A model for continuously degrading systems with outsourcing maintenance service

Allen H. Tai

Department of Applied Mathematics,
The Hong Kong Polytechnic University,
Hung Hom, Hong Kong
E-mail: allen.tai@polyu.edu.hk

Abstract: It is a common practice in industry that maintenance service is outsourced to external suppliers. In this paper, a model for a continuously degrading system is developed such that an optimal inspection-maintenance strategy can be derived. The model is capable of handling the situation when there is deferment of the maintenance services. Hence, the system manager can decide whether to stop system operation during the waiting time for the maintenance services. The system is subjected to two degradation sources: (i) the degradation due to the operation of the system and (ii) the degradation due to the operation environment. An optimal value of the inter-inspection time and an optimal maintenance threshold are then obtained by numerical methods such that the average availability of the system is maximized. Illustrative examples and some special cases are also provided.

Keywords: Maintenance outsourcing; Degrading system; Periodic inspection; Availability

Reference to this paper should be made as follows: Tai, A.H. (xxxx) 'A model for continuously degrading systems with outsourcing maintenance service', *Int. J. Mathematics in Operational Research*, Vol. x, No. x, pp.xxx–xxx.

Biographical notes: Allen H. Tai received his BSc and MPhil in Mathematics and PhD in Industrial Engineering at The University of Hong Kong. He is now an Instructor at the Department of Applied Mathematics, The Hong Kong Polytechnic University. His current research interests are in the area of mathematical and statistical modeling, specifically optimization problems in manufacturing systems, inventory systems and supply chain management.

1 Introduction

System maintenance problem is always a challenging problem to manufacturers. Unexpected system failure may incur loss of production time, and therefore lead to extra operation costs. Carefully planned maintenance scheme can reduce the chance of system failure. In order to avoid system failure, inspections can be performed such that timely maintenance action can be carried out. In this paper, we consider a condition-based maintenance (CBM) problem, in which maintenance decisions are made based on the condition of the system. In real industrial practice, CBM is more realistic and applicable than the time-based maintenance (TBM) technique when maintenance decisions are determined (Ahmad and Kamaruddin, 2012).

Operating systems face degradation over time because of corrosion, material fatigue, wearing out or fracturing on the moving parts of the systems (Bogdanoff and Kozin, 1985). As stated in Castro et al. (2015), typical studies investigate the degradation process of such systems by modeling the corresponding process with just one single degradation source. However, in real situation, degradation may be caused by multiple sources. Recently, Alaswad and Xiang (2017) provide a review on condition-based maintenance optimization models for stochastically deteriorating system. They point out that only few models consider multiple independent degradation processes. In this paper, an integrated degradation process which consists of two degradation sources is considered. The sources of degradation of the system can be classified as the following types:

(a) The degradation due to the operation of the system, which is referred as the degradation due to internal factors in this study. Such degradation is mainly due to the wear of moving parts of the system. Here we assume that this type of degradation only incurs during the operation of the system. When operation stops, the degradation level of the system remains at the same level until the system resumes its operation.

(b) The degradation due to the operation environment, which is referred as the degradation due to external factors in this study. Here we assume that this type of degradation incurs once the system starts its operation and stops when the system is replaced.

Our objective is to develop a model that incorporates two sources of degradation and determine an optimal inspection-maintenance policy for a continuously degrading system such that the average availability of the system is optimized. The remainder of this paper is organized as follows. In Section 2, we give a literature review on related research studies. We then give a description of a continuously degrading system as well as the assumptions and notations used in this paper in Section 3. In Section 4, we present mathematical models for the system such that the average availability of the system can be maximized by choosing an optimal inspection-maintenance policy. The main theorems and some special cases are analyzed in Section 5 and a numerical example is presented in Section 6. Finally, the paper is concluded in Section 7 with future research directions.

2 Literature Review

Inspection and maintenance model for continuously degrading systems were studied intensively in literature. The inspections can be done continuously, see for example (Bloch-Mercier, 2002; Gürler and Kaya, 2002), or periodically, see for example (Chen et al., 2002; Lin and Makis, 2006; Tai et al., 2009). Marseguerra et al. (2002) consider a continuously monitored multi-component system and applied genetic algorithm to determine the optimal degradation level for maintenance actions. Grall et al. (2002) propose a decision model which optimizes the cost engaged by failure and unavailability by considering two decision variables: the preventive replacement threshold and the inspection schedule based on the system state. Sheu et al. (2006) consider three types of preventive maintenance policies: imperfect preventive maintenance (IPM), perfect preventive maintenance (PPM) and failed preventive maintenance (FPM). The system availability is maximized by choosing the optimal preventive maintenance time. Tai and Chan (2010) propose a maintenance model for a continuously degrading system, in which the inter-maintenance time and the maintenance time depend on the system state at which maintenance is carried out. Park et al. (2015) propose an RFID monitoring and investment decision model for semiconductor raw material maintenance operation.

Stochastic process is commonly used in modeling the degradation process of a system, see for example (Hong et al., 2014; Yusuf et al., 2014). Besides the degradation due to operation, other sources of degradation such as random shocks are also considered in literature. Deloux et al. (2009) consider a continuously degrading system subject to stress. Two failure mechanisms, which are due to an excessive deterioration level and a shock respectively, were considered. The optimal policy were combining statistical process control (SPC) and condition-based maintenance (CBM). A survey on modeling the relationship between failure time data and covariate data like internal degradation and external environmental processes can be found in Lehmann (2009). Huynh et al. (2011) then consider CBM of single-unit systems under dependent deterioration and traumatic shocks. A periodic inspection and replacement policy were developed to minimize the maintenance costs. Van Noortwijk et al. (2007) also consider combining two stochastic processes as modeling the condition of the system, one is gamma process and the other is a Poisson process. Lewandowski and Scholz-Reiter (2013) introduce a framework of methods and tools that enable the systematic design of CBM systems with application at a sea port.

