

# Convergence of finite elements on an evolving surface driven by diffusion on the surface

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**Abstract** For a parabolic surface partial differential equation coupled to surface evolution, convergence of the spatial semidiscretization is studied in this paper. The velocity of the evolving surface is not given explicitly, but depends on the solution of the parabolic equation on the surface. Various velocity laws are considered: elliptic regularization of a direct pointwise coupling, a regularized mean curvature flow and a dynamic velocity law. A novel stability and convergence analysis for evolving surface finite elements for the coupled problem of surface diffusion and surface evolution is developed. The stability analysis works with the matrix-vector formulation of the method and does not use geometric arguments. The geometry enters only into the consistency estimates. Numerical experiments complement the theoretical results.

**Keywords** diffusion-driven surface · velocity law · evolving surface FEM · stability · convergence analysis

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## 1 Introduction

Starting from a paper by Dziuk and Elliott [10], much insight into the stability and convergence properties of finite elements on evolving surfaces has been obtained by studying a linear parabolic equation on a given moving closed surface  $\Gamma(t)$ . The strong formulation of this model problem is to find a solution

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$u(x, t)$  (for  $x \in \Gamma(t)$  and  $0 \leq t \leq T$ ) with given initial data  $u(x, 0) = u_0(x)$  to the linear partial differential equation

$$\partial^\bullet u(x, t) + u(x, t) \nabla_{\Gamma(t)} \cdot v(x, t) - \Delta_{\Gamma(t)} u(x, t) = 0, \quad x \in \Gamma(t), \quad 0 < t \leq T,$$

where  $\partial^\bullet$  denotes the material time derivative,  $\Delta_{\Gamma(t)}$  is the Laplace–Beltrami operator on the surface, and  $\nabla_{\Gamma(t)} \cdot v$  is the tangential divergence of the *given* velocity  $v$  of the surface. We refer to [12] for an excellent review article (up to 2012) on the numerical analysis of this and related problems. Optimal-order  $L^2$  error bounds for piecewise linear finite elements are shown in [13] and maximum-norm error bounds in [23]. Stability and convergence of full discretizations obtained by combining the evolving surface finite element method (ESFEM) with various time discretizations are shown in [11, 15, 24]. Convergence of semi- and full discretizations using high-order evolving surface finite elements is studied in [20]. Arbitrary Euler–Lagrangian (ALE) variants of the ESFEM method for this equation are studied in [16, 17, 21]. Convergence properties of the ESFEM and of full discretizations for quasilinear parabolic equations on prescribed moving surfaces are studied in [22].

Beyond the above model problem, there is considerable interest in cases where the velocity of the evolving surface is *not given explicitly*, but depends on the solution  $u$  of the parabolic equation; see, e.g., [1, 6, 16, 18] for physical and biological models where such situations arise. Contrary to the case of surfaces with prescribed motion, there exists so far no numerical analysis for solution-driven surfaces in  $\mathbb{R}^3$ , to the best of our knowledge.

For the case of evolving *curves* in  $\mathbb{R}^2$ , there are recent preprints by Pozzi & Stinner [25] and Barrett, Deckelnick & Styles [2], who couple the curve-shortening flow with diffusion on the curve and study the convergence of finite element discretizations without and with a tangential part in the discrete velocity, respectively. The analogous problem for two- or higher-dimensional surfaces would be to couple mean curvature flow with diffusion on the surface. Studying the convergence of finite elements for these coupled problems, however, remains illusive as long as the convergence of ESFEM for mean curvature flow of closed surfaces is not understood. This has remained an open problem since Dziuk’s formulation of such a numerical method for mean curvature flow in his 1990 paper [9].

In this paper we consider different velocity laws for coupling the surface motion with the diffusion on the surface. Conceivably the simplest velocity law would be to prescribe the normal velocity at any surface point as a function of the solution value and possibly its tangential gradient at this point:  $v(x, t) = g(u(x, t), \nabla_{\Gamma(t)} u(x, t)) \nu_{\Gamma(t)}(x)$  for  $x \in \Gamma(t)$ , where  $\nu_{\Gamma(t)}(x)$  denotes the outer normal vector and  $g$  is a given smooth scalar-valued function. This does, however, not appear to lead to a well-posed problem, and in fact we found no mention of this seemingly obvious choice in the literature. Here we study instead a *regularized velocity law*:

$$v(x, t) - \alpha \Delta_{\Gamma(t)} v(x, t) = g(u(x, t), \nabla_{\Gamma(t)} u(x, t)) \nu_{\Gamma(t)}(x), \quad x \in \Gamma(t),$$

with a fixed regularization parameter  $\alpha > 0$ . This elliptic regularization will turn out to permit us to give a complete stability and convergence analysis of the ESFEM semidiscretization, for finite elements of polynomial degree at least two. The case of linear finite elements is left open in the theory of this paper, but will be considered in our numerical experiments. The stability and convergence results can be extended to full discretizations with linearly implicit backward difference time-stepping, as we plan to show in later work.

Our approach also applies to the ESFEM discretization of coupling a *regularized mean curvature flow* and diffusion on the surface:

$$v - \alpha \Delta_{\Gamma(t)} v = \left( -H + g(u, \nabla_{\Gamma(t)} u) \right) \nu_{\Gamma(t)},$$

where  $H$  denotes mean curvature on the surface  $\Gamma(t)$ .

The error analysis is further extended to a *dynamic velocity law*

$$\partial^\bullet v + v \nabla_{\Gamma(t)} \cdot v - \alpha \Delta_{\Gamma(t)} v = g(u, \nabla_{\Gamma(t)} u) \nu_{\Gamma(t)}.$$

A physically more relevant dynamic velocity law would be based on momentum and mass balance, such as incompressible Navier–Stokes motion of the surface coupled to diffusion on the surface. We expect that our analysis extends to such a system, but this is beyond the scope of this paper. Surface evolutions under Navier–Stokes equations and under Willmore flow have recently been considered in [3–5].

The paper is organized as follows.

In Section 2 we describe the considered problems and give the weak formulation. We recall the basics of the evolving surface finite element method and describe the semidiscrete problem. Its matrix-vector formulation is useful not only for the implementation, but will play a key role in the stability analysis of this paper.

In Section 3 we present the main result of the paper, which gives convergence estimates for the ESFEM semidiscretization with finite elements of polynomial degree at least 2. We further outline the main ideas and the organization of the proof.

In Section 4 we present auxiliary results that are used to relate different surfaces to one another. They are the key technical results used later on in the stability analysis. Section 5 contains the stability analysis for the regularized velocity law with a prescribed driving term. In Section 6 this is extended to the stability analysis for coupling surface PDEs and surface motion. The stability analysis works with the matrix-vector formulation of the ESFEM semidiscretization and does not use geometric arguments.

In Section 7 we briefly recall some geometric estimates used for estimating the consistency errors, which are the defects obtained on inserting the interpolated exact solution into the scheme. Section 8 deals with the defect estimates. Section 9 proves the main result by combining the results of the previous sections.

In Section 10 we give extensions to other velocity laws: the regularized mean curvature flow and the dynamic velocity law addressed above.

Section 11 presents numerical experiments that are complementary to our theoretical results in that they show the numerical behaviour of piecewise linear finite elements on some examples.

We use the notational convention to denote vectors in  $\mathbb{R}^3$  by italic letters, but to denote finite element nodal vectors in  $\mathbb{R}^N$  and  $\mathbb{R}^{3N}$  by boldface lowercase letters and finite element mass and stiffness matrices by boldface capitals. All boldface symbols in this paper will thus be related to the matrix-vector formulation of the ESFEM.

## 2 Problem formulation and evolving surface finite element semidiscretization

### 2.1 Basic notions and notation

We consider the evolving two-dimensional closed surface  $\Gamma(t) \subset \mathbb{R}^3$  as the image

$$\Gamma(t) = \{X(p, t) : p \in \Gamma^0\}$$

of a sufficiently regular vector-valued function  $X : \Gamma^0 \times [0, T] \rightarrow \mathbb{R}^3$ , where  $\Gamma^0$  is the smooth closed initial surface, and  $X(p, 0) = p$ . In view of the subsequent numerical discretization, it is convenient to think of  $X(p, t)$  as the position at time  $t$  of a moving particle with label  $p$ , and of  $\Gamma(t)$  as a collection of such particles. To indicate the dependence of the surface on  $X$ , we will write

$$\Gamma(t) = \Gamma(X(\cdot, t)), \quad \text{or briefly} \quad \Gamma(X)$$

when the time  $t$  is clear from the context. The *velocity*  $v(x, t) \in \mathbb{R}^3$  at a point  $x = X(p, t) \in \Gamma(t)$  equals

$$\partial_t X(p, t) = v(X(p, t), t). \quad (2.1)$$

Note that for a known velocity field  $v : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3$ , the position  $X(p, t)$  at time  $t$  of the particle with label  $p$  is obtained by solving the ordinary differential equation (2.1) from 0 to  $t$  for a fixed  $p$ .

For a function  $u(x, t)$  ( $x \in \Gamma(t)$ ,  $0 \leq t \leq T$ ) we denote the *material derivative* as

$$\partial^\bullet u(x, t) = \frac{d}{dt} u(X(p, t), t) \quad \text{for } x = X(p, t).$$

At  $x \in \Gamma(t)$  and  $0 \leq t \leq T$ , we denote by  $\nu_{\Gamma(X)}(x, t)$  the outer normal, by  $\nabla_{\Gamma(X)} u(x, t)$  the tangential gradient of  $u$ , by  $\Delta_{\Gamma(X)} u(x, t)$  the Laplace–Beltrami operator applied to  $u$ , and by  $\nabla_{\Gamma(X)} \cdot v(x, t)$  the tangential divergence of  $v$ ; see, e.g., [12] for these notions.

## 2.2 Surface motion coupled to a surface PDE: strong and weak formulation

As outlined in the introduction, we consider a parabolic equation on an evolving surface that moves according to an elliptically regularized velocity law:

$$\begin{aligned}\partial^\bullet u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ v - \alpha \Delta_{\Gamma(X)} v &= g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)}.\end{aligned}\tag{2.2}$$

Here,  $f : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  are given continuously differentiable functions, and  $\alpha > 0$  is a fixed parameter. This system is considered together with the collection of ordinary differential equations (2.1) for every label  $p$ . Initial values are specified for  $u$  and  $X$ .

On applying the Leibniz formula as in [10], the weak formulation reads as follows: Find  $u(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))$  and  $v(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))^3$  such that for all test functions  $\varphi(\cdot, t) \in H^1(\Gamma(X(\cdot, t)))$  with  $\partial^\bullet \varphi = 0$  and  $\psi(\cdot, t) \in H^1(\Gamma(X(\cdot, t)))^3$ ,

$$\begin{aligned}\frac{d}{dt} \int_{\Gamma(X)} u \varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)} \varphi &= \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u) \varphi, \\ \int_{\Gamma(X)} v \cdot \psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)} \cdot \psi,\end{aligned}\tag{2.3}$$

alongside with the ordinary differential equations (2.1) for the positions  $X$  determining the surface  $\Gamma(X)$ .

We assume throughout this paper that the problem (2.2) or (2.3) admits a unique solution with sufficiently high Sobolev regularity on the time interval  $[0, T]$  for the given initial data  $u(\cdot, 0)$  and  $X(\cdot, 0)$ . We assume further that the flow map  $X(\cdot, t) : \Gamma_0 \rightarrow \Gamma(t) \subset \mathbb{R}^3$  is non-degenerate for  $0 \leq t \leq T$ , so that  $\Gamma(t)$  is a regular surface.

## 2.3 Evolving surface finite elements

We describe the surface finite element discretization of our problem, following [8] and [7]. We use simplicial elements and continuous piecewise polynomial basis functions of degree  $k$ , as defined in [7, Section 2.5].

We triangulate the given smooth surface  $\Gamma^0$  by an admissible family of triangulations  $\mathcal{T}_h$  of decreasing maximal element diameter  $h$ ; see [10] for the notion of an admissible triangulation, which includes quasi-uniformity and shape regularity. For a momentarily fixed  $h$ , we denote by  $\mathbf{x}^0 = (x_1^0, \dots, x_N^0)$  the vector in  $\mathbb{R}^{3N}$  that collects all  $N$  nodes of the triangulation. By piecewise polynomial interpolation of degree  $k$ , the nodal vector defines an approximate surface  $\Gamma_h^0$  that interpolates  $\Gamma^0$  in the nodes  $x_j^0$ . We will evolve the  $j$ th node in time, denoted  $x_j(t)$  with  $x_j(0) = x_j^0$ , and collect the nodes at time  $t$  in a vector

$$\mathbf{x}(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{R}^{3N}.$$

Provided that  $x_j(t)$  is sufficiently close to the exact position  $x_j^*(t) := X(p_j, t)$  (with  $p_j = x_j^0$ ) on the exact surface  $\Gamma(t) = \Gamma(X(\cdot, t))$ , the nodal vector  $\mathbf{x}(t)$  still corresponds to an admissible triangulation. In the following discussion we omit the omnipresent argument  $t$  and just write  $\mathbf{x}$  for  $\mathbf{x}(t)$  when the dependence on  $t$  is not important.

By piecewise polynomial interpolation on the plane reference triangle that corresponds to every curved triangle of the triangulation, the nodal vector  $\mathbf{x}$  defines a closed surface denoted by  $\Gamma_h(\mathbf{x})$ . We can then define finite element *basis functions*

$$\phi_j[\mathbf{x}] : \Gamma_h(\mathbf{x}) \rightarrow \mathbb{R}, \quad j = 1, \dots, N,$$

which have the property that on every triangle their pullback to the reference triangle is polynomial of degree  $k$ , and which satisfy

$$\phi_j[\mathbf{x}](x_k) = \delta_{jk} \quad \text{for all } j, k = 1, \dots, N.$$

These functions span the finite element space on  $\Gamma_h(\mathbf{x})$ ,

$$S_h(\mathbf{x}) = \text{span}\{\phi_1[\mathbf{x}], \phi_2[\mathbf{x}], \dots, \phi_N[\mathbf{x}]\}.$$

For a finite element function  $u_h \in S_h(\mathbf{x})$  the tangential gradient  $\nabla_{\Gamma_h(\mathbf{x})} u_h$  is defined piecewise.

We set

$$X_h(p_h, t) = \sum_{j=1}^N x_j(t) \phi_j[\mathbf{x}(0)](p_h), \quad p_h \in \Gamma_h^0,$$

which has the properties that  $X_h(p_j, t) = x_j(t)$  for  $j = 1, \dots, N$ , that  $X_h(p_h, 0) = p_h$  for all  $p_h \in \Gamma_h^0$ , and

$$\Gamma_h(\mathbf{x}(t)) = \Gamma(X_h(\cdot, t)).$$

The *discrete velocity*  $v_h(x, t) \in \mathbb{R}^3$  at a point  $x = X_h(p_h, t) \in \Gamma(X_h(\cdot, t))$  is given by

$$\partial_t X_h(p_h, t) = v_h(X_h(p_h, t), t).$$

A key property of the basis functions is the *transport property* [10]:

$$\frac{d}{dt} \left( \phi_j[\mathbf{x}(t)](X_h(p_h, t)) \right) = 0,$$

which by integration from 0 to  $t$  yields

$$\phi_j[\mathbf{x}(t)](X_h(p_h, t)) = \phi_j[\mathbf{x}(0)](p_h).$$

This implies that the discrete velocity is simply

$$v_h(x, t) = \sum_{j=1}^N v_j(t) \phi_j[\mathbf{x}(t)](x) \quad \text{for } x \in \Gamma_h(\mathbf{x}(t)), \quad \text{with } v_j(t) = \dot{x}_j(t),$$

where the dot denotes the time derivative  $d/dt$ .

The *discrete material derivative* of a finite element function

$$u_h(x, t) = \sum_{j=1}^N u_j(t) \phi_j[\mathbf{x}(t)](x), \quad x \in \Gamma_h(\mathbf{x}(t)),$$

is defined as

$$\partial_h^\bullet u_h(x, t) = \frac{d}{dt} u_h(X_h(p_h, t), t) \quad \text{for } x = X_h(p_h, t).$$

By the transport property of the basis functions, this is just

$$\partial_h^\bullet u_h(x, t) = \sum_{j=1}^N \dot{u}_j(t) \phi_j[\mathbf{x}(t)](x), \quad x \in \Gamma_h(\mathbf{x}(t)).$$

## 2.4 Semidiscretization of the evolving surface problem

The finite element spatial semidiscretization of the problem (2.3) reads as follows: Find the unknown nodal vector  $\mathbf{x}(t) \in \mathbb{R}^{3N}$  and the unknown finite element functions  $u_h(\cdot, t) \in S_h(\mathbf{x}(t))$  and  $v_h(\cdot, t) \in S_h(\mathbf{x}(t))^3$  such that, for all  $\varphi_h(\cdot, t) \in S_h(\mathbf{x}(t))$  with  $\partial_h^\bullet \varphi_h = 0$  and all  $\psi_h(\cdot, t) \in S_h(\mathbf{x}(t))^3$ ,

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma_h(\mathbf{x})} u_h \varphi_h + \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} u_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \varphi_h &= \int_{\Gamma_h(\mathbf{x})} f(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h) \varphi_h, \\ \int_{\Gamma_h(\mathbf{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} v_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \psi_h &= \int_{\Gamma_h(\mathbf{x})} g(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h) \nu_{\Gamma_h(\mathbf{x})} \cdot \psi_h, \end{aligned} \quad (2.4)$$

and

$$\partial_t X_h(p_h, t) = v_h(X_h(p_h, t), t), \quad p_h \in \Gamma_h^0. \quad (2.5)$$

The initial values for the nodal vector  $\mathbf{u}$  corresponding to  $u_h$  and the nodal vector  $\mathbf{x}$  of the initial positions are taken as the exact initial values at the nodes  $x_j^0$  of the triangulation of the given initial surface  $\Gamma^0$ :

$$x_j(0) = x_j^0, \quad u_j(0) = u(x_j^0, 0), \quad (j = 1, \dots, N).$$

## 2.5 Differential-algebraic equations of the matrix-vector formulation

We now show that the nodal vectors  $\mathbf{u} \in \mathbb{R}^N$  and  $\mathbf{v} \in \mathbb{R}^{3N}$  of the finite element functions  $u_h$  and  $v_h$ , respectively, together with the surface nodal vector  $\mathbf{x} \in \mathbb{R}^{3N}$  satisfy a system of differential-algebraic equations (DAEs). Using the above finite element setting, we set (omitting the argument  $t$ )

$$\begin{aligned} u_h &= \sum_{j=1}^N u_j \phi_j[\mathbf{x}], & u_h(x_j) &= u_j \in \mathbb{R}, \\ v_h &= \sum_{j=1}^N v_j \phi_j[\mathbf{x}], & v_h(x_j) &= v_j \in \mathbb{R}^3, \end{aligned}$$

and collect the nodal values in column vectors  $\mathbf{u} = (u_j) \in \mathbb{R}^N$  and  $\mathbf{v} = (v_j) \in \mathbb{R}^{3N}$ .

