Global well-posedness of the time-dependent Ginzburg–Landau superconductivity model in curved polyhedra [*]

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Abstract

We prove global existence and uniqueness of weak solutions for the time-dependent Ginzburg–Landau equations in a three-dimensional curved polyhedron which is not necessarily convex, where the gradient of the magnetic potential may not be square integrable. Preceding analyses all required the gradient of the solution to be square integrable, which is only true in convex or smooth domains.

Key words. superconductivity, curved polyhedron, corner, singularity, well-posedness

1   Introduction

The Ginzburg–Landau theory has been widely accepted to describe macroscopic superconductivity phenomena in superconductors. Based on the Ginzburg–Landau theory [15] and its extension to the time-dependent model [17], the macroscopic state of a superconductor can be described by a complex-valued order parameter ψ whose modulus represents the superconductivity density, and a real vector-valued magnetic potential $\mathbf{A}$ whose curl represents the magnetic induction field. If a superconductor occupies a domain Ω, then ψ and $\mathbf{A}$ are governed by the time-dependent Ginzburg–Landau equations (TDGL) which, in a nondimensionalization form, can be written as

$$\eta\frac{\partial\psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + (|\psi|^2 - 1)\psi + i\eta\kappa\psi\phi = 0, \tag{1.1}$$

$$\frac{\partial\mathbf{A}}{\partial t} + \nabla\times(\nabla\times\mathbf{A}) + \nabla\phi + \mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right] = \nabla\times\mathbf{H}, \tag{1.2}$$

where η and κ are physical constants and $\mathbf{H}$ is the external magnetic field, ψ^* denotes the complex conjugate of ψ, and the electric potential ϕ is an unknown real scalar-valued function. Physically there holds $0 \leq |\psi| \leq 1$, where $|\psi| = 0$ and $|\psi| = 1$ correspond to the normal state and superconducting state, respectively, and $0 < |\psi| < 1$ represents a mixed state. More detailed discussion of the physics can be found in [3, 8, 14, 26]. Clearly, the three unknown

variables ψ, $\mathbf{A}$ and ϕ cannot be determined uniquely by the two equations above. Thus an additional gauge condition is required for the well-posedness.

Under the Lorentz gauge $\phi = -\nabla \cdot \mathbf{A}$, (1.1)-(1.2) reduce to

$$\eta \frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right)^2 \psi + (|\psi|^2 - 1)\psi - i\eta\kappa\psi\nabla \cdot \mathbf{A} = 0, \tag{1.3}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \nabla \times (\nabla \times \mathbf{A}) - \nabla(\nabla \cdot \mathbf{A}) + \mathrm{Re}\left[\psi^* \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right] = \nabla \times \mathbf{H}, \tag{1.4}$$

with the natural boundary and initial conditions

$$\nabla\psi \cdot \mathbf{n} = 0, \quad \mathbf{A} \cdot \mathbf{n} = 0, \quad (\nabla \times \mathbf{A}) \times \mathbf{n} = \mathbf{H} \times \mathbf{n}, \quad \text{on} \ \ \partial\Omega \times (0, T), \tag{1.5}$$

$$\psi(x, 0) = \psi_0(x), \quad \mathbf{A}(x, 0) = \mathbf{A}_0(x), \quad \text{in} \ \ \Omega, \tag{1.6}$$

where $\mathbf{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$. Existence and uniqueness of solutions of (1.3)-(1.6) have been proved in [6] in smooth domains, and the long time behavior of the solutions and vortex dynamics have been studied in [25]. Numerical approximations of (1.3)-(1.6) and their convergence have also been studied in many works, e.g., [4, 5, 13]. In both the theoretical and numerical analyses, the proofs presented in these works have used the equivalence relation $\|\nabla \times \mathbf{A}\|_{\mathbf{L}^2}^2 + \|\nabla \cdot \mathbf{A}\|_{L^2}^2 \sim \|\nabla \mathbf{A}\|_{\mathbf{L}^2}^2$, which holds for any $\mathbf{A} \in H^1(\Omega)^3$ such that $\mathbf{A} \cdot \mathbf{n} = 0$ on $\partial\Omega$. The previous analyses based on this equivalence relation showed that $\mathbf{A} \in L^\infty(0, T; H^1(\Omega)^3) \hookrightarrow L^\infty(0, T; L^6(\Omega)^3)$, and this regularity guarantees uniqueness of the solution. However, when the domain contains edges or corners whose interior angles are larger than $180°$, the equivalence relation no longer holds and $\mathbf{A}$ does not have the regularity above. Thus the preceding analyses cannot be extended to nonconvex curved polyhedra directly. Since numerical simulations of the TDGL in nonsmooth domains are important for physicists to study the surface effects in superconductivity [24, 28], it is important to know whether the TDGL is well-posed in such nonsmooth domains and what is the natural regularity of the solution.

Other gauges have also been studied in literature, which are all equivalent in producing the physical quantities $|\psi|$ and $\nabla \times \mathbf{A}$. In particular, the zero electric potential gauge $\phi = 0$ is widely used by physicists [12, 16, 20, 30]. Under this gauge, existence and uniqueness of solutions have been proved in [9] when the superconductor occupies a smooth domain. Finite element approximations of the equations and its convergence have also been studied in many works, e.g., [10, 22, 23]. Existence and uniqueness of weak solutions have also been proved in [27] under the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ (also in smooth domains), which is less used in numerical simulations as it increases the computational cost. Well-posedness of the TDGL under these gauges also remains open in general curved polyhedra.

In this paper, we prove existence and uniqueness of weak solutions for the initial-boundary value problem (1.3)-(1.6) in a general three-dimensional curved polyhedron where the magnetic potential $\mathbf{A}$ may not be in $L^2(0, T; H^1(\Omega)^3)$, based on the weaker embedding inequality $\|\mathbf{A}\|_{\mathbf{L}^{3+\delta}(\Omega)} \leq C\|\mathbf{A}\|_{\mathbf{H}_\mathbf{n}(\mathrm{curl,div})}$. Due to the low regularity of the solution in general curved polyhedra, we construct approximating solutions which preserve the physical property $0 \leq |\psi| \leq 1$ in order to prove the convergence of the approximating solutions.

We would like to mention that under the natural regularity we proved in this paper many standard numerical methods, such as the widely used continuous Galerkin finite element method, may produce numerical solutions which do not converge to the true solution.

Numerical approximation of (1.3)-(1.6) in general three-dimensional curved polyhedra is challenging under either gauge. The current paper provides a theoretical foundation for numerical analysis of the TDGL in such domains.

2 Main results

Let Ω be a curved polyhedron, i.e. a bounded Lipschitz domain with piecewise smooth boundary such that any boundary point is contained in a neighborhood which is C^∞-diffeomorphic to a neighborhood of a boundary point of a polyhedron (unlike a polyhedron, we allow a curved polyhedron to contain holes); see section 4 of [2]. For any nonnegative integer k, we denote by $H^k(\Omega)$ and $\mathcal{H}^k(\Omega)$ the conventional Sobolev spaces of real-valued and complex-valued functions defined in Ω, respectively, with $L^2(\Omega) = H^0(\Omega)$ and $\mathcal{L}^2(\Omega) = \mathcal{H}^0(\Omega)$; see [1]. To simplify the notations, we denote $H^k = H^k(\Omega)$, $\mathcal{H}^k = \mathcal{H}^k(\Omega)$, $L^p = L^p(\Omega)$, $\mathcal{L}^p = \mathcal{L}^p(\Omega)$ and $\mathbf{L}^p := L^p(\Omega)^3$ for $1 \le p \le \infty$, and we define the function spaces

$$\mathbf{H}_{\mathrm{n}}^1(\Omega) := \{\mathbf{a} \in H^1(\Omega)^3 : \mathbf{a} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \mathbf{H}(\mathrm{curl}) := \{\mathbf{a} \in \mathbf{L}^2 : \nabla \times \mathbf{a} \in L^2\},$$

$$\mathbf{H}_{\mathrm{n}}(\mathrm{div}) := \{\mathbf{a} \in \mathbf{L}^2 : \nabla \cdot \mathbf{a} \in L^2 \text{ and } \mathbf{a} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}) := \{\mathbf{a} \in \mathbf{L}^2 : \nabla \times \mathbf{a} \in \mathbf{L}^2, \ \nabla \cdot \mathbf{a} \in L^2 \text{ and } \mathbf{a} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

where $C^\infty(\overline{\Omega})$ is dense in $\mathbf{H}_{\mathrm{n}}(\mathrm{div})$ but may not be dense in $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$ (depending on the convexity of the domain). For any two functions $f, g \in \mathcal{L}^2$ we define $(f, g) = \int_\Omega f(x)g(x)^* \, \mathrm{d}x$, and for any two vector fields $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2$ we define $(\mathbf{f}, \mathbf{g}) = \int_\Omega \mathbf{f}(x) \cdot \mathbf{g}(x)^* \, \mathrm{d}x$.