It is a common practice in various industries that system maintenance service are outsourced, which means that maintenance service are performed by external maintenance suppliers (Martin, 1997). Bowman and Schmee (2001) develop a pricing model for a large corporation which provides maintenance and repair services provider for aircraft engines. Outsourcing maintenance has been applied also in business activities such as in telecommunications providers (Gómez et al., 2009). Lugtigheid et al. (2007) construct a finite horizon model of a repairable system to decide when to repair or replace the system under a maintenance contract. Lisnianski et al. (2008) consider a piecewise constant approximation for an increasing system failure rate function. Decision can then be made for choosing an optimal maintenance contract from those which have different repair rate and cost of repair. Wang (2010) presents a model for maintenance service contract in which maintenance supplier provides different level of maintenance services. A detail review of maintenance outsourcing can be found in (Espino-Rodríguez and Padrón-Robaina, 2006).

Besides considering the total operational cost, availability/reliability can also be used to measure the performance of a system. In some situations, cost parameters are difficult to obtain. In contrast, it is practically more convenient to measure the system uptime/downtime. Zhu et al. (2010) construct an availability optimization model for a competing risk maintenance situation, in which the system is subjected to sudden failure. Hajeeh (2012) examines several maintenance/repair options and develop an analytical expression for the availability of the general system. Jain and Rani (2013) investigate the availability characteristics for three different configurations: warm standby, switching failure and delay of reboot. Yusuf (2015) considers a system with minor deterioration and imperfect repairs and then determines the effect of failure, repair rate and number of states on system availability. Das Adhikary et al. (2016) present a preventive maintenance scheduling model to improve the availability with lower cost. The effectiveness of the model is demonstrated through a case study on a coal-fired thermal power plant.

3 Description of the System

In this section, we first provide a description for a continuously degrading system. We then give the details of the maintenance contract from external maintenance supplier.

3.1 The system states

In this study, we consider a continuously degrading system. Suppose that the system starts to operate at time $t = 0$ and the degradation level of the system at time t can be quantified by a continuous function $X(t)$, where $X(0) = 0$. Here we also refer the degradation level of the system as the state of the system. The value of $X(t)$ increases as the system operates and deteriorates. When $X(t)$ reaches a failure threshold B (> 0), the system fails immediately and corrective maintenance is required. A corrective maintenance order will then be issued to the maintenance supplier of the system.

In order to prevent the system from sudden failure, system inspections are carried out periodically such that the values of $X(t)$ can be observed. An inspection is carried out whenever the system has operated for a time period of length T since the completion of the last inspection. In other words, T is the inter-inspection time. The operation of the system is stopped during inspection and each inspection will take time T_I. Under condition-based maintenance (CBM), a preventive maintenance threshold L $(< B)$ is determined such that whenever the observation of $X(t)$ is above L, a preventive maintenance order will be issued to the maintenance supplier of the system.

On the other hand, if the observation of $X(t)$ is below L, since the system has stopped operation, an operational maintenance can be carried out to the system before the system resumes its operation. In contrast to corrective or preventive maintenance, such operational maintenance normally does not require detailed technical knowledge of the system and can be performed relatively easily. Operational maintenance can be performed by lubricating, cleaning, or replacing worn parts of the system.

The aim of this study is to formulate optimization problems for obtaining an optimal value of the inter-inspection time T and an optimal maintenance threshold L such that the average availability of the system is maximized. In this paper, we take the average availability of the system as the ratio of the expected system uptime in a given operation cycle to the expected length of the cycle.

3.2 Description of the maintenance contract

Suppose that after a maintenance order is issued to the supplier, it takes time T_W to wait for the maintenance service (either corrective or preventive) to arrive. If the maintenance order is preventive, which means the system remains operable, the system manager has two options during the waiting time:

- Option 1: The system stops its operation until a maintenance action is carried out. In this case, the chance of system failure can be reduced.
- Option 2: The system continues to operate until the maintenance carries out. In this case, the ability of the system can be fully utilized but it is at risk that the system will break down in this waiting period.

Due to practical reasons or other operational constraints, the decision of which option is used can be made

(i) before the beginning of the operation cycle;

(ii) at the time when the preventive maintenance order is issued.

We also suppose there is an agreement in the maintenance contract with the supplier such that when the system has been installed for a fixed time interval T_S, the system will be maintained. Once the time reaches T_S, then the operation stops and maintenance action will be carried out immediately. In other words, T_S is the maximum operational time of the system. Hence, the maximum number of inspections performed before maintenance, which is denoted by N, must satisfy

$$N(T+T_I) < T_S \text{ and } (N+1)(T+T_I) \geq T_S. \tag{1}$$

We denote the time required to perform preventive maintenance by $T_P\ (> 0)$ and the time required to perform corrective maintenance by $T_C\ (> T_P)$. A cycle starts when a new system starts to operate, and ends when the system is maintained.

3.3 Assumptions and notations

The following assumptions are made in the development of the model.

1. No system failure occurs during inspection. If $X(t)$ reaches B during the inspection period, it will fail immediately when it resumes operation after the inspection period.

2. Inspections are perfect in the sense that it will correctly indicate the state of the system.

3. Maintenance service from supplier are perfect. After the completion of maintenance action (preventive or corrective), the system is as-good-as-new.

4. During the inspection time period T_I, only the degradation due to external factor incurs. The internal degradation level remains at the same level as the system is not operating in this time interval.