We define the surface-dependent mass matrix  $\mathbf{M}(\mathbf{x})$  and stiffness matrix  $\mathbf{A}(\mathbf{x})$  on the surface determined by the nodal vector  $\mathbf{x}$ :

$$\begin{aligned}\mathbf{M}(\mathbf{x})|_{jk} &= \int_{\Gamma_h(\mathbf{x})} \phi_j[\mathbf{x}] \phi_k[\mathbf{x}], \\ \mathbf{A}(\mathbf{x})|_{jk} &= \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h} \phi_j[\mathbf{x}] \cdot \nabla_{\Gamma_h} \phi_k[\mathbf{x}],\end{aligned}\quad (j, k = 1, \dots, N).$$

We further let (with the identity matrix  $I_3 \in \mathbb{R}^{3 \times 3}$ )

$$\mathbf{K}(\mathbf{x}) = I_3 \otimes \left( \mathbf{M}(\mathbf{x}) + \alpha \mathbf{A}(\mathbf{x}) \right). \quad (2.6)$$

The right-hand side vectors  $\mathbf{f}(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^N$  and  $\mathbf{g}(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{3N}$  are given by

$$\begin{aligned}\mathbf{f}(\mathbf{x}, \mathbf{u})|_j &= \int_{\Gamma_h(\mathbf{x})} f(u_h, \nabla_{\Gamma_h} u_h) \phi_j[\mathbf{x}], \\ \mathbf{g}(\mathbf{x}, \mathbf{u})|_{3(j-1)+\ell} &= \int_{\Gamma_h(\mathbf{x})} g(u_h, \nabla_{\Gamma_h} u_h) (\nu_{\Gamma_h(\mathbf{x})})_\ell \phi_j[\mathbf{x}],\end{aligned}$$

for  $j = 1, \dots, N$ , and  $\ell = 1, 2, 3$ .

We then obtain from (2.4)–(2.5) the following coupled DAE system for the nodal values  $\mathbf{u}, \mathbf{v}$  and  $\mathbf{x}$ :

$$\begin{aligned}\frac{d}{dt} \left( \mathbf{M}(\mathbf{x}) \mathbf{u} \right) + \mathbf{A}(\mathbf{x}) \mathbf{u} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{K}(\mathbf{x}) \mathbf{v} &= \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \dot{\mathbf{x}} &= \mathbf{v}.\end{aligned}\quad (2.7)$$

With the auxiliary vector  $\mathbf{w} = \mathbf{M}(\mathbf{x}) \mathbf{u}$ , this system becomes

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v}, \\ \dot{\mathbf{w}} &= -\mathbf{A}(\mathbf{x}) \mathbf{u} + \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{0} &= -\mathbf{K}(\mathbf{x}) \mathbf{v} + \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \mathbf{0} &= -\mathbf{M}(\mathbf{x}) \mathbf{u} + \mathbf{w}.\end{aligned}$$

This is of a form to which standard DAE time discretization can be applied; see, e.g., [19, Chap. VI].

As will be seen in later sections, the matrix-vector formulation is very useful in the stability analysis of the ESFEM, beyond its obvious role for practical computations.



## 2.6 Lifts

In the error analysis we need to compare functions on three different surfaces: the *exact surface*  $\Gamma(t) = \Gamma(X(\cdot, t))$ , the *discrete surface*  $\Gamma_h(t) = \Gamma_h(\mathbf{x}(t))$ , and the *interpolated surface*  $\Gamma_h^*(t) = \Gamma_h(\mathbf{x}^*(t))$ , where  $\mathbf{x}^*(t)$  is the nodal vector collecting the grid points  $x_j^*(t) = X(p_j, t)$  on the exact surface. In the following definitions we omit the argument  $t$  in the notation.

A finite element function  $w_h : \Gamma_h \rightarrow \mathbb{R}^m$  ( $m = 1$  or  $3$ ) on the discrete surface, with nodal values  $w_j$ , is related to the finite element function  $\widehat{w}_h$  on the interpolated surface that has the same nodal values:

$$\widehat{w}_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}^*].$$

The transition between the interpolated surface and the exact surface is done by the *lift operator*, which was introduced for linear surface approximations in [8]; see also [10, 13]. Higher-order generalizations have been studied in [7]. The lift operator  $l$  maps a function on the interpolated surface  $\Gamma_h^*$  to a function on the exact surface  $\Gamma$ , provided that  $\Gamma_h^*$  is sufficiently close to  $\Gamma$ .

The exact regular surface  $\Gamma(X(\cdot, t))$  can be represented by a (sufficiently smooth) signed distance function  $d : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$ , cf. [10, Section 2.1], such that

$$\Gamma(X(\cdot, t)) = \{x \in \mathbb{R}^3 \mid d(x, t) = 0\} \subset \mathbb{R}^3. \quad (2.8)$$

Using this distance function, the lift of a continuous function  $\eta_h : \Gamma_h^* \rightarrow \mathbb{R}$  is defined as

$$\eta_h^l(y) := \eta_h(x), \quad x \in \Gamma_h^*,$$

where for every  $x \in \Gamma_h^*$  the point  $y = y(x) \in \Gamma$  is uniquely defined via

$$y = x - \nu(y)d(x).$$

For functions taking values in  $\mathbb{R}^3$  the lift is componentwise. By  $\eta^{-l}$  we denote the function on  $\Gamma_h^*$  whose lift is  $\eta$ .

We denote the composed lift  $L$  from finite element functions on  $\Gamma_h$  to functions on  $\Gamma$  via  $\Gamma_h^*$  by

$$w_h^L = (\widehat{w}_h)^l.$$

## 3 Statement of the main result: semidiscrete error bound

We are now in the position to formulate the main result of this paper, which yields optimal-order error bounds for the finite element semidiscretization of a surface PDE on a solution-driven surface as specified in (2.2), for finite elements of polynomial degree  $k \geq 2$ . We denote by  $\Gamma(t) = \Gamma(X(\cdot, t))$  the exact surface and by  $\Gamma_h(t) = \Gamma(X_h(\cdot, t)) = \Gamma_h(\mathbf{x}(t))$  the discrete surface at time  $t$ . We introduce the notation

$$x_h^L(x, t) = X_h^L(p, t) \in \Gamma_h(t) \quad \text{for } x = X(p, t) \in \Gamma(t).$$

**Theorem 3.1** *Consider the space discretization (2.4)–(2.5) of the coupled problem (2.1)–(2.2), using evolving surface finite elements of polynomial degree  $k \geq 2$ . We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for  $u$ . Suppose that the problem admits an exact solution  $(u, v, X)$  that is sufficiently smooth (say, in the Sobolev class  $H^{k+1}$ ) on the time interval  $0 \leq t \leq T$ , and that the flow map  $X(\cdot, t) : \Gamma_0 \rightarrow \Gamma(t) \subset \mathbb{R}^3$  is non-degenerate for  $0 \leq t \leq T$ , so that  $\Gamma(t)$  is a regular surface.*

*Then, there exists  $h_0 > 0$  such that for all mesh widths  $h \leq h_0$  the following error bounds hold over the exact surface  $\Gamma(t) = \Gamma(X(\cdot, t))$  for  $0 \leq t \leq T$ :*

$$\left( \|u_h^L(\cdot, t) - u(\cdot, t)\|_{L^2(\Gamma(t))}^2 + \int_0^t \|u_h^L(\cdot, s) - u(\cdot, s)\|_{H^1(\Gamma(s))}^2 ds \right)^{\frac{1}{2}} \leq Ch^k$$

and

$$\left( \int_0^t \|v_h^L(\cdot, s) - v(\cdot, s)\|_{H^1(\Gamma(s))^3}^2 ds \right)^{1/2} \leq Ch^k,$$

$$\|x_h^L(\cdot, t) - \text{id}_{\Gamma(t)}\|_{H^1(\Gamma(t))^3} \leq Ch^k.$$

*The constant  $C$  is independent of  $t$  and  $h$ , but depends on bounds of the  $H^{k+1}$  norms of the solution  $(u, v, X)$ , on local Lipschitz constants of  $f$  and  $g$ , on the regularization parameter  $\alpha > 0$  and on the length  $T$  of the time interval.*

We note that the last error bound is equivalent to

$$\|X_h^L(\cdot, t) - X(\cdot, t)\|_{H^1(\Gamma_0)^3} \leq ch^k.$$

Moreover, in the case of a coupling function  $g$  in (2.2) that is independent of the solution gradient, so that  $g = g(u)$ , we obtain an error bound for the velocity that is pointwise in time: uniformly for  $0 \leq t \leq T$ ,

$$\|v_h^L(\cdot, t) - v(\cdot, t)\|_{H^1(\Gamma(t))^3} \leq Ch^k.$$

A key issue in the proof is to ensure that the  $W^{1,\infty}$  norm of the position error of the curves remains small. The  $H^1$  error bound and an inverse estimate yield an  $O(h^{k-1})$  error bound in the  $W^{1,\infty}$  norm. This is small only for  $k \geq 2$ , which is why we impose the condition  $k \geq 2$  in the above result.

Since the exact flow map  $X(\cdot, t) : \Gamma_0 \rightarrow \Gamma(t)$  is assumed to be smooth and non-degenerate, it is locally close to an invertible linear transformation, and (using compactness) it therefore preserves the admissibility of grids with sufficiently small mesh width  $h \leq h_0$ . Our assumptions therefore guarantee that the triangulations formed by the nodes  $x_j^*(t) = X(p_j, t)$  remain admissible uniformly for  $t \in [0, T]$  for sufficiently small  $h$  (though the bounds in the admissibility inequalities and the largest possible mesh width may deteriorate with growing time). Since  $k \geq 2$ , the position error estimate implies that for sufficiently small  $h$  also the triangulations formed by the numerical nodes  $x_j(t)$  remain admissible uniformly for  $t \in [0, T]$ . This cannot be concluded for  $k = 1$ .

The error bound will be proven by clearly separating the issues of consistency and stability. The consistency error is the defect on inserting a projection (interpolation or Ritz projection) of the exact solution into the discretized equation. The defect bounds involve *geometric estimates* that were obtained for the time dependent case and for higher order  $k \geq 2$  in [20], by combining techniques of Dziuk & Elliott [10,13] and Demlow [7]. This is done with the ESFEM formulation of Section 2.4.

The main issue in the proof of Theorem 3.1 is to prove *stability* in the form of an  $h$ -independent bound of the error in terms of the defect. The stability analysis is done in the matrix-vector formulation of Section 2.5. It uses energy estimates and transport formulae that relate the mass and stiffness matrices and the coupling terms for different nodal vectors  $\mathbf{x}$ . *No geometric estimates* enter in the proof of stability.

In Section 4 we prove important auxiliary results for the stability analysis. The stability is first analysed for the discretized velocity law without coupling to the surface PDE in Section 5 and is then extended to the coupled problem in Section 6. The necessary geometric estimates for the consistency analysis are collected in Section 7, and the defects are then bounded in Section 8. The proof of Theorem 3.1 is then completed in Section 9 by putting together the results on stability, defect bounds and interpolation error bounds.

#### 4 Auxiliary results for the stability analysis: relating different surfaces

The finite element matrices of Section 2.5 induce discrete versions of Sobolev norms. For any  $\mathbf{w} = (w_j) \in \mathbb{R}^N$  with corresponding finite element function  $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}] \in S_h(\mathbf{x})$  we note

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x})}^2 := \mathbf{w}^T \mathbf{M}(\mathbf{x}) \mathbf{w} = \|w_h\|_{L^2(\Gamma_h(\mathbf{x}))}^2, \quad (4.1)$$

$$\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x})}^2 := \mathbf{w}^T \mathbf{A}(\mathbf{x}) \mathbf{w} = \|\nabla_{\Gamma_h(\mathbf{x})} w_h\|_{L^2(\Gamma_h(\mathbf{x}))}^2. \quad (4.2)$$

In our stability analysis we need to relate finite element matrices corresponding to different nodal vectors. We use the following setting. Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3N}$  be two nodal vectors defining discrete surfaces  $\Gamma_h(\mathbf{x})$  and  $\Gamma_h(\mathbf{y})$ , respectively. We let  $\mathbf{e} = (e_j) = \mathbf{x} - \mathbf{y} \in \mathbb{R}^{3N}$ . For the parameter  $\theta \in [0, 1]$ , we consider the intermediate surface  $\Gamma_h^\theta = \Gamma_h(\mathbf{y} + \theta \mathbf{e})$  and the corresponding finite element functions given as

$$e_h^\theta = \sum_{j=1}^N e_j \phi_j[\mathbf{y} + \theta \mathbf{e}]$$

and, for any vectors  $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$ ,

$$w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta \mathbf{e}] \quad \text{and} \quad z_h^\theta = \sum_{j=1}^N z_j \phi_j[\mathbf{y} + \theta \mathbf{e}].$$

**Lemma 4.1** *In the above setting the following identities hold:*

$$\begin{aligned}\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta d\theta, \\ \mathbf{w}^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{y}))\mathbf{z} &= \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} w_h^\theta \cdot (D_{\Gamma_h^\theta} e_h^\theta) \nabla_{\Gamma_h^\theta} z_h^\theta d\theta,\end{aligned}$$

with  $D_{\Gamma_h^\theta} e_h^\theta = \text{trace}(E)I_3 - (E + E^T)$  for  $E = \nabla_{\Gamma_h^\theta} e_h^\theta \in \mathbb{R}^{3 \times 3}$ .

*Proof* Using the fundamental theorem of calculus and the Leibniz formula we write

$$\begin{aligned}\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} &= \int_{\Gamma_h(\mathbf{x})} w_h^1 z_h^1 - \int_{\Gamma_h(\mathbf{y})} w_h^0 z_h^0 = \int_0^1 \frac{d}{d\theta} \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta d\theta \\ &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta d\theta.\end{aligned}$$

In the last formula we used that the material derivatives (with respect to  $\theta$ ) of  $w_h^\theta$  and  $z_h^\theta$  vanish, thanks to the transport property of the basis functions. The second identity is shown in the same way, using the formula for the derivative of the Dirichlet integral; see [10] and also [15, Lemma 3.1].  $\square$

A direct consequence of Lemma 4.1 is the following conditional equivalence of norms:

**Lemma 4.2** *If  $\|\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \leq \mu$  for  $0 \leq \theta \leq 1$ , then*

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\mathbf{e})} \leq e^{\mu/2} \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})}.$$

*If  $\|D_{\Gamma_h^\theta} e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \leq \eta$  for  $0 \leq \theta \leq 1$ , then*

$$\|\mathbf{w}\|_{\mathbf{A}(\mathbf{y}+\mathbf{e})} \leq e^{\eta/2} \|\mathbf{w}\|_{\mathbf{A}(\mathbf{y})}.$$

*Proof* By Lemma 4.1 we have for  $0 \leq \tau \leq 1$

$$\begin{aligned}\|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\tau\mathbf{e})}^2 - \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})}^2 &= \mathbf{w}^T(\mathbf{M}(\mathbf{y}+\tau\mathbf{e}) - \mathbf{M}(\mathbf{y}))\mathbf{w} \\ &= \int_0^\tau \int_{\Gamma_h^\theta} w_h^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) w_h^\theta d\theta \leq \mu \int_0^\tau \|w_h^\theta\|_{L^2(\Gamma_h^\theta)}^2 d\theta \\ &= \mu \int_0^\tau \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\theta\mathbf{e})}^2 d\theta,\end{aligned}$$

and the first result follows from Gronwall's inequality. The second result is proved in the same way.  $\square$

The following result, when used with  $w_h^\theta$  equal to components of  $e_h^\theta$ , reduces the problem of checking the conditions of the previous lemma for  $0 \leq \theta \leq 1$  to checking the condition just for the case  $\theta = 0$ .