Definition 2.1. (Weak solution) *The pair $(\psi, \mathbf{A})$ is called a weak solution of (1.3)-(1.6) if*

$$\psi \in C([0, T]; \mathcal{L}^2) \cap L^\infty(0, T; \mathcal{H}^1), \quad \partial_t \psi \in L^2(0, T; \mathcal{L}^2), \quad |\psi| \le 1 \ \ a.e. \ in \ \Omega \times (0, T),$$

$$\mathbf{A} \in C([0, T]; \mathbf{L}^2) \cap L^\infty(0, T; \mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})),$$

$$\partial_t \mathbf{A} \in L^2(0, T; \mathbf{L}^2), \quad \nabla \times \mathbf{A} \in L^2(0, T; \mathbf{H}(\mathrm{curl})), \quad \nabla \cdot \mathbf{A} \in L^2(0, T; H^1),$$

with $\psi(\cdot, 0) = \psi_0$, $\mathbf{A}(\cdot, 0) = \mathbf{A}_0$, and the variational equations

$$\int_0^T \left[\left(\eta \frac{\partial \psi}{\partial t}, \varphi \right) + \left(\left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi, \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \varphi \right) \right] \mathrm{d}t$$

$$+ \int_0^T \left[\left((|\psi|^2 - 1)\psi - i\eta\kappa\psi\nabla \cdot \mathbf{A}, \varphi \right) \right] \mathrm{d}t = 0, \tag{2.1}$$

$$\int_0^T \left[\left(\frac{\partial \mathbf{A}}{\partial t}, \mathbf{a} \right) + (\nabla \times \mathbf{A}, \nabla \times \mathbf{a}) + (\nabla \cdot \mathbf{A}, \nabla \cdot \mathbf{a}) \right] \mathrm{d}t$$

$$= \int_0^T \left[(\mathbf{H}, \nabla \times \mathbf{a}) - \left(\mathrm{Re} \left[\psi^* \left(\frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right], \mathbf{a} \right) \right] \mathrm{d}t, \tag{2.2}$$

hold for all $\varphi \in L^2(0, T; \mathcal{H}^1)$ and $\mathbf{a} \in L^2(0, T; \mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}))$.

The main result of this paper is the following theorem, whose proof is presented in the next section.

Theorem 2.1. (Global well-posedness of the TDGL in a curved polyhedron)
For any given $T > 0$, if $\psi_0 \in \mathcal{H}^1$, $|\psi_0| \leq 1$ a.e. in Ω, $\mathbf{A}_0 \in \mathbf{H}_n(\mathrm{curl}, \mathrm{div})$ and the external magnetic field $\mathbf{H}$ is a divergence-free vector field such that $\mathbf{H} \in L^\infty(0, T; \mathbf{L}^2) \cap L^2(0, T; \mathbf{H}(\mathrm{curl}))$, then the problem (1.3)-(1.6) has a unique weak solution in the sense of Definition 2.1.

3   Proof of Theorem 2.1

In this section, we prove existence of a weak solution for the TDGL by constructing approximating solutions and applying a compactness argument. Uniqueness is then proved based on the regularity of the weak solution. To simplify the notations, we denote by $C_{p_1, \cdots, p_l}$ a generic positive constant which may be different at each occurrence, depending on the parameters $p_1, \cdots, p_l$, and denote by ϵ a small generic positive constant.

3.1   Some preliminary lemmas

Here we present some preliminary lemmas which will be used in the next subsection in proving existence of weak solutions. Lemma 3.1 is concerned with the embedding of $\mathbf{H}_n(\mathrm{curl}, \mathrm{div})$ into $\mathbf{L}^{3+\delta}$, which is used into control the magnetic potential, and Lemma 3.2 is used to control the order parameter pointwisely.

Lemma 3.1. *There exists a positive constant δ (which depends on the domain Ω) such that $\mathbf{H}_n(\mathrm{curl}, \mathrm{div}) \hookrightarrow \mathbf{L}^{3+\delta}$.*

Proof.   It is proved in [2] that any $\mathbf{u} \in \mathbf{H}_n(\mathrm{curl}, \mathrm{div})$ admits a decomposition $\mathbf{u} = \mathbf{v} + \nabla\varphi$, where $\varphi \in H^1, \Delta\varphi \in L^2$, and $\partial_n\varphi = 0$ on $\partial\Omega$, and

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|\nabla\varphi\|_{\mathbf{L}^2} + \|\Delta\varphi\|_{L^2} \leq C\|\mathbf{u}\|_{\mathbf{H}_n(\mathrm{curl},\mathrm{div})}.$$

From [18] we know that

$$\|\nabla\varphi\|_{\mathbf{L}^{3+\delta}} \leq C\|\Delta\varphi\|_{L^2}$$

in any bounded Lipschitz domain Ω, where δ depends on the domain. It follows that

$$\|\mathbf{u}\|_{\mathbf{L}^{3+\delta}} \leq \|\mathbf{v}\|_{\mathbf{L}^{3+\delta}} + \|\nabla\varphi\|_{\mathbf{L}^{3+\delta}} \leq C\|\mathbf{v}\|_{\mathbf{H}^1} + C\|\Delta\varphi\|_{L^2} \leq C\|\mathbf{u}\|_{\mathbf{H}_n(\mathrm{curl},\mathrm{div})}. \qquad \square$$

Lemma 3.2. *For any given $\mathbf{A} \in L^\infty(0, T; \mathbf{H}_n(\mathrm{curl}, \mathrm{div}))$, the nonlinear equation (1.3) has a unique weak solution $\psi \in L^\infty(0, T; \mathcal{L}^\infty) \cap L^2(0, T; \mathcal{H}^1) \cap H^1(0, T; (\mathcal{H}^1)')$ in the sense of (2.1). Moreover, the solution satisfies that $|\psi| \leq 1$ a.e. in $\Omega \times (0, T)$.*

Proof.   It suffices to construct approximating solutions which preserve the pointwise estimate

$$|\psi| \leq 1 \text{ a.e. in } \Omega \times (0, T). \tag{3.1}$$

Since $\mathbf{H}_n(\mathrm{curl}, \mathrm{div}) \hookrightarrow \mathbf{L}^{3+\delta}$, it follows that $\mathbf{A} \in L^\infty(0, T; \mathbf{L}^{3+\delta})$. Let $\mathbf{A}_m$, $m = 1, 2, \cdots$, be a sequence of smooth functions which converges to $\mathbf{A}$ in $\mathbf{L}^{3+\delta} \cap \mathbf{H}_n(\mathrm{div})$. Since $\mathbf{A}_m$ is smooth for any fixed m. By the theory of parabolic equations [11], the equation

$$\eta\frac{\partial\psi_m}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}_m\right)^2\psi_m + (|\psi_m|^2 - 1)\psi_m - i\eta\kappa\psi_m\nabla \cdot \mathbf{A}_m = 0 \tag{3.2}$$

has a unique weak solution $\psi_m \in L^\infty(0, T; H^1 \cap L^\infty) \cap H^1(0, T; L^2)$ in the sense of (2.1). Let $(|\psi_m|^2 - 1)_+$ denote the positive part of $|\psi_m|^2 - 1$ and integrate this equation against $\psi_m^*(|\psi_m|^2 - 1)_+$. By considering the real part of the result, for any $t' \in (0, T)$ we derive

$$
\int_\Omega \left(\frac{\eta}{4} (|\psi_m(x, t')|^2 - 1)_+^2 \right) dx + \int_0^{t'} \int_\Omega (|\psi_m|^2 - 1)_+^2 |\psi_m|^2 \, dx \, dt
$$

$$
= -\int_0^{t'} \operatorname{Re} \int_\Omega \left(\frac{i}{\kappa} \nabla \psi_m + \mathbf{A}_m \psi \right) \left(-\frac{i}{\kappa} \nabla + \mathbf{A}_m \right) [\psi_m^*(|\psi_m|^2 - 1)_+] \, dx \, dt
$$

