Equivalent Structures of Interval Sets and Fuzzy Interval Sets

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Ideas of interval sets come from the lower and upper approximations of rough sets in order to study a unified structure of rough sets and their generalizations. Starting from the interval sets and their operations, this paper summarizes and analyzes other sets which have similarities with the interval sets or fuzzy interval sets. Our conclusions are that interval sets are mathematically equivalent to shadowed sets and flou sets respectively, and fuzzy interval sets are mathematically equivalent to interval-valued fuzzy sets and intuitionistic fuzzy sets respectively.

1. INTRODUCTION

The concept of rough sets was proposed by Pawlak in 1982.[30] Subsequently, rough sets were extended to rough fuzzy sets,[10] fuzzy rough sets,[10] generalized rough fuzzy sets,[39] generalized fuzzy rough sets,[39-40] generalized fuzzy rough sets based on triangle norm,[27] generalized fuzzy rough sets based on logic operators,[28,33,38] interval-valued fuzzy rough sets,[35] generalized interval-valued fuzzy rough sets[18,21,22] and so on.

In the extension process, equivalence relations with regard to attributives are extended to fuzzy equivalence relations, general fuzzy relations and interval-valued fuzzy relations; Classic sets with regard to objects are extended to fuzzy sets and interval-valued fuzzy sets; operators are extended to interval-valued fuzzy t-norm, interval-valued fuzzy R-implication and interval-valued implication[21,28].

No matter what kind of rough sets is considered, there are lower and upper approximations in the conceptual formulation. All kinds of concepts are approximately expressed by their lower approximation or upper approximation. After studying rough set and its various expansions, Yao introduced the concepts of interval sets from approximation structures of rough sets.[45-46] Particularly in [46], Yao clearly described the interval sets by the lower and upper approximation of rough sets. Interval set features of a variety of rough sets, for which the lower and upper bounds are their lower and upper approximations respectively, show that researches on rough sets can be unified to the interval set. This is one of the motivations for the research on interval sets.

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There are many researches on interval sets now, such as concept of interval sets and their algebraic structures,[14,37,41-42,45-47,61-62] extension of interval sets,[61] comparison of interval sets,[14,48,53-54] reasoning researches of interval sets,[49,54,56-58] applications of interval sets[8,25,55] and so on.

In our studies, we find a lot of similar concepts such as interval-valued fuzzy sets, flou sets, shadowed sets and intuitionistic fuzzy sets, etc. What relationships are there between interval sets and these concepts? This paper wants to delve into this problem. In the course of study we know that interval sets are mathematically equivalent to shadowed sets and flou sets respectively, and the fuzzy interval sets are mathematically equivalent to interval-valued fuzzy sets and intuitionistic fuzzy sets respectively.

The rest of this paper is organized as follows. Section 2 studies (fuzzy) interval sets and their algebraic structures. In this section, first, L-fuzzy interval sets are defined and their operations are discussed, such that interval sets and fuzzy interval sets are their special cases. And also we give measurement methods of interval sets and discuss interval sets based on partition. In Section 3, we study the relationships between interval sets and other similar sets, such as interval-valued fuzzy sets, shadowed sets, flou sets, intuitionistic fuzzy sets, rough sets, and three-way decisions. The last section concludes this paper.

2. INTERVAL SETS AND FUZZY INTERVAL SETS

In this paper, $I=[0,1]$ or $I=([0,1],\vee,\wedge,c,0,1)$, where $\wedge$, $\vee$, and c are, respectively, defined by

$x\vee y=\max\{x,y\}$,

$x\wedge y=\min\{x,y\}$, and

$x^c=1-x$.

$(L,\vee,\wedge,N,0,1)$ denotes a fuzzy lattice[26] or complete De Morgan algebra,[1] where 0 and 1 are the minimum and maximum in L respectively. Further $\le$ is its order relation and N is an involution negator over L, i.e.

(1) $N(N(x))=x$, $\forall x\in L$ and

(2) $x\le y\Rightarrow N(y)\le N(x)$, $\forall x,y\in L$.

Let $\mathscr{F}_L(X)$ be a family of all L-fuzzy sets over X, i.e. $\mathscr{F}_L(X)=\{A\mid A:X\to L\}$.[13] While $L=[0,1]$, $\mathscr{F}(X)$ denotes a class of all fuzzy sets over X, i.e. $\mathscr{F}(X)=\{A\mid A:X\to[0,1]\}$.[59] While $L=\{0,1\}$, $\mathscr{P}(X)$ denotes a family of all subsets of X.

In the following we assume that $\bigcup,\bigcap$, $\cdot^N$ and $\subseteq$ are union, intersection, complement and order relation over $\mathscr{F}_L(X)$ respectively, i.e. $\forall x\in L$

$(A\bigcap B)(x)=A(x)\wedge B(x)$,

$(A\bigcup B)(x)=A(x)\vee B(x)$,

$A^N(x)=N(A(x))$,

$A\subseteq B\Leftrightarrow A(x)\le B(x)$,

and $A-B=A\bigcap B^N$ for $A,B\in\mathscr{F}_L(X)$.

While $L=[0,1]$, $N(x)=1-x$, A^N is written as A^c, i.e. $A^c(x)=1-A(x)$, $\forall x\in[0,1]$.

2.1. Fuzzy interval sets and their operations

Yao introduced the concept of interval set on finite set.[45] The following interval set is not limited in finite universe.

DEFINITION 1. [45, 47] *Let X be a universe and* $A_l, A_u \subseteq X, A_l \subseteq A_u$. *Then*

$$\mathcal{A} = [A_l, A_u] = \{A \subseteq X \mid A_l \subseteq A \subseteq A_u\}$$

is called an interval set of X. The family of all interval sets of X is shown by $I(\mathscr{P}(X))$.

Naturally the concept of interval set can be generalized to fuzzy sets (L-fuzzy sets).

DEFINITION 2. *Let X be a universe and* $A_l, A_u \in \mathscr{F}_L(X)$ *with* $A_l \subseteq A_u$. *Then*

$$\mathcal{A} = [A_l, A_u] = \{A \in \mathscr{F}_L(X) \mid A_l \subseteq A \subseteq A_u\}$$

is called an L-fuzzy interval set of X. The family of all L-fuzzy interval sets of X is shown by $I(\mathscr{F}_L(X))$. *Specially if* $L = [0,1]$, *an L-fuzzy interval set of X is called as a fuzzy interval set of X and the family of all fuzzy interval sets of X is shown by* $I(\mathscr{F}(X))$.

For two L-fuzzy interval sets $\mathcal{A} = [A_l, A_u]$ and $\mathcal{B} = [B_l, B_u]$, we define

$$\mathcal{A} \sqcap \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\mathcal{A} \sqcup \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\mathcal{A} \setminus \mathcal{B} = \{A - B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\neg\mathcal{A} = [X, X] \setminus \mathcal{A}.$$

It follows from Definition 2 that

$$\mathcal{A} \sqcap \mathcal{B} = [A_l \cap B_l, A_u \cap B_u],$$
$$\mathcal{A} \sqcup \mathcal{B} = [A_l \cup B_l, A_u \cup B_u],$$
$$\mathcal{A} \setminus \mathcal{B} = [A_l - B_u, A_u - B_l],$$
$$\neg\mathcal{A} = [(A_u)^N, (A_l)^N].$$

Inclusion relation of L-fuzzy interval sets $\sqsubseteq$ is defined as follows.

$$[A_l, A_u] \sqsubseteq [B_l, B_u]$$
$$\Leftrightarrow \quad A_l \subseteq B_l \text{ and } A_u \subseteq B_u$$
$$\Leftrightarrow (\forall A \in [A_l, A_u], \exists B \in [B_l, B_u], s.t.\ A \subseteq B) \text{ and } (\forall B \in [B_l, B_u], \exists A \in [A_l, A_u], s.t.\ A \subseteq B).$$

For L-fuzzy interval sets $\mathcal{A}_i = [A_l^{(i)}, A_u^{(i)}]$, $i \in \Lambda$ (any index set), we define

$$\bigsqcup_{i\in\Lambda} \mathcal{A}_i = \bigsqcup_{i\in\Lambda} [A_l^{(i)}, A_u^{(i)}] = \left[\bigcup_{i\in\Lambda} A_l^{(i)}, \bigcup_{i\in\Lambda} A_u^{(i)}\right],$$
$$\bigsqcap_{i\in\Lambda} \mathcal{A}_i = \bigsqcap_{i\in\Lambda} [A_l^{(i)}, A_u^{(i)}] = \left[\bigcap_{i\in\Lambda} A_l^{(i)}, \bigcap_{i\in\Lambda} A_u^{(i)}\right].$$

The following is immediate from the definition of interval sets and their operations.