**Lemma 4.3** *In the above setting, assume that*

$$\|\nabla_{\Gamma_h[\mathbf{y}]} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \leq \frac{1}{2}. \quad (4.3)$$

*Then, for  $0 \leq \theta \leq 1$  the function  $w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta \mathbf{e}]$  on  $\Gamma_h^\theta = \Gamma[\mathbf{y} + \theta \mathbf{e}]$  is bounded by*

$$\|\nabla_{\Gamma_h^\theta} w_h^\theta\|_{L^p(\Gamma_h^\theta)} \leq c_p \|\nabla_{\Gamma_h^0} w_h^0\|_{L^p(\Gamma_h^0)} \quad \text{for } 1 \leq p \leq \infty,$$

*where  $c_p$  depends only on  $p$  (we have  $c_\infty = 2$ ).*

*Proof* We describe the finite element parametrization of the discrete surfaces  $\Gamma_h^\theta$  in the same way as in Section 2.3, with  $\theta$  instead of  $t$  in the role of the time variable. We set

$$Y_h^\theta(q_h) = Y_h(q_h, \theta) = \sum_{j=1}^N (y_j + \theta e_j) \phi_j[\mathbf{y}](q_h), \quad q_h \in \Gamma_h[\mathbf{y}], \quad (4.4)$$

so that

$$\Gamma(Y_h^\theta) = \Gamma_h[\mathbf{y} + \theta \mathbf{e}] = \Gamma_h^\theta.$$

Since  $Y_h^0(q_h) = q_h$  for all  $q_h \in \Gamma_h^0 = \Gamma_h[\mathbf{y}]$ , the above formula can be rewritten as

$$Y_h^\theta(q_h) = q_h + \theta e_h^0(q_h).$$

Tangent vectors to  $\Gamma_h^\theta$  at  $y_h^\theta = Y_h^\theta(q_h)$  are therefore of the form

$$\delta y_h^\theta = DY_h^\theta(q_h) \delta q_h = \delta q_h + \theta (\nabla_{\Gamma_h^0} e_h^0(q_h))^T \delta q_h,$$

where  $\delta q_h$  is a tangent vector to  $\Gamma_h^0$  at  $q_h$ , or written more concisely,  $\delta q_h \in T_{q_h} \Gamma_h^0$ .

Letting  $|\cdot|$  denote the Euclidean norm of a vector in  $\mathbb{R}^3$ , we have at  $y_h^\theta = Y_h^\theta(q_h)$

$$\begin{aligned} |\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta)| &= \sup_{\delta y_h^\theta \in T_{y_h^\theta} \Gamma_h^\theta} \frac{(\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta))^T \delta y_h^\theta}{|\delta y_h^\theta|} = \sup_{\delta y_h^\theta \in T_{y_h^\theta} \Gamma_h^\theta} \frac{Dw_h^\theta(y_h^\theta) \delta y_h^\theta}{|\delta y_h^\theta|} \\ &= \sup_{\delta q_h \in T_{q_h} \Gamma_h^0} \frac{Dw_h^\theta(y_h^\theta) DY_h^\theta(q_h) \delta q_h}{|DY_h^\theta(q_h) \delta q_h|}. \end{aligned}$$

By construction of  $w_h^\theta$  and the transport property of the basis functions, we have

$$w_h^\theta(Y_h^\theta(q_h)) = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta \mathbf{e}](Y_h^\theta(q_h)) = \sum_{j=1}^N w_j \phi_j[\mathbf{y}](q_h) = w_h^0(q_h).$$

By the chain rule, this yields

$$Dw_h^\theta(y_h^\theta) DY_h^\theta(q_h) = Dw_h^0(q_h).$$

Under the imposed condition  $\|\nabla_{\Gamma_h^0} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \leq \frac{1}{2}$  we have for  $0 \leq \theta \leq 1$

$$|DY_h^\theta(q_h) \delta q_h| \geq |\delta q_h| - \theta |(\nabla_{\Gamma_h^0} e_h^0(q_h))^T \delta q_h| \geq \frac{1}{2} |\delta q_h|.$$

Hence we obtain

$$\begin{aligned} |\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta)| &= \sup_{\delta q_h \in T_{q_h} \Gamma_h^0} \frac{Dw_h^0(q_h) \delta q_h}{|DY_h^\theta(q_h) \delta q_h|} \\ &\leq \sup_{\delta q_h \in T_{q_h} \Gamma_h^0} \frac{Dw_h^0(q_h) \delta q_h}{\frac{1}{2} |\delta q_h|} = 2 |\nabla_{\Gamma_h^0} w_h^0(q_h)|. \end{aligned}$$

This yields the stated result for  $p = \infty$ . For  $1 \leq p < \infty$  we note in addition that in using the integral transformation formula we have a uniform bound between the surface elements, since  $DY_h^\theta$  is close to the identity matrix by our smallness assumption on  $\nabla_{\Gamma_h^0} e_h^0$ .  $\square$

The arguments of the previous proof are also used in estimating the changes of the normal vectors on the various surfaces  $\Gamma_h^\theta = \Gamma_h[\mathbf{y} + \theta \mathbf{e}]$ .

**Lemma 4.4** *Suppose that condition (4.3) is satisfied. Let  $y_h^\theta = Y_h^\theta(q_h) \in \Gamma_h^\theta$  be related by the parametrization (4.4) of  $\Gamma_h^\theta$  over  $\Gamma_h^0$ , for  $0 \leq \theta \leq 1$ . Then, the corresponding unit normal vectors differ by no more than*

$$|\nu_{\Gamma_h^\theta}(y_h^\theta) - \nu_{\Gamma_h^0}(y_h^0)| \leq C\theta |\nabla_{\Gamma_h^0} e_h^0(y_h^0)|,$$

with some constant  $C$ .

*Proof* Let  $\delta q_h^1$  and  $\delta q_h^2$  be two linearly independent tangent vectors of  $\Gamma_h^0$  at  $q_h \in \Gamma_h^0$  (which may be chosen orthogonal to each other and of unit length with respect to the Euclidean norm). With  $\delta y_h^{\theta,i} = DY_h^\theta(q_h) \delta q_h^i = \delta q_h^i + \theta (\nabla_{\Gamma_h^0} e_h^0(q_h))^T \delta q_h^i$  for  $i = 1, 2$  we then have, for  $0 \leq \theta \leq 1$ ,

$$\nu_{\Gamma_h^\theta}(y_h^\theta) = \frac{\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}}{|\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}|}.$$

Since this expression is a locally Lipschitz continuous function of the two vectors, the result follows. (The imposed bound (4.3) is sufficient to ensure the linear independence of the vectors  $\delta y_h^{\theta,i}$ .)  $\square$

We denote by  $\partial_\theta^\bullet f$  the material derivative of a function  $f = f(y_h^\theta, \theta)$  depending on  $\theta \in [0, 1]$  and  $y_h^\theta \in \Gamma_h^\theta$ :

$$\partial_\theta^\bullet f = \frac{d}{d\theta} f(y_h^\theta, \theta).$$

From Lemma 4.4 together with Lemma 4.3 we obtain the following bound:

**Lemma 4.5** *If condition (4.3) is satisfied, then*

$$\|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}\|_{L^p(\Gamma_h^\theta)} \leq C \|\nabla_{\Gamma_h^0} e_h^0\|_{L^p(\Gamma_h^0)},$$

where  $C$  is independent of  $0 \leq \theta \leq 1$  and  $1 \leq p \leq \infty$ .

*Proof* By Lemma 4.4 with  $\Gamma_h^\theta$  in the role of  $\Gamma_h^0$ , we obtain

$$|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}(y_h^\theta)| = \left| \lim_{\tau \rightarrow 0} (\nu_{\Gamma_h^{\theta+\tau}}(y_h^{\theta+\tau}) - \nu_{\Gamma_h^\theta}(y_h^\theta)) / \tau \right| \leq C |\nabla_{\Gamma_h^\theta} e_h^\theta(y_h^\theta)|,$$

which implies

$$\|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}\|_{L^p(\Gamma_h^\theta)} \leq C \|\nabla_{\Gamma_h^\theta} e_h^\theta\|_{L^p(\Gamma_h^\theta)},$$

and Lemma 4.3 completes the proof.  $\square$

We finally need a result that bounds the time derivatives of the mass and stiffness matrices corresponding to nodes on the exact smooth surface  $\Gamma(t)$ . The following result is a direct consequence of [15, Lemma 4.1].

**Lemma 4.6** *Let  $\Gamma(t) = \Gamma(X(\cdot, t))$ ,  $t \in [0, T]$ , be a smoothly evolving family of smooth closed surfaces, and let the vector  $\mathbf{x}^*(t) \in \mathbb{R}^{3N}$  collect the nodes  $x_j^*(t) = X(p_j, t)$ . Then,*

$$\begin{aligned} \mathbf{w}^T \frac{d}{dt} \mathbf{M}(\mathbf{x}^*(t)) \mathbf{z} &\leq C \|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}^*(t))} \|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}^*(t))}, \\ \mathbf{w}^T \frac{d}{dt} \mathbf{A}(\mathbf{x}^*(t)) \mathbf{z} &\leq C \|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}^*(t))} \|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}^*(t))}, \end{aligned}$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$ . The constant  $C$  depends only on a bound of the  $W^{1,\infty}$  norm of the surface velocity.

## 5 Stability of discretized surface motion under a prescribed driving-term

In this section we begin the stability analysis by first studying the stability of the spatially discretized velocity law with a given inhomogeneity instead of a coupling to the surface PDE. This allows us to present, in a technically simpler setting, some of the basic arguments that are used in our approach to stability estimates, which works with the matrix-vector formulation. The stability of the spatially discretized problem including coupling with the surface PDE is then studied in Section 6 by similar, but more elaborate arguments.

### 5.1 Uncoupled velocity law and its semidiscretization

In this section we consider the velocity law without coupling to a surface PDE:

$$v - \alpha \Delta_{\Gamma(X)} v = g \nu_{\Gamma(X)},$$

where  $g : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  is a given [smooth](#) function of  $(x, t)$ , and  $\alpha > 0$  is a fixed parameter. This problem is considered together with the ordinary differential equations (2.1) for the positions  $X$  determining the surface  $\Gamma(X)$ . Initial values are specified for  $X$ .

The weak formulation is given by the second formula of (2.3) with the function  $g$  considered here. This is considered together with the ordinary differential equations (2.1) for the positions  $X$ .

Then the finite element spatial semidiscretization of this problem reads as: Find the unknown nodal vector  $\mathbf{x}(t) \in \mathbb{R}^{3N}$  and the unknown finite element function  $v_h(\cdot, t) \in S_h(\mathbf{x}(t))^3$  such that the following semidiscrete equation holds for every  $\psi_h \in S_h(\mathbf{x}(t))^3$ :

$$\int_{\Gamma_h(\mathbf{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} v_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \psi_h = \int_{\Gamma_h(\mathbf{x})} g \nu_{\Gamma_h(\mathbf{x})} \cdot \psi_h, \quad (5.1)$$

together with the ordinary differential equations (2.5). As before, the nodal vector of the initial positions  $\mathbf{x}(0)$  is taken from the exact initial values at the nodes  $x_j^0$  of the triangulation of the given initial surface  $\Gamma^0$ :  $x_j(0) = x_j^0$  for  $j = 1, \dots, N$ .

As in Section 2.5, the nodal vectors  $\mathbf{v} \in \mathbb{R}^{3N}$  of the finite element function  $v_h$  together with the surface nodal vector  $\mathbf{x} \in \mathbb{R}^{3N}$  satisfy a system of differential-algebraic equations (DAEs). We obtain from (5.1) and (2.5) the following coupled DAE system for the nodal values  $\mathbf{v}$  and  $\mathbf{x}$ :

$$\begin{aligned} \mathbf{K}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x}, t), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \quad (5.2)$$

Here the matrix  $\mathbf{K}(\mathbf{x}) = I_3 \otimes (\mathbf{M}(\mathbf{x}) + \alpha \mathbf{A}(\mathbf{x}))$  is from (2.6), and the driving term  $\mathbf{g}(\mathbf{x}, t)$  is given by

$$\mathbf{g}(\mathbf{x}, t)|_{3(j-1)+\ell} = \int_{\Gamma_h(\mathbf{x})} g(\cdot, t) (\nu_{\Gamma_h(\mathbf{x})})_\ell \phi_j[\mathbf{x}], \quad (j = 1, \dots, N, \ell = 1, 2, 3).$$

### 5.2 Error equations

We denote by

$$\mathbf{x}^*(t) = (x_j^*(t)) \in \mathbb{R}^{3N} \quad \text{with} \quad x_j^*(t) = X(p_j, t), \quad (j = 1, \dots, N)$$

the nodal vector of the *exact* positions on the surface  $\Gamma(X(\cdot, t))$ . This defines a discrete surface  $\Gamma_h(\mathbf{x}^*(t))$  that interpolates the exact surface  $\Gamma(X(\cdot, t))$ .



We consider the interpolated exact velocity

$$v_h^*(\cdot, t) = \sum_{j=1}^N v_j^*(t) \phi_j[\mathbf{x}^*(t)] \quad \text{with} \quad v_j^*(t) = \dot{x}_j^*(t),$$

with the corresponding nodal vector

$$\mathbf{v}^*(t) = (v_j^*(t)) = \dot{\mathbf{x}}^*(t) \in \mathbb{R}^{3N}.$$

Inserting  $v_h^*$  and  $\mathbf{x}^*$  in place of the numerical solution  $v_h$  and  $\mathbf{x}$  into (5.1) yields a defect  $d_h(\cdot, t) \in S_h(\mathbf{x}^*(t))^3$ : for every  $\psi_h \in S_h(\mathbf{x}^*(t))^3$ ,

$$\int_{\Gamma_h(\mathbf{x}^*)} v_h^* \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} v_h^* \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \psi_h = \int_{\Gamma_h(\mathbf{x}^*)} g \nu_{\Gamma_h(\mathbf{x}^*)} \cdot \psi_h + \int_{\Gamma_h(\mathbf{x}^*)} d_h \cdot \psi_h.$$

With  $d_h(\cdot, t) = \sum_{j=1}^N d_j(t) \phi_j[\mathbf{x}^*(t)]$  and the corresponding nodal vector  $\mathbf{d}_v(t) = (d_j(t)) \in \mathbb{R}^{3N}$  we then have  $(I_3 \otimes \mathbf{M}(\mathbf{x}^*(t))) \mathbf{d}_v(t)$  as the defect on inserting  $\mathbf{x}^*$  and  $\mathbf{v}^*$  in the first equation of (5.2). With  $\mathbf{M}^{[3]}(\mathbf{x}^*) = I_3 \otimes \mathbf{M}(\mathbf{x}^*)$ , we thus have

$$\begin{aligned} \mathbf{K}(\mathbf{x}^*) \mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*) + \mathbf{M}^{[3]}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{x}}^* &= \mathbf{v}^*. \end{aligned} \tag{5.3}$$

We denote the errors in the surface nodes and in the velocity by  $\mathbf{e}_x = \mathbf{x} - \mathbf{x}^*$  and  $\mathbf{e}_v = \mathbf{v} - \mathbf{v}^*$ , respectively. We rewrite the velocity law in (5.2) as

$$\mathbf{K}(\mathbf{x}^*) \mathbf{v} = -(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* - (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{e}_v + \mathbf{g}(\mathbf{x}).$$

Then, by subtracting (5.3) from the above version of (5.2), we obtain the following error equations for the uncoupled problem:

$$\begin{aligned} \mathbf{K}(\mathbf{x}^*) \mathbf{e}_v &= -(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* - (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{e}_v \\ &\quad + (\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{M}^{[3]}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{e}}_x &= \mathbf{e}_v. \end{aligned} \tag{5.4}$$

When no confusion can arise, we write in the following  $\mathbf{M}(\mathbf{x}^*)$  for  $\mathbf{M}^{[3]}(\mathbf{x}^*)$  and  $\|\cdot\|_{H^1(\Gamma)}$  for  $\|\cdot\|_{H^1(\Gamma)^3}$ , etc.

### 5.3 Norms

We recall that  $\mathbf{K}(\mathbf{x}^*) = I_3 \otimes (\mathbf{M}(\mathbf{x}^*) + \alpha \mathbf{A}(\mathbf{x}^*))$  and, for  $\mathbf{w} \in \mathbb{R}^{3N}$  and the corresponding finite element function  $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}^*] \in S_h(\mathbf{x}^*)^3$ , we consider the norm

$$\begin{aligned} \|\mathbf{w}\|_{\mathbf{K}(\mathbf{x}^*)}^2 &:= \mathbf{w}^T \mathbf{K}(\mathbf{x}^*) \mathbf{w} \\ &= \|w_h\|_{L^2(\Gamma_h(\mathbf{x}^*))}^2 + \alpha \|\nabla_{\Gamma_h(\mathbf{x}^*)} w_h\|_{L^2(\Gamma_h(\mathbf{x}^*))}^2 \sim \|w_h\|_{H^1(\Gamma_h(\mathbf{x}^*))}^2. \end{aligned}$$

For convenience, we will take  $\alpha = 1$  in the remainder of this section, so that the last norm equivalence becomes an equality. For the defect  $d_h \in S_h(\mathbf{x}^*)^3$  we use the dual norm (cf. [24, Proof of Theorem 5.1])

$$\begin{aligned} \|d_h\|_{H_h^{-1}(\Gamma_h(\mathbf{x}^*))} &:= \sup_{0 \neq \psi_h \in S_h(\mathbf{x}^*)^3} \frac{\int_{\Gamma_h(\mathbf{x}^*)} d_h \cdot \psi_h}{\|\psi_h\|_{H^1(\Gamma_h(\mathbf{x}^*))}^3} \\ &= \sup_{0 \neq \mathbf{z} \in \mathbb{R}^{3N}} \frac{\mathbf{d}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{z}}{(\mathbf{z}^T \mathbf{K}(\mathbf{x}^*) \mathbf{z})^{\frac{1}{2}}} = \sup_{0 \neq \mathbf{w} \in \mathbb{R}^{3N}} \frac{\mathbf{d}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{K}(\mathbf{x}^*)^{-\frac{1}{2}} \mathbf{w}}{(\mathbf{w}^T \mathbf{w})^{\frac{1}{2}}} \\ &= \|\mathbf{K}(\mathbf{x}^*)^{-\frac{1}{2}} \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v\|_2 = (\mathbf{d}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{K}(\mathbf{x}^*)^{-1} \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v)^{\frac{1}{2}}. \end{aligned} \quad (5.5)$$

We denote

$$\|\mathbf{d}_v\|_{*, \mathbf{x}^*}^2 := \mathbf{d}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{K}(\mathbf{x}^*)^{-1} \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v,$$

so that

$$\|\mathbf{d}_v\|_{*, \mathbf{x}^*} = \|d_h\|_{H_h^{-1}(\Gamma_h(\mathbf{x}^*))}.$$

#### 5.4 Stability estimate

The following stability result holds for the errors  $\mathbf{e}_v$  and  $\mathbf{e}_x$ , under an assumption of small defects. It will be shown in Section 8 that this assumption is satisfied if the exact solution is sufficiently smooth.