$$
= -\int_0^{t'} \int_\Omega \left| \frac{i}{\kappa} \nabla \psi_m + \mathbf{A}_m \psi \right|^2 (|\psi_m|^2 - 1)_+ \, dx \, dt
$$

$$
+ \int_0^{t'} \operatorname{Re} \int_{\{|\psi_m|^2 > 1\}} \left(\frac{i}{\kappa} \nabla \psi_m + \mathbf{A}_m \psi \right) \psi_m^* \left(\frac{i}{\kappa} \psi_m \nabla \psi_m^* + \frac{i}{\kappa} \psi_m^* \nabla \psi_m \right) \, dx \, dt
$$

$$
= -\int_0^{t'} \int_\Omega \left| \frac{i}{\kappa} \nabla \psi_m + \mathbf{A}_m \psi_m \right|^2 (|\psi_m|^2 - 1)_+ \, dx \, dt
$$

$$
- \int_0^{t'} \operatorname{Re} \int_{\{|\psi_m|^2 > 1\}} (|\psi_m|^2 |\nabla \psi_m|^2 + (\psi_m^*)^2 \nabla \psi_m \cdot \nabla \psi_m) \, dx \, dt
$$

$$
\leq 0,
$$

which implies that $\int_\Omega (|\psi_m(x, t')|^2 - 1)_+^2 \, dx = 0$. Thus $|\psi_m| \leq 1$ a.e. in $\Omega \times (0, T)$.

Integrating (3.2) against ψ_m^* and considering the real part of the result, we derive that

$$
\frac{d}{dt} \left(\frac{\eta}{2} \|\psi_m\|_{L^2}^2 \right) + \left\| i\kappa^{-1} \nabla \psi_m + \mathbf{A}_m \psi_m \right\|_{L^2}^2 + \|\psi_m\|_{L^4}^4 = \|\psi_m\|_{L^2}^2 \leq C,
$$

which further implies

$$
\|\psi_m\|_{L^2(0,T;\mathcal{H}^1)} \leq C, \tag{3.3}
$$

where C is independent of m. From (3.2) we also see that

$$
\left| \int_0^T (\partial_t \psi_m, \varphi) \, dt \right| = \left| \int_0^T \left[-\frac{1}{\kappa^2} (\nabla \psi_m, \nabla \varphi) - \frac{i}{\kappa^2} (\nabla \psi_m, \mathbf{A}_m \varphi) + \frac{i}{\kappa^2} (\mathbf{A}_m \psi_m, \nabla \varphi) \right. \right.
$$

$$
\left. \left. - (|\mathbf{A}_m|^2 \psi_m, \varphi) - ((|\psi_m|^2 - 1)\psi_m, \varphi) + (i\eta\kappa\psi_m \nabla \cdot \mathbf{A}_m, \varphi) \right] dt \right|
$$

$$
\leq C\|\psi_m\|_{L^2(0,T;\mathcal{H}^1)} \|\varphi\|_{L^2(0,T;\mathcal{H}^1)}
$$

$$
+ C\|\psi_m\|_{L^2(0,T;\mathcal{H}^1)} \|\mathbf{A}_m\|_{L^\infty(0,T;\mathbf{L}^3)} \|\varphi\|_{L^2(0,T;\mathcal{L}^6)}
$$

$$
+ C\|\psi_m\|_{L^2(0,T;\mathcal{L}^6)} \|\mathbf{A}_m\|_{L^\infty(0,T;\mathbf{L}^3)} \|\varphi\|_{L^2(0,T;\mathcal{H}^1)}
$$

$$
+ \|\mathbf{A}_m\|_{L^\infty(0,T;\mathbf{L}^3)}^2 \|\psi_m\|_{L^2(0,T;\mathcal{L}^6)} \|\varphi\|_{L^2(0,T;\mathcal{L}^6)}
$$

$$
+ C\|(|\psi_m|^2 - 1)\psi_m\|_{L^\infty(0,T;\mathcal{L}^\infty)} \|\varphi\|_{L^1(0,T;\mathcal{L}^1)}
$$

$$
+ C\|\nabla \cdot \mathbf{A}_m\|_{L^\infty(0,T;L^2)} \|\varphi\|_{L^1(0,T;\mathcal{L}^2)}
$$

$$
\leq C\|\varphi\|_{L^2(0,T;\mathcal{H}^1)},
$$

which implies (via a duality argument)

$$
\|\partial_t \psi_m\|_{L^2(0,T;(\mathcal{H}^1)')} \leq C, \tag{3.4}
$$

where C is independent of m. With the estimates (3.1)-(3.4), it is easy to prove that there exists $\psi \in L^{\infty}(0, T; \mathcal{L}^{\infty}) \cap L^2(0, T; \mathcal{H}^1) \cap H^1(0, T; (\mathcal{H}^1)')$ and a subsequence of ψ_m, $m = 1, 2, \cdots$, which converges to ψ weakly in $L^2(0, T; \mathcal{H}^1) \cap H^1(0, T; (\mathcal{H}^1)')$, strongly in $L^p(0, T; \mathcal{L}^p)$ for any $1 < p < \infty$, and convergence is pointwise a.e. in $\Omega \times (0, T)$. It is easy to see that ψ is a weak solution of (1.3), with $|\psi| \leq 1$ a.e. in $\Omega \times (0, T)$.

Uniqueness of the weak solution can be proved easily based on the regularity of ψ. $\qquad \square$

3.2 Construction of approximating solutions

In this subsection, we construct approximating solutions of (1.3)-(1.6) in semi-finite dimensional spaces.

Let $M : \mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}) \to (\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}))'$ be defined by

$$(M\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}), \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}).$$

It is easy to see that the bilinear form on the right-hand side is coercive on $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$, which is compactly embedded into $\mathbf{L}^2$ (see [29]). In this case, the spectrum of the operator M consists of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N, \cdots$, which tend to infinity, and the corresponding eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots$ form a Hilbert basis of $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$ (see Theorem 2.37 of [21]). Let $\mathbf{X}_N = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\}$, which is a finite dimensional subspace of $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$, equipped with the norm of $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$. We look for $\Psi_N(t) \in \mathcal{H}^1$ and $\mathbf{\Lambda}_N(t) \in \mathbf{X}_N$ such that

$$\left(\eta \frac{\partial \Psi_N}{\partial t}, \varphi\right) + \left(\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\Psi_N, \left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\varphi\right)$$
$$+ \left((|\Psi_N|^2 - 1)\Psi_N - i\eta\kappa\Psi_N\nabla \cdot \mathbf{\Lambda}_N, \varphi\right) = 0, \tag{3.5}$$

$$\left(\frac{\partial \mathbf{\Lambda}_N}{\partial t}, \mathbf{a}\right) + \left(\nabla \times \mathbf{\Lambda}_N, \nabla \times \mathbf{a}\right) + \left(\nabla \cdot \mathbf{\Lambda}_N, \nabla \cdot \mathbf{a}\right)$$
$$+ \left(\mathrm{Re}\left[\Psi_N^*\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\Psi_N\right], \mathbf{a}\right) = (\mathbf{H}, \nabla \times \mathbf{a}), \tag{3.6}$$

for any $\varphi \in \mathcal{H}^1$ and $\mathbf{a} \in \mathbf{X}_N$ at any $t \in (0, T)$, with the initial conditions $\Psi(0) = \psi_0 \in \mathcal{H}^1$ and $\mathbf{\Lambda}(0) = \Pi_N \mathbf{A}_0 \in \mathbf{H}_n(\mathrm{curl}, \mathrm{div})$, where Π_N is the projection of $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$ onto its subspace $\mathbf{X}_N$. We have the following lemma concerning the approximating solution $(\Psi_N, \mathbf{\Lambda}_N)$.

Lemma 3.3. *For a given positive integer N, the system (3.5)-(3.6) has a unique solution*

$$\Psi_N \in L^{\infty}(0, T; \mathcal{H}^1) \cap H^1(0, T; \mathcal{L}^2) \cap L^{\infty}(\Omega \times (0, T)), \quad |\Psi_N| \leq 1 \ \text{a.e. in } \ \Omega \times (0, T),$$
$$\mathbf{\Lambda}_N \in W^{1,\infty}(0, T; \mathbf{X}_N)$$

Moreover, we have the uniform estimate:

$$\|\Psi_N\|_{L^{\infty}(0,T;\mathcal{H}^1)} + \|\partial_t\Psi_N\|_{L^2(0,T;\mathcal{L}^2)} + \|\mathbf{\Lambda}_N\|_{L^{\infty}(0,T;\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))} + \|\partial_t\mathbf{\Lambda}_N\|_{L^2(0,T;L^2)} \leq C, \tag{3.7}$$

where the constant C does not depend on N.