5. The operational maintenance after each inspection can only reduce the degradation due to internal factor. We assume that by an operational maintenance, the internal degradation level incurred between successive inspections will reduce by a factor θ, where $\theta \in [0, 1]$. Here $\theta = 0$ implies that there is no operational maintenance while $\theta = 1$ implies that the effect of internal degradation is removed completely by an operational maintenance. We also assume that the time needed to perform an operational maintenance is negligible.

6. No inspection will be performed after the Nth inspection. We assume that $N \geq 1$ in the development of our model in Section 4. The case when $N = 0$ is considered as a special case in Section 5.

The following notations are used throughout the paper.

L	preventive maintenance threshold
B	system failure threshold
T	inter-inspection time
N	maximum number of inspections performed
$X(t)$	the state of a degrading system at time t
$Y(t)$	the internal degradation level at time t
$M(t)$	deterministic process describing the internal degradation process (without any operational maintenance) at time t
$Z(t)$	the external degradation level at time t
$X^{(n)}(T)$	observation of $X(t)$ at the nth inspection, $n = 1, \dots, N$
$Y^{(n)}(T)$	observation of $Y(t)$ at the nth inspection, $n = 1, \dots, N$
$Z^{(n)}(T)$	observation of $Z(t)$ at the nth inspection, $n = 1, \dots, N$
$f_s(z)$	probability density function of $Z(t+s) - Z(s)$
$F_s(z)$	cumulative distribution function of $Z(t+s) - Z(s)$
T_I	time required to perform inspection
T_W	time required to wait for maintenance action
T_S	maximum operation time since installation of the system
T_C	time required to perform corrective maintenance action
T_P	time required to perform preventive maintenance action
$P_U^{(n)}(T)$	probability that the system continues to operate after the nth inspection
$P_C^{(n)}(T)$	probability that a corrective maintenance order is issued after the nth inspection
$P_P^{(n)}(T)$	probability that a preventive maintenance order is issued after the nth inspection
$P_C(T_S)$	probability that a corrective maintenance order is issued at time T_S
$P_P(T_S)$	probability that a preventive maintenance order is issued at time T_S

4 The Mathematical Models

In this section, we first present the mathematical model for the system. We then give the average availabilities for the two options defined in Section 3.2 and hence the optimization problems can be formulated.

4.1 Model formulation

Suppose that the state of the system can be represented by a function $X(t)$, where t is the time since the system starts to operate. The system state is the total degradation level of the system, which is given by the sum of two functions:

$$X(t) = Y(t) + Z(t).$$

The function $Y(t)$ is the degradation level due to internal factors at time t, which is referred as internal degradation level. For a production system, degradation is mainly due to the production process. Since production is a routine process, degradation due to wear and fatigue of the moving parts of the machines will accumulate as production proceeds. Hence, it is reasonable to assume that the degradation due to production is deterministic. A similar model is also used in modeling the deterioration of large infrastructure systems (Sanchez-Silva et al., 2011). Since operational maintenance actions are carried out during inspections, $Y(t)$ can be modeled by a deterministic function depends on the inter-inspection time T.

Suppose that when the system has operated for time t, the internal degradation process can be modeled by a known non-decreasing function $M(t)$, where $M(0) = 0$. In other words, if there is no inspection and operational maintenance in the cycle then we have $Y(t) = M(t)$ for all $t \geq 0$. The internal degradation level of a system at any time during a given cycle is illustrated in Figure 1. The first inspection is performed at time $t = T$. At the first inspection, the internal degradation level is

$$Y^{(1)}(T) = M(T).$$

It takes T_I unit of time to finish each inspection, which means the first inspection finishes at time $t = T + T_I$. The internal degradation level remains at the same level during inspection. Hence, we have

$$Y(T + T_I) = M(T).$$

After the first inspection, an operational maintenance is carried out. This operational maintenance will reduce the degradation level by $\theta M(T)$. The degradation level $Y(t)$ starts at $(1-\theta)M(T)$ following the operational maintenance. Hence, in the second operational period of length T, we have

$$\begin{aligned} Y(T + T_I + t_1) &= (1-\theta)M(T) + [M(T + t_1) - M(T)] \\ &= M(T + t_1) - \theta M(T), \end{aligned}$$

for $0 \leq t_1 \leq T$. The second inspection is performed at time $t = 2T + T_I$. At the second inspection, the internal degradation level is

$$Y^{(2)}(T) = (1-\theta)M(T) + [M(2T) - M(T)] = M(2T) - \theta M(T).$$

The internal degradation incurs during the second operational period is $M(2T) - M(T)$. After the second inspection, the degradation level starts with

$$(1-\theta)M(T) + (1-\theta)[M(2T) - M(T)] = (1-\theta)M(2T),$$

following the operational maintenance at time $t = 2(T + T_I)$. Hence, in the next operational period of length T, we have

$$\begin{aligned} Y(2(T + T_I) + t_2) &= (1-\theta)M(2T) + [M(2T + t_2) - M(2T)] \\ &= M(2T + t_2) - \theta M(2T). \end{aligned}$$

for $0 \leq t_2 \leq T$. By similar arguments, the internal degradation level at the nth inspection, where $n = 1, 2, \ldots, N$, is given by

$$Y^{(n)}(T) = M(nT) - \theta M\big((n-1)T\big), \tag{2}$$

with, for $0 \leq t_n \leq T$,

$$Y(n(T + T_I) + t_n) = M(nT + t_n) - \theta M(nT).$$

The following proposition gives a condition that the internal degradation level is increasing in n.

Proposition 1: *$Y^{(n)}(T)$ is increasing in n if $M(t)$ is strictly convex.*

Proof. See Appendix.