PROPOSITION 1. *L-interval sets algebra* $(I(\mathscr{F}_L(X)), \sqcup, \sqcap, \neg, \varnothing, \mathcal{X})$ *is a fuzzy lattice. Specially, interval sets algebra* $(I(\mathscr{P}(X)), \sqcup, \sqcap, \neg, \varnothing, \mathcal{X})$ [53] *and fuzzy interval sets algebra* $(I(\mathscr{F}(X)), \sqcup, \sqcap, \neg, \varnothing, \mathcal{X})$ *are fuzzy lattices.*

The operations $\sqcap$ and $\sqcup$ are different from intersection $\cap$ and union $\cup$ of classic sets respectively. For two interval sets $\mathcal{A} = [A_l, A_u]$ and $\mathcal{B} = [B_l, B_u]$, then

$$\mathcal{A} \cap \mathcal{B} = [A_l, A_u] \cap [B_l, B_u] = \begin{cases} [A_l \cup B_l, A_u \cup B_u], & A_l \cup B_l \subseteq A_u \cap B_u \\ [\varnothing, \varnothing], & otherwise \end{cases}$$

$$\mathcal{A}\cup\mathcal{B}=[A_l,A_u]\cup[B_l,B_u].$$

$\mathcal{A}\cap\mathcal{B}$ is an interval set. But $\mathcal{A}\cup\mathcal{B}$ is not necessarily an interval set. The following example illustrates this problem.

EXAMPLE 1. *Let* $X=\{a,b,c\}$ *and consider classic inclusion relation of* $\mathscr{P}(X)$, *as shown in Fig. 1. Then*

$$\mathcal{A}=\left[\{a\},\{a,b,c\}\right]=\{\{a\},\{a,b\},\{a,c\},\{a,b,c\}\} \text{ and}$$

$$\mathcal{B}=\left[\{c\},\{a,b,c\}\right]=\{\{c\},\{a,c\},\{b,c\},\{a,b,c\}\}$$

are interval sets on X. But

$$\mathcal{A}\cup\mathcal{B}=\{\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$$

is not an interval set.

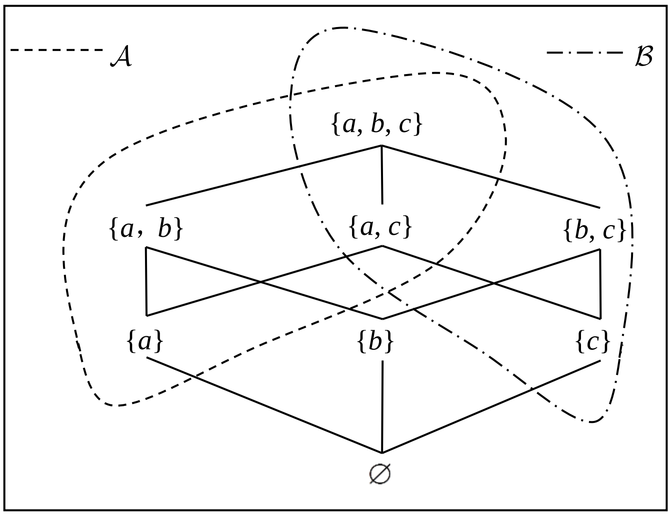

Fig. 1. Inclusion relationship of power sets for classic sets

DEFINITION 3. *Let X be a universe and* $A_l,A_u\subseteq\mathscr{F}_L(X)$. *Then* $[\varnothing,A_u]=\left\{A\subseteq X\mid A\subseteq A_u\right\}$ *and* $[A_l,X]=\left\{A\subseteq X\mid A_l\subseteq A\right\}$ *are called a u-interval set and l-interval set respectively, written as* $(A_u]$ *and* $[A_l)$ *respectively. The family of all u-interval sets and l-interval sets of X are denoted as* $I_u(\mathscr{F}_L(X))$ *and* $I_l(\mathscr{F}_L(X))$ *respectively.*

It follows from Definition 3 that if $(A_u]\in I_u(\mathscr{F}_L(X))$, $[A_l)\in I_l(\mathscr{F}_L(X))$ and $A_l\subseteq A_u$, then $[A_l,A_u]\in I(\mathscr{F}_L(X))$.

DEFINITION 4. *An interval set* $[A_l,A_u]$ *of X is called definable, if*

$$A_l\subseteq A\subseteq A_u\Rightarrow A=A_l \text{ or } A=A_u.$$

Specially, a definable interval set $(A_u]$ *(* $A_u\neq\varnothing$ *) of X is called an atom interval set of X.*

Semantically speaking, the real extension of concept shown by a definable interval set $[A_l,A_u]$ could only be A_l or A_u.

For example, $[\{a\},\{a,b\}]$ is a definable interval set and $(\{a\}]$ is an atom interval set of *X*. It is clear $(X]=\mathscr{P}(X)=\sqcup_{a\in X}(\{a\}]$.

2.2. Implication of interval sets

Logic implication is an important operation in logical algebra. In the following, we discuss implication operation over L-fuzzy interval sets. First we consider implication over L-fuzzy sets as follows.

For two L-fuzzy sets $A, B \in \mathscr{F}_L(X)$, we define

$A \to B = \bigcup\{C \in \mathscr{F}_L(X) : A \cap C \subseteq B\}$.

For two L-fuzzy interval sets $\mathcal{A} = [A_l, A_u]$ and $\mathcal{B} = [B_l, B_u]$ on X, we define

$\mathcal{A} \to \mathcal{B} = \sqcup\{[C_l, C_u] \in I(\mathscr{F}_L(X)) : [A_l, A_u] \sqcap [C_l, C_u] \sqsubseteq [B_l, B_u]\}$.

PROPOSION 2. For two L-fuzzy interval sets $\mathcal{A} = [A_l, A_u]$ and $\mathcal{B} = [B_l, B_u]$ on X, the following holds.

$\mathcal{A} \to \mathcal{B} = \left[(A_l \to B_l) \cap (A_u \to B_u), A_u \to B_u\right]$.

Proof. $\mathcal{A} \to \mathcal{B} = \sqcup\{[C_l, C_u] \in I(\mathscr{F}_L(X)) : [A_l, A_u] \sqcap [C_l, C_u] \sqsubseteq [B_l, B_u]\}$

$$= \sqcup\{[C_l, C_u] \in I(\mathscr{F}_L(X)) : A_l \cap C_l \subseteq B_l, A_u \cap C_u \subseteq B_u\}$$

$$= \Big[\left(\bigcup\{C_l \in F_L(X) : A_l \cap C_l \subseteq B_l\}\right) \cap \left(\bigcup\{C_u \in F_L(X) : A_u \cap C_u \subseteq B_u\}\right),$$

$$\bigcup\{C_u \in F_L(X) : A_u \cap C_u \subseteq B_u\}\Big]$$

$$= \left[(A_l \to B_l) \cap (A_u \to B_u), A_u \to B_u\right].$$ ■

It is worth noting that

$\mathcal{A} \to \mathcal{B} = [A_l \to B_l, A_u \to B_u]$

is not always true, since $\bigcup\{C_l \in \mathscr{F}_L(X) : A_l \cap C_l \subseteq B_l\} \subseteq \bigcup\{C_u \in \mathscr{F}_L(X) : A_u \cap C_u \subseteq B_u\}$ is not always true. It can be illustrated by the following example.

EXAMPLE 2. *Consider Example 1 again, i.e.* $X = \{a, b, c\}$ *and* $\mathscr{P}(X)$. *For two interval sets*

$\mathcal{A} = [\{a\}, \{a, b, c\}] = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ *and*

$\mathcal{B}_1 = [\{c\}, \{a, b, c\}] = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$,

We have

$\{a\} \to \{c\} = \bigcup\{C_l \in \mathscr{P}(X) : \{a\} \cap C_l \subseteq \{c\}\} = \bigcup\{\{b\}, \{c\}, \{b, c\}\} = \{b, c\}$ *and*

$\{a, b, c\} \to \{a, b, c\} = \bigcup\{C_u \in \mathscr{P}(X) : \{a, b, c\} \cap C_u \subseteq \{a, b, c\}\} = \{a, b, c\}$.

Thus $\mathcal{A} \to \mathcal{B}_1 = [\{b, c\}, \{a, b, c\}]$.