6

To prove Lemma 3.4, we apply Schaefer's fixed point theorem [11]:

Lemma 3.4. (Schaefer's fixed point theorem) *Let $\mathcal{N} : L^\infty(0, T; \mathbf{X}_N) \to L^\infty(0, T; \mathbf{X}_N)$ be a continuous and compact mapping such that the set*

$$\mathbf{V}_N := \{\mathbf{\Lambda}_N^0 \in L^\infty(0, T; \mathbf{X}_N) : \exists\, s \in [0, 1] \text{ satisfying } \mathbf{\Lambda}_N^0 = s\mathcal{N}\mathbf{\Lambda}_N^0\} \qquad (3.8)$$

is bounded in $L^\infty(0, T; \mathbf{X}_N)$. Then the mapping $\mathcal{N}$ has a fixed point in $L^\infty(0, T; \mathbf{X}_N)$: there exists $\mathbf{\Lambda} \in L^\infty(0, T; \mathbf{X}_N)$ satisfying $\mathbf{\Lambda} = \mathcal{N}\mathbf{\Lambda}$.

Proof of Lemma 3.3. For a given $\mathbf{\Lambda}_N^0 \in L^\infty(0, T; \mathbf{X}_N)$, we define $\Psi_N \in L^\infty(0, T; L^\infty) \cap L^2(0, T; \mathcal{H}^1)$ as the unique weak solution of the equation

$$\left(\eta\frac{\partial \Psi_N}{\partial t}, \varphi\right) + \left(\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N^0\right)\Psi_N, \left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N^0\right)\varphi\right)$$
$$+ \left((|\Psi_N|^2 - 1)\Psi_N - i\eta\kappa\Psi_N\nabla \cdot \mathbf{\Lambda}_N^0, \varphi\right) = 0, \qquad \forall\, \varphi \in L^2(0, T; \mathcal{H}^1). \qquad (3.9)$$

Existence and uniqueness of the solution Ψ_N follow Lemma 3.2. Then we define $\mathbf{\Lambda}_N$ as the solution of

$$\left(\frac{\partial \mathbf{\Lambda}_N}{\partial t}, \mathbf{a}\right) + \left(\nabla \times \mathbf{\Lambda}_N, \nabla \times \mathbf{a}\right) + \left(\nabla \cdot \mathbf{\Lambda}_N, \nabla \cdot \mathbf{a}\right)$$
$$+ \left(\text{Re}\left[\Psi_N^*\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N^0\right)\Psi_N\right], \mathbf{a}\right) = \left(\mathbf{H}, \nabla \times \mathbf{a}\right), \qquad (3.10)$$

Since $\mathbf{X}_N$ is a finite dimensional space, the equation above can be reduced to an ordinary differential equation. Therefore, existence and uniqueness of a solution $\mathbf{\Lambda}_N \in W^{1,\infty}(0, T; \mathbf{X}_N)$ are obvious. We denote the mapping from $\mathbf{\Lambda}_N^0 \in L^\infty(0, T; \mathbf{X}_N)$ to $\Psi_N \in L^\infty(0, T; \mathcal{L}^\infty) \cap L^2(0, T; \mathcal{H}^1)$ by $\mathcal{N}_1$, and denote the mapping from $\Psi_N \in L^\infty(0, T; \mathcal{L}^\infty) \cap L^2(0, T; \mathcal{H}^1)$ to $\mathbf{\Lambda}_N \in L^\infty(0, T; \mathbf{X}_N)$ by $\mathcal{N}_2$, and then define $\mathcal{N} := \mathcal{N}_1\mathcal{N}_2$. We shall prove that the mapping $\mathcal{N}$ satisfies the conditions of Lemma 3.4 and thus has a fixed point in $L^\infty(0, T; \mathbf{X}_N)$.

Firstly, we prove that the mapping $\mathcal{N} : L^\infty(0, T; \mathbf{X}_N) \to L^\infty(0, T; \mathbf{X}_N)$ is continuous and compact. To prove the continuity of the mapping, we let $\Psi_N = \mathcal{N}_1\mathbf{\Lambda}_N^0$, $\widetilde{\Psi}_N = \mathcal{N}_1\widetilde{\mathbf{\Lambda}}_N^0$, $\mathbf{\Lambda}_N = \mathcal{N}\mathbf{\Lambda}_N^0$ and $\widetilde{\mathbf{\Lambda}}_N = \mathcal{N}\widetilde{\mathbf{\Lambda}}_N^0$, and assume that $\mathbf{\Lambda}_N^0$ and $\widetilde{\mathbf{\Lambda}}_N^0$ are bounded in $L^\infty(0, T; \mathbf{X}_N)$:

$$\|\mathbf{\Lambda}_N^0\|_{L^\infty(0,T;\mathbf{X}_N)} + \|\widetilde{\mathbf{\Lambda}}_N^0\|_{L^\infty(0,T;\mathbf{X}_N)} \leq K, \qquad (3.11)$$

where K is some positive constant. From Lemma 3.2 we see that the inequality above implies

$$\|\Psi_N\|_{L^2(0,T;\mathcal{H}^1)} + \|\widetilde{\Psi}_N\|_{L^2(0,T;\mathcal{H}^1)} \leq C_K, \qquad (3.12)$$
$$|\Psi_N| \leq 1 \quad \text{and} \quad |\widetilde{\Psi}_N| \leq 1, \quad \text{a.e. in } \Omega \times (0, T). \qquad (3.13)$$

If we define $e_N = \widetilde{\Psi}_N - \Psi_N$ and $\mathbf{E}_N^0 = \widetilde{\mathbf{\Lambda}}_N^0 - \mathbf{\Lambda}_N^0$, then (3.9) implies

$$\int_0^T \left[(\eta\partial_t e_N, \varphi) + \frac{1}{\kappa^2}(\nabla e_N, \nabla\varphi) + (|\widetilde{\mathbf{\Lambda}}_N^0|^2 e_N, \varphi)\right] \mathrm{d}t$$
$$= \int_0^T \left[-\frac{i}{\kappa}(\widetilde{\mathbf{\Lambda}}_N^0 \cdot \nabla e_N, \varphi) - \frac{i}{\kappa}(\mathbf{E}_N^0 \cdot \nabla\Psi_N, \varphi) + \frac{i}{\kappa}(e_N \widetilde{\mathbf{\Lambda}}_N^0, \nabla\varphi) + \frac{i}{\kappa}(\Psi_N \mathbf{E}_N^0, \nabla\varphi)\right]$$

$$- \big((|\widetilde{\mathbf{\Lambda}}_N^0|^2 - |\mathbf{\Lambda}_N^0|^2)\Psi_N, \varphi\big) - \big((|\widetilde{\Psi}_N|^2 - 1)\widetilde{\Psi}_N - (|\Psi_N|^2 - 1)\Psi_N, \varphi\big)\Big]\, \mathrm{d}t$$
$$- \int_0^T \big(i\eta\kappa\widetilde{\Psi}_N\nabla\cdot\mathbf{E}_N^0 + i\eta\kappa e_N\nabla\cdot\mathbf{\Lambda}_N^0, \varphi\big)\,\mathrm{d}t.$$