The function $Z(t)$ is the degradation level due to external factors at time t, which is referred as external degradation level. As the external factors are hardly controllable by the system manager, $Z(t)$ is assumed to be a non-negative random variable depends on t. To simplify the calculations, we assume that $Z(t)$ has statistically independent increments: For any starting time $t > 0$ and time period $s > 0$, the increment of Z from t to the time $t + s$, $Z(t + s) - Z(t)$, depends on the time period s only and is independent of the starting time t. We denote the probability density function (pdf) of $Z(t + s) - Z(t)$ by $f_s(z)$ and cumulative distribution function (cdf) of $Z(t + s) - Z(t)$ by $F_s(z)$. At the nth inspection, the external degradation level is

$$Z^{(n)}(T) = Z(n(T + T_I)).$$

Therefore, the observed system state at the nth inspection is

$$X^{(n)}(T) = Y^{(n)}(T) + Z^{(n)}(T).$$

The above model provides another upper bound for N besides (1). Since N is the maximum number of inspections performed before maintenance action, we have

$$Y^{(N)}(T) \leq B \text{ and } Y^{(N+1)}(T) > B, \tag{3}$$

which means the system will definitely fail before the $(N + 1)$th inspection. Therefore, in what follows we set N by

$$N = \min\{N_1, N_2\}, \tag{4}$$

where N_1, N_2 are positive integers satisfying (1) and (3) respectively.

It is easy to see that if $Y^{(n)}(T)$ is increasing in n and $Z(t)$ is increasing in t, then $X^{(n)}(T)$ is increasing in n. In this study, we assume $X^{(n)}(T)$ is increasing in n so that the event $X^{(n)}(T) < L$ is equivalent to

$$(X^{(n)}(T) < L) \cap (X^{(n-1)}(T) < L) \cap \cdots \cap (X^{(1)}(T) < L).$$

In what follows, we present the probabilities for different actions after the nth inspection.

(1) $\underline{n = 1:}$

The probability that the system continues to operate after the first inspection is given by

$$\begin{aligned} P_U^{(1)}(T) &= \Pr(X^{(1)}(T) < L) \\ &= \Pr(Z^{(1)}(T) < L - Y^{(1)}(T)) \\ &= F_{T+T_I}(L - Y^{(1)}(T)). \end{aligned}$$

The probability that after the first inspection a corrective maintenance order is issued is given by

$$\begin{aligned} P_C^{(1)}(T) &= \Pr(X^{(1)}(T) \geq B) \\ &= \Pr(Z^{(1)}(T) \geq B - Y^{(1)}(T)) \\ &= 1 - F_{T+T_I}(B - Y^{(1)}(T)). \end{aligned}$$

The probability that after the first inspection a preventive maintenance order is issued is then given by

$$P_P^{(1)}(T) = 1 - P_U^{(1)}(T) - P_C^{(1)}(T).$$

(2) $\underline{n = 2, \ldots, N:}$

The probability that the system continues to operate after the nth inspection is given by

$$\begin{aligned} P_U^{(n)}(T) &= \Pr(X^{(n)}(T) < L) \\ &= \Pr(Z^{(n)}(T) < L - Y^{(n)}(T)) \\ &= F_{n(T+T_I)}(L - Y^{(n)}(T)). \end{aligned}$$

The probability that after the nth inspection a corrective maintenance order is issued is given by

$$\begin{aligned} &P_C^{(n)}(T) \\ &= \Pr\big((X^{(n)}(T) \geq B) \cap (X^{(n-1)}(T) < L)\big) \\ &= \Pr\big((Z^{(n)}(T) \geq B - Y^{(n)}(T)) \cap (Z^{(n-1)}(T) < L - Y^{(n-1)}(T))\big) \\ &= \Pr\big((Z^{(n)}(T) - Z^{(n-1)}(T) \geq B - Y^{(n)}(T) - Z^{(n-1)}(T)) \cap (Z^{(n-1)}(T) < L - Y^{(n-1)}(T))\big) \\ &= \int_0^{L-Y^{(n-1)}(T)} f_{(n-1)(T+T_I)}(y) \cdot (1 - F_{T+T_I}(B - Y^{(n)}(T) - y))\, dy. \end{aligned}$$

The probability that after the nth inspection a preventive maintenance order is issued is then given by

$$\begin{aligned} P_P^{(n)}(T) &= \Pr\big((L \leq X^{(n)}(T) < B) \cap (X^{(n-1)}(T) < L)\big) \\ &= \Pr(X^{(n-1)}(T) < L) - \Pr\big((X^{(n)}(T) < L) \cap (X^{(n-1)}(T) < L)\big) \\ &\quad - \Pr\big((X^{(n)}(T) \geq B) \cap (X^{(n-1)}(T) < L)\big) \\ &= P_U^{(n-1)}(T) - P_U^{(n)}(T) - P_C^{(n)}(T). \end{aligned}$$

If N inspections are performed to the system, then the maintenance action would be carried out at time T_S. In this case, the probability that a corrective maintenance action is carried out at time T_S is given by

$$\begin{aligned} &P_C(T_S) \\ &= \Pr\big((X(T_S) \geq B) \cap (X^{(N)}(T) < L)\big) \\ &= \Pr\big((Z(T_S) \geq B - Y(T_S)) \cap (Z^{(N)}(T) < L - Y^{(N)}(T))\big) \\ &= \Pr\big((Z(T_S) - Z^{(N)}(T) \geq B - Y(T_S) - Z^{(N)}(T)) \cap (Z^{(N)}(T) < L - Y^{(N)}(T))\big) \\ &= \int_0^{L-Y^{(N)}(T)} f_{N(T+T_I)}(y) \cdot (1 - F_{T_S - N(T+T_I)}(B - Y(T_S) - y))\, dy, \end{aligned}$$

where $Y(T_S)$ is given by

$$\begin{aligned} Y(T_S) &= Y^{(N)}(T) - \theta[M(NT) - M((N-1)T)] + M(T_S - NT_I) - M(NT) \\ &= M(T_S - NT_I) - \theta M(NT). \end{aligned}$$