**Proposition 5.1** *Suppose that the defect is bounded as follows, with  $\kappa > 1$ :*

$$\|\mathbf{d}_v(t)\|_{*, \mathbf{x}^*(t)} \leq ch^\kappa, \quad t \in [0, T].$$

*Then there exists  $h_0 > 0$  such that the following error bounds hold for  $h \leq h_0$  and  $0 \leq t \leq T$ :*

$$\|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*(t))}^2 \leq C \int_0^t \|\mathbf{d}_v(s)\|_{*, \mathbf{x}^*}^2 ds, \quad (5.6)$$

$$\|\mathbf{e}_v(t)\|_{\mathbf{K}(\mathbf{x}^*(t))}^2 \leq C \|\mathbf{d}_v(t)\|_{*, \mathbf{x}^*}^2 + C \int_0^t \|\mathbf{d}_v(s)\|_{*, \mathbf{x}^*}^2 ds. \quad (5.7)$$

*The constant  $C$  is independent of  $t$  and  $h$ , but depends on the final time  $T$  and on the regularization parameter  $\alpha$ .*

We note that the error functions  $e_v(\cdot, t), e_x(\cdot, t) \in S_h(\mathbf{x}^*(t))^3$  with nodal vectors  $\mathbf{e}_v(t)$  and  $\mathbf{e}_x(t)$ , respectively, are then bounded by

$$\|e_v(\cdot, t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))} \leq Ch^\kappa \quad \text{and} \quad \|e_x(\cdot, t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))} \leq Ch^\kappa, \quad t \in [0, T].$$

*Proof* The proof uses energy estimates for the error equations (5.4) in the matrix-vector formulation, and it relies on the results of Section 4. In the course of this proof  $c$  and  $C$  will be generic constants that take on different values on different occurrences.

In view of condition (4.3) for  $\mathbf{y} = \mathbf{x}^*(t)$ , we will need to control the  $W^{1,\infty}$  norm of the position error  $e_x(\cdot, t)$ . Let  $0 < t^* \leq T$  be the maximal time such that

$$\|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot, t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} \leq h^{(\kappa-1)/2} \quad \text{for } t \in [0, t^*]. \quad (5.8)$$

At  $t = t^*$  either this inequality becomes an equality, or else we have  $t^* = T$ .

We will first prove the stated error bounds for  $0 \leq t \leq t^*$ . Then the proof will be finished by showing that in fact  $t^*$  coincides with  $T$ .

By testing the first equation in (5.4) with  $\mathbf{e}_v$ , and dropping the omnipresent argument  $t \in [0, t^*]$ , we obtain:

$$\begin{aligned} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 &= \mathbf{e}_v^T \mathbf{K}(\mathbf{x}^*) \mathbf{e}_v = -\mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* \\ &\quad - \mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{e}_v \\ &\quad + \mathbf{e}_v^T (\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{e}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v. \end{aligned}$$

We separately estimate the four terms on the right-hand side in an appropriate way, with Lemmas 4.1 – 4.4 as our main tools.

(i) We denote, for  $0 \leq \theta \leq 1$ , by  $e_v^\theta$  and  $v_h^{*,\theta}$  the finite element functions in  $S_h(\Gamma_h^\theta)^3$  for  $\Gamma_h^\theta = \Gamma_h(\mathbf{x}^* + \theta \mathbf{e}_x)$  with nodal vectors  $\mathbf{e}_v$  and  $\mathbf{v}^*$ , respectively. Lemma 4.1 then gives us

$$\begin{aligned} \mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* &= \int_0^1 \int_{\Gamma_h^\theta} e_v^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) v_h^{*,\theta} d\theta + \alpha \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_v^\theta \cdot (D_{\Gamma_h^\theta} e_x^\theta) \nabla_{\Gamma_h^\theta} v_h^{*,\theta} d\theta. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we estimate the integral with the product of the  $L^2 - L^2 - L^\infty$  norms of the three factors. We thus have

$$\begin{aligned} \mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* &\leq \int_0^1 \|e_v^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} d\theta \\ &\quad + \alpha \int_0^1 \|\nabla_{\Gamma_h^\theta} e_v^\theta\|_{L^2(\Gamma_h^\theta)} \|D_{\Gamma_h^\theta} e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} d\theta \\ &\leq c \int_0^1 \|e_v^\theta\|_{H^1(\Gamma_h^\theta)} \|e_x^\theta\|_{H^1(\Gamma_h^\theta)} \|v_h^{*,\theta}\|_{W^{1,\infty}(\Gamma_h^\theta)} d\theta. \end{aligned}$$

By (5.8) and Lemma 4.3, this is bounded by

$$\mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* \leq c \|e_v\|_{H^1(\Gamma_h(\mathbf{x}^*))} \|e_x\|_{H^1(\Gamma_h(\mathbf{x}^*))} \|v_h^*\|_{W^{1,\infty}(\Gamma_h(\mathbf{x}^*))},$$

where the last factor is bounded independently of  $h$ . By the Young inequality, we thus obtain

$$\begin{aligned} \mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* &\leq \frac{1}{6} \|e_v\|_{H^1(\Gamma_h(\mathbf{x}^*))}^2 + C \|e_x\|_{H^1(\Gamma_h(\mathbf{x}^*))}^2 \\ &= \frac{1}{6} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 + C \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2. \end{aligned}$$

(ii) Similarly, estimating the three factors in the integrals by  $L^2 - L^\infty - L^2$ , we obtain

$$\begin{aligned} \mathbf{e}_\mathbf{v}^T(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*))\mathbf{e}_\mathbf{v} &\leq c\|\mathbf{e}_\mathbf{v}\|_{L^2(\Gamma_h(\mathbf{x}^*))}^2 \|\nabla_{\Gamma_h} \cdot \mathbf{e}_\mathbf{x}\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\quad + c\alpha \|\nabla_{\Gamma_h} \mathbf{e}_\mathbf{v}\|_{L^2(\Gamma_h(\mathbf{x}^*))}^2 \|D_{\Gamma_h} \mathbf{e}_\mathbf{x}\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\leq ch^{(\kappa-1)/2} \|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2, \end{aligned}$$

where in the last inequality we used the bound (5.8).

(iii) In the following bound we use Lemma 4.5. Again with the finite element function  $e_v^\theta = \sum_{j=1}^N (\mathbf{e}_\mathbf{v})_j \phi_j[\mathbf{x}^* + \theta \mathbf{e}_\mathbf{x}]$  on the surface  $\Gamma_h^\theta = \Gamma_h(\mathbf{x}^* + \theta \mathbf{e}_\mathbf{x})$ , for  $0 \leq \theta \leq 1$ , we write

$$\mathbf{e}_\mathbf{v}^T(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) = \int_{\Gamma_h^1} g \nu_{\Gamma_h^1} \cdot \mathbf{e}_v^1 - \int_{\Gamma_h^0} g \nu_{\Gamma_h^0} \cdot \mathbf{e}_v^0 = \int_0^1 \frac{d}{d\theta} \int_{\Gamma_h^\theta} g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta d\theta.$$

Using the Leibniz formula, this becomes

$$\mathbf{e}_\mathbf{v}^T(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) = \int_0^1 \int_{\Gamma_h^\theta} \left( \partial_\theta^\bullet (g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta) + (g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta) (\nabla_{\Gamma_h^\theta} \cdot \mathbf{e}_x^\theta) \right) d\theta.$$

Here we have, noting that  $\partial_\theta^\bullet \mathbf{e}_v^\theta = 0$ ,

$$\partial_\theta^\bullet (g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta) = g' e_x^\theta \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta + g \partial_\theta^\bullet \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta.$$

With Lemmas 4.3 and 4.5 we therefore obtain via the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\Gamma_h^\theta} \partial_\theta^\bullet (g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta) &\leq c_2^2 \|g'\|_{L^\infty} \|e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))} \|\mathbf{e}_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \\ &\quad + c_2^2 \|g\|_{L^\infty} \|\nabla_{\Gamma_h(\mathbf{x}^*)} \mathbf{e}_x\|_{L^2(\Gamma_h(\mathbf{x}^*))} \|\mathbf{e}_v\|_{L^2(\Gamma_h(\mathbf{x}^*))}, \end{aligned}$$

and again with Lemma 4.3,

$$\int_{\Gamma_h^\theta} (g \nu_{\Gamma_h^\theta} \cdot \mathbf{e}_v^\theta) (\nabla_{\Gamma_h^\theta} \cdot \mathbf{e}_x^\theta) \leq c_2^2 \|g\|_{L^\infty} \|\mathbf{e}_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \|\nabla_{\Gamma_h(\mathbf{x}^*)} \cdot \mathbf{e}_x\|_{L^2(\Gamma_h(\mathbf{x}^*))}.$$

In total, we obtain a bound of the same type as for the terms in (i) and (ii):

$$\begin{aligned} \mathbf{e}_\mathbf{v}^T(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) &\leq c\|e_x\|_{H^1(\Gamma_h(\mathbf{x}^*))} \|\mathbf{e}_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \\ &= c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)} \|\mathbf{e}_\mathbf{v}\|_{\mathbf{M}(\mathbf{x}^*)} \leq \frac{1}{6} \|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + C\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2. \end{aligned}$$

The combination of the estimates of the three terms (i)–(iii) with absorptions (for sufficiently small  $h \leq h_0$ ), and a simple dual norm estimate, based on (5.5), for the defect term, yield the bound

$$\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{d}_\mathbf{v}\|_{\star, \mathbf{x}^*}^2. \quad (5.9)$$

Using this estimate, together with taking the  $\|\cdot\|_{\mathbf{K}(\mathbf{x}^*)}$  norm of both sides of the second equation in (5.4), we obtain

$$\|\dot{\mathbf{e}}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 = \|\mathbf{e}_{\mathbf{v}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq c\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{d}_{\mathbf{v}}\|_{\star, \mathbf{x}^*}^2. \quad (5.10)$$

In order to apply Gronwall's inequality, we connect  $\frac{d}{dt}\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2$  and  $\|\dot{\mathbf{e}}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2$  as follows:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 &= \mathbf{e}_{\mathbf{x}}^T \mathbf{K}(\mathbf{x}^*) \dot{\mathbf{e}}_{\mathbf{x}} + \frac{1}{2} \mathbf{e}_{\mathbf{x}}^T \left( \frac{d}{dt} \mathbf{K}(\mathbf{x}^*) \right) \mathbf{e}_{\mathbf{x}} \\ &\leq \|\dot{\mathbf{e}}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2, \end{aligned}$$

where we use the Cauchy-Schwarz inequality and Lemma 4.6 in the estimate. Inserting (5.10), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq c\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{d}_{\mathbf{v}}\|_{\star, \mathbf{x}^*}^2.$$

A Gronwall inequality then yields (5.6), using  $e_j(0) = x_j(0) - x_j^0 = 0$  for  $j = 1, \dots, N$ . Inserting this estimate in (5.9), we can bound  $\mathbf{e}_{\mathbf{v}}(t)$  for  $0 \leq t \leq t^*$  by (5.7).

Now it only remains to show that  $t^* = T$  for  $h$  sufficiently small. For  $0 \leq t \leq t^*$  we use an inverse inequality and (5.6) to bound the left-hand side in (5.8):

$$\begin{aligned} \|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot, t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} &\leq ch^{-1} \|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot, t)\|_{L^2(\Gamma_h(\mathbf{x}^*(t)))} \\ &\leq ch^{-1} \|\mathbf{e}_{\mathbf{x}}(t)\|_{\mathbf{K}(\mathbf{x}^*(t))} \leq cCh^{\kappa-1} \leq \frac{1}{2}h^{(\kappa-1)/2} \end{aligned}$$

for sufficiently small  $h$ . Hence, we can extend the bound (5.8) beyond  $t^*$ , which contradicts the maximality of  $t^*$  unless we have already  $t^* = T$ .  $\square$

## 6 Stability of coupling surface PDEs to surface motion

Now we turn to the stability bounds of the original problem (2.4)–(2.5), or in DAE form (2.7), which is the formulation we will actually use for the stability analysis.

### 6.1 Error equations

Similarly as before, in order to derive stability estimates we consider the DAE system when we insert the nodal values  $\mathbf{u}^*(t) \in \mathbb{R}^N$  of the exact solution  $u(\cdot, t)$ , the nodal values  $\mathbf{x}^*(t) \in \mathbb{R}^{3N}$  of the exact positions  $X(\cdot, t)$ , and the nodal values  $\mathbf{v}^*(t) \in \mathbb{R}^{3N}$  of the exact velocity  $v(\cdot, t)$ . Inserting them into (2.7)

yields defects  $\mathbf{d}_u(t) \in \mathbb{R}^N$  and  $\mathbf{d}_v(t) \in \mathbb{R}^{3N}$ : omitting the argument  $t$  in the notation, we have

$$\begin{aligned} \frac{d}{dt} \left( \mathbf{M}(\mathbf{x}^*) \mathbf{u}^* \right) + \mathbf{A}(\mathbf{x}^*) \mathbf{u}^* &= \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*) + \mathbf{M}(\mathbf{x}^*) \mathbf{d}_u, \\ \mathbf{K}(\mathbf{x}^*) \mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*, \mathbf{u}^*) + \mathbf{M}^{[3]}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{x}}^* &= \mathbf{v}^*, \end{aligned} \quad (6.1)$$

where again  $\mathbf{M}^{[3]}(\mathbf{x}^*) = I_3 \otimes \mathbf{M}(\mathbf{x}^*)$ . As no confusion can arise, we write again  $\mathbf{M}(\mathbf{x}^*)$  for  $\mathbf{M}^{[3]}(\mathbf{x}^*)$ .

We denote the PDE error by  $\mathbf{e}_u = \mathbf{u} - \mathbf{u}^*$ , and as in the previous section,  $\mathbf{e}_v = \mathbf{v} - \mathbf{v}^*$  and  $\mathbf{e}_x = \mathbf{x} - \mathbf{x}^*$  denote the velocity error and surface error, respectively. Subtracting (6.1) from (2.7), we obtain the following error equation:

$$\begin{aligned} \frac{d}{dt} \left( \mathbf{M}(\mathbf{x}^*) \mathbf{e}_u \right) + \mathbf{A}(\mathbf{x}^*) \mathbf{e}_u &= - \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{u}^* \right) \\ &\quad - \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\ &\quad - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{u}^* \\ &\quad - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{e}_u \\ &\quad + (\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) - \mathbf{M}(\mathbf{x}^*) \mathbf{d}_u, \\ \mathbf{K}(\mathbf{x}^*) \mathbf{e}_v &= - (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* - (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{e}_v \\ &\quad + (\mathbf{g}(\mathbf{x}, \mathbf{u}) - \mathbf{g}(\mathbf{x}^*, \mathbf{u}^*)) - \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{e}}_x &= \mathbf{e}_v. \end{aligned} \quad (6.2)$$

## 6.2 Stability estimate

We now formulate the stability result for the errors  $\mathbf{e}_u$ ,  $\mathbf{e}_v$  and  $\mathbf{e}_x$  of the surface motion coupled to the surface PDE. Here, we use the norms (4.1)-(4.2) and those of Section 5.3.

**Proposition 6.1** *Assume that the following bounds hold for the defects, for some  $\kappa > 1$ :*

$$\|\mathbf{d}_u(t)\|_{*, \mathbf{x}^*(t)} \leq ch^\kappa, \quad \|\mathbf{d}_v(t)\|_{*, \mathbf{x}^*(t)} \leq ch^\kappa, \quad \text{for } t \in [0, T].$$

*Then there exists  $h_0 > 0$  such that the following stability estimate holds for all  $h \leq h_0$  and  $0 \leq t \leq T$ :*

$$\begin{aligned} \|\mathbf{e}_u(t)\|_{\mathbf{M}(\mathbf{x}^*)}^2 &+ \int_0^t \|\mathbf{e}_u(s)\|_{\mathbf{A}(\mathbf{x}^*)}^2 ds + \|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*)}^2 + \int_0^t \|\mathbf{e}_v(s)\|_{\mathbf{K}(\mathbf{x}^*)}^2 ds \\ &\leq C \int_0^t \left( \|\mathbf{d}_u(s)\|_{*, \mathbf{x}^*}^2 + \|\mathbf{d}_v(s)\|_{*, \mathbf{x}^*}^2 \right) ds. \end{aligned} \quad (6.3)$$

The constant  $C$  is independent of  $t$  and  $h$ , but depends on the final time  $T$  and on the regularization parameter  $\alpha$ .