Substituting $\varphi(x,t) = e_N(x,t)1_{(0,t')}(t)$ into the equation above and considering the real part, we obtain

$$\frac{\eta}{2}\|e_N(\cdot,t')\|_{\mathcal{L}^2}^2 + \int_0^{t'}\Big(\frac{1}{\kappa^2}\|\nabla e_N\|_{\mathcal{L}^2}^2 + \|\widetilde{\mathbf{\Lambda}}_N^0 e_N\|_{\mathcal{L}^2}^2\Big)\,\mathrm{d}t$$

$$\leq \int_0^{t'}\Big(C\|\widetilde{\mathbf{\Lambda}}_N^0\|_{\mathbf{L}^{3+\delta}}\|\nabla e_N\|_{\mathcal{L}^2}\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}} + C\|\mathbf{E}_N^0\|_{\mathbf{L}^{3+\delta}}\|\nabla\Psi_N\|_{\mathcal{L}^2}\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}$$

$$+ C\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}\|\widetilde{\mathbf{\Lambda}}_N^0\|_{\mathbf{L}^{3+\delta}}\|\nabla e_N\|_{\mathcal{L}^2}$$

$$+ C\|\Psi_N\|_{\mathcal{L}^\infty}\|\mathbf{E}_N^0\|_{\mathbf{L}^2}\|\nabla e_N\|_{\mathcal{L}^2} + C(\|\widetilde{\mathbf{\Lambda}}_N^0\|_{\mathbf{L}^{3+\delta}} + \|\mathbf{\Lambda}_N^0\|_{\mathbf{L}^{3+\delta}})\|\mathbf{E}_N^0\|_{\mathbf{L}^2}\|\Psi_N\|_{\mathcal{L}^\infty}\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}$$

$$+ C\|e_N\|_{\mathcal{L}^2}^2 + C\|\widetilde{\Psi}_N\|_{\mathcal{L}^\infty}\|\nabla\cdot\mathbf{E}_N^0\|_{L^2}\|e_N\|_{\mathcal{L}^2}\Big)\,\mathrm{d}t$$

$$\leq \int_0^{t'}\Big(CK\|\nabla e_N\|_{\mathcal{L}^2}(C_\epsilon\|e_N\|_{\mathcal{L}^2} + \epsilon\|\nabla e_N\|_{\mathcal{L}^2})$$

$$+ C\|\nabla\Psi_N\|_{\mathcal{L}^2}\|\mathbf{E}_N^0\|_{\mathbf{H}_n(\mathrm{curl},\mathrm{div})}(C_\epsilon\|e_N\|_{\mathcal{L}^2} + \epsilon\|\nabla e_N\|_{\mathcal{L}^2})$$

$$+ CK\|\nabla e_N\|_{\mathcal{L}^2}(C_\epsilon\|e_N\|_{\mathcal{L}^2} + \epsilon\|\nabla e_N\|_{\mathcal{L}^2}) + C\|\mathbf{E}_N^0\|_{L^2}\|\nabla e_N\|_{\mathcal{L}^2}$$

$$+ CK\|\mathbf{E}_N^0\|_{L^2}(C_\epsilon\|e_N\|_{\mathcal{L}^2} + \epsilon\|\nabla e_N\|_{L^2}) + C\|e_N\|_{L^2}^2 + C\|\nabla\cdot\mathbf{E}_N^0\|_{\mathbf{L}^2}\|e_N\|_{\mathcal{L}^2}\Big)\,\mathrm{d}t$$

$$\leq \int_0^{t'}\Big((1+CK)\epsilon\|\nabla e_N\|_{\mathcal{L}^2}^2 + C_\epsilon\|e_N\|_{\mathcal{L}^2}^2 + (C_{\epsilon,K} + C_{\epsilon,K}\|\nabla\Psi_N\|_{\mathcal{L}^2}^2)\|\mathbf{E}_N^0\|_{\mathbf{H}_n(\mathrm{curl},\mathrm{div})}^2\Big)\,\mathrm{d}t,$$

which holds for any $\epsilon \in (0,1)$. Since $e_N(x,0) = 0$, by choosing $\epsilon < 1/(2\kappa^2 + 2CK\kappa^2)$ in the inequality above, applying Gronwall's inequality and using (3.12), we derive

$$\|e_N\|_{L^\infty(0,T;\mathcal{L}^2)}^2 + \|e_N\|_{L^2(0,T;\mathcal{H}^1)}^2$$
$$\leq e^{C_K T}\bigg(\int_0^T (C_K + C_K\|\nabla\Psi_N\|_{\mathcal{L}^2}^2)\,\mathrm{d}t\bigg)\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl},\mathrm{div}))}^2$$
$$\leq C_K\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl},\mathrm{div}))}^2. \tag{3.14}$$

If we define $\mathbf{E}_N = \widetilde{\mathbf{\Lambda}}_N - \mathbf{\Lambda}_N$, then (3.10) implies

$$\int_0^T\Big[\big(\partial_t\mathbf{E}_N, \mathbf{a}\big) + \big(\nabla\times\mathbf{E}_N, \nabla\times\mathbf{a}\big) + \big(\nabla\cdot\mathbf{E}_N, \nabla\cdot\mathbf{a}\big)\Big]\,\mathrm{d}t$$
$$= -\int_0^T \mathrm{Re}\Big(\frac{i}{\kappa}(\widetilde{\Psi}_N^*\nabla\widetilde{\Psi}_N - \Psi_N^*\nabla\Psi_N) + \widetilde{\mathbf{\Lambda}}_N^0(|\widetilde{\Psi}_N|^2 - |\Psi_N|^2) + |\Psi_N|^2\mathbf{E}_N\,,\,\mathbf{a}\Big)\,\mathrm{d}t. \tag{3.15}$$

Since $\mathbf{X}_N$ is a finite dimensional space, any two norms on $\mathbf{X}_N$ are equivalent. This implies that $\|\mathbf{E}_N\|_{\mathbf{H}_n(\mathrm{curl},\mathrm{div})} \leq C_N\|\mathbf{E}_N\|_{\mathbf{L}^2}$. Substituting $\mathbf{a}(x,t) = \mathbf{E}_N(x,t)1_{(0,t')}(t)$ into the equation above, we obtain

$$\frac{1}{2}\|\mathbf{E}_N(\cdot,t')\|_{L^2}^2 + \int_0^{t'}\Big(\|\nabla\times\mathbf{E}_N\|_{\mathbf{L}^2}^2 + \|\nabla\cdot\mathbf{E}_N\|_{\mathbf{L}^2}^2\Big)\,\mathrm{d}t$$

8

$$\leq \int_0^{t'} \Big(C\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}\|\nabla\widetilde{\Psi}_N\|_{\mathcal{L}^2}\|\mathbf{E}_N\|_{\mathbf{L}^{3+\delta}} + C\|\nabla e_N\|_{\mathcal{L}^2}\|\mathbf{E}_N\|_{\mathbf{L}^2}$$

$$+ C\|\widetilde{\mathbf{\Lambda}}_N^0\|_{\mathbf{L}^{3+\delta}}\|e_N\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}\|\mathbf{E}_N\|_{\mathbf{L}^2} + \|\mathbf{E}_N\|_{\mathbf{L}^2}^2 \Big)\,\mathrm{d}t$$

$$\leq C\|e_N\|_{L^2(0,T;\mathcal{H}^1)}\|\widetilde{\Psi}_N\|_{L^2(0,T;\mathcal{H}^1)}\|\mathbf{E}_N\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))} + \|e_N\|_{L^2(0,T;\mathcal{H}^1)}\|\mathbf{E}_N\|_{L^2(0,T;\mathbf{L}^2)}$$

$$+ \|\widetilde{\mathbf{\Lambda}}_N^0\|_{L^\infty(0,T;\mathbf{L}^{3+\delta})}\|e_N\|_{L^2(0,T;\mathcal{H}^1)}\|\mathbf{E}_N\|_{L^2(0,T;\mathcal{L}^2)} + \|\mathbf{E}_N\|_{L^2(0,T;\mathcal{L}^2)}^2$$

$$\leq C_N C_K\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}\|\mathbf{E}_N\|_{L^\infty(0,T;\mathbf{L}^2)} + C\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}\|\mathbf{E}_N\|_{L^2(0,T;\mathbf{L}^2)}$$

$$+ CK\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}\|\mathbf{E}_N\|_{L^2(0,T;\mathbf{L}^2)} + \|\mathbf{E}_N\|_{L^2(0,T;\mathcal{L}^2)}^2$$

$$\leq C_N C_K C_\epsilon\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}^2 + \epsilon\|\mathbf{E}_N\|_{L^\infty(0,T;\mathbf{L}^2)}^2 + C\|\mathbf{E}_N\|_{L^2(0,T;\mathbf{L}^2)}^2$$

Since $\mathbf{E}_N(x,0) = 0$, by applying Gronwall's inequality we derive

$$\|\mathbf{E}_N\|_{L^\infty(0,T;\mathbf{L}^2)} \leq C_N C_K\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}.$$

which further implies that

$$\|\mathbf{E}_N\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))} \leq C_N C_K\|\mathbf{E}_N^0\|_{L^\infty(0,T;\mathbf{H}_n(\mathrm{curl,div}))}. \tag{3.16}$$

This prove that the mapping $\mathcal{N}: L^\infty(0,T;\mathbf{X}_N) \to L^\infty(0,T;\mathbf{X}_N)$ is continuous. Since $\mathbf{X}_N$ is finite dimensional, any continuous mapping is also compact.