The probability that a preventive maintenance action is carried out at time T_S is then given by

$$\begin{aligned} P_P(T_S) &= \Pr\big((X(T_S) < B) \cap (X^{(N)}(T) < L)\big) \\ &= \Pr(X^{(N)}(T) < L) - \Pr\big((X(T_S) \geq B) \cap (X^{(N)}(T) < L)\big) \\ &= P_U^{(N)}(T) - P_C(T_S). \end{aligned}$$

It is clear that from the definitions of probabilities above, we have

$$\sum_{n=1}^{N}[P_C^{(n)}(T) + P_P^{(n)}(T)] + P_C(T_S) + P_P(T_S) = 1. \tag{5}$$

The following lemma gives the cdf of time to failure after the nth inspection.

Lemma 2: *The cdf of time to failure starting from the completion of the nth inspection with $Z^{(n)}(T) = z$ is given by*

$$G_z^{(n)}(\tau) = 1 - F_\tau(B - Y(n(T+T_I)+\tau) - z),$$

with the corresponding pdf

$$g_z^{(n)}(\tau) = \frac{d}{d\tau} G_z^{(n)}(\tau).$$

Proof. See Appendix.

For simplicity, we let $H_z^{(n)}(t) = \displaystyle\int_0^t G_z^{(n)}(\tau)\, d\tau$. The following lemma is also needed in our discussion.

Lemma 3: *If $G'(x) = g(x)$ and $\int_0^u G(x)\, dx = H(u)$, then*

$$\int_0^u x g(x)\, dx = uG(u) - H(u).$$

Proof. See Appendix.

Given that a corrective maintenance order is issued after the nth inspection, it is clear that system failure occurs after the $(n-1)$th inspection. The following proposition gives the expected time to system failure after the $(n-1)$th inspection.

Proposition 4: *Given that a corrective maintenance order is issued after the nth inspection, the expected time to system failure starting from the completion of the $(n-1)$th inspection is given by*

$$E\left[\overline{T_B^{(n-1)}}\right] = \begin{cases} T + T_I - \dfrac{H_0^{(0)}(T+T_I)}{G_0^{(0)}(T+T_I)}, & n = 1; \\ T + T_I - \displaystyle\int_0^{L-Y^{(n-1)}(T)} \frac{H_z^{(n-1)}(T+T_I)}{G_z^{(n-1)}(T+T_I)} \cdot \frac{f_{(n-1)(T+T_I)}(z)}{F_{(n-1)(T+T_I)}(L - Y^{(n-1)}(T))}\, dz, & n = 2, \ldots, N. \end{cases}$$

Proof. See Appendix.

For the case when the system is maintained at time T_S, we have the following proposition.

Proposition 5: *Given that a corrective maintenance action is carried out at time T_S, the expected time to system failure starting from the completion of the Nth inspection is given by*

$$E[\overline{T_B^{(N)}}] = T_S - N(T+T_I) - \int_0^{L-Y^{(N)}(T)} \frac{H_z^{(N)}(T_S - N(T+T_I))}{G_z^{(N)}(T_S - N(T+T_I))} \cdot \frac{f_{N(T+T_I)}(z)}{F_{N(T+T_I)}(L - Y^{(N)}(T))}\, dz.$$

The proof is similar to that of Proposition 4 and hence is skipped here. We are now ready to give the average availability of the system for the two options.

4.2 Average availability for Option 1

For the case when the maintenance manager chooses Option 1 after the preventive maintenance order is issued to the supplier, the system stops its operation. The average system availability A_1 is the ratio of the expected uptime of the system in a given cycle $E[T_{U1}]$ to the expected length of a given cycle $E[T_{L1}]$:

$$A_1(T, L) = \frac{E[T_{U1}]}{E[T_{L1}]}.$$

The expected uptime of the system in a given cycle is

$$E[T_{U1}] = \sum_{n=1}^{N} P_C^{(n)}(T)\Big((n-1)T + E[\overline{T_B^{(n-1)}}]\Big) + \sum_{n=1}^{N} P_P^{(n)}(T)(nT) + P_C(T_S)\Big(NT + E[\overline{T_B^{(N)}}]\Big) + P_P(T_S)(T_S - NT_I). \quad (6)$$

The first two terms of Equation (6) correspond to the cases where corrective and preventive maintenance actions are carried out after n inspections respectively, where $n = 1, \ldots, N$. The last two terms of Equation (6) correspond to the cases where corrective and preventive maintenance actions are carried out at time T_S respectively.

For the case when maintenance action is carried out after n inspections $(n = 1, \ldots, N)$, the time from the beginning of the cycle to the completion of the nth inspection is $n(T + T_I)$. After the inspection, the cycle ends after $T_W + T_C$ if the maintenance is corrective or $T_W + T_P$ if the maintenance is preventive. For the case when maintenance action is carried out at time T_S, it is easy to see that the cycle length is $T_S + T_C$ for corrective maintenance and $T_S + T_P$ for preventive maintenance. Hence, the expected length of a given cycle is

$$E[T_{L1}] = \sum_{n=1}^{N} P_C^{(n)}(T)\Big(n(T+T_I) + T_W + T_C\Big) + \sum_{n=1}^{N} P_P^{(n)}(T)\Big(n(T+T_I) + T_W + T_P\Big) + P_C(T_S)(T_S + T_C) + P_P(T_S)(T_S + T_P). \quad (7)$$

The optimization problem is to obtain the optimal inter-inspection time T and the maintenance threshold L such that the average system availability is maximized, which is given by

$$\max A_1(T, L),$$
$$\text{subject to } 0 < T \le T_S,\ 0 < L \le B.$$

4.3 Average availability for Option 2

For the case when the maintenance manager chooses Option 2 after the maintenance order is issued to the supplier, the system continues its operation during the waiting time. To find the expected uptime in a given cycle, we first give the following lemma.