If we consider $\mathcal{B}_2 = [\{c\}, \{c\}]$, *then*

$\{a, b, c\} \to \{c\} = \bigcup\{C_u \in \mathscr{P}(X) : \{a, b, c\} \cap C_u \subseteq \{c\}\} = \bigcup\{\{c\}\} = \{c\}$.

So $(\{a\} \to \{c\}) \supset (\{a, b, c\} \to \{c\})$.

Thus $\mathcal{A} \to \mathcal{B}_1 = [\{b, c\} \cap \{c\}, \{c\}] = [\{c\}, \{c\}]$. ■

2.3. Measurement of interval sets

Inaccuracy of interval sets is caused by the existence of boundary domain. The greater boundary domain of interval sets is, the lower the accuracy of its conceptual description is. In order to more accurately express it, we introduce the concept of information certainty degree and probability certainty degree.

DEFINITION 5. *Let* $\mathcal{A} = [A_l, A_u] \in I(\mathscr{F}(X))$ *and* $X = \{x_1, x_2, \cdots, x_n\}$. *Then*

$$\alpha_I(\mathcal{A}) = \begin{cases} 1, & A_u = \varnothing \\ \dfrac{|A_l|}{|A_u|}, & A_u \neq \varnothing \end{cases}$$

is called the information certainty degree of $\mathcal{A}$ *, where* $|A| = \sum_{i=1}^{n} A(x_i)$ *for all* $A \in \mathscr{F}(X)$.

$$\rho_I(\mathcal{A}) = 1 - \alpha_I(\mathcal{A}) = \begin{cases} 0, & A_u = \varnothing \\ \dfrac{|A_u - A_l|}{|A_u|} & A_u \neq \varnothing \end{cases}$$

is called the information uncertainty degree of $\mathcal{A}$.

Consider Example 1 again, we have $\alpha_I([\{a\},\{a,b,c\}]) = \frac{1}{3}, \rho_I([\{a\},\{a,b,c\}]) = \frac{2}{3}$. Information certainty degree reflects "level" to express the concept of approximation of interval set. The smaller $\alpha_I(\mathcal{A})$ is (the larger $\rho_I(\mathcal{A})$ is), the less known information is. It follows from Definition 5 that $\alpha_I([\varnothing, A]) = 0$ for any nonempty subset *A* of *X*. This means that information certainty degree does not reflect true "size' of boundary domain. To do it, the following concept is introduced.

DEFINITION 6. $\alpha_P(\mathcal{A}) = \frac{1}{|\mathcal{A}|}$ *is called the probability certainty degree of* $\mathcal{A}$.

Magnitude of boundary can be measured by the probability certainty degree. For Example 1, we have $\alpha_P([\{a\},\{a,b,c\}]) = \frac{1}{4}$, which means that interval set $[\{a\},\{a,b,c\}]$ is a concept represented by the four possible approximations. For a definable interval set $\mathcal{A}$, it is easy to see $\alpha_P(\mathcal{A}) = \frac{1}{2}$ or 1. Obviously, $\alpha_P([A_l, A_u]) = 1$ iff $A_l = A_u$, which shows the interval set expresses an accurate concept.

2.4. Interval sets based on partition

Let $\mathscr{C} = \{X_1, X_2, \cdots, X_n\}$ be a partition of *X*, i.e. $\forall i \in \{1,2,\cdots,n\}, X_i \neq \varnothing, X_i \subseteq X$, $X_i \cap X_j = \varnothing, i \neq j$ and $\bigcup_{i=1}^{n} X_i = X$.

DEFUNITION 7. *Let* $\{X_1, X_2, \cdots, X_n\}$ *be a partition of X. For* $A \subseteq X$ *, if* $A = \varnothing$ *, or there is* $X_{i_j} \in \{X_1, X_2, \cdots, X_n\}, j = 1,2,\cdots,k$ *, such that* $A = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}$ *, then A is called a certain concept based on the partition* $\mathscr{C}$ *. If* A_l *and* A_u *are certain concepts based on the partition* $\mathscr{C}$ *, then interval set* $\mathcal{A} = [A_l, A_u] \in I(\mathscr{P}(X))$ *is called a* $\mathscr{C}$ *-certain interval set.*

PROPOSITION 3. *Let* $\mathscr{C} = \{X_1, X_2, \cdots, X_n\}$ *be a partition of X. If* $\mathcal{A} = [A_l, A_u]$ *and* $\mathcal{B} = [B_l, B_u] \in I(\mathscr{P}(X))$ *are* $\mathscr{C}$ *-certain interval sets, then so are* $\mathcal{A} \sqcap \mathcal{B}$ *,* $\mathcal{A} \sqcup \mathcal{B}$ *,* $\mathcal{A} \setminus \mathcal{B}$ *and* $\neg\mathcal{A}$.

Proof. In fact we only need to prove that if *A* and *B* are certain concepts based on the partition $\mathscr{C}$, then so are $A \cap B$, $A \cup B$, $A - B$ and A^c .

(1) Let *A* and *B* be certain concepts based on the partition $\mathscr{C} = \{X_1, X_2, \cdots, X_n\}$. Without loss of generality, let us assume that $A \neq \varnothing$ and $B \neq \varnothing$. Then there are $X_{i_p} \in \{X_1, X_2, \cdots, X_n\}, p = 1,2,\cdots,m_1$ ($1 \leq m_1 \leq n$) and $X_{j_q} \in \{X_1, X_2, \cdots, X_n\}, q = 1,2,\cdots,m_2$ ($1 \leq m_2 \leq n$), such that

$$A = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_{m_1}} \quad \text{and} \quad B = X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{m_2}} ,$$

respectively. Thus

$$A \cap B = \left(X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_{m_1}} \right) \cap \left(X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{m_2}} \right)$$
$$= X_{k_1} \cup X_{k_2} \cup \cdots \cup X_{k_{m_3}} ,$$

where $X_{k_r} \in \{X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_{m_1}}\}$, $1 \le r \le m_3 \le m_1$, i.e. $A \cap B$ is a certain concept based on the partition $\mathscr{C}$.

(2) For $A \cup B$, conclusion is obvious.

(3) If $A = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_m}$, then $A^c = \varnothing$ or $A^c = X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_{n-m}}$, where $X_{j_r} \in \{X_1, X_2, \cdots, X_n\} - \{X_{i_1}, X_{i_2}, \cdots, X_{i_m}\}$.

(4) Since $A - B = A \cap B^c$, it is straightforward from (1) and (3). ■

3. RELATIONSHIP BETWEEN (FUZZY) INTERVAL SETS AND THE SIMILAR CONCEPTS

Idea of interval sets seems to be everywhere. What relationship does interval set have with interval-valued fuzzy sets, shadowed sets, flou sets, intuitionistic fuzzy sets, rough sets and three-way decisions? This problem is discussed in the next section.

3.1 Fuzzy interval sets and interval-valued fuzzy sets

There are a lot of researches on interval-valued fuzzy sets.[9,43,60] Interval-valued fuzzy set is called the grey set in [44] and ϕ–fuzzy set in [34]. An interval type-2 fuzzy set is an interval-valued fuzzy set.[19] An interval-valued type-2 fuzzy set is a fuzzy set with membership of interval-valued fuzzy set.[20] Fuzzy-valued fuzzy set is a fuzzy set with membership of fuzzy value.[36]

The following notations are introduced for the ease of exposition.

$I^{(2)} = \left\{ [a^-, a^+] \mid 0 \le a^- \le a^+ \le 1 \right\}$,

$\bar{a} = [a, a]$,

$[a^-, a^+] \le [b^-, b^+] \Leftrightarrow a^- \le b^-, a^+ \le b^+$.

If $[a_i^-, a_i^+] \in I^{(2)}$, $i \in \Lambda$ (any index set), then we define

$$\sup_{i\in\Lambda}[a_i^-, a_i^+] = [\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+], \quad \inf_{i\in\Lambda}[a_i^-, a_i^+] = [\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+].$$

DEFINITION 8.[60] *A mapping* $A: X \to I^{(2)}$ *is called an IVF set of X and* $A(x)$ *is its membership function.* $\mathscr{F}_{I^{(2)}}(X)$ *is a family of all IVF sets of X.*

Fig.2 shows interval set characteristic of interval-valued fuzzy set.