Secondly, we prove that the set $\mathbf{V}_N$ defined in (3.8) is bounded in $L^\infty(0,T;\mathbf{X}_N)$. Suppose that $\mathbf{\Lambda}_N^0 \in \mathbf{V}_N$, then we define $\Psi_N = \mathcal{N}_1\mathbf{\Lambda}_N^0$ and $\mathbf{\Lambda}_N = \mathcal{N}\mathbf{\Lambda}_N^0$, which satisfy $\mathbf{\Lambda}_N^0 = s\mathbf{\Lambda}_N$ and the following equations:

$$\left(\eta\frac{\partial\Psi_N}{\partial t}, \varphi \right) + \left(\left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right)\Psi_N, \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right)\varphi \right)$$

$$+ \left((|\Psi_N|^2 - 1)\Psi_N - is\eta\kappa\Psi_N\nabla\cdot\mathbf{\Lambda}_N, \varphi \right) = 0, \tag{3.17}$$

$$\left(\frac{\partial\mathbf{\Lambda}_N}{\partial t}, \mathbf{a} \right) + \left(\nabla\times\mathbf{\Lambda}_N, \nabla\times\mathbf{a} \right) + \left(\nabla\cdot\mathbf{\Lambda}_N, \nabla\cdot\mathbf{a} \right)$$

$$+ \left(\mathrm{Re}\left[\Psi_N^*\left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right)\Psi_N \right], \mathbf{a} \right) = (\mathbf{H}, \nabla\times\mathbf{a}). \tag{3.18}$$

It suffices to prove the boundedness of $\mathbf{\Lambda}_N$ in $L^\infty(0,T;\mathbf{X}_N)$. Substituting $\varphi = \Psi_N$ into (3.17) and considering the real part, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta}{2}\|\Psi_N\|_{\mathcal{L}^2}^2 \right) + \left\| \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right)\Psi_N \right\|_{\mathcal{L}^2}^2 \leq \|\Psi_N\|_{\mathcal{L}^2}^2 \leq C,$$

which implies

$$\|\Psi_N\|_{L^\infty(0,T;\mathcal{L}^2)}^2 + \left\| \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right)\Psi_N \right\|_{L^2(0,T;\mathcal{L}^2)}^2 \leq C. \tag{3.19}$$

Substituting $\mathbf{a} = \mathbf{\Lambda}_N$ into (3.18), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2 \right) + \|\nabla\times\mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2 + \|\nabla\cdot\mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2$$

9

$$= (\mathbf{H}, \nabla \times \mathbf{\Lambda}_N) - \left(\mathrm{Re}\left[\Psi_N^* \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right) \Psi_N \right], \mathbf{\Lambda}_N \right)$$

$$\leq \|\mathbf{H}\|_{\mathbf{L}^2}\|\nabla \times \mathbf{\Lambda}_N\|_{\mathbf{L}^2} + \left\| \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right) \Psi_N \right\|_{\mathcal{L}^2} \|\mathbf{\Lambda}_N\|_{\mathbf{L}^2}$$

$$\leq \frac{1}{2}\|\mathbf{H}\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\nabla \times \mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2 + \frac{1}{2}\left\| \left(\frac{i}{\kappa}\nabla + s\mathbf{\Lambda}_N \right) \Psi_N \right\|_{\mathcal{L}^2}^2 + \frac{1}{2}\|\mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2$$

$$\leq C + \frac{1}{2}\|\mathbf{H}\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\nabla \times \mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2 + \frac{1}{2}\|\mathbf{\Lambda}_N\|_{\mathbf{L}^2}^2,$$

which implies (via Gronwall's inequality)

$$\|\mathbf{\Lambda}_N\|_{L^\infty(0,T;\mathcal{L}^2)} \leq C. \tag{3.20}$$

Since $\mathbf{X}_N$ is finite dimensional, we have

$$\|\mathbf{\Lambda}_N\|_{L^\infty(0,T;\mathbf{X}_N)} \leq C_N \|\mathbf{\Lambda}_N\|_{L^\infty(0,T;\mathcal{L}^2)} \leq C_N. \tag{3.21}$$

Therefore, the set $\mathbf{V}_N$ defined in (3.8) is bounded in $L^\infty(0,T;\mathbf{X}_N)$.

To conclude, we have proved that the mapping $\mathcal{N}$ satisfies the conditions of Lemma 3.4, which implies that the mapping $\mathcal{N}$ has a fixed point $\mathbf{\Lambda}_N \in L^\infty(0,T;\mathbf{X}_N)$. If we define $\Psi_N = \mathcal{N}_1 \mathbf{\Lambda}_N$, then $(\Psi_N, \mathbf{\Lambda}_N)$ is a solution of (3.5)-(3.6). It remains to present estimates on the regularity of Ψ_N and $\mathbf{\Lambda}_N$ (uniformly with respect to N).

Substituting $\varphi = \partial_t \Psi$ and $\mathbf{a} = \partial_t \mathbf{\Lambda}$ into the equations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}\left(\left| \frac{i}{\kappa}\nabla \Psi_N + \mathbf{\Lambda}_N \Psi_N \right|^2 + \frac{1}{2}(|\Psi_N|^2 - 1)^2 + |\nabla \times \mathbf{\Lambda}_N - \mathbf{H}|^2 + |\nabla \cdot \mathbf{\Lambda}_N|^2 \right) \mathrm{d}x$$

$$+ \int_\Omega \left(\left| \frac{\partial \mathbf{\Lambda}_N}{\partial t} \right|^2 + \eta \left| \frac{\partial \Psi_N}{\partial t} \right|^2 \right) \mathrm{d}x$$

$$= \eta\kappa \int_\Omega \mathrm{Im}\left(\Psi_N \frac{\partial \Psi_N^*}{\partial t} \right) \nabla \cdot \mathbf{\Lambda}_N \, \mathrm{d}x$$

$$\leq \frac{1}{2}\int_\Omega \eta \left| \frac{\partial \Psi_N}{\partial t} \right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \eta\kappa^2 |\nabla \cdot \mathbf{\Lambda}_N|^2 \, \mathrm{d}x,$$

which implies (3.7) via Gronwall's inequality. The proof of Lemma 3.3 is completed. $\qquad\square$

3.3 Existence of a weak solution

We show that a subsequence of the approximating solutions constructed in the last subsection converges to a weak solution of (1.3)-(1.4). We need the following lemma [19].

Lemma 3.5. (Aubin–Lions) *Let $B_1 \hookrightarrow\hookrightarrow B_2 \hookrightarrow B_3$ be reflexive and separable Banach spaces. Then*

$$\{u \in L^p(I; B_1)| \, u_t \in L^q(I; B_3)\} \hookrightarrow\hookrightarrow L^p(I; B_2), \quad 1 < p,q < \infty,$$

where the symbol "$\hookrightarrow\hookrightarrow$" indicates compact embedding.

On one hand, by choosing $B_1 = \mathcal{H}^1$, $B_2 = B_3 = \mathcal{L}^2$ and $1 < p = q < \infty$ in Lemma 3.5, from (3.7) we see that the set $\{\Psi_N : N = 1, 2, \cdots\}$ is compact in $L^p(0, T; \mathcal{L}^2)$. Therefore, there exists a subsequence Ψ_{N_m}, $m = 1, 2, \cdots$, which converges to a function ψ in $L^p(0, T; \mathcal{L}^2)$ and the convergence is also pointwise a.e. in $\Omega \times (0, T)$. Since $|\Psi_{N_m}| \leq 1$ a.e. in $\Omega \times (0, T)$, it follows that $|\psi| \leq 1$ a.e. in $\Omega \times (0, T)$. By the Lebesgue dominated convergence theorem, the sequence Ψ_{N_m} also converges to ψ in $L^p(0, T; \mathcal{L}^p)$. On the other hand, by the Eberlein-Shmulyan theorem (see Page 141 of [31]), by passing to a subsequence if necessary, Ψ_{N_m} converges to ψ weakly in $L^p(0, T; \mathcal{H}^1)$ and weakly* in $L^\infty(0, T; \mathcal{H}^1)$, and $\partial_t \Psi_{N_m}$ converges to $\partial_t \psi$ weakly in $L^2(0, T; \mathcal{L}^2)$. In a similar way, by passing to a subsequence if necessary, we can derive the convergence of $\mathbf{\Lambda}_{N_m}$ to a function $\mathbf{A} \in L^\infty(0, T; \mathbf{H}_n(\text{curl}, \text{div})) \cap H^1(0, T; \mathbf{L}^2)$. For the reader's convenience, we summarize the convergence of Ψ_{N_m} and $\mathbf{\Lambda}_{N_m}$ below: there exist