Lemma 6: *The pdf of $Z^{(1)}(T)$ under the condition that $L - Y^{(1)}(T) \leq Z^{(1)}(T) < B - Y^{(1)}(T)$ is given by*

$$\phi^{(1)}(z) = \frac{f_{T+T_I}(z)}{F_{T+T_I}(B - Y^{(1)}(T)) - F_{T+T_I}(L - Y^{(1)}(T))}.$$

For $n = 2, \dots, N$, the pdf of $Z^{(n)}(T)$ under the condition that $L - Y^{(n)}(T) \leq Z^{(n)}(T) < B - Y^{(n)}(T)$ and $Z^{(n-1)}(T) < L - Y^{(n-1)}(T)$ is given by

$$\phi^{(n)}(z) = k \int_0^{L-Y^{(n-1)}(T)} \frac{f_{T+T_I}(z-u) \cdot f_{(n-1)(T+T_I)}(u)}{F_{(n-1)(T+T_I)}(L - Y^{(n-1)}(T))}\, du$$

and k is the normalizing constant that makes

$$\int_{L-Y^{(n)}(T)}^{B-Y^{(n)}(T)} \phi^{(n)}(z)\, dz = 1.$$

Proof. See Appendix.

We then have the following proposition.

Proposition 7: *Given that a preventive maintenance order is issued after the nth inspection, the expected system uptime starting from the completion of the nth inspection is given by*

$$E\big[\overline{T_U^{(n)}}\big] = T_W - \int_{L-Y^{(n)}(T)}^{B-Y^{(n)}(T)} H_z^{(n)}(T_W)\phi^{(n)}(z)\, dz.$$

Proof. See Appendix.

The maintenance action after the waiting time T_W is either preventive or corrective, depends on whether the system fails or not in T_W. Hence, similar to the above method of finding $E\big[\overline{T_U^{(n)}}\big]$, the expected maintenance time is given by

$$E\big[\overline{T_D^{(n)}}\big] = \int_{L-Y^{(n)}(T)}^{B-Y^{(n)}(T)} [T_C G_z^{(n)}(T_W) + T_P(1 - G_z^{(n)}(T_W))]\phi^{(n)}(z)\, dz.$$

For Option 2, the average system availability A_2 is the ratio of the expected uptime of the system in a given cycle $E[T_{U2}]$ to the expected length of a given cycle $E[T_{L2}]$:

$$A_2(T, L) = \frac{E[T_{U2}]}{E[T_{L2}]}.$$

The expected uptime of the system in a given cycle is

$$E[T_{U2}] = \sum_{n=1}^{N} P_C^{(n)}(T)\Big((n-1)T + E\big[\overline{T_B^{(n-1)}}\big]\Big) + \sum_{n=1}^{N} P_P^{(n)}(T)\Big(nT + E\big[\overline{T_U^{(n)}}\big]\Big)$$
$$+ P_C(T_S)\Big(NT + E\big[\overline{T_B^{(N)}}\big]\Big) + P_P(T_S)(T_S - NT_I). \tag{8}$$

Equation (8) is the same as equation (6) except the second term, which corresponds to the case where a preventive maintenance order is issued after the n inspection, where $1 \leq n \leq N$. The extra expected uptime is given by Proposition 4.

The expected length of a given cycle can be obtained by replacing T_P in the second term of equation (7) with $E\big[\overline{T_D^{(n)}}\big]$ given above,

$$E[T_{L2}] = \sum_{n=1}^{N} P_C^{(n)}(T)\Big(n(T+T_I) + T_W + T_C\Big) + \sum_{n=1}^{N} P_P^{(n)}(T)\Big(n(T+T_I) + T_W + E\big[\overline{T_D^{(n)}}\big]\Big)$$
$$+ P_C(T_S)(T_S + T_C) + P_P(T_S)(T_S + T_P).$$

The optimization problem is to obtain the optimal inspection time interval T and the maintenance threshold L such that the average system availability is maximized, which is given by

$$\max A_2(T, L),$$
$$\text{subject to } \ 0 < T < T_S, \ 0 < L \leq B.$$

5 Main results

The following theorems give the conditions that after a preventive maintenance order is issued, the system should continue to operate until the maintenance carries out. Theorem 8 is for the case that decision is made before the beginning of the cycle.

Theorem 8: *Suppose that A_1 and A_2 are maximized at (T_1^*, L_1^*) and (T_2^*, L_2^*) respectively. To maximize the average system availability in a cycle, after the preventive maintenance order is issued the system should continue to operate until the maintenance carries out if*

$$A_2(T_2^*, L_2^*) > A_1(T_1^*, L_1^*).$$

If (T_1^*, L_1^*) is close to (T_2^*, L_2^*), then the above condition can be reduced to

$$E[T_{L1}] \sum_{n=1}^{N} P_P^{(n)} E\big[\overline{T_U^{(n)}}\big] > E[T_{U1}] \sum_{n=1}^{N} P_P^{(n)} \big(E\big[\overline{T_D^{(n)}}\big] - T_P\big).$$

The proof of Theorem 8 is directly followed from the derivations of A_1 and A_2. Theorem 9 is for the case that decision is made at the time when the preventive maintenance order is issued.

