le C. \tag{A.9}$$

From (A.5) and (A.4), we see that

$$\begin{aligned}\|\phi_t\|_{L^2((0,T);H^1)} &\le C\|w\nabla\phi\|_{L^2((0,T);L^2)} + C\|g_t\|_{L^2((0,T);H^1)} \\ &\le C\|w\|_{L^2((0,T);L^4)}\|\nabla\phi\|_{L^\infty((0,T);L^4)} + C\|g_t\|_{L^2((0,T);H^1)} \\ &\le C \end{aligned} \tag{A.10}$$

and

$$\begin{aligned}\|w\|_{L^2((0,T);H^1)} &\le C\|w\nabla\phi\|_{L^2((0,T);L^2)} + C\|\phi_t\nabla\phi\|_{L^2((0,T);L^2)} \\ &\quad + \|\nabla\phi_t\|_{L^2((0,T);L^2)} + \|w_0\|_{L^2}) \\ &\le C. \end{aligned}$$

By the Sobolev embedding theorem, we have further

$$\begin{aligned}&\|w\|_{L^2((0,T);L^{p_3})} + \|\phi_t\|_{L^2((0,T);L^{p_3})} \\ &\le C(\|w\|_{L^2((0,T);H^1)} + \|\phi_t\|_{L^2((0,T);H^1)}) \le C,\end{aligned} \tag{A.11}$$

where p_3 is determined by $1/p_3 + 1/p_1 = 1/4$. Then

$$\|w\nabla\phi\|_{L^2((0,T);L^4)} \le C\|w\|_{L^2((0,T);L^{p_3})}\|\nabla\phi\|_{L^\infty((0,T);L^{p_1})} \le C, \tag{A.12}$$

$$\|\phi_t\nabla\phi\|_{L^2((0,T);L^4)} \le C\|\phi_t\|_{L^2((0,T);L^{p_3})}\|\nabla\phi\|_{L^\infty((0,T);L^{p_1})} \le C\,. \tag{A.13}$$

Moreover, by applying Lemma A.1 to the equation (A.5), we see that

$$\|\phi_t\|_{L^2((0,T);W^{1,4})} \le C\|w\nabla\phi\|_{L^2((0,T);L^4)} + C\|g_t\|_{L^2((0,T);W^{1,4})} \le C \tag{A.14}$$

and by Lemma A.2, (A.4) implies that

$$\begin{aligned}\|w\|_{L^2((0,T);W^{1,4})} &\le C\|w\nabla\phi\|_{L^2((0,T);L^4)} + C\|\phi_t\nabla\phi\|_{L^2((0,T);L^4)} \\ &\quad + C\|\nabla\phi_t\|_{L^2((0,T);L^4)} + C\|w_0\|_{H^{1/2}}) \\ &\le C.\end{aligned} \tag{A.15}$$

Using the Sobolev embedding theorem again, we arrive at

$$\|w\|_{L^2((0,T);L^\infty)} \le C\|w\|_{L^2((0,T);W^{1,4})} \le C. \tag{A.16}$$

Differentiating (1.2) with respect to t, we get

$$\partial_t w - \Delta w = \sigma'(u)w|\nabla\phi|^2 + 2\sigma(u)\nabla\phi\cdot\nabla\phi_t. \tag{A.17}$$

and the above equation times Δw gives

$$\begin{aligned}&\|w\|_{L^\infty((0,T);H^1)} + \|\partial_t w\|_{L^2((0,T);L^2)} + \|\Delta w\|_{L^2((0,T);L^2)} \\ &\le C(\|w|\nabla\phi|^2\|_{L^2((0,T);L^2)} + \|\nabla\phi\cdot\nabla\phi_t\|_{L^2((0,T);L^2)} + \|w_0\|_{H^1}) \\ &\le C,\end{aligned}$$

which further shows that

$$\|w\|_{L^\infty((0,T);L^{p_3})} + \|u_{tt}\|_{L^2((0,T);L^2)} + \|u_t\|_{L^2((0,T);H^{1+s})} \le C. \tag{A.18}$$

It follows that

$$\|u\|_{C([0,T];H^{1+s})} \le C\|u\|_{L^2((0,T);H^{1+s})} + C\|u_t\|_{L^2((0,T);H^{1+s})} \le C. \tag{A.19}$$

From (A.5) we see that

$$\begin{aligned}\|\phi_t\|_{L^\infty((0,T);W^{1,4})} &\le C\|w\nabla\phi\|_{L^\infty((0,T);L^4)} + C\|g_t\|_{L^\infty((0,T);W^{1,4})} \\ &\le C.\end{aligned} \tag{A.20}$$

Furthermore, we rewrite (1.3) by

$$-\Delta\phi = \frac{\sigma'(u)}{\sigma(u)}\nabla u\cdot\nabla\phi. \tag{A.21}$$

With (A.19), we obtain

$$\begin{aligned}\|\phi\|_{L^\infty((0,T);H^{1+s})} &\le C\|\Delta\phi\|_{L^\infty((0,T);L^2)} + C\|g\|_{L^\infty((0,T);H^2)}\\ &\le C\|\nabla u\cdot\nabla\phi\|_{L^\infty((0,T);L^2)} + C \le C_2. \end{aligned} \tag{A.22}$$

Since $H^{1+s_1} \hookrightarrow\hookrightarrow H^{1+s} \hookrightarrow W^{1,4}$, we have the following estimate (see Lemma 1.1, pp. 106 of [35])

$$\begin{aligned}&\|\phi(\cdot,t_1)-\phi(\cdot,t_2)\|_{H^{1+s}}\\ &\le \frac{1}{2}C_2^{-1}\epsilon\|\phi(\cdot,t_1)-\phi(\cdot,t_2)\|_{H^{1+s_1}} + C_\epsilon\|\phi(\cdot,t_1)-\phi(\cdot,t_2)\|_{W^{1,4}}\\ &\le \epsilon + C_\epsilon|t_1-t_2|\|\phi_t\|_{L^\infty((0,T);W^{1,4})}\\ &\le \epsilon + C_\epsilon|t_1-t_2|, \qquad \forall\ \epsilon>0,\end{aligned}$$

which implies that $\phi \in C([0,T];H^{1+s})$.

The proof of Lemma 1 is completed. ∎

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