We note that the error functions  $e_u(\cdot, t) \in S_h(\mathbf{x}^*(t))$  and  $e_v(\cdot, t), e_x(\cdot, t) \in S_h(\mathbf{x}^*(t))^3$  with nodal vectors  $\mathbf{e}_u(t)$  and  $\mathbf{e}_v(t), \mathbf{e}_x(t)$ , respectively, are then bounded by

$$\begin{aligned} & \|e_u(\cdot, t)\|_{L^2(\Gamma_h(\mathbf{x}^*(t)))} + \left( \int_0^t \|e_u(\cdot, s)\|_{H^1(\Gamma_h(\mathbf{x}^*(s)))}^2 ds \right)^{1/2} \leq Ch^\kappa, \\ & \left( \int_0^t \|e_v(\cdot, s)\|_{H^1(\Gamma_h(\mathbf{x}^*(s)))^3}^2 ds \right)^{1/2} \leq Ch^\kappa, \\ & \|e_x(\cdot, t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))^3} \leq Ch^\kappa, \end{aligned} \quad t \in [0, T]. \quad (6.4)$$

*Proof* The proof is an extension of the proof of Proposition 5.1, again based on the matrix-vector formulation and the auxiliary results of Section 4. We handle the surface PDE and the surface equations separately: we first estimate the errors of the PDE, while those for the surface equation are based on Section 5. Finally we will combine the results to obtain the stability estimates for the coupled problem. In the course of this proof  $c$  and  $C$  will be generic constants that take on different values on different occurrences.

Let  $0 < t^* \leq T$  be the maximal time such that the following inequalities hold:

$$\begin{aligned} & \|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot, t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} \leq h^{(\kappa-1)/2}, \\ & \|e_u(\cdot, t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} \leq 1, \end{aligned} \quad \text{for } t \in [0, t^*]. \quad (6.5)$$

Note that  $t^* > 0$  since initially both  $e_x(\cdot, 0) = 0$  and  $e_u(\cdot, 0) = 0$ .

We first prove the stated error bounds for  $0 \leq t \leq t^*$ . At the end, the proof will be finished by showing that in fact  $t^*$  coincides with  $T$ .

Testing the first two equations of (6.2) with  $\mathbf{e}_u$  and  $\mathbf{e}_v$ , and dropping the omnipresent argument  $t \in [0, t^*]$ , we obtain:

$$\begin{aligned} \mathbf{e}_u^T \frac{d}{dt} (\mathbf{M}(\mathbf{x}^*) \mathbf{e}_u) + \mathbf{e}_u^T \mathbf{A}(\mathbf{x}^*) \mathbf{e}_u &= -\mathbf{e}_u^T \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{u}^* \right) \\ &\quad -\mathbf{e}_u^T \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\ &\quad -\mathbf{e}_u^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{u}^* \\ &\quad -\mathbf{e}_u^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{e}_u \\ &\quad +\mathbf{e}_u^T (\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) - \mathbf{e}_u^T \mathbf{M}(\mathbf{x}^*) \mathbf{d}_u, \\ \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 &= -\mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{v}^* - \mathbf{e}_v^T (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)) \mathbf{e}_v \\ &\quad +\mathbf{e}_v^T (\mathbf{g}(\mathbf{x}, \mathbf{u}) - \mathbf{g}(\mathbf{x}^*, \mathbf{u}^*)) - \mathbf{e}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{e}}_x &= \mathbf{e}_v. \end{aligned}$$

(A) *Estimates for the surface PDE:* We estimate the terms separately, with Lemmas 4.1 – 4.3 as our main tools.

(i) The symmetry of  $\mathbf{M}(\mathbf{x}^*)$  and a simple calculation yield

$$\begin{aligned} \mathbf{e}_u^T \frac{d}{dt} (\mathbf{M}(\mathbf{x}^*) \mathbf{e}_u) &= \frac{1}{2} \frac{d}{dt} (\mathbf{e}_u^T \mathbf{M}(\mathbf{x}^*) \mathbf{e}_u) + \frac{1}{2} \mathbf{e}_u^T \left( \frac{d}{dt} \mathbf{M}(\mathbf{x}^*) \right) \mathbf{e}_u \\ &= \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \frac{1}{2} \mathbf{e}_u^T \left( \frac{d}{dt} \mathbf{M}(\mathbf{x}^*) \right) \mathbf{e}_u, \end{aligned}$$

where the last term is bounded by Lemma 4.6 as

$$\left| \mathbf{e}_u^T \frac{d}{dt} \mathbf{M}(\mathbf{x}^*) \mathbf{e}_u \right| \leq c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2.$$

(ii) By the definition of the  $\mathbf{A}$ -norm we have

$$\mathbf{e}_u^T \mathbf{A}(\mathbf{x}^*) \mathbf{e}_u = \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2.$$

(iii) With the product rule we write

$$\begin{aligned} \mathbf{e}_u^T \frac{d}{dt} ((\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{u}^*) \\ = \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \dot{\mathbf{u}}^* + \mathbf{e}_u^T \left( \frac{d}{dt} (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \right) \mathbf{u}^*. \end{aligned} \quad (6.6)$$

With  $\Gamma_h^\theta(t) = \Gamma_h[\mathbf{x}^*(t) + \theta \mathbf{e}_x(t)]$  and with the finite element functions

$$e_u^\theta(\cdot, t), u_h^{*,\theta}(\cdot, t) \in S_h(\mathbf{x}^*(t) + \theta \mathbf{e}_x(t)) \text{ with nodal vectors } \mathbf{e}_u(t), \mathbf{u}^*(t), \text{ resp.,}$$

Lemma 4.1 (with  $\mathbf{x}^*(t)$  in the role of  $\mathbf{y}$ ) yields for the first term, omitting again the argument  $t$ ,

$$\mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \dot{\mathbf{u}}^* = \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \partial_h^\bullet u_h^{*,\theta} d\theta.$$

Using the Cauchy-Schwarz inequality we obtain

$$|\mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \dot{\mathbf{u}}^*| \leq \int_0^1 \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|\partial_h^\bullet u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} d\theta.$$

Under condition (6.5) we obtain from Lemmas 4.2 and 4.3 that for  $0 \leq t \leq t^*$ ,

$$|\mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \dot{\mathbf{u}}^*| \leq c \|e_u^0\|_{L^2(\Gamma_h^0)} \|e_x^0\|_{H^1(\Gamma_h^0)} \|\partial_h^\bullet u_h^{*,0}\|_{L^\infty(\Gamma_h^0)}.$$

Now, the last factor is bounded by

$$\|\partial_h^\bullet u_h^{*,0}\|_{L^\infty(\Gamma_h^0)} \leq c \|\dot{\mathbf{u}}^*\|_\infty \leq C$$

because of the assumed smoothness of the exact solution  $u$  and hence of its material derivative  $\partial^\bullet u(\cdot, t)$ , whose values at the nodes are the entries of the vector  $\dot{\mathbf{u}}^*(t)$ . Hence we obtain, on recalling the definitions of the discrete norms,

$$-\mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \dot{\mathbf{u}}^* \leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}.$$



Using Lemma 4.1 together with the Leibniz formula, the last term in (6.6) becomes

$$\begin{aligned} \mathbf{e}_{\mathbf{u}}^T \left( \frac{d}{dt} (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \right) \mathbf{u}^* &= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \partial_h^\bullet (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) u_h^{*,\theta} d\theta \\ &\quad + \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) u_h^{*,\theta} (\nabla_{\Gamma_h^\theta} \cdot v_h^\theta) d\theta, \end{aligned}$$

where  $v_h^\theta$  is the velocity of  $\Gamma_h^\theta$  (as a function of  $t$ ), which is the finite element function in  $S_h(\mathbf{x}^* + \theta \mathbf{e}_{\mathbf{x}})$  with nodal vector  $\dot{\mathbf{x}}^* + \theta \dot{\mathbf{e}}_{\mathbf{x}} = \mathbf{v}^* + \theta \mathbf{e}_{\mathbf{v}}$ . Thus

$$v_h^\theta = v_h^{*,\theta} + \theta e_v^\theta, \quad (6.7)$$

where  $v_h^{*,\theta}$  and  $e_v^\theta$  are the finite element functions on  $\Gamma_h^\theta$  with nodal vectors  $\mathbf{v}^*$  and  $\mathbf{e}_{\mathbf{v}}$ , respectively. In the first integral we further use, cf. [14, Lemma 2.6],

$$\partial_h^\bullet (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) = \nabla_{\Gamma_h^\theta} \cdot \partial_h^\bullet e_x^\theta - ((I_3 - \nu_h^\theta (\nu_h^\theta)^T) \nabla_{\Gamma_h^\theta} v_h^\theta) : \nabla_{\Gamma_h^\theta} e_x^\theta,$$

where  $:$  symbolizes the Euclidean inner product of the vectorization of two matrices. Here we note that  $\partial_h^\bullet e_x^\theta$  is the finite element function on  $\Gamma_h^\theta$  with nodal vector  $\dot{\mathbf{e}}_{\mathbf{x}} = \mathbf{e}_{\mathbf{v}}$ , so that  $\partial_h^\bullet e_x^\theta = e_v^\theta$ .

We then estimate, using the Cauchy-Schwarz inequality in the first step, Lemmas 4.2 and 4.3 in the second step (using (6.5) to ensure the smallness condition in these lemmas), the definition of the discrete norms in the third step, and using the first bound of (6.5) and the boundedness of the discrete gradient of the interpolated exact velocity  $\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*$  and of the interpolated exact solution  $u_h^*$  in the fourth step,

$$\begin{aligned} &\left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \partial_h^\bullet (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) u_h^{*,\theta} d\theta \right| \\ &\leq \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \left( \|\nabla_{\Gamma_h^\theta} \cdot e_v^\theta\|_{L^2(\Gamma_h^\theta)} \right. \\ &\quad \left. + \|\nabla_{\Gamma_h^\theta} v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \cdot \|\nabla_{\Gamma_h^\theta} e_x^\theta\|_{L^2(\Gamma_h^\theta)} \right. \\ &\quad \left. + \|\nabla_{\Gamma_h^\theta} e_v^\theta\|_{L^2(\Gamma_h^\theta)} \cdot \|\nabla_{\Gamma_h^\theta} e_x^\theta\|_{L^\infty(\Gamma_h^\theta)} \right) \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} d\theta \\ &\leq c \|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} \left( \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \right. \\ &\quad \left. + \|\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \cdot \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))} \right. \\ &\quad \left. + \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \cdot \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \right) \|u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\leq c \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_{\mathbf{v}}\|_{\mathbf{A}(\mathbf{x}^*)} + \|\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{A}(\mathbf{x}^*)} \right. \\ &\quad \left. + \|\mathbf{e}_{\mathbf{v}}\|_{\mathbf{A}(\mathbf{x}^*)} \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \right) \|\mathbf{u}^*\|_\infty \\ &\leq c \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_{\mathbf{v}}\|_{\mathbf{K}(\mathbf{x}^*)} + C \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)} h^{(\kappa-1)/2} \right) C \\ &\leq C' \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_{\mathbf{v}}\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)} \right). \end{aligned}$$

With the same arguments we estimate, on inserting (6.7),

$$\begin{aligned}
& \left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) u_h^{*,\theta} (\nabla_{\Gamma_h^\theta} \cdot v_h^\theta) d\theta \right| \\
& \leq \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} d\theta \\
& \quad + \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^\infty(\Gamma_h^\theta)} \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_v^\theta\|_{L^2(\Gamma_h^\theta)} d\theta \\
& \leq c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \|\mathbf{u}^*\|_\infty \|\nabla_{\Gamma_h(\mathbf{x}^*)} \cdot v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
& \quad + c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \|\mathbf{u}^*\|_\infty \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} \\
& \leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right).
\end{aligned}$$

Altogether we obtain the bound

$$-\mathbf{e}_u^T \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{u}^* \right) \leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right).$$

(iv) We obtain similarly

$$\begin{aligned}
& -\mathbf{e}_u^T \frac{d}{dt} \left( (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\
& = -\frac{1}{2} \mathbf{e}_u^T \left( \frac{d}{dt} (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \right) \mathbf{e}_u - \frac{1}{2} \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\
& \leq c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right) \|\mathbf{e}_u\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
& \quad - \frac{1}{2} \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\
& \leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right) - \frac{1}{2} \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right),
\end{aligned}$$

where we used the second bound of (6.5) in the last inequality.

(v) Lemma 4.1, the Cauchy-Schwarz inequality and Lemma 4.3 yield

$$\begin{aligned}
& -\mathbf{e}_u^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{u}^* \\
& = - \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_u^\theta \cdot (D_{\Gamma_h^\theta} e_x^\theta) \nabla_{\Gamma_h^\theta} u_h^{*,\theta} d\theta \\
& \leq c \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{A}(\mathbf{x}^*)} \|\nabla_{\Gamma_h(\mathbf{x}^*)} u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
& \leq C \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}.
\end{aligned}$$

(vi) Similarly we estimate

$$\begin{aligned}
-\mathbf{e}_u^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{e}_u & \leq c \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 \|D_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
& \leq C h^{(\kappa-1)/2} \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2,
\end{aligned}$$

where we used the first bound of (6.5).

(vii) The coupling term is estimated similarly to (iii) in the proof of Proposition 5.1:

$$\mathbf{e}_u^T(\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) = \int_{\Gamma_h^1} f(u_h, \nabla_{\Gamma_h^1} u_h) e_u^1 - \int_{\Gamma_h^0} f(u_h^*, \nabla_{\Gamma_h^0} u_h^*) e_u^0.$$

With

$$u_h^\theta = \sum_{j=1}^N (u_j^* + \theta(\mathbf{e}_u)_j) \phi_j[\mathbf{x}^* + \theta \mathbf{e}_x] = u_h^{*,\theta} + \theta e_u^\theta \quad (6.8)$$

we therefore have

$$\mathbf{e}_u^T(\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) = \int_0^1 \frac{d}{d\theta} \int_{\Gamma_h^\theta} f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) e_u^\theta d\theta$$

and with the Leibniz formula (noting that  $e_x^\theta$  is the velocity of the surface  $\Gamma_h^\theta$  considered as a function of  $\theta$ ), we rewrite this as

$$\begin{aligned} & \mathbf{e}_u^T(\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) \\ &= \int_0^1 \int_{\Gamma_h^\theta} \left( \partial_\theta^\bullet f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) e_u^\theta + f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) e_u^\theta (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \right) d\theta. \end{aligned}$$

Here we use the chain rule

$$\partial_\theta^\bullet f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) = \partial_1 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) \partial_\theta^\bullet u_h^\theta + \partial_2 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) \partial_\theta^\bullet \nabla_{\Gamma_h^\theta} u_h^\theta$$

and observe the following: by the assumed smoothness of  $f$  and the exact solution  $u$ , and by the bound (6.5) for  $e_u$  (and hence for  $e_u^\theta$  by Lemmas 4.2 and 4.3), we have on recalling (6.8)

$$\|\partial_i f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\|_{L^\infty(\Gamma_h^\theta)} \leq C, \quad i = 1, 2.$$

We note

$$\partial_\theta^\bullet u_h^\theta = e_u^\theta$$

and the relation, see [14, Lemma 2.6],

$$\partial_\theta^\bullet \nabla_{\Gamma_h^\theta} u_h^\theta = \nabla_{\Gamma_h^\theta} \partial_\theta^\bullet u_h^\theta - \nabla_{\Gamma_h^\theta} e_x^\theta \nabla_{\Gamma_h^\theta} u_h^\theta + \nu_h^\theta (\nu_h^\theta)^T (\nabla_{\Gamma_h^\theta} e_x^\theta)^T \nabla_{\Gamma_h^\theta} u_h^\theta.$$

We then have, on inserting (6.8) and using once again Lemmas 4.2 and 4.3 and the bound (6.5),

$$\begin{aligned}
& \mathbf{e}_u^T(\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*)) \\
&= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \left( \partial_1 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) e_u^\theta \right. \\
&\quad \left. + \partial_2 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) (\nabla_{\Gamma_h^\theta} e_u^\theta - \nabla_{\Gamma_h^\theta} e_x^\theta \nabla_{\Gamma_h^\theta} u_h^\theta + \nu_h^\theta (\nu_h^\theta)^T (\nabla_{\Gamma_h^\theta} e_x^\theta)^T \nabla_{\Gamma_h^\theta} u_h^\theta) \right) d\theta \\
&\leq c \|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} \left( \|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} \right. \\
&\quad \left. + \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} + \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))} \|\nabla_{\Gamma_h(\mathbf{x}^*)} u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \right. \\
&\quad \left. + \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} \right) \\
&\leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} \right) \\
&\leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right).
\end{aligned}$$

Combined, the above estimates yield the following inequality:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 &\leq C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right) \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} + c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)} \\
&\quad - \frac{1}{2} \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \\
&\quad + Ch^{(\kappa-1)/2} \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} \left( \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)} \right) \\
&\quad + C \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} \|\mathbf{d}_u\|_{*, \mathbf{x}^*}.
\end{aligned}$$

Estimating further, using Young's inequality and absorptions into  $\|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2$  (using  $h \leq h_0$  for a sufficiently small  $h_0$ ), we obtain the following estimate, where we can choose  $\rho > 0$  small at the expense of enlarging the constant in front of  $\|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2$ :

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \frac{1}{2} \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 &\leq c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + \rho \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 \\
&\quad + \frac{1}{2} \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) + c \|\mathbf{d}_u\|_{*, \mathbf{x}^*}^2.
\end{aligned} \tag{6.9}$$

(B) *Estimates in the surface equation:* Based on Section 5, we obtain

$$\|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + |\mathbf{e}_v^T(\mathbf{g}(\mathbf{x}, \mathbf{u}) - \mathbf{g}(\mathbf{x}^*, \mathbf{u}^*))| + c \|\mathbf{d}_v\|_{*, \mathbf{x}^*}^2,$$

where the coupling term can be estimated based on (iii) in the proof of Proposition 5.1 and (vii) above:

$$|\mathbf{e}_v^T(\mathbf{g}(\mathbf{x}, \mathbf{u}) - \mathbf{g}(\mathbf{x}^*, \mathbf{u}^*))| \leq \|\mathbf{e}_v\|_{\mathbf{M}(\mathbf{x}^*)} (\|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}).$$