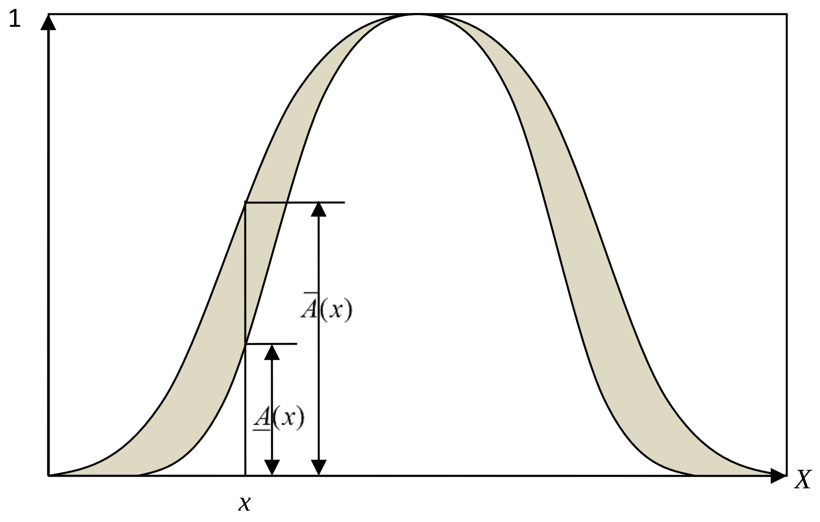

Fig. 2. Interval-valued fuzzy set $A(x)=[\underline{A}(x),\overline{A}(x)]$

For $A=[A^-,A^+], B=[B^-,B^+]\in\mathscr{F}_{I^{(2)}}(X)$, we define union, intersection and complement pointwise by the formulas

$(A\bigcup B)(x)=[A^-(x)\vee B^-(x), A^+(x)\vee B^+(x)]$,

$(A\bigcap B)(x)=[A^-(x)\wedge B^-(x), A^+(x)\wedge B^+(x)]$,

$(A^c)(x)=\overline{1}-A(x)=[1-A^+(x),1-A^-(x)]$.

The order relation $\subseteq$ in $\mathscr{F}_{I^{(2)}}(X)$ is defined by $A\subseteq B$ if and only if $A(x)\le B(x)$, i.e., $A^-(x)\le B^-(x)$ and $A^+(x)\le B^+(x)$ for all $x\in X$.

And $\varnothing, X\in\mathscr{F}_{I^{(2)}}(X)$ with membership functions $\varnothing(x)=\overline{0}$ and $X(x)=\overline{1}$, $\forall x\in X$, respectively.

It follows from Definition 1 that an interval set $[A_l, A_u]$ of *X* is an interval-valued fuzzy set $[\chi_{A_l},\chi_{A_u}]$ of *X*，where χ_A is a characteristic function of *A*. It is easy to see the following proposition which shows the relationship between fuzzy interval sets and interval-valued fuzzy sets through the mapping $f: I(\mathscr{F}(X))\to\mathscr{F}_{I^{(2)}}(X)$, $[A_l,A_u]\mapsto[\chi_{A_l},\chi_{A_u}]$.

PROPOSITION 4. (1) *Interval-valued fuzzy sets algebra* $(\mathscr{F}_{I^{(2)}}(X),\bigcup,\bigcap,c,\varnothing,X)$ *is a fuzzy lattice.*

(2) *Fuzzy interval sets algebra* $(I(\mathscr{F}(X)),\sqcup,\sqcap,\neg,\varnothing,\mathcal{X})$ *is isomorphic to interval-valued fuzzy sets algebra* $(\mathscr{F}_{I^{(2)}}(X),\bigcup,\bigcap,c,\varnothing,X)$.

3.2 Interval sets and shadowed sets

If $L=\{0,1,[0,1]\}$ and its order relation $\le_s$ is defined as $0\le_s[0,1]\le_s 1$, then $(L,\le_s)$ is a fuzzy lattice and its algebraic operations are written as $\wedge_s$, $\vee_s$ and $\neg_s$. Based on $(L,\le_s)$ Pedrycz introduced the following concept of shadowed set.[31]

DEFINITION 9.[31-32] *Let X be a universe. Then a set-valued mapping* $A: X\to\{0,1,[0,1]\}$ *is a shadowed set of X.* $\mathscr{F}_s(X)$ *is a family of all shadowed sets of X.*

It follows from Definition 9 that a shadowed set *A* of *X* is an interval-valued fuzzy set $[A^-,A^+]$, where：

$$A^-(x)=\begin{cases}A(x), & A(x)=0,1\\ 0, & A(x)=[0,1]\end{cases},\quad A^+(x)=\begin{cases}A(x), & A(x)=0,1\\ 1, & A(x)=[0,1]\end{cases}.$$

PROPOSITION 5. *An interval set is a shadowed set; conversely a shadowed set is an interval set.*

Proof. Let $[A_l, A_u]$ be an interval set of *X*. Define

$$A(x) = \begin{cases} 1, & x \in A_l \\ [0,1], & x \in A_u - A_l \\ 0, & \text{otherwise} \end{cases}$$

Then *A* is a shadowed set of *X*. Conversely, let *A* be a shadowed set of *X*. Define

$$A_l = \{x \in X \mid A(x) = 1\}, \quad A_u = \{x \in X \mid A(x) = 1 \; or \; [0,1]\}.$$

Then $[A_l, A_u]$ is an interval set of *X*. ■

In $\mathscr{F}_s(X)$, we define the following operations, for all $A, B \in \mathscr{F}_S(X)$,

(1) $(A \bigcup_s B)(x) = A(x) \vee_s B(x)$,

(2) $(A \bigcap_s B)(x) = A(x) \wedge_s B(x)$,

(3) $(\neg_s A)(x) = \neg_s (A(x))$.

The following proposition shows the relationship between interval sets and shadow sets.

PROPOSITION 6. (1) *A shadowed sets algebra* $(\mathscr{F}_s(X), \bigcup_s, \bigcap_s, \neg_s, \varnothing, X)$ *is a fuzzy lattice.*

(2) *Interval sets algebra* $(I(\mathscr{P}(X)), \sqcup, \sqcap, \neg, \varnothing, \mathcal{X})$ *is isomorphic to shadowed sets algebra* $(\mathscr{F}_s(X), \bigcup_s, \bigcap_s, \neg_s, \varnothing, X)$.

Proof. Item (1) is straightforward from the definition and operations of shadowed sets. We only prove item (2) in the following.

Let $f : I(\mathscr{P}(X)) \to \mathscr{F}_s(X)$

$$f([A_l, A_u])(x) = \begin{cases} 1, & x \in A_l \\ [0,1], & x \in A_u - A_l \\ 0, & \textit{otherwise.} \end{cases}$$

It is clear to see that *f* is a one to one mapping (injection and surjection) from $I(\mathscr{P}(X))$ to $\mathscr{F}_s(X)$. And for $\mathcal{A} = [A_l, A_u], \mathcal{B} = [B_l, B_u] \in I(\mathscr{P}(X))$, we have

$$f([A_l, A_u] \sqcup [B_l, B_u])(x) = f([A_l \cup B_l, A_u \cup B_u])(x)$$

$$= \begin{cases} 1, & x \in A_l \cup B_l \\ [0,1], & x \in A_u \cup B_u - A_l \cup B_l \\ 0, & \textit{otherwise.} \end{cases}$$

The following discussions are from three cases (see Fig. 3).

(i) $x \in A_l \cup B_l$

Here $x \in A_l$ or $x \in B_l$. $f([A_l, A_u])(x) = 1$, $f([B_l, B_u])(x) \leq_s 1$ or $f([A_l, A_u])(x) \leq_s 1$, $f([B_l, B_u])(x) = 1$. Thus $f([A_l, A_u] \sqcup [B_l, B_u])(x) = 1 = f([A_l, A_u])(x) \vee_s f([B_l, B_u])(x)$.

(ii) $x \in A_u \cup B_u - A_l \cup B_l$

$f([A_l, A_u])(x) = [0,1]$, $f([B_l, B_u])(x) \leq_s [0,1]$ or $f([A_l, A_u])(x) \leq_s [0,1]$, $f([B_l, B_u])(x) = [0,1]$. Thus $f([A_l, A_u] \sqcup [B_l, B_u])(x) = [0,1] = f([A_l, A_u])(x) \vee_s f([B_l, B_u])(x)$.