$$\psi \in L^\infty(0, T; \mathcal{H}^1) \cap H^1(0, T; \mathcal{L}^2), \quad |\psi| \leq 1 \text{ a.e. in } \Omega \times (0, T),$$
$$\mathbf{A} \in L^\infty(0, T; \mathbf{H}_n(\text{curl}, \text{div})) \cap H^1(0, T; \mathbf{L}^2),$$

such that

$$\Psi_{N_m} \rightharpoonup \psi \quad \text{weakly* in } L^\infty(0, T; \mathcal{H}^1),$$
$$\Psi_{N_m} \rightharpoonup \psi \quad \text{weakly in } L^p(0, T; \mathcal{H}^1) \text{ for any } 1 < p < \infty,$$
$$\partial_t \Psi_{N_m} \rightharpoonup \partial_t \psi \quad \text{weakly in } L^2(0, T; \mathcal{L}^2),$$
$$\Psi_{N_m} \to \psi \quad \text{strongly in } L^p(0, T; \mathcal{L}^p) \text{ for any } 1 < p < \infty,$$
$$\mathbf{\Lambda}_{N_m} \rightharpoonup \mathbf{A} \quad \text{weakly* in } L^\infty(0, T; \mathbf{H}_n(\text{curl}, \text{div})),$$
$$\mathbf{\Lambda}_{N_m} \rightharpoonup \mathbf{A} \quad \text{weakly in } L^p(0, T; \mathbf{H}_n(\text{curl}, \text{div})) \text{ for any } 1 < p < \infty,$$
$$\partial_t \mathbf{\Lambda}_{N_m} \rightharpoonup \partial_t \mathbf{A} \quad \text{weakly in } L^2(0, T; \mathbf{L}^2),$$
$$\mathbf{\Lambda}_{N_m} \to \mathbf{A} \quad \text{strongly in } L^p(0, T; \mathbf{L}^{3+\delta}) \text{ for any } 1 < p < \infty.$$

It is easy to see that the convergences described above imply

$$\Psi_{N_m} \mathbf{\Lambda}_{N_m} \to \psi \mathbf{A} \quad \text{strongly in } L^2(0, T; (\mathcal{L}^2)^3),$$
$$\nabla \Psi_{N_m} \cdot \mathbf{\Lambda}_{N_m} \rightharpoonup \nabla \psi \cdot \mathbf{A} \quad \text{weakly in } L^2(0, T; \mathcal{L}^{6/5}),$$
$$\Psi_{N_m} |\mathbf{\Lambda}_{N_m}|^2 \to \psi |\mathbf{A}|^2 \quad \text{strongly in } L^2(0, T; \mathcal{L}^{3/2}),$$
$$\Psi_{N_m} \nabla \cdot \mathbf{\Lambda}_{N_m} \rightharpoonup \psi \nabla \cdot \mathbf{A} \quad \text{weakly in } L^2(0, T; \mathcal{L}^{3/2}),$$
$$\Psi_{N_m}^* \left(\frac{i}{\kappa} \nabla + \mathbf{\Lambda}_{N_m}\right) \Psi_{N_m} \rightharpoonup \psi^* \left(\frac{i}{\kappa} \nabla + \mathbf{A}\right) \psi \quad \text{weakly in } L^2(0, T; (\mathcal{L}^{3/2})^3).$$

For any given $\varphi \in L^2(0, T; \mathcal{H}^1) \hookrightarrow L^2(0, T; \mathcal{L}^6)$ and $\mathbf{a} \in L^2(0, T; \mathbf{X}_N) \hookrightarrow L^2(0, T; \mathbf{L}^{3+\delta})$, integrating (3.5)-(3.6) with respect to time, we obtain

$$\int_0^T \left[\left(\eta \frac{\partial \Psi_{N_m}}{\partial t}, \varphi\right) + \left(\left(\frac{i}{\kappa} \nabla + \mathbf{\Lambda}_{N_m}\right) \Psi_{N_m}, \left(\frac{i}{\kappa} \nabla + \mathbf{\Lambda}_{N_m}\right) \varphi\right)\right] \mathrm{d}t$$
$$+ \int_0^T \left[\left((|\Psi_{N_m}|^2 - 1)\Psi_{N_m} - i\eta\kappa \Psi_{N_m} \nabla \cdot \mathbf{\Lambda}_{N_m}, \varphi\right)\right] \mathrm{d}t = 0,$$

$$\int_0^T \left[\left(\frac{\partial \mathbf{\Lambda}_{N_m}}{\partial t}, \mathbf{a}\right) + \left(\nabla \times \mathbf{\Lambda}_{N_m}, \nabla \times \mathbf{a}\right) + \left(\nabla \cdot \mathbf{\Lambda}_{N_m}, \nabla \cdot \mathbf{a}\right)\right] \mathrm{d}t$$

11

$$= \int_0^T \left[(\mathbf{H}, \nabla \times \mathbf{a}) - \left(\mathrm{Re} \left[\Psi_{N_m}^* \left(\frac{i}{\kappa} \nabla + \mathbf{\Lambda}_{N_m} \right) \Psi_{N_m} \right], \mathbf{a} \right) \right] \mathrm{d}t.$$

Letting $m \to \infty$ in the equations above, we derive that (2.1)-(2.2) hold for any $\varphi \in L^2(0, T; \mathcal{H}^1)$ and $\mathbf{a} \in L^2(0, T; \mathbf{X}_N)$. Since $L^2(0, T; \mathbf{X}_N)$ is dense in $L^2(0, T; \mathbf{H}_n(\mathrm{curl}, \mathrm{div}))$, it follows that (2.1)-(2.2) also hold for any $\varphi \in L^2(0, T; \mathcal{H}^1)$ and $\mathbf{a} \in L^2(0, T; \mathbf{H}_n(\mathrm{curl}, \mathrm{div}))$.

It remains to prove $\nabla \times \mathbf{A} \in L^2(0, T; \mathbf{H}_n(\mathrm{curl}, \mathrm{div}))$ and $\nabla \cdot \mathbf{A} \in L^2(0, T; H^1)$. Then $(\varphi, \mathbf{A})$ is a solution of (2.1)-(2.2) in the sense of Definition 2.1. For this purpose, we consider (1.4), which implies

$$\|\nabla \times (\nabla \times \mathbf{A}) - \nabla(\nabla \cdot \mathbf{A})\|_{L^2(0,T;\mathbf{L}^2)}$$
$$\leq C\|\partial_t \mathbf{A}\|_{L^2(0,T;\mathbf{L}^2)} + C\|\psi^*(i\kappa^{-1}\nabla\psi + \mathbf{A}\psi)\|_{L^2(0,T;\mathbf{L}^2)} + C\|\nabla \times \mathbf{H}\|_{L^2(0,T;\mathbf{L}^2)}$$
$$\leq C\|\partial_t \mathbf{A}\|_{L^2(0,T;\mathbf{L}^2)} + C\|\nabla\psi\|_{L^2(0,T;\mathcal{L}^2)} + C\|\mathbf{A}\|_{L^2(0,T;\mathbf{L}^2)} + C\|\nabla \times \mathbf{H}\|_{L^2(0,T;\mathbf{L}^2)}$$
$$\leq C.$$

If we define $\mathbf{Q} = \nabla \times \mathbf{A} - \mathbf{H}$, then $\mathbf{Q}$ is a divergence-free vector fields which satisfies the equation

$$\begin{cases} \nabla \times (\nabla \times \mathbf{Q}) = \nabla \times \mathbf{f} & \text{in } \Omega, \\ \mathbf{Q} \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathbf{f} = \nabla \times \mathbf{Q} - \nabla(\nabla \cdot \mathbf{A}) = \nabla \times (\nabla \times \mathbf{A}) - \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \mathbf{H} \in L^2(0, T; \mathbf{L}^2).$$

The standard energy estimate of the equation above gives $\nabla \times \mathbf{Q} \in L^2(0, T; \mathbf{L}^2)$, which implies $\mathbf{Q} \in L^2(0, T; \mathbf{H}(\mathrm{curl}))$. Then we further derive $\nabla(\nabla \cdot \mathbf{A}) = \nabla \times \mathbf{Q} - \mathbf{f} \in L^2(0, T; \mathbf{L}^2)$.

Existence of a weak solution is proved.