We then obtain

$$\|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq C(\|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 + \|\mathbf{d}_v\|_{\star, \mathbf{x}^*}^2). \quad (6.10)$$

As in Section 5, this provides the estimate

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq C(\|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 + \|\mathbf{d}_v\|_{\star, \mathbf{x}^*}^2). \quad (6.11)$$

(C) *Combination:* We first insert (6.10) into (6.9), where we can choose  $\rho > 0$  so small that  $C\rho \leq 1/2$  for the constant  $C$  in (6.10). Then we take a linear combination of (6.9) and (6.11) to obtain, for a sufficiently small  $\sigma > 0$ ,

$$\begin{aligned} \frac{d}{dt} \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \frac{1}{2} \|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2 + \sigma \frac{d}{dt} \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 \\ \leq c \|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}^2 + c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + \frac{d}{dt} \left( \mathbf{e}_u^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u \right) \\ + c \|\mathbf{d}_u\|_{\star, \mathbf{x}^*}^2 + c \|\mathbf{d}_v\|_{\star, \mathbf{x}^*}^2. \end{aligned}$$

We integrate both sides over  $[0, t]$ , for  $0 \leq t \leq t^*$ , to get

$$\begin{aligned} \|\mathbf{e}_u(t)\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \frac{1}{2} \int_0^t \|\mathbf{e}_u(s)\|_{\mathbf{A}(\mathbf{x}^*)}^2 ds + \sigma \|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*)}^2 \\ \leq \|\mathbf{e}_u(0)\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \|\mathbf{e}_x(0)\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c \int_0^t \left( \|\mathbf{e}_u(s)\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \|\mathbf{e}_x(s)\|_{\mathbf{K}(\mathbf{x}^*)}^2 \right) ds \\ - \mathbf{e}_u(t)^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u(t) \\ + c \int_0^t \left( \|\mathbf{d}_u(s)\|_{\star, \mathbf{x}^*}^2 + \|\mathbf{d}_v(s)\|_{\star, \mathbf{x}^*}^2 \right) ds. \end{aligned}$$

The middle term can be further bounded using Lemmas 4.1–4.3 and an  $L^2 - L^\infty - L^2$  estimate, as

$$\begin{aligned} \mathbf{e}_u(t)^T (\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_u(t) &= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) e_u^\theta d\theta \\ &\leq c \|\mathbf{e}_u(t)\|_{\mathbf{M}(\mathbf{x}^*)}^2 \|\nabla_{\Gamma_h(\mathbf{x}^*)} \cdot e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\leq Ch^{(\kappa-1)/2} \|\mathbf{e}_u(t)\|_{\mathbf{M}(\mathbf{x}^*)}^2, \end{aligned}$$

where we used the first bound from (6.5) in the last inequality.

Absorbing this to the left-hand side and using Gronwall's inequality yields the stability estimate

$$\begin{aligned}
& \|\mathbf{e}_u(t)\|_{\mathbf{M}(\mathbf{x}^*)}^2 + \int_0^t \|\mathbf{e}_u(s)\|_{\mathbf{A}(\mathbf{x}^*)}^2 ds + \|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*)}^2 \\
& \leq c \int_0^t \left( \|\mathbf{d}_u(s)\|_{\star, \mathbf{x}^*}^2 + \|\mathbf{d}_v(s)\|_{\star, \mathbf{x}^*}^2 \right) ds. \tag{6.12}
\end{aligned}$$

Inserting this bound in (6.10), squaring and integrating from 0 to  $t$  yields

$$\int_0^t \|\mathbf{e}_v(s)\|_{\mathbf{K}(\mathbf{x}^*)}^2 ds \leq c \int_0^t \left( \|\mathbf{d}_u(s)\|_{\star, \mathbf{x}^*}^2 + \|\mathbf{d}_v(s)\|_{\star, \mathbf{x}^*}^2 \right) ds.$$

With the assumed bounds of the defects, we obtain  $O(h^k)$  error estimates for  $0 \leq t \leq t^*$ . Finally, to show that  $t^* = T$ , we use the same argument as at the end of the proof of Proposition 5.1.  $\square$

*Remark 6.1* If the coupling function  $g = g(u)$  in (2.2) does not depend on the tangential gradient of  $u$ , then the term  $\|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}^2$  does not appear in the bound (6.10). Therefore, inserting the estimate (6.12) into (6.10) then yields a pointwise stability estimate for  $\mathbf{e}_v$ : uniformly for  $0 \leq t \leq T$ ,

$$\|\mathbf{e}_v(t)\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq C \|\mathbf{d}_v(t)\|_{\star, \mathbf{x}^*}^2 + C \int_0^t \left( \|\mathbf{d}_u(s)\|_{\star, \mathbf{x}^*}^2 + \|\mathbf{d}_v(s)\|_{\star, \mathbf{x}^*}^2 \right) ds.$$

## 7 Geometric estimates

In this section we give further notations and some technical lemmas from [20] that will be used later on. Most of the results are high-order and time-dependent extensions of geometric approximation estimates shown in [8, 10, 13] and [7].

### 7.1 The interpolating surface

We return to the setting of Section 2, where  $X(\cdot, t)$  defines a smooth surface  $\Gamma(t) = \Gamma(X(\cdot, t))$ . For an admissible triangulation of  $\Gamma(t)$  with nodes  $x_j^*(t) = X(p_j, t)$  and the corresponding nodal vector  $\mathbf{x}^*(t) = (x_j^*(t))$ , we define the interpolating surface by

$$X_h^*(p_h, t) = \sum_{j=1}^N x_j^*(t) \phi_j[\mathbf{x}(0)](p_h), \quad p_h \in \Gamma_h^0,$$

which has the properties that  $X_h^*(p_j, t) = x_j^*(t) = X(p_j, t)$  for  $j = 1, \dots, N$ , and

$$\Gamma_h^*(t) := \Gamma_h(\mathbf{x}^*(t)) = \Gamma(X_h^*(\cdot, t)).$$

In the following we drop the argument  $t$  when it is not essential. The velocity of the interpolating surface  $\Gamma_h^*$ , defined as in Section 2.3, is denoted by  $v_h^*$ .

## 7.2 Approximation results

The lift of a function  $\eta_h : \Gamma_h^*(t) \rightarrow \mathbb{R}$  is again denoted by  $\eta_h^l : \Gamma(t) \rightarrow \mathbb{R}$ , defined via the oriented distance function  $d$  between  $\Gamma_h^*(t)$  and  $\Gamma(t)$  provided that the surfaces are sufficiently close (which is the case if  $h$  is sufficiently small).

**Lemma 7.1 (Equivalence of norms [8, Lemma 3], [7])** *Let  $\eta_h : \Gamma_h^*(t) \rightarrow \mathbb{R}$  with lift  $\eta_h^l : \Gamma(t) \rightarrow \mathbb{R}$ . Then the  $L^p$  and  $W^{1,p}$  norms on the discrete and continuous surfaces are equivalent for  $1 \leq p \leq \infty$ , uniformly in the mesh size  $h \leq h_0$  (with sufficiently small  $h_0 > 0$ ) and in  $t \in [0, T]$ .*

In particular, there is a constant  $c$  such that for  $h \leq h_0$  and  $0 \leq t \leq T$ ,

$$\begin{aligned} c^{-1} \|\eta_h\|_{L^2(\Gamma_h^*(t))} &\leq \|\eta_h^l\|_{L^2(\Gamma(t))} \leq c \|\eta_h\|_{L^2(\Gamma_h^*(t))}, \\ c^{-1} \|\eta_h\|_{H^1(\Gamma_h^*(t))} &\leq \|\eta_h^l\|_{H^1(\Gamma(t))} \leq c \|\eta_h\|_{H^1(\Gamma_h^*(t))}. \end{aligned}$$

Later on the following estimates will be used. They have been shown in [20], based on [7] and [13].

**Lemma 7.2** *Let  $\Gamma(t)$  and  $\Gamma_h^*(t)$  be as above in Section 7.1. Then, for  $h \leq h_0$  with a sufficiently small  $h_0 > 0$ , we have the following estimates for the distance function  $d$  from (2.8), and for the error in the normal vector:*

$$\|d\|_{L^\infty(\Gamma_h^*(t))} \leq ch^{k+1}, \quad \|\nu_{\Gamma(t)} - \nu_{\Gamma_h^*(t)}^l\|_{L^\infty(\Gamma(t))} \leq ch^k,$$

with constants independent of  $h \leq h_0$  and  $t \in [0, T]$ .

## 7.3 Bilinear forms and their estimates

We use surface-dependent bilinear forms defined similarly as in [13]: Let  $X$  be a given surface with velocity  $v$ , with interpolation surface  $X_h^*$  with velocity  $v_h^*$ . For arbitrary  $z, \varphi \in H^1(\Gamma(X))$  and for their discrete analogs  $Z_h, \phi_h \in S_h(\mathbf{x}^*)$ :

$$\begin{aligned} m(X; z, \varphi) &= \int_{\Gamma(X)} z \varphi, & m(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} Z_h \phi_h, \\ a(X; z, \varphi) &= \int_{\Gamma(X)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\ q(X; v; z, \varphi) &= \int_{\Gamma(X)} (\nabla_\Gamma \cdot v) z \varphi, & q(X_h^*; v_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} (\nabla_{\Gamma_h} \cdot v_h^*) Z_h \phi_h, \end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense. For more details see [13, Lemma 2.1] (and the references in the proof), or [12, Lemma 5.2].

We start by defining a discrete velocity on the smooth surface, denoted by  $\hat{v}_h$ . We follow Section 5.3 of [20], where the high-order ESFEM generalization

of the discrete velocity on  $\Gamma(X)$  from Sections 4.3 and 5.3 of [13] is discussed. Using the lifted elements,  $\Gamma(X)$  is decomposed into curved elements whose Lagrange points move with the velocity  $\hat{v}_h$  defined by

$$\hat{v}_h((X_h^*)^l(\cdot, t), t) = \frac{d}{dt}(X_h^*)^l(\cdot, t).$$

Discrete material derivatives on  $\Gamma(X_h^*)$  and  $\Gamma(X)$  are given by

$$\begin{aligned} \partial_{v_h^*}^\bullet \varphi_h &= \partial_t \varphi_h + v_h^* \cdot \nabla \varphi_h, \\ \partial_{\hat{v}_h}^\bullet \varphi_h^l &= \partial_t \varphi_h^l + \hat{v}_h \cdot \nabla \varphi_h^l, \end{aligned} \quad (\varphi_h \in S_h(\mathbf{x}^*)).$$

In [13, Lemma 4.1] it was shown that the transport property of the basis functions carries over to the lifted basis functions  $\phi_j[\mathbf{x}^*]$ :

$$\partial_{\hat{v}_h}^\bullet \phi_j[\mathbf{x}^*]^l = (\partial_{v_h^*}^\bullet \phi_j[\mathbf{x}^*])^l = 0, \quad (j = 1, \dots, N).$$

Therefore, the above discrete material derivatives and the lift operator satisfy, for  $\varphi_h \in S_h(X_h^*)$ ,

$$\partial_{\hat{v}_h}^\bullet \varphi_h^l = (\partial_{v_h^*}^\bullet \varphi_h)^l. \quad (7.1)$$

**Lemma 7.3 (Transport properties [13, Lemma 4.2])** *For any  $z(t), \varphi(t) \in H^1(\Gamma(X(\cdot, t)))$ ,*

$$\frac{d}{dt} m(X; z, \varphi) = m(X; \partial^\bullet z, \varphi) + m(X; z, \partial^\bullet \varphi) + q(X; v; z, \varphi).$$

*The same formulas hold when  $\Gamma(X)$  is considered as the lift of the discrete surface  $\Gamma(X_h^*)$  (i.e.  $\Gamma(X)$  can be decomposed into curved elements which are lifts of the elements of  $\Gamma(X_h^*)$ ), moving with the velocity  $\hat{v}_h$ :*

$$\frac{d}{dt} m(X; z, \varphi) = m(X; \partial_{\hat{v}_h}^\bullet z, \varphi) + m(X; z, \partial_{\hat{v}_h}^\bullet \varphi) + q(X; \hat{v}_h; z, \varphi).$$

*Similarly, in the discrete case, for arbitrary  $z_h(t), \varphi_h(t), \partial_{v_h^*}^\bullet z_h(t), \partial_{v_h^*}^\bullet \varphi_h(t) \in S_h(\mathbf{x}^*(t))$  we have:*

$$\frac{d}{dt} m(X_h^*; z_h, \varphi_h) = m(X_h^*; \partial_{v_h^*}^\bullet z_h, \varphi_h) + m(X_h^*; z_h, \partial_{v_h^*}^\bullet \varphi_h) + q(X_h^*; v_h^*; z_h, \varphi_h),$$

*where  $v_h^*$  is the velocity of the surface  $\Gamma(X_h^*)$ .*

The following estimates, proved in Lemma 5.6 of [20], will play a crucial role in the defect bounds later on.

**Lemma 7.4 (Geometric perturbation errors)** *For any  $Z_h, \psi_h \in S_h(\mathbf{x}^*)$  where  $\Gamma(X_h^*)$  is the interpolation surface of piecewise polynomial degree  $k$ , we have the following bounds, for  $h \leq h_0$  with a sufficiently small  $h_0 > 0$ ,*

$$\begin{aligned} |m(X; Z_h^l, \varphi_h^l) - m(X_h^*; Z_h, \varphi_h)| &\leq ch^{k+1} \|Z_h^l\|_{L^2(\Gamma(X))} \|\varphi_h^l\|_{L^2(\Gamma(X))}, \\ |a(X; Z_h^l, \varphi_h^l) - a(X_h^*; Z_h, \varphi_h)| &\leq ch^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(X))} \|\nabla_\Gamma \varphi_h^l\|_{L^2(\Gamma(X))}, \\ |q(X; \hat{v}_h; Z_h^l, \varphi_h^l) - q(X_h^*; v_h^*; Z_h, \varphi_h)| &\leq ch^{k+1} \|Z_h^l\|_{L^2(\Gamma(X))} \|\varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

*The constant  $c$  is independent of  $h$  and  $t \in [0, T]$ .*



#### 7.4 Interpolation error estimates for evolving surface finite element functions

For any  $u \in H^{k+1}(\Gamma(X))$ , there is a unique piecewise polynomial surface finite element interpolation of degree  $k$  in the nodes  $x_j^*$ , denoted by  $\tilde{I}_h u \in S_h(\mathbf{x}^*)$ . We set  $I_h u := (\tilde{I}_h u)^l : \Gamma(X) \rightarrow \mathbb{R}$ . Error estimates for this interpolation are obtained from [7, Proposition 2.7] by carefully studying the time dependence of the constants, cf. [20].

**Lemma 7.5** *There exists a constant  $c > 0$  independent of  $h \leq h_0$ , with a sufficiently small  $h_0 > 0$ , and  $t$  such that for  $u(\cdot, t) \in H^{k+1}(\Gamma(t))$ , for  $0 \leq t \leq T$ ,*

$$\|u - I_h u\|_{L^2(\Gamma(X))} + h \|\nabla_\Gamma(u - I_h u)\|_{L^2(\Gamma(X))} \leq ch^{k+1} \|u\|_{H^{k+1}(\Gamma(X))}.$$

The same result holds for vector valued functions. As it will always be clear from the context we do not distinguish between interpolations for scalar and vector valued functions.

### 8 Defect bounds

In this section we show that the assumed defect estimates of Proposition 5.1 and 6.1 are indeed fulfilled when the projection  $\Pi_h$  is chosen to be the piecewise  $k$ th-degree polynomial interpolation operator  $I_h$  for  $k \geq 2$ .

The interpolations satisfy the discrete problem (2.4)–(2.5) only up to some defects. These defects are denoted by  $d_u \in S_h(\mathbf{x}^*)$ ,  $d_v \in S_h(\mathbf{x}^*)^3$ , with  $\mathbf{x}^*(t)$  the vector of exact nodal values  $x_j^*(t) = X(p_j, t) \in \Gamma(t)$ , and are given as follows: for all  $\varphi_h \in S_h(\mathbf{x}^*)$  with  $\partial_{v_h^*}^\bullet \varphi_h = 0$  and  $\psi_h \in S_h(\mathbf{x}^*)^3$ ,

$$\begin{aligned} \int_{\Gamma_h(\mathbf{x}^*)} d_u \varphi_h &= \frac{d}{dt} \int_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h u \varphi_h + \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h u \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \varphi_h \\ &\quad - \int_{\Gamma_h(\mathbf{x}^*)} f(\tilde{I}_h u, \nabla_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h u) \varphi_h, \\ \int_{\Gamma_h(\mathbf{x}^*)} d_v \cdot \psi_h &= \int_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h v \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h v \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \psi_h \\ &\quad - \int_{\Gamma_h(\mathbf{x}^*)} g(\tilde{I}_h u, \nabla_{\Gamma_h(\mathbf{x}^*)} \tilde{I}_h u) \nu_{\Gamma_h(\mathbf{x}^*)} \cdot \psi_h. \end{aligned}$$

Later on the vectors of nodal values of the defects  $d_u$  and  $d_v$  are denoted by  $\mathbf{d}_u \in \mathbb{R}^N$  and  $\mathbf{d}_v \in \mathbb{R}^{3N}$ , respectively. These vectors satisfy (6.1).