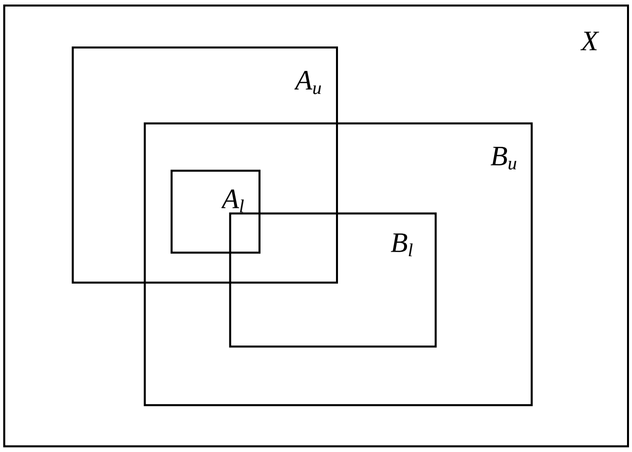

Fig. 3. The relationship diagram of two interval sets

(iii) $x \notin A_u \cup B_u$

$\left(f([A_l, A_u]) \cup_s f([B_l, B_u])\right)(x) = 0 = f([A_l, A_u])(x) = f([B_l, B_u])(x)$, i.e.,

$f([A_l, A_u] \sqcup [B_l, B_u]) = f([A_l, A_u]) \cup_s f([B_l, B_u])$.

It can be proved that $f([A_l, A_u] \sqcap [B_l, B_u]) = f([A_l, A_u]) \cap_s f([B_l, B_u])$ in a similar way.

$$f(\neg[A_l, A_u])(x) = f([(A_u)^c, (A_l)^c])(x) = \begin{cases} 1, & x \in (A_u)^c \\ [0,1], & x \in (A_l)^c - (A_u)^c \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \neg_s 0, & x \in (A_u)^c \\ \neg_s [0,1], & x \in A_u - A_l \\ \neg_s 1, & x \in A_l \end{cases}$$

$$= \neg_s (f([A_l, A_u]))(x) .$$

This completes the proof that *f* is an isomorphic mapping from $I(\mathscr{P}(X))$ to $\mathscr{F}_s(X)$. ■

Cattanco and Ciucci [6] adopted membership value 0.5 instead of [0, 1] which is equivalent to Pedrycz’s method.

3.3 Interval sets and flou sets

Flou set was introduced by Gentilhomme in 1968 [12] through considering the natural language words. For example, starting from English word “act”, one can form other words by adding prefix such as “in”, “un”, “re”, “dis” etc., and/or by adding suffix such as “ive”, “ivity”, “ion”, “ionability”, “able” etc. That is to say, we can consider the following tree structure, shown as Fig. 4.

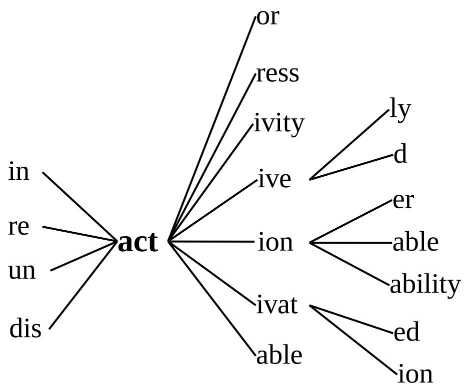

Fig. 4. The tree structure of English word "act"

Some combination from the tree produces some common words such as inactive, action, activate, actable etc. Some combinations are clearly less common words such as inactable. Some combinations seem to be acceptable, but no one can find them in the dictionary. Through this consideration from Gentilhomme, a universe is divided into three categories: the first class is the central elements, i.e. these elements must satisfy certain properties; the second is the surrounding elements, i.e. suspicious elements; the third is non-elements, i.e. these elements do not satisfy the given property. It formalizes these ideas as follows.

DEFINITION 10. [12] *Let X be a universe. Then the pair* (E,F) *is called a flou set of X, where* $E,F\subseteq X$, $E\subseteq F$ *. E is a certain domain, F is a maximum domain and* $F-E$ *is called a flou area.* $\mathscr{H}(X)$ *is a family of all flou sets of X, i.e.*

$$\mathscr{H}(X)=\{(E,F):E,F\in\mathscr{P}(X),E\subseteq F\}.$$

Obviously, crisp set *A* of *X* is a special flou set, written as (A,A).

Flou set (E,F) may be interpreted as：*E* is a set of "core" elements of *A*; $F-E$ is a set of "periphery" elements of *A*; elements of *E* are more likely to belong to *A* than elements of $F-E$.

EXAMPLE 3. (1) Let *X*=[0, 100] be a universe of age. Old age can be represented by a flou set. For example, interval set ([60, 100], [50,100]) indicates, every people, not less than 60 years of age, must be aged, every one below 50 years of age is certainly not old people and there is a question whether they are old or not for people having the ages from 50 to 60.

(2) Let *R* be an equivalence relation on finite universe, *A* be a subset of *X*, $\underline{R}(A)$ and $\overline{R}(A)$ be Pawlak lower and upper approximation of *A* respectively. Then the pair $(\underline{R}(A),\overline{R}(A))$ is a flou set. Elements of $\underline{R}(A)$ must belong to *A* (called positive domain in rough sets theory), elements of $\left(\overline{R}(A)\right)^c$ do not belong to *A* certainly (called negative domain in rough sets theory) and it is doubtful whether elements of $\overline{R}(A)-\underline{R}(A)$ belong to *A* (called boundary domain in rough sets theory).

DEFINITION 11. Let $A=(E,F),B=(E',F')\in\mathscr{FL}(X)$. Then we define

(1) $(E,F)\subseteq(E',F')\Leftrightarrow E\subseteq E',F\subseteq F'$.

(2) $(E,F)\bigcup(E',F')=(E\bigcup E',F\bigcup F')$.

(3) $(E,F)\bigcap(E',F')=(E\bigcap E',F\bigcap F')$.

(4) $(E,F)^c = (F^c, E^c)$.

The following are immediate consequences of Definition 11.

PROPOSITION 7. (1) *Flou sets algebra* $(\mathscr{H}(X), \bigcup, \bigcap, c, (\varnothing,\varnothing), (X,X))$ *is a fuzzy lattice.*

(2) *Interval sets algebra* $(I(\mathscr{P}(X)), \sqcup, \sqcap, \neg, \varnothing, \mathcal{X})$ *is isomorphic to shadowed sets algebra* $(\mathscr{H}(X), \bigcup, \bigcap, c, (\varnothing,\varnothing), (X,X))$.

Interval set and flou set are two equivalent concepts from the mathematical point of view. However interval set stresses not only results of three-way decisions, i.e. acceptance, rejection and non-promise, and also emphasizes the extension of concept described by the sets between the given two sets. But flou set emphasizes only a result of three-way decisions.

Flou set can be generalized to *n*-flou set.

DEFINITION 12. *Let X be a universe. Then an n-flou set of X is an n-tuple* $(E_1, E_2, \cdots, E_n)$, *where* $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq X$. $\mathscr{H}_n(X)$ *is a family of all n-flou sets of X.*

DEFINITION 13. *Let* $(E_1, E_2, \cdots, E_n), (F_1, F_2, \cdots, F_n) \in \mathscr{FL}_n(X)$. *Then*

(1) $(E_1, E_2, \cdots, E_n) \subseteq (F_1, F_2, \cdots, F_n) \Leftrightarrow (\forall i \in \{1, 2, \cdots, n\})(E_i \subseteq F_i)$.

(2) $(E_1, E_2, \cdots, E_n) \bigcup (F_1, F_2, \cdots, F_n) = (E_1 \bigcup F_1, E_2 \bigcup F_2, \cdots, E_n \bigcup F_n)$.

(3) $(E_1, E_2, \cdots, E_n) \bigcap (F_1, F_2, \cdots, F_n) = (E_1 \bigcap F_1, E_2 \bigcap F_2, \cdots, E_n \bigcap F_n)$.

(4) $(E_1, E_2, \cdots, E_n)^c = (E_n^c, E_{n-1}^c, \cdots, E_1^c)$.

It is trivial to show the following proposition.

PROPOSITION 8. $(\mathscr{FL}_n(X), \bigcup, \bigcap, c)$ *is a fuzzy lattice.*

Negoita and Ralescu[29] introduced *L*-flou set and proved the equivalency between *L*-Fuzzy set and *L*-flou set.

DEFINITION 14.[29] *Let X be a universe. Then an L-flou set of X is a mapping from L to* $\mathscr{P}(X)$, *i.e. subset of X, which satisfies the following conditions:*

(1) $E_0 = \varnothing$;

(2) $E_{\sup \alpha_i} = \bigcup_{i \in I} E_{\alpha_i}$, $\forall (\alpha_i)_{i \in I}, \alpha_i \in L$.

$\mathscr{H}_L(X)$ *is a family of all L-flou sets of X.*

It is clear to see that an *n*-flou set is a special *L*-flou set, where $L = \{0, 1, 2, \cdots, n\}$ with nature order, and $E_0 = \varnothing$, $(E_\alpha)_{\alpha \in \{1, 2, \cdots, n\}} \in \mathscr{H}_n(X)$.