Theorem 9: *Suppose that the preventive maintenance order is issued after n inspections. To maximize the average system availability in a cycle, after the preventive maintenance order is issued the system should continue to operate until the maintenance carries out if*

$$E\big[\overline{T_U^{(n)}}\big]\big[n(T+T_I) + T_W + T_P\big] > nT\big(E\big[\overline{T_D^{(n)}}\big] - T_P\big),$$

where the values of T and L are chosen at the beginning of the cycle.

Proof. The result follows from Theorem 8 by putting $P_P^{(n)} = 1$ and the remaining probabilities in equation (5) equal 0. □

We remark that Theorem 9 is applicable irrespective to the value of $X^{(n)}(T)$.

In what follows, we analyze some special cases for the model.

1. $N = 0$: In this case, there is no inspection and the maintenance action depends on the system state at time T_S. By Lemma 2 and Lemma 3, the expected uptime in a given cycle is
$$\begin{aligned} E[\tilde{T}_U] &= \int_0^{T_S} \tau g_0^{(0)}(\tau)\, d\tau + \int_{T_S}^{\infty} T_S g_0^{(0)}(\tau)\, d\tau \\ &= T_S - H_0^{(0)}(T_S), \end{aligned}$$
and the expected length of the cycle is
$$\begin{aligned} E[\tilde{T}_L] &= \int_0^{T_S} (T_S + T_C) g_0^{(0)}(\tau)\, d\tau + \int_{T_S}^{\infty} (T_S + T_P) g_0^{(0)}(\tau)\, d\tau \\ &= T_S + T_C G_0^{(0)}(T_S) + T_P(1 - G_0^{(0)}(T_S)). \end{aligned}$$
The average availability is then given by
$$\begin{aligned} A(T, L) &= \frac{E[\tilde{T}_U]}{E[\tilde{T}_L]} \\ &= \frac{T_S - H_0^{(0)}(T_S)}{T_S + T_C G_0^{(0)}(T_S) + T_P(1 - G_0^{(0)}(T_S))}. \end{aligned}$$

2. $T_W \to 0$: In this case, the waiting time for maintenance action T_W is approaching 0. Then $E\left[\overline{T_U^{(n)}}\right] \to 0$ and $E\left[\overline{T_D^{(n)}}\right] \to T_P$. By Theorem 8 or Theorem 9, it is indifferent to choose Option 1 or Option 2.

3. $B \to \infty$: In this case, the failure threshold is so high such that the system is unlikely to fail. Therefore, to maximize the average availability there should be no inspection, which means we should set both $T, L \to \infty$ and the system maintenance action is carried out at T_S. The average availability is then
$$A(T, L) = \frac{T_S}{T_S + T_P}.$$

4. $T_I \to 0$: In this case, each inspection takes nearly no time. Therefore we should set $T \to 0$ so that the system is continuously monitored. To avoid system failure the maintenance threshold L can be set close to B. By equation (2), we have
$$Y(t) = \lim_{\Delta t \to 0} M(t) - \theta M(t - \Delta t) = (1 - \theta) M(t).$$
By Lemma 2 and Lemma 3, the expected uptime in a given cycle is the same as that in Case 1 and the expected length of the cycle is
$$\begin{aligned} E[\tilde{T}_L] &= \int_0^{T_S} (\tau + T_W + T_P) g_0^{(0)}(\tau)\, d\tau + \int_{T_S}^{\infty} (T_S + T_P) g_0^{(0)}(\tau)\, d\tau \\ &= T_W G_0^{(0)}(T_S) + T_P + T_S - H_0^{(0)}(T_S). \end{aligned}$$

The average availability is then

$$A(T, L) = \frac{E[\tilde{T}_U]}{E[\tilde{T}_L]} = \frac{T_S - H_0^{(0)}(T_S)}{T_W G_0^{(0)}(T_S) + T_P + T_S - H_0^{(0)}(T_S)}.$$

6 Numerical Example

In this section, we first present an illustrative example to demonstrate the use of the theorems we develop in Section 5. In this example, we use a gamma process to model the degradation process due to external factor. Gamma process is a commonly used stochastic process in modeling monotonically increasing (or decreasing) degradation, see for example (van Noortwijk, 2009). Mathematically, a stochastic process $Z(t)$ is a gamma process such that for any $s, t > 0$, the increment $Z(t+s) - Z(t)$ is independent of t and has gamma pdf

$$f_s(z) = \frac{z^{\alpha s - 1} \exp(-z/\beta)}{\Gamma(\alpha s) \beta^{\alpha s}}, \quad z \geq 0,$$

where $\alpha, \beta > 0$ and $\Gamma(\cdot)$ is the gamma function.

Follow from Lemma 2, $G_z^{(n)}(\tau)$, the cdf of time to failure after the nth inspection with $Z^{(n)}(T) = z$, can be expressed by (Park and Padgett, 2005)

$$G_z^{(n)}(\tau) = \frac{\Gamma\big(\alpha\tau, (B - Y(n(T + T_I) + \tau) - z)/\beta\big)}{\Gamma(\alpha\tau)},$$

where $\Gamma(\cdot, \cdot)$ is the incomplete gamma function defined by $\Gamma(a, b) = \int_0^b u^{a-1} \exp(-u)\, du$.

Suppose that the system considered is under a degradation process characterized by $M(t) = t$ and $Z(t)$ is governed by a gamma process with the parameters $\alpha = 4$ and $\beta = 0.5$. The failure threshold of the system is assumed to be $B = 10$ and the factor for operational maintenance is assumed to be $\theta = 0.6$. Suppose that after a cycle starts the maintenance from supplier comes by time $T_S = 8$ and after a maintenance order is issued to the maintenance supplier the waiting time is $T_W = 3$. We further suppose that each inspection takes time $T_I = 0.5$ and the time needed for corrective and preventive maintenance actions are respectively $T_C = 6$ and $T_P = 2$.