**Lemma 8.1** *Let the solution  $u$ , the surface  $X$  and its velocity  $v$  be all sufficiently smooth. Then there exists a constant  $c > 0$  such that for all  $h \leq h_0$ , with a sufficiently small  $h_0 > 0$ , and for all  $t \in [0, T]$ , the defects  $d_u$  and  $d_v$  of the  $k$ th-degree finite element interpolation are bounded as*

$$\begin{aligned} \|\mathbf{d}_u\|_{*, \mathbf{x}^*} &= \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))} \leq ch^k, \\ \|\mathbf{d}_v\|_{*, \mathbf{x}^*} &= \|d_v\|_{H_h^{-1}(\Gamma(X_h^*))} \leq ch^k, \end{aligned}$$

where the  $H_h^{-1}$ -norm is defined in (5.5). The constant  $c$  is independent of  $h$  and  $t \in [0, T]$ .

*Proof* (i) We start from an identity for the dual norm as in (5.5), (omitting the argument  $\mathbf{x}^*$  of the matrices):

$$\|\mathbf{d}_u\|_{*, \mathbf{x}^*} = (\mathbf{d}_u^T \mathbf{M}(\mathbf{M} + \mathbf{A})^{-1} \mathbf{M} \mathbf{d}_u)^{\frac{1}{2}} = \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))}.$$

In order to estimate the defect in  $u$ , we subtract (2.3) from the above equation, and perform almost the same proof as in [13, Section 7]. We use the bilinear forms and the discrete versions of the transport properties from Lemma 7.3. We obtain, for any  $\varphi_h \in S_h(\mathbf{x}^*)$  with  $\partial_{v_h^*}^\bullet \varphi_h = 0$ ,

$$\begin{aligned} m(X_h^*; d_u, \varphi_h) &= \frac{d}{dt} m(X_h^*; \tilde{I}_h u, \varphi_h) + a(X_h^*; \tilde{I}_h u, \varphi_h) \\ &\quad - m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u), \varphi_h) \\ &= m(X_h^*; \partial_{v_h^*}^\bullet \tilde{I}_h u, \varphi_h) + q(X_h^*; v_h^*; \tilde{I}_h u, \varphi_h) + a(X_h^*; \tilde{I}_h u, \varphi_h) \\ &\quad - m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u), \varphi_h), \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{d}{dt} m(X; u, \varphi_h^l) + a(X; u, \varphi_h^l) - m(X; f(u, \nabla_{\Gamma(X)} u), \varphi_h^l) \\ &= m(X; \partial_{v_h}^\bullet u, \varphi_h^l) + q(X; \hat{v}_h; u, \varphi_h^l) + a(X; u, \varphi_h^l) - m(X; f(u, \nabla_{\Gamma(X)} u), \varphi_h^l). \end{aligned}$$

Subtracting the two equation yields

$$\begin{aligned} m(X_h^*; d_u, \varphi_h) &= m(X_h^*; \partial_{v_h^*}^\bullet \tilde{I}_h u, \varphi_h) - m(X; \partial_{v_h}^\bullet u, \varphi_h^l) \\ &\quad + q(X_h^*; v_h^*; \tilde{I}_h u, \varphi_h) - q(X; \hat{v}_h; u, \varphi_h^l) \\ &\quad + a(X_h^*; \tilde{I}_h u, \varphi_h) - a(X; u, \varphi_h^l) \\ &\quad - \left( m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_{\Gamma(X)} u), \varphi_h^l) \right). \end{aligned}$$

We bound all the terms pairwise, by using the interpolation estimates of Lemma 7.5 and the estimates for the geometric perturbation errors of the bilinear forms of Lemma 7.4. For the first pair, using that  $(\partial_{v_h^*}^\bullet \tilde{I}_h u)^l = \partial_{v_h}^\bullet I_h u$ , we obtain

$$\begin{aligned} |m(X_h^*; \partial_{v_h^*}^\bullet \tilde{I}_h u, \varphi_h) - m(X; \partial_{v_h}^\bullet u, \varphi_h^l)| &\leq |m(X_h^*; \partial_{v_h^*}^\bullet \tilde{I}_h u, \varphi_h) - m(X; \partial_{v_h}^\bullet I_h u, \varphi_h^l)| \\ &\quad + |m(X; I_h \partial_{v_h}^\bullet u - \partial_{v_h}^\bullet u, \varphi_h^l)| \\ &\leq ch^{k+1} \|\varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

For the second pair we obtain

$$\begin{aligned} |q(X_h^*; v_h^*; \tilde{I}_h u, \varphi_h) - q(X; \hat{v}_h; u, \varphi_h^l)| &\leq |q(X_h^*; v_h^*; \tilde{I}_h u, \varphi_h) - q(X; \hat{v}_h; I_h u, \varphi_h^l)| \\ &\quad + |q(X; v_h^*; I_h u - u, \varphi_h^l)| \\ &\leq ch^{k+1} \|\varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

The third pair is estimated by

$$\begin{aligned} |a(X_h^*; \tilde{I}_h u, \varphi_h) - a(X; u, \varphi_h^l)| &\leq |a(X_h^*; \tilde{I}_h u, \varphi_h) - a(X; I_h u, \varphi_h^l)| \\ &\quad + |a(X; I_h u - u, \varphi_h^l)| \\ &\leq ch^k \|\nabla_{\Gamma} \varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

For the last pair we use the fact that  $(f(u, \nabla_{\Gamma} u))^{-l} = f(u^{-l}, (\nabla_{\Gamma} u)^{-l})$  and the local Lipschitz continuity of the function  $f$ , to obtain

$$\begin{aligned} &|m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_{\Gamma} u), \varphi_h^l)| \\ &\leq |m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma} u)^{-l}), \varphi_h)| \\ &\quad + |m(X_h^*; f(u, \nabla_{\Gamma} u)^{-l}, \varphi_h) - m(X; f(u, \nabla_{\Gamma} u), \varphi_h^l)| \\ &\leq c \|f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma} u)^{-l})\|_{L^2(\Gamma(X_h^*))} \|\varphi_h^l\|_{L^2(\Gamma(X))} \\ &\quad + ch^{k+1} \|\varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

The first term is estimated, using the local Lipschitz continuity of  $f$  and equivalence of norms, by

$$\begin{aligned} &\|f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma} u)^{-l})\|_{L^2(\Gamma(X_h^*))} \\ &\leq \|f\|_{W^{1,\infty}} \left( c \|I_h u - u\|_{L^2(\Gamma(X))} + c \|\nabla_{\Gamma}(I_h u - u)\|_{L^2(\Gamma(X))} \right. \\ &\quad \left. + c \|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_{\Gamma} u\|_{L^2(\Gamma(X))} \right), \end{aligned}$$

where the first two terms are bounded by  $O(h^k)$  using interpolation estimates, while the third term is bounded, using Remark 4.1 in [13] and Lemma 7.2, as

$$\|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_{\Gamma} u\|_{L^2(\Gamma(X))} \leq ch^k.$$

Thus for the fourth pair we obtained

$$|m(X_h^*; f(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_{\Gamma} u), \varphi_h^l)| \leq ch^k \|\varphi_h^l\|_{L^2(\Gamma(X))}.$$

Altogether, we have

$$m(X_h^*; d_u, \varphi_h) \leq ch^k \|\varphi_h^l\|_{H^1(\Gamma(X))},$$

which, by the equivalence of norms given by Lemma 7.1, shows the first bound of the stated lemma.

(ii) In order to estimate the defect in  $v$ , similarly as previously we subtract (2.3) from the above equation and use the bilinear forms to obtain

$$\begin{aligned} &m(X_h^*; d_v, \psi_h) \\ &= m(X_h^*; \tilde{I}_h v, \psi_h) - m(X; v, \psi_h^l) \\ &\quad + \alpha \left( a(X_h^*; \tilde{I}_h v, \psi_h) - a(X; v, \psi_h^l) \right) \\ &\quad + m(X_h^*; g(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) \nu_{\Gamma(X_h^*)}, \psi_h) - m(X; g(u, \nabla_{\Gamma} u) \nu_{\Gamma(X)}, \psi_h^l). \end{aligned}$$

Similarly as in the previous part, these three pairs are bounded pairwise. For the first pair we have

$$\begin{aligned} |m(X_h^*; \tilde{I}_h v, \psi_h) - m(X; v, \psi_h^l)| &\leq |m(X_h^*; \tilde{I}_h v, \psi_h) - m(X; I_h v, \psi_h^l)| \\ &\quad + |m(X; I_h v - v, \psi_h^l)| \\ &\leq ch^{k+1} \|\psi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

For the second pair we use the interpolation estimate to bound

$$\begin{aligned} |a(X_h^*; \tilde{I}_h v, \psi_h) - a(X; v, \psi_h^l)| &\leq |a(X_h^*; \tilde{I}_h v, \psi_h) - a(X; I_h v, \psi_h^l)| \\ &\quad + |a(X; I_h v - v, \psi_h^l)| \\ &\leq ch^k \|\nabla_\Gamma \psi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

The third pair we estimate, similarly to the nonlinear pair above, by

$$\begin{aligned} &|m(X_h^*; g(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) \nu_{\Gamma(X_h^*)}, \psi_h) - m(X; g(u, \nabla_\Gamma u) \nu_{\Gamma(X)}, \psi_h^l)| \\ &\leq |m(X_h^*; (g(\tilde{I}_h u, \nabla_{\Gamma_h} \tilde{I}_h u) - g(u, \nabla_\Gamma u)^{-l}) \nu_{\Gamma(X_h^*)}, \psi_h)| \\ &\quad + |m(X_h^*; g(u, \nabla_\Gamma u)^{-l} (\nu_{\Gamma(X_h^*)} - \nu_{\Gamma(X)}^{-l}), \psi_h)| \\ &\quad + |m(X_h^*; g(u, \nabla_\Gamma u)^{-l} \nu_{\Gamma(X)}^{-l}, \psi_h) - m(X; g(u, \nabla_\Gamma u) \nu_{\Gamma(X)}, \psi_h^l)| \\ &\leq ch^k \|g\|_{W^{1,\infty}} \|\psi_h^l\|_{L^2(\Gamma(X))} + c \|\nabla_\Gamma (X - X_h^*)\|_{L^2(\Gamma(X))} \|\psi_h^l\|_{L^2(\Gamma(X))} \\ &\quad + ch^{k+1} \|g\|_{L^2} \|\psi_h^l\|_{L^2(\Gamma(X))} \\ &\leq ch^k \|g\|_{W^{1,\infty}} \|\psi_h^l\|_{L^2(\Gamma(X))} + ch^k \|\psi_h^l\|_{L^2(\Gamma(X))} \\ &\leq ch^k \|\psi_h^l\|_{L^2(\Gamma(X))}, \end{aligned}$$

where we have used the local Lipschitz boundedness of the function  $g$ , the interpolation estimate, Lemma 7.2, and Lemma 7.4, through a similar argument as above for the semilinear term with  $f$ .

Finally, the combination of these bounds yields

$$m(X_h^*; d_v, \psi_h) \leq ch^k \|\psi_h^l\|_{H^1(\Gamma(X))},$$

providing the asserted bound on  $\mathbf{d}_v$ .  $\square$

## 9 Proof of Theorem 3.1

The errors are decomposed using interpolations and the definition of lifts from Section 2.6: omitting the argument  $t$ ,

$$\begin{aligned} u_h^L - u &= (\hat{u}_h - \tilde{I}_h u)^l + (I_h u - u), \\ v_h^L - v &= (\hat{v}_h - \tilde{I}_h v)^l + (I_h v - v), \\ X_h^L - X &= (\hat{X}_h - \tilde{I}_h X)^l + (I_h X - X). \end{aligned}$$

The last terms in these formulas can be bounded in the  $H^1(\Gamma)$  norm by  $Ch^k$ , using the interpolation bounds of Lemma 7.5.

To bound the first terms on the right-hand sides, we first use the defect bounds of Lemma 8.1, which then together with the stability estimate of Proposition 6.1 proves the result, since by the norm equivalences of Lemma 7.1 and equations (4.1)–(4.2) we have (again omitting the argument  $t$ )

$$\begin{aligned} \|(\hat{u}_h - \tilde{I}_h u)^l\|_{L^2(\Gamma)} &\leq c\|\hat{u}_h - \tilde{I}_h u\|_{L^2(\Gamma_h^*)} = c\|\mathbf{e}_u\|_{\mathbf{M}(\mathbf{x}^*)}, \\ \|\nabla_\Gamma(\hat{u}_h - \tilde{I}_h u)^l\|_{L^2(\Gamma)} &\leq c\|\nabla_{\Gamma_h^*}(\hat{u}_h - \tilde{I}_h u)\|_{L^2(\Gamma_h^*)} = c\|\mathbf{e}_u\|_{\mathbf{A}(\mathbf{x}^*)}, \end{aligned}$$

and similarly for  $\hat{v}_h - \tilde{I}_h v$  and  $\hat{X}_h - \tilde{I}_h X$ .

## 10 Extension to other velocity laws

In this section we consider the extension of our results to different velocity laws: adding a mean curvature term to the regularized velocity law considered so far, and a dynamic velocity law. We concentrate on the velocity laws without coupling to the surface PDE, since the coupling can be dealt with in the same way as previously. We only consider the stability of the evolving surface finite element discretization, since bounds for the consistency error are obtained by the same arguments as before.

### 10.1 Regularized mean curvature flow

We next extend our results to the case where the velocity law contains a mean curvature term:

$$v - \alpha \Delta_{\Gamma(X)} v - \beta \Delta_{\Gamma(X)} X = g(\cdot, t) \nu_{\Gamma(X)}, \quad (10.1)$$

where  $g : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  is a given Lipschitz continuous function of  $(x, t)$ , and  $\alpha > 0$  and  $\beta > 0$  are fixed parameters. Here  $\Delta_{\Gamma(X)} X$  is a suggestive notation for  $-H\nu$ , where  $H$  denotes the mean curvature of the surface  $\Gamma(X)$ . (More precisely,  $\Delta_{\Gamma(X)} \text{id} = -H\nu_{\Gamma(X)}$ .)

The corresponding differential-algebraic system reads

$$\mathbf{K}(\mathbf{x})\mathbf{v} + \mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{g}(\mathbf{x}), \quad (10.2)$$

where  $\mathbf{K}(\mathbf{x})$  is again defined by (2.6) and where we now write  $\mathbf{A}(\mathbf{x})$  for the matrix  $\beta I_3 \otimes \mathbf{A}(\mathbf{x})$  with  $\mathbf{A}(\mathbf{x})$  of Section 2.5.

Similarly as before the corresponding error equation is given as

$$\begin{aligned} \mathbf{K}(\mathbf{x}^*)\mathbf{e}_v + \mathbf{A}(\mathbf{x}^*)\mathbf{e}_x = & -(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*))\mathbf{v}^* - (\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*))\mathbf{e}_v \\ & -(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{x}^* - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{e}_x \\ & + (\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{M}(\mathbf{x}^*)\mathbf{d}_v \end{aligned}$$

together with  $\dot{\mathbf{e}}_x = \mathbf{e}_v$ .

**Proposition 10.1** *Under the assumptions of Proposition 5.1, there exists  $h_0 > 0$  such that the following stability estimate holds for all  $h \leq h_0$ , for  $0 \leq t \leq T$ :*

$$\begin{aligned}\|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*(t))}^2 &\leq C \int_0^t \|\mathbf{d}_v(s)\|_{*,\mathbf{x}^*}^2 ds, \\ \|\mathbf{e}_v(t)\|_{\mathbf{K}(\mathbf{x}^*(t))}^2 &\leq C \|\mathbf{d}_v(t)\|_{*,\mathbf{x}^*}^2 + C \int_0^t \|\mathbf{d}_v(s)\|_{*,\mathbf{x}^*}^2 ds.\end{aligned}$$

The constant  $C$  is independent of  $t$  and  $h$ , but depends on the final time  $T$ , and on the parameters  $\alpha$  and  $\beta$ .

*Proof* We detail only those parts of the proof of Proposition 5.1 where the mean curvature term introduces differences, otherwise exactly the same proof applies.

In order to prove the stability estimate we again test with  $\mathbf{e}_v$ , and obtain

$$\begin{aligned}\|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 &= -\mathbf{e}_v^T(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*))\mathbf{v}^* - \mathbf{e}_v^T(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*))\mathbf{e}_v \\ &\quad - \mathbf{e}_v^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{x}^* - \mathbf{e}_v^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{e}_x - \mathbf{e}_v^T\mathbf{A}(\mathbf{x}^*)\mathbf{e}_x \\ &\quad + \mathbf{e}_v^T(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{e}_v^T\mathbf{M}(\mathbf{x}^*)\mathbf{d}_v.\end{aligned}$$

Every term is estimated exactly as previously in the proof of Proposition 5.1, except the terms corresponding to the mean curvature term, involving the stiffness matrix  $\mathbf{A}$ . They are estimated by the same techniques as previously:

$$\begin{aligned}\mathbf{e}_v^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{x}^* + \mathbf{e}_v^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*))\mathbf{e}_x &\leq \frac{1}{6} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2, \\ \mathbf{e}_v^T\mathbf{A}(\mathbf{x}^*)\mathbf{e}_x &\leq \frac{1}{6} \|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2.\end{aligned}$$

Altogether we obtain the error bound

$$\|\mathbf{e}_v\|_{\mathbf{K}(\mathbf{x}^*)}^2 \leq c \|\mathbf{e}_x\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c \|\mathbf{d}_v\|_{*,\mathbf{x}^*}^2,$$

which is exactly (5.9). The proof is then completed as before.  $\square$

With Proposition 10.1 and the appropriate defect bounds, Theorem 3.1 extends directly to the system with mean curvature term in the regularized velocity law.