3.4 Uniqueness of the weak solution

Suppose that there are two weak solutions $(\psi, \mathbf{A})$ and $(\Psi, \mathbf{\Lambda})$ for the system (1.3)-(1.6) in the sense of Definition 2.1. Then we define $e = \psi - \Psi$ and $\mathbf{E} = \mathbf{A} - \mathbf{\Lambda}$ and consider the difference equations

$$\int_0^T \left[(\eta \partial_t e, \varphi) + \frac{1}{\kappa^2}(\nabla e, \nabla \varphi) + (|\mathbf{A}|^2 e, \varphi) \right] \mathrm{d}t$$
$$= \int_0^T \left[-\frac{i}{\kappa}(\mathbf{A} \cdot \nabla e, \varphi) - \frac{i}{\kappa}(\mathbf{E} \cdot \nabla \Psi, \varphi) + \frac{i}{\kappa}(e\mathbf{A}, \nabla \varphi) + \frac{i}{\kappa}(\Psi\mathbf{E}, \nabla \varphi) \right.$$
$$\left. - ((|\mathbf{A}|^2 - |\mathbf{\Lambda}|^2)\Psi, \varphi) - ((|\psi|^2 - 1)\psi - (|\Psi|^2 - 1)\Psi, \varphi) \right] \mathrm{d}t$$
$$- \int_0^T \left(i\eta\kappa\psi\nabla \cdot \mathbf{E} + i\eta\kappa e\nabla \cdot \mathbf{\Lambda}, \varphi \right) \mathrm{d}t, \tag{3.22}$$

and

$$\int_0^T \left[(\partial_t \mathbf{E}, \mathbf{a}) + (\nabla \times \mathbf{E}, \nabla \times \mathbf{a}) + (\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{a}) \right] \mathrm{d}t$$

$$= -\int_0^T \mathrm{Re}\left(\frac{i}{\kappa}(\psi^* \nabla \psi - \Psi^* \nabla \Psi) + \mathbf{A}(|\psi|^2 - |\Psi|^2) + |\Psi|^2 \mathbf{E}, \mathbf{a}\right) \mathrm{d}t, \qquad (3.23)$$

which hold for any $\varphi \in L^2(0, T; \mathcal{H}^1)$ and $\mathbf{a} \in L^2(0, T; \mathbf{H}_n(\mathrm{curl}, \mathrm{div}))$. Choosing $\varphi(x, t) = e(x, t)1_{(0, t')}(t)$ in (3.22) and considering the real part, we obtain

$$\frac{\eta}{2}\|e(\cdot, t')\|_{\mathcal{L}^2}^2 + \int_0^{t'} \left(\frac{1}{\kappa^2}\|\nabla e\|_{\mathcal{L}^2}^2 + \|\mathbf{A}e\|_{\mathbf{L}^2}^2\right) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(C\|\mathbf{A}\|_{\mathbf{L}^{3+\delta}}\|\nabla e\|_{\mathcal{L}^2}\|e\|_{\mathcal{L}^{6-4\delta/(1+\delta)}} + C\|\mathbf{E}\|_{\mathbf{L}^{3+\delta}}\|\nabla \Psi\|_{\mathcal{L}^2}\|e\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}$$

$$+ C\|e\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}\|\mathbf{A}\|_{\mathbf{L}^{3+\delta}}\|\nabla e\|_{\mathcal{L}^2} + C\|\mathbf{E}\|_{\mathbf{L}^2}\|\nabla e\|_{\mathcal{L}^2}$$

$$+ C(\|\mathbf{A}\|_{\mathbf{L}^{3+\delta}} + \|\mathbf{\Lambda}\|_{\mathbf{L}^{3+\delta}})\|\mathbf{E}\|_{\mathbf{L}^2}\|e\|_{\mathcal{L}^{6-4\delta/(1+\delta)}} + C\|e\|_{\mathcal{L}^2}^2 + C\|\nabla \cdot \mathbf{E}\|_{L^2}\|e\|_{\mathcal{L}^2}\Big) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(C\|\nabla e\|_{L^2}(C_\epsilon\|e\|_{\mathcal{L}^2} + \epsilon\|\nabla e\|_{\mathcal{L}^2}) + C\|\mathbf{E}\|_{\mathbf{H}_n(\mathrm{curl}, \mathrm{div})}(C_\epsilon\|e\|_{\mathcal{L}^2} + \epsilon\|\nabla e\|_{\mathcal{L}^2})$$

$$+ C\|\nabla e\|_{\mathcal{L}^2}(C_\epsilon\|e\|_{\mathcal{L}^2} + \epsilon\|\nabla e\|_{\mathcal{L}^2}) + C\|\mathbf{E}\|_{\mathbf{L}^2}\|\nabla e\|_{\mathcal{L}^2}$$

$$+ C\|\mathbf{E}\|_{\mathbf{L}^2}(C_\epsilon\|e\|_{\mathcal{L}^2} + \epsilon\|\nabla e\|_{\mathcal{L}^2}) + C\|e\|_{\mathcal{L}^2}^2 + C\|\nabla \cdot \mathbf{E}\|_{L^2}\|e\|_{\mathcal{L}^2}\Big) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(\epsilon\|\nabla e\|_{\mathcal{L}^2}^2 + \epsilon\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2}^2 + \epsilon\|\nabla \cdot \mathbf{E}\|_{L^2}^2 + C_\epsilon\|e\|_{\mathcal{L}^2}^2 + C_\epsilon\|\mathbf{E}\|_{\mathbf{L}^2}^2\Big) \mathrm{d}t,$$

where ϵ can be arbitrarily small. By choosing $\mathbf{a}(x, t) = \mathbf{E}(x, t)1_{(0, t')}(t)$ in (3.23), we get

$$\frac{1}{2}\|\mathbf{E}(\cdot, t')\|_{\mathbf{L}^2}^2 + \int_0^{t'} \left(\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2}^2 + \|\nabla \cdot \mathbf{E}\|_{\mathbf{L}^2}^2\right) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(C\|e\|_{\mathcal{L}^{6-4\delta/(1+\delta)}}\|\nabla \psi\|_{L^2}\|\mathbf{E}\|_{\mathbf{L}^{3+\delta}} + C\|\nabla e\|_{\mathcal{L}^2}\|\mathbf{E}\|_{\mathbf{L}^2}$$

$$+ (\|e\|_{L^{6-4\delta/(1+\delta)}}\|\mathbf{A}\|_{\mathbf{L}^{3+\delta}} + \|\mathbf{E}\|_{\mathbf{L}^2})\|\mathbf{E}\|_{L^2}\Big) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(C(C_\epsilon\|e\|_{\mathcal{L}^2} + \epsilon\|\nabla e\|_{\mathcal{L}^2})\|\mathbf{E}\|_{\mathbf{H}_n(\mathrm{curl}, \mathrm{div})} + \|\nabla e\|_{\mathcal{L}^2}\|\mathbf{E}\|_{L^2}$$

$$+ (\|e\|_{\mathcal{L}^2} + \|\nabla e\|_{\mathcal{L}^2} + \|\mathbf{E}\|_{\mathbf{L}^2})\|\mathbf{E}\|_{\mathbf{L}^2}\Big) \mathrm{d}t$$

$$\leq \int_0^{t'} \Big(\epsilon\|\nabla e\|_{\mathcal{L}^2}^2 + \epsilon\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2} + \epsilon\|\nabla \cdot \mathbf{E}\|_{\mathbf{L}^2} + C_\epsilon\|e\|_{\mathcal{L}^2}^2 + C_\epsilon\|\mathbf{E}\|_{\mathbf{L}^2}^2\Big) \mathrm{d}t,$$

where ϵ can be arbitrarily small. By choosing $\epsilon < \frac{1}{4}\min(1, \kappa^{-2})$ and summing up the two inequalities above, we have

$$\frac{\eta}{2}\|e(\cdot, t')\|_{L^2}^2 + \frac{1}{2}\|\mathbf{E}(\cdot, t')\|_{\mathbf{L}^2}^2 \leq \int_0^{t'} \Big(C\|e\|_{L^2}^2 + C\|\mathbf{E}\|_{L^2}^2\Big) \mathrm{d}t,$$

which implies

$$\max_{t \in (0, T)} \left(\frac{\eta}{2}\|e\|_{L^2}^2 + \frac{1}{2}\|\mathbf{E}\|_{L^2}^2\right) = 0$$

via Gronwall's inequality. Uniqueness of the weak solution is proved.

The proof of Theorem 2.1 is completed. □

13

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