3.4 Fuzzy interval sets and intuitionistic fuzzy sets

The concept of intuitionistic fuzzy sets was introduced by Atanassov,[2, 3] as a generalization of fuzzy sets which were developed by Zadeh.

DEFINITION 15.[2] *The object* $A = (\mu_A, \nu_A)$ *is called an intuitionistic fuzzy set on X, where* $\mu_A, \nu_A \in \mathscr{F}(X)$, $\mu_A(x)$ *is membership degree of x to A and* $\nu_A(x)$ *non-membership degree of x to A with* $\mu_A(x) + \nu_A(x) \le 1$.

Union, intersection, complement and order relation of intuitionitic fuzzy sets are defined as follows.

DEFINITION 16. *Let* $A=(\mu_A,\nu_A)$ *and* $B=(\mu_B,\nu_B)$ *be two intuitionstic fuzzy sets of X. Then the following statements hold.*

(1) $A\bigcup B=\{(x,\max\{\mu_A(x),\mu_B(x)\},\min\{\nu_A(x),\nu_B(x)\})\mid x\in X\}$.

(2) $A\bigcap B=\{(x,\min\{\mu_A(x),\mu_B(x)\},\max\{\nu_A(x),\nu_B(x)\})\mid x\in X\}$.

(3) $A^c=\{(x,\nu_A(x),\mu_A(x))\mid x\in X\}$.

(4) $A\subseteq B \Leftrightarrow \mu_A\subseteq\mu_B$ *and* $\nu_A\supseteq\nu_B$.

(5) $A=B\Leftrightarrow A\subseteq B$ *and* $B\subseteq A$.

(6) $A\preceq B\Leftrightarrow \mu_A\subseteq\mu_B$ *and* $\nu_A\subseteq\nu_B$.

The following proposition shows the relationship between fuzzy interval sets and intuitionistic fuzzy sets.

PROPOSITION 9. (1) *Intuitionistic fuzzy sets algebra* $(\mathscr{IF}(X),\bigcup,\bigcap,c,\varnothing,X)$ *is a fuzzy lattice.*

(2) *Fuzzy interval sets algebra* $(I(\mathscr{F}(X)),\sqcup,\sqcap,\neg,\varnothing,\mathcal{X})$ *is isomorphic to intuitionistic fuzzy sets algebra* $(\mathscr{IF}(X),\bigcup,\bigcap,c,\varnothing,X)$.

Proof. Let $f: I\left(\mathscr{F}(X)\right)\rightarrow\mathscr{IF}(X)$,

$[A_l,A_u]\rightarrow(A_l,(A_u)^c)$.

Obviously $(\forall x\in X)(A_l(x)\le A_u(x)\Leftrightarrow A_l(x)+(A_u)^c(x)\le 1)$ and f is a bijection (injection and surjection).

$$f([A_l,A_u]\sqcup[B_l,B_u])=f([A_l\cup B_l,A_u\cup B_u])$$
$$=\left(A_l\cup B_l,(A_u\cup B_u)^c\right)$$
$$=\left(A_l\cup B_l,(A_u)^c\cap(B_u)^c\right)$$
$$=\left(A_l,(A_u)^c\right)\cup\left(B_l,(B_u)^c\right)$$
$$=f([A_l,A_u])\cup f([B_l,B_u]).$$
$$f(\neg[A_l,A_u])=f([A_u^c,A_l^c])=(A_u^c,A_l)=(A_l,A_u^c)^c=(f([A_l,A_u]))^c.$$ ■

Gau and Buehrer[11] introduced vague sets. Bustince and Burillo[7] proved that vague sets are intuitionistic fuzzy sets soon after. Atanassov et. al discussed interval-valued intuitionistic fuzzy sets[4] and intuitionistic *L*-fuzzy sets[5].

3.5 Interval sets and rough sets

Let *R* be an equivalence relation on a finite universe, *A* be a subset of *X*, $\underline{R}(A)$ and $\overline{R}(A)$ be Pawlak's lower and upper approximation of *A* respectively. Then $[\underline{R}(A),\overline{R}(A)]$ is an interval set. The lower approximation and upper approximation are extended to all kinds of rough sets, which interval set features are listed in Table 1.

Table 1. Interval set features of all kinds of rough sets（IVF stands for interval-valued fuzzy and RS stands for rough set）

RS's family	Universe	relation (R) /objet (A)	lower and upper approximation
$(\perp,\top)$ - generalized IVF RS [18,21,22]	$X\times Y$	IVF relation / IVF set	$\underline{R}_{\perp}(A)=\left[\underline{R^{+}}_{\perp_1}(A^{-}),\underline{R^{-}}_{\perp_2}(A^{+})\right]$, $\overline{R}_{\top}(A)=\left[\overline{R^{-}}_{\top_1}(A^{-}),\overline{R^{+}}_{\top_2}(A^{+})\right]$.
$(\perp,\top)$ - generalized fuzzy RS [27]	$X\times Y$	Fuzzy relation / fuzzy set	$\underline{R}_{\perp}(A)(x)=\inf_{y\in Y}\perp\left(N(R(x,y)),A(y)\right)$, $\overline{R}_{\top}(A)(x)=\sup_{y\in Y}\top\left(R(x,y),A(y)\right)$.
$(\theta_{\top},\sigma_{\perp})$ - generalized fuzzy RS [28]	$X\times Y$	Fuzzy relation / fuzzy set	$\underline{R}_{\theta_{\top}}(A)(x)=\inf_{y\in Y}\theta_{\top}\left(R(x,y),A(y)\right)$, $\overline{R}_{\sigma_{\perp}}(A)(x)=\sup_{y\in Y}\sigma_{\perp}\left(N(R(x,y)),A(y)\right)$.
$(I,\top)$ - generalized fuzzy RS [38]	$X\times Y$	Fuzzy relation /fuzzy set of Y	$\underline{R}_{I}(A)(x)=\inf_{y\in Y}I\left(R(x,y),A(y)\right)$ $\overline{R}_{\top}(A)(x)=\sup_{y\in Y}\top\left(R(x,y),A(y)\right)$
IVF RS [35]	$X\times X$	IVF relation / IVF set of X	$\underline{R}(A)(x)=[\wedge_{y\in X}\{A^{-}(y)\vee(1-R^{+}(x,y))\},\wedge_{y\in X}\{A^{+}(y)\vee(1-R^{-}(x,y))\}]$ $\overline{R}(A)(x)=\left[\vee_{y\in X}\{A^{-}(y)\wedge R^{-}(x,y)\},\vee_{y\in X}\{A^{+}(y)\wedge R^{+}(x,y)\}\right]$
Interval-valued rough fuzzy set [35]	$X\times X$	Equivalence relation / IVF set of X	$\underline{R}(A)(x)=\left[\wedge\{A^{-}(y)\mid y\in[x]_R\},\wedge\{A^{+}(y)\mid y\in[x]_R\}\right]$ $\overline{R}(A)(x)=\left[\vee\{A^{-}(y)\mid y\in[x]_R\},\vee\{A^{+}(y)\mid y\in[x]_R\}\right]$
generalized fuzzy RS [39,40]	$X\times Y$	Fuzzy relation / fuzzy set	$\underline{R}(A)(x)=\inf_{y\in X}\left(A(y)\vee(1-R(x,y))\right)$, $\overline{R}(A)(x)=\sup_{y\in X}\left(A(y)\wedge R(x,y)\right)$.
generalized rough fuzzy set [39]	$X\times Y$	General relation / fuzzy set	$\underline{R}(A)(x)=\wedge\{A(y)\mid y\in F(x)\}$ $\overline{R}(A)(x)=\vee\{A(y)\mid y\in F(x)\}$ $F(x)=\{y\in Y\mid(x,y)\in R\},x\in X$
generalized RS [46]	$X\times Y$	General relation / Cantor set	$\underline{R}(A)=\{x\in X\mid F(x)\subseteq A\}$ $\overline{R}(A)=\{x\in X\mid F(x)\cap A\neq\varnothing\}$ $F(x)=\{y\in Y\mid(x,y)\in R\},x\in X$
Fuzzy RS [10]	$X\times X$	Fuzzy relation / fuzzy set	$\underline{R}(A)(x)=\inf_{y\in X}\left(A(y)\vee(1-R(x,y))\right)$, $\overline{R}(A)(x)=\sup_{y\in X}\left(A(y)\wedge R(x,y)\right)$.
Fuzzy RS [10]	$X\times X$	Fuzzy relation / Cantor set	$\underline{R}(A)(x)=\wedge_{y\notin A}\{1-R(x,y)\}$ $\overline{R}(A)(x)=\vee_{y\in A}\{R(x,y)\}$
Rough fuzzy set [10]	$X\times X$	Equivalence relation / fuzzy set	$\underline{R}(A)(x)=\inf\{A(y)\mid y\in[x]_R\}$, $\overline{R}(A)(x)=\sup\{A(y)\mid y\in[x]_R\}$.
Pawlak RS [30]	$X\times X$	Equivalence relation / Cantor set	$\underline{R}(A)=\{x\in X:[x]_R\subseteq A\}=\bigcup\{[x]_R\in X/R:[x]_R\subseteq A\}$, $\overline{R}(A)=\{x\in X:[x]_R\cap A\neq\varnothing\}=\bigcup\{[x]_R\in X/R:[x]_R\cap A\neq\varnothing\}$.