## 10.2 A dynamic velocity law

Let us consider the dynamic velocity law, again without coupling to a surface PDE:

$$\partial^\bullet v + v \nabla_{\Gamma(X)} \cdot v - \alpha \Delta_{\Gamma(X)} v = g(\cdot, t) \nu_{\Gamma(X)},$$

where again  $g : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  is a given Lipschitz continuous function of  $(x, t)$ , and  $\alpha > 0$  is a fixed parameter. This problem is considered together with the ordinary differential equations (2.1) for the positions  $X$  determining the surface  $\Gamma(X)$ . Initial values are specified for  $X$  and  $v$ .

The weak formulation and the semidiscrete problem can be obtained by a similar argument as for the PDE on the surface in Section 6. Therefore we immediately present the ODE formulation of the semidiscretization. As in Section 2.5, the nodal vectors  $\mathbf{v} \in \mathbb{R}^{3N}$  of the finite element function  $v_h$ , together with the surface nodal vector  $\mathbf{x} \in \mathbb{R}^{3N}$  satisfy a system of ODEs with matrices and driving term as in Section 5:

$$\begin{aligned} \frac{d}{dt} (\mathbf{M}(\mathbf{x}) \mathbf{v}) + \mathbf{A}(\mathbf{x}) \mathbf{v} &= \mathbf{g}(\mathbf{x}, t), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{10.3}$$

By using the same notations for the exact positions  $\mathbf{x}^*(t) \in \mathbb{R}^{3N}$ , for the interpolated exact velocity  $\mathbf{v}^*(t) \in \mathbb{R}^{3N}$ , and for the defect  $\mathbf{d}_v(t)$ , we obtain that they fulfill the following equation

$$\begin{aligned} \frac{d}{dt} (\mathbf{M}(\mathbf{x}^*) \mathbf{v}^*) + \mathbf{A}(\mathbf{x}^*) \mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*, t) + \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v, \\ \dot{\mathbf{x}}^* &= \mathbf{v}^*. \end{aligned}$$

By subtracting this from (10.3), and using similar arguments as before, we obtain the error equations for the surface nodes and velocity:

$$\begin{aligned} \frac{d}{dt} (\mathbf{M}(\mathbf{x}^*) \mathbf{e}_v) + \mathbf{A}(\mathbf{x}^*) \mathbf{e}_v &= - \frac{d}{dt} ((\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{v}^*) \\ &\quad - \frac{d}{dt} ((\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_v) \\ &\quad - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{v}^* \\ &\quad - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{e}_v \\ &\quad + (\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v \\ \dot{\mathbf{e}}_x &= \mathbf{e}_v. \end{aligned}$$

We then have the following stability result.

**Proposition 10.2** *Under the assumptions of Proposition 5.1, there exists  $h_0 > 0$  such that the following error estimate holds for all  $h \leq h_0$ , uniformly for  $0 \leq t \leq T$ :*

$$\|\mathbf{e}_x(t)\|_{\mathbf{K}(\mathbf{x}^*(t))}^2 + \|\mathbf{e}_v(t)\|_{\mathbf{M}(\mathbf{x}^*(t))}^2 + \int_0^t \|\mathbf{e}_v(s)\|_{\mathbf{A}(\mathbf{x}^*(s))}^2 ds \leq C \int_0^t \|\mathbf{d}_v(s)\|_{*, \mathbf{x}^*}^2 ds.$$

The constant  $C > 0$  is independent of  $t$  and  $h$ , but depends on the final time  $T$  and the parameter  $\alpha$ .

*Proof* By testing the error equation with  $\mathbf{e}_v$  we obtain

$$\begin{aligned} \mathbf{e}_v^T \frac{d}{dt} (\mathbf{M}(\mathbf{x}^*) \mathbf{e}_v) + \mathbf{e}_v^T \mathbf{A}(\mathbf{x}^*) \mathbf{e}_v = & -\mathbf{e}_v^T \frac{d}{dt} ((\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{v}^*) \\ & -\mathbf{e}_v^T \frac{d}{dt} ((\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)) \mathbf{e}_v) \\ & -\mathbf{e}_v^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{v}^* \\ & -\mathbf{e}_v^T (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)) \mathbf{e}_v \\ & +\mathbf{e}_v^T (\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)) - \mathbf{e}_v^T \mathbf{M}(\mathbf{x}^*) \mathbf{d}_v. \end{aligned}$$

The terms are bounded in the same way as the corresponding terms in the proofs of Propositions 5.1 and 6.1. With these estimates, a Gronwall inequality yields the result.  $\square$

With Proposition 10.2 and the appropriate defect bounds, Theorem 3.1 extends directly to the parabolic surface PDE coupled with the dynamic velocity law.

## 11 Numerical results

In this section we complement Theorem 3.1 by showing the numerical behaviour of piecewise linear finite elements, which are not covered by Theorem 3.1, but nevertheless perform remarkably well. Moreover, we compare our regularized velocity law with regularization by mean curvature flow.

### 11.1 A coupled problem

Our test problem is a combination of (2.2) with a mean curvature term as in (10.1):

$$\begin{aligned} \partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u &= f(t, x), \\ v - \alpha \Delta_\Gamma v - \beta \Delta_\Gamma X &= \delta u \nu_\Gamma + g(t, x) \nu_\Gamma, \end{aligned} \quad (11.1)$$

for non-negative parameters  $\alpha, \beta, \delta$ . The velocity law here is a special case of (2.2) for  $\beta = 0$ , and reduces to (10.1) for  $\delta = 0$ . The matrix-vector form reads

$$\begin{aligned} \frac{d}{dt} (\mathbf{M}(\mathbf{x}(t)) \mathbf{u}(t)) + \mathbf{A}(\mathbf{x}(t)) \mathbf{u}(t) &= \mathbf{f}(t, \mathbf{x}(t)), \quad t \in [0, T], \\ \mathbf{K}(\mathbf{x}(t)) \dot{\mathbf{x}}(t) + \beta \mathbf{A}(\mathbf{x}(t)) \mathbf{x}(t) &= \delta \mathbf{N}(\mathbf{x}(t)) \mathbf{u}(t) + \mathbf{g}(t, \mathbf{x}(t)), \quad t \in [0, T], \end{aligned}$$

for given  $\mathbf{x}(0)$  and  $\mathbf{u}(0)$ , where

$$\mathbf{N}(\mathbf{x}) \mathbf{u}|_{3(j-1)+\ell} = \int_{\Gamma_h(\mathbf{x})} (\nu_{\Gamma_h})_\ell \mathbf{u}_j \phi_j[\mathbf{x}],$$



for  $j = 1, \dots, N$  and  $\ell = 1, 2, 3$ .

In our numerical experiments we used a linearly implicit Euler discretization of this system with step sizes chosen so small that the error is dominated by the spatial discretization error.

*Example 11.1* We consider (11.1) and choose  $f$  and  $g$  such that  $X(p, t) = r(t)p$  with

$$r(t) = \frac{r_0 r_K}{r_K e^{-kt} + r_0(1 - e^{-kt})}$$

and  $u(X, t) = X_1 X_2 e^{-6t}$  are the exact solution of the problem. The parameters are set to be  $T = 1$ ,  $\alpha = 1$ ,  $\beta = 0$ ,  $\delta = 0.4$ ,  $r_0 = 1$ ,  $r_K = 2$  and  $k = 0.5$ .

We choose  $(\mathcal{T}_k)$  as a series of meshes such that  $2h_k \approx h_{k-1}$ . In Table 11.1 we report on the errors and the corresponding experimental orders of convergence (EOC). Using the notation of Section 2.6, the following norms are used:

$$\begin{aligned} \|\text{err}_u\|_{L^\infty(L^2)} &= \sup_{[0, T]} \|\widehat{u}_h(\cdot, t) - \widetilde{I}_h u(\cdot, t)\|_{L^2(\Gamma_h^*(t))}, \\ \|\text{err}_u\|_{L^2(H^1)} &= \left( \int_0^T \left\| \widehat{u}_h(\cdot, s) - \widetilde{I}_h u(\cdot, s) \right\|_{H^1(\Gamma_h^*(s))}^2 ds \right)^{\frac{1}{2}}, \\ \|\text{err}_v\|_{L^\infty(H^1)} &= \sup_{[0, T]} \|\widehat{v}_h(\cdot, t) - \widetilde{I}_h v(\cdot, t)\|_{H^1(\Gamma_h^*(t))}, \\ \|\text{err}_x\|_{L^\infty(H^1)} &= \sup_{[0, T]} \|\widehat{x}_h(\cdot, t) - \text{id}_{\Gamma_h^*(t)}\|_{H^1(\Gamma_h^*(t))}. \end{aligned}$$

The EOCs for the errors  $E(h_{k-1})$  and  $E(h_k)$  with mesh sizes  $h_{k-1}, h_k$  are given via

$$EOC(h_{k-1}, h_k) = \frac{\log \left( \frac{E(h_{k-1})}{E(h_k)} \right)}{\log \left( \frac{h_{k-1}}{h_k} \right)}, \quad (k = 2, \dots, n).$$

The degree of freedoms (DOF) and maximum mesh size at time  $T$  are also reported in the tables.

In Table 11.1 we report on the errors and EOCs observed using Example 11.1. The EOCs in the PDE are expected to be 2 for the  $L^\infty(L^2)$  norm and 1 for the  $L^2(H^1)$  norm, while the errors in the surface and in the surface velocity are expected to be 1 in the  $L^\infty(H^1)$  norm.

*Example 11.2* Again we consider (11.1), but this time we quantitatively compare the two different regularized velocity laws. Hence, we let  $\delta$  vanish. We use a  $g$  like in Example 11.1 and run two tests with the common parameters  $T = 2$ ,  $r_0 = 1$ ,  $r_K = 2$  and  $k = 0.5$ , and use the same mesh and time step levels as before. The first test uses  $\alpha = 0$  and  $\beta = 1$  and the second test uses  $\alpha = 1$  and  $\beta = 0$ . The results are captured in Table 11.2. Our regularized velocity law provides smaller errors as regularizing with mean curvature flow. The EOCs in the errors in the surface and in the errors for the surface velocity are expected to be 1 in  $L^\infty(H^1)_v$  and  $L^\infty(H^1)_x$  norm, see Table 11.2.b. While it can be observed that for this particular example the convergence rates for  $\alpha \neq 0$  are higher then for  $\beta \neq 0$ .

level	DOF	$h(T)$	$\ \text{err}_u\ _{L^\infty(L^2)}$	EOC	$\ \text{err}_u\ _{L^2(H^1)}$	EOC
1	126	0.6664	0.1519165	-	0.2727214	-
2	516	0.4088	0.0896624	1.08	0.1498895	1.22
3	2070	0.1799	0.0222349	1.70	0.0344362	1.79
4	8208	0.0988	0.0070552	1.91	0.0109074	1.92
5	32682	0.0499	0.0018319	1.98	0.0029375	1.92

(a) Errors for  $u$ 

level	DOF	$h(T)$	$\ \text{err}_v\ _{L^\infty(H^1)}$	EOC	$\ \text{err}_x\ _{L^\infty(H^1)}$	EOC
1	126	0.6664	0.2260428	-	0.1473157	-
2	516	0.4088	0.0595755	2.73	0.0298673	3.27
3	2070	0.1799	0.0158342	1.61	0.0106836	1.25
4	8208	0.0988	0.0053584	1.81	0.0042312	1.54
5	32682	0.0499	0.0019341	1.50	0.0017838	1.27

(b) Surface and velocity errors

Table 11.1: Errors and EOCs for Example 11.1

level	DOF	$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
1	126	0.6664	0.756045	-	1.31532	-	1.601255	-
2	516	0.4088	0.393067	1.34	0.78538	1.06	0.522342	2.29
3	2070	0.1799	0.095914	1.72	0.96206	-0.25	0.137396	1.63
4	8208	0.0988	0.035166	1.67	1.48784	-0.73	0.044666	1.87
5	32682	0.0499	0.019755	0.85	2.73584	-0.89	0.013507	1.75

(a) Surface and velocity errors with parameters  $\alpha = 0$  and  $\beta = 1$ .

level	DOF	$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
1	126	0.6664	0.149836	-	0.225114	-	0.143419	-
2	516	0.4088	0.036118	2.91	0.058147	2.77	0.024087	3.65
3	2070	0.1799	0.009286	1.65	0.015843	1.58	0.009702	1.11
4	8208	0.0988	0.002705	2.06	0.005361	1.81	0.003990	1.48
5	32682	0.0499	0.000686	2.01	0.001935	1.49	0.001746	1.21

(b) Surface and velocity errors with parameters  $\alpha = 1$  and  $\beta = 0$ .

Table 11.2: Errors and EOCs for Example 11.2.

## 11.2 A model for tumor growth

Our next test problem is the coupled system of equations

$$\begin{aligned}
\partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u &= f_1(u, w), \\
\partial^\bullet w + w \nabla_\Gamma \cdot v - D_c \Delta_\Gamma w &= f_2(u, w), \\
v - \alpha \Delta_\Gamma v - \beta \Delta_\Gamma X &= \delta w \nu_\Gamma,
\end{aligned} \tag{11.2}$$

where

$$f_1(u, w) = \gamma(a - u + u^2 w), \quad f_2(u, w) = \gamma(b - u^2 w),$$

with non-negative parameters  $D_c, \gamma, a, b, \alpha, \beta$ .

For  $\alpha = 0$  this system has been used as a simplified model for tumor growth; see Barreira, Elliott and Madzvamuse [1] and [16, 6]. These authors used the mean curvature term with a small parameter  $\beta > 0$  to regularize their velocity law.

We used piecewise linear finite elements and the same time discretization scheme as in [1, 16].

*Example 11.3* We consider (11.2) and want to compare qualitatively the two different regularized velocity laws  $\alpha \neq 0$  and  $\beta \neq 0$ . As common parameters we use  $D_c = 10$ ,  $\gamma = 100$ ,  $a = 0.1$ ,  $b = 0.9$  and  $T = 5$ . The initial surface is a sphere and the initial values  $u_0$  and  $w_0$  are calculated by solving an auxiliary surface PDE as follows. We take small perturbations around the steady state

$$\begin{pmatrix} \tilde{u}_0 \\ \tilde{w}_0 \end{pmatrix} = \begin{pmatrix} a + b + \varepsilon_1(x) \\ \frac{b}{(a+b)^2} + \varepsilon_2(x) \end{pmatrix},$$

where  $\varepsilon_1(x), \varepsilon_2(x) \in [0, 0.01]$  take random values. We solve the auxiliary coupled diffusion equations with the stationary initial surface until time  $\tilde{T} = 5$ . We set  $u_0 = \tilde{u}(\tilde{T})$  and  $w_0 = \tilde{w}(\tilde{T})$ , which we used as initial values for (11.2).

We perform two experiments with  $(\alpha, \beta) = (0, 0.01)$  and  $(\alpha, \beta) = (0.01, 0)$ . We present snapshots in Figure 11.1. We observe that both velocity laws display the same qualitative behavior, also agreeing with [16].

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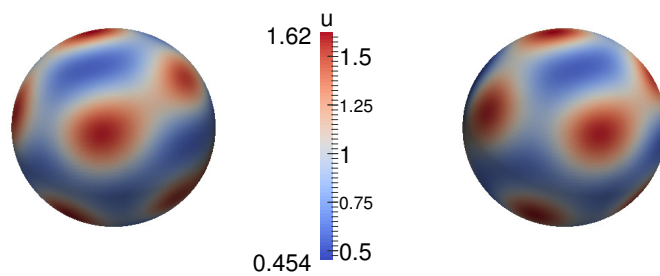
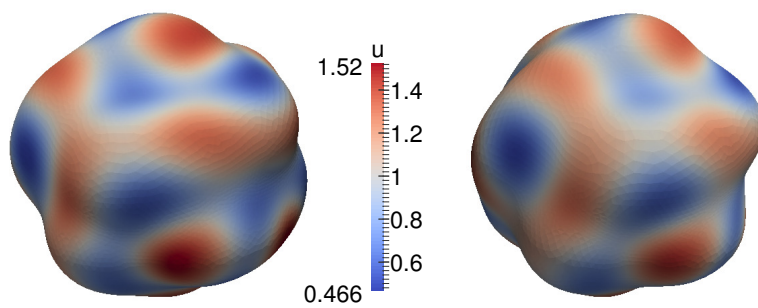
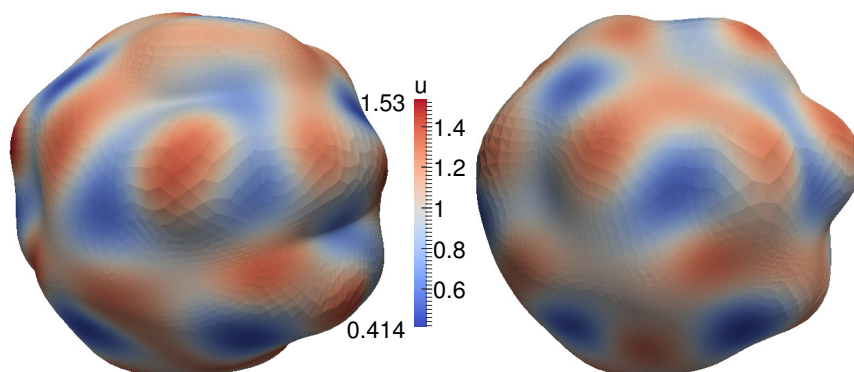
(a) time  $t = 0$ (b) time  $t = 1$ (c) time  $t = 2$ 

Fig. 11.1: Simulation for Example 11.3. The first column corresponds to  $(\alpha, \beta) = (0, 0.01)$  and the second column to  $(\alpha, \beta) = (0.01, 0)$ .

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