There are material differences in operations of rough sets and classic sets, especially in the definition of set equation. In the classic set, if the two sets have the same elements, then these two sets are equal. In rough sets theory, we need another concept on equality of sets, i.e. proximately (or roughly) equation. Two sets are not equal in the classic sets, but there may be approximately equal in rough sets. Whether two sets are approximately equal or not is based on our judgments on knowledge we have obtained.

In the following, we introduce the concepts on approximately equal and inclusion of sets.

DEFINITION 17. *Let (X, R) be a Pawlak approximation space and* $A, B \subseteq X$.

(1) *If* $\underline{R}A = \underline{R}B$ *(resp.* $\overline{R}A = \overline{R}B$*), then A and B are called R-lower equality (resp. R-upper equality), written as* $A =_{\underline{R}} B$ *(resp.* $A =_{\overline{R}} B$*).*

(2) *If* $A =_{\underline{R}} B$ *and* $A =_{\overline{R}} B$ *, then A and B are called R-equality, written as* $A =_R B$.

(3) *If* $\underline{R}A \subseteq \underline{R}B$ *(resp.* $\overline{R}A \subseteq \overline{R}B$ *), then A is said to be R-lower-contained in B (R-upper-contained in B), written as* $A \subseteq_{\underline{R}} B$ *(resp.* $A \subseteq_{\overline{R}} B$ *).*

(4) *If* $A \subseteq_{\underline{R}} B$ *and* $A \subseteq_{\overline{R}} B$ *, then A is called R-contained in B, written as* $A \subseteq_R B$.

Obviously, for any equivalence relation of *X*, $=_{\underline{R}}$, $=_{\overline{R}}$ and $=_R$ are equivalence relations on $\mathscr{P}(X)$ and $\subseteq_{\underline{R}}, \subseteq_{\overline{R}}$ and $\subseteq_R$ are preorder relations on $\mathscr{P}(X)$.

PROPOSITION 10. $\forall A \in \mathscr{P}(X)$ *, interval set* $\left[\underline{R}(A), \overline{R}(A)\right]$ *possess the following properties.*

(1) $\left[\underline{R}(A), \overline{R}(A)\right] \supseteq [A]_{=_R} = \{B \in \mathscr{P}(X) \mid B =_R A\}$;

(2) $\left[\underline{R}(A), \overline{R}(A)\right] = \{B \in \mathscr{P}(X) \mid A \subseteq_{\underline{R}} B \subseteq_{\overline{R}} A\}$.

Proof. It is easy to show (1) and we only prove (2). $\forall B \in \left[\underline{R}(A), \overline{R}(A)\right]$,

$$\underline{R}(A) \subseteq B \subseteq \overline{R}(A) \Rightarrow \underline{R}\left(\underline{R}(A)\right) \subseteq \underline{R}(B), \overline{R}(B) \subseteq \overline{R}\left(\overline{R}(A)\right)$$
$$\Rightarrow \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(B) \subseteq \overline{R}(A)$$
$$\Rightarrow A \subseteq_{\underline{R}} B \subseteq_{\overline{R}} A .$$

Conversely, $A \subseteq_{\underline{R}} B \subseteq_{\overline{R}} A$ implies $\underline{R}(A) \subseteq \underline{R}(B) \subseteq B \subseteq \overline{R}(B) \subseteq \overline{R}(A)$. ■

Fig.5 shows the relationship between interval sets constituted by rough sets and *R*-equality (*R*-contained).

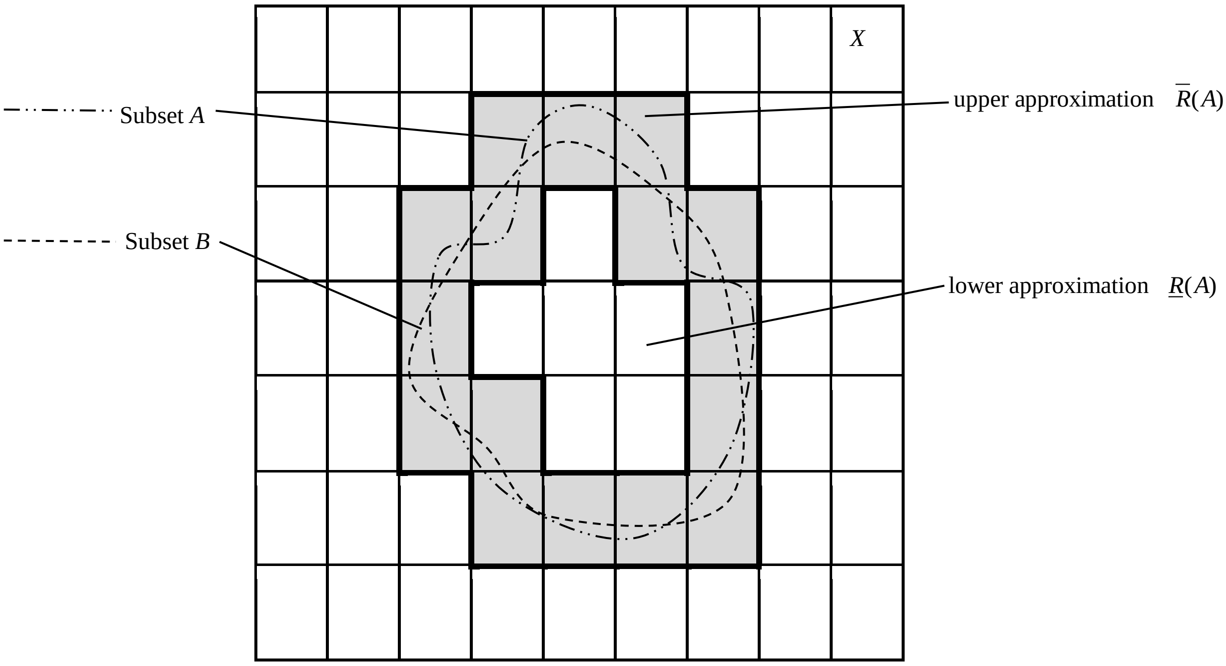

Fig. 5. Equivalence classes determined by interval sets constituted by rough sets

From properties of rough sets, we have the following proposition.

PROPOSITION 11. *Let (X, R) be a Pawlak approximation space and* $A, B \subseteq X$. *Then the followings hold.*

(1) $[\underline{R}(A), \overline{R}(A)] \sqcup [\underline{R}(B), \overline{R}(B)] \sqsubseteq [\underline{R}(A \cup B), \overline{R}(A \cup B)]$;

(2) $[\underline{R}(A), \overline{R}(A)] \sqcap [\underline{R}(B), \overline{R}(B)] \sqsupseteq [\underline{R}(A \cap B), \overline{R}(A \cap B)]$;

(3) $\neg[\underline{R}(A), \overline{R}(A)] = [\underline{R}(A^c), \overline{R}(A^c)]$;

(4) $[\underline{R}(\emptyset), \overline{R}(\emptyset)] = \emptyset, [\underline{R}(X), \overline{R}(X)] = \mathcal{X}$;

(5) $A \subseteq B \Rightarrow [\underline{R}(A), \overline{R}(A)] \sqsubseteq [\underline{R}(B), \overline{R}(B)]$.

Proof. We prove only (1), and others are similar in their proof.

$[\underline{R}(A), \overline{R}(A)] \sqcup [\underline{R}(B), \overline{R}(B)] = [\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B)]$

$= [\underline{R}(A) \cup \underline{R}(B), \overline{R}(A \cup B)]$

$\sqsubseteq [\underline{R}(A \cup B), \overline{R}(A \cup$. ■

From the discussion above, for any rough set $(\underline{R}(A), \overline{R}(A))$, it forms an interval set $[\underline{R}(A), \overline{R}(A)]$. It is nature to ask if there is an equivalence relation R over X and a subset A of X such that $[A_l, A_u] = [\underline{R}(A), \overline{R}(A)]$ for any interval set $[A_l, A_u]$ over X. The following proposition is an answer.

PROPOSITION 12. *Let* $[A_l, A_u]$ *be an interval set over X with* $A_l \neq A_u$ *and*

$$R = \{(x, y) : x, y \in A_l \text{ or } x, y \in A_u \setminus A_l \text{ or } x, y \in A_u^c\}.$$

Then $[A_l, A_u] = [\underline{R}(A), \overline{R}(A)]$ *if and only if* $A \in [A_l, A_u]$ *and* $A \neq A_l$, $A \neq A_u$.

Proof. Supposed $A \in [A_l, A_u]$ and $A \neq A_l$, $A \neq A_u$. Then $\underline{R}(A) = \{x \in X : [x]_R \subseteq A\} = A_l$ because $A \in [A_l, A_u]$ and $A \neq A_u$. $\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \emptyset\} = A_u$ because $A \in [A_l, A_u]$ and $A \neq A_l$.

Conversely, let $[A_l, A_u] = [\underline{R}(A), \overline{R}(A)]$, i.e. $\underline{R}(A) = \{x \in X : [x]_R \subseteq A\} = A_l$ and $\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \emptyset\} = A_u$.

(i) If $A_l \not\subseteq A$, then $\underline{R}(A) = \{x \in X : [x]_R \subseteq A\} \neq A_l$. This is a contradiction.

(ii) If $A \not\subseteq A_u$, $\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \emptyset\} \neq A_u$. This is a contradiction.

Thus, $A \in [A_l, A_u]$ *and* $A \neq A_l$, $A \neq A_u$. ■

3.6. Interval sets and three-way decisions

Another description of the lower and supper approximation of rough sets is three-way decisions (3WD) proposed by Yao.[50-52] For theoretical research of three-way decisions, Hu established three-way decision space such that the researches on 3WD are unified to a theoretical framework.[15-17,23-24] In the following, we discuss the relationship between three-way decisions and interval sets.

Semantically speaking, an interval set describes a concept partly known. Despite the extension of the concept is a subset of *X*, it is difficult to precisely present the subset owing to the incompleteness of information. One possible approach is to describe the concept by a lower bound A_l and an upper bound A_u. For any subset A of X, if $A_l \subseteq A \subseteq A_u$, then, as it is, A is a real extension of the concept, shown as Fig. 6.

Three-way decisions of interval set $\mathcal{A} = [A_l, A_u] = \{A \subseteq X \mid A_l \subseteq A \subseteq A_u\}$ are

(1) Acceptance： A_l ;

(2) Rejection： $(A_u)^c$;

(3) Uncertain： $A_u - A_l$.

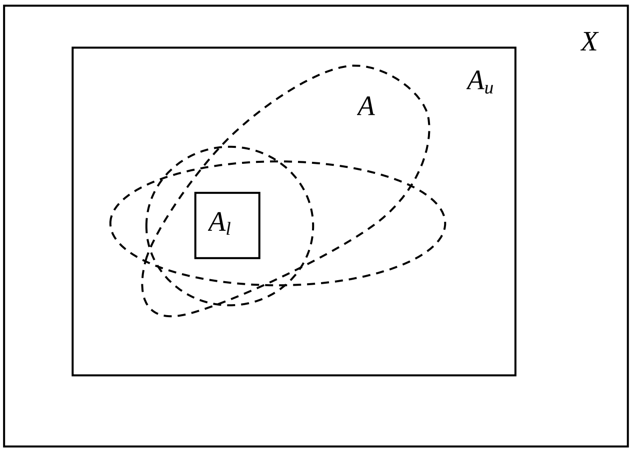

Fig. 6 Interval set description of the concept

For $[A_l, A_u] \in I(\mathscr{P}(X))$ (resp. $[A_l, A_u] \in I(\mathscr{F}(X))$, $[A_l, A_u] \in I(\mathscr{F}_L(X))$), A_l and A_u are called the lower and upper bound of (resp. fuzzy, *L*-fuzzy) interval set $[A_l, A_u]$ respectively. A_l, $(A_u)^N$ and $A_u - A_l$ are called acceptance region, rejection region and uncertain region of $[A_l, A_u]$ respectively, which be denoted by ACP($[A_l, A_u]$), REJ($[A_l, A_u]$) and UNC($[A_l, A_u]$).

EXAMPLE 4. Consider the course evaluation for students, we use an interval set

$$([60,100],[50,100]) = \{[x,100] \mid 50 \le x \le 60\} .$$

Acceptance region is [60,100], i.e., if course exam grade of a student is not less than 60, he/she passes. Rejection region is $[0,50)$, i.e., if course exam grade of a student is less than 50, he/she does not passes. Uncertain region is $[50,60)$, i.e., if course exam grade of a student is not less than 50 and less than 60, it is not sure whether he/she passes the exam and further evaluation is needed.

4. CONCLUSIONS

This paper discusses relationship between the interval sets and sets with similar concepts. Integrating the above results we draw the following relations, as shown in Fig. 7.

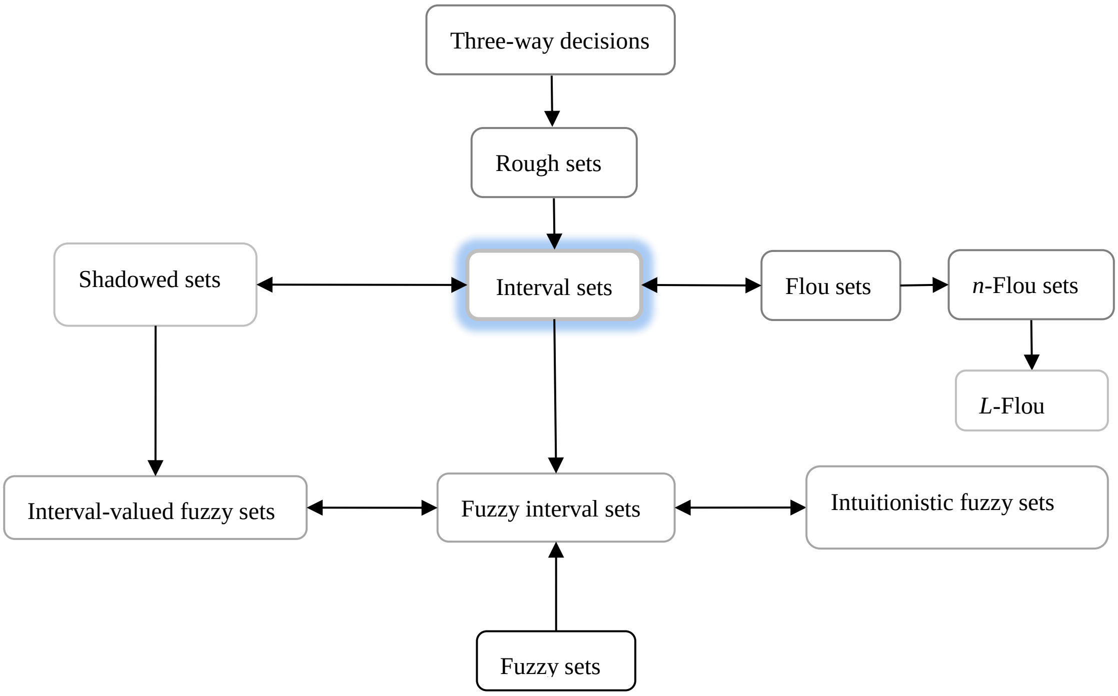

Fig. 7. The relationship between interval sets and their likeness

As has been said that the concepts of interval sets are mathematically equivalent to shadowed sets and flou sets respectively, and the concepts of fuzzy interval sets are mathematically equivalent to interval-valued fuzzy sets and intuitionistic fuzzy sets respectively. These concepts can be discussed uniformly on the interval sets.

Acknowledgements

The work of Hu was supported by grants from the National Natural Science Foundation of China (Grant nos.11571010 and 61179038). The research of Wong was supported by a grant from the Research Committee of The Hong Kong Polytechnic University (Grant no. G-YBCV). The research of Yiu was supported by a grant from the Research Committee of The Hong Kong Polytechnic University (Grant no. 152016/15E) and the GRF grant PolyU (Grant No. 152200/14E).

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