For illustrative purpose, suppose we set that inspection is carried out once the system has operated continuously for time $T = 1$ and the preventive maintenance threshold is taken as $L = 8$. Then by equation (4) the maximum number of inspections performed is $N = 5$. The probabilities of different actions after the nth inspection are given in Table 1, where $n = 1, 2, \ldots, 5$.

If the system continues to operate without failure after 5 inspections, the system is then maintained at time T_S. The corresponding probabilities of corrective and preventive maintenance at time T_S are

$$P_C(T_S) = 6.4478 \times 10^{-8} \quad \text{and} \quad P_P(T_S) = 1.3573 \times 10^{-4}.$$

By Propositions 3 and 4, the expected values $E\big[\overline{T_B^{(n-1)}}\big]$ and $E\big[\overline{T_B^{(N)}}\big]$ are obtained as in Table 2.

We now compare the average availability for the two maintenance options. For the case when Option 1 is chosen, the expected uptime of the system in a given cycle and the expected length of a given cycle are respectively

$$E[T_{U1}] = 3.0135 \quad \text{and} \quad E[T_{L1}] = 11.0655.$$

Hence, the average system availability A_1 is

$$A_1(1,8) = \frac{E[T_{U1}]}{E[T_{L1}]} = 0.2723.$$

For the case when Option 2 is chosen, by Proposition 7, the expected values $E\big[\overline{T_U^{(n)}}\big]$ and $E\big[\overline{T_D^{(n)}}\big]$ are obtained as in Table 3. The expected uptime of the system in a given cycle and the expected length of a given cycle are respectively

$$E[T_{U2}] = 3.5635 \quad \text{and} \quad E[T_{L2}] = 13.3740.$$

Hence, the average system availability A_2 is

$$A_2(1,8) = \frac{E[T_{U2}]}{E[T_{L2}]} = 0.2665,$$

which is less than $A_1(1,8)$. Therefore, if we set $T = 1$ and $L = 8$, after the preventive maintenance order is issued the system should stop operation and wait for the maintenance arrives.

We next provide a method of obtaining the optimal pair of the inter-inspection time T and preventive maintenance threshold L for the above system. To obtain the optimal pairs (T_1^*, L_1^*) and (T_2^*, L_2^*), we use the two-dimensional golden section search method (Chang, 2009). Golden section search is widely used in many fields because of its simplicity as it does not require finding the derivatives of the objective function. In this example, we set the stopping criteria of the optimal pairs to be $|T^{(i+1)} - T^{(i)}| < 5 \times 10^{-5}$ and $|L^{(i+1)} - L^{(i)}| < 5 \times 10^{-5}$. After 19 iterations, we obtain $(T_1^*, L_1^*) = (2.684, 6.516)$ with corresponding average availability $A_1(T_1^*, L_1^*) = 0.293$ and $(T_2^*, L_2^*) = (1.168, 2.361)$ with corresponding average availability $A_2(T_2^*, L_2^*) = 0.334$. Therefore, we can conclude that in order to maximize the average system availability, we should inspect the system every 1.168 unit of time since its last inspection. If the system state is between 2.361 and 10 then a preventive maintenance order is issued and the system should continue to operate until the maintenance is carried out.

7 Conclusions

In this paper, we first develop a model for a continuously degrading system which subject to internal and external degradations. The probabilities of different actions after inspections are derived and some expressions of the expected operational time of the system are then obtained. Two average availabilities are obtained with respect to the two options after a preventive maintenance order is issued. Hence, system managers can decide the optimal inter-inspection time and preventive maintenance threshold for the system. In addition, the optimal operation strategy while waiting for maintenance service can be obtained. An illustrative example is also presented, in which it is assumed that the external degradation process is a gamma process. Golden section search method is used to obtain the optimal inspection-maintenance strategy.

In this study, it is assumed that the system is inspected every fixed period T. One possible extension of this work is to consider condition-based inspection policy, in which the next inspection is planned based on the observed degradation level. Another research direction is considering imperfect system inspection. In future research, different maintenance contracts provided by maintenance suppliers, such as different combinations of values of T_S, T_W, T_C and T_P, can also be considered.

Acknowledgment

The author would like to thank the two anonymous referees for their helpful comments and suggestions.

References

Ahmad, R., and Kamaruddin, S. (2012). An overview of time-based and condition-based maintenance in industrial application. Computers and Industrial Engineering, 63(1), 135–149.

Alaswad, S., and Xiang, Y. (2017). A review on condition-based maintenance optimization models for stochastically deteriorating system. Reliability Engineering and System Safety, 157, 54–63.

Bloch-Mercier, S. (2002). A preventive maintenance policy with sequential checking procedure for a Markov deteriorating system. European Journal of Operational Research, 142(3), 548–576.

Bogdanoff, J.L., and Kozin, F. (1985). Probabilistic models of cumulative damage. John Wiley and Sons.

Bowman, R.A., and Schmee, J. (2001). Pricing and managing a maintenance contract for a fleet of aircraft engines. Simulation, 76(2), 69–77.

Castro, I.T., Caballé, N.C., and Pérez, C.J. (2015). A condition-based maintenance for a system subject to multiple degradation processes and external shocks. International Journal of Systems Science, 46(9), 1692–1704.

Chang, Y.C. (2009). N-dimension golden section search: Its variants and limitations. In 2009 2nd International Conference on Biomedical Engineering and Informatics (pp. 1–6). IEEE.

Chen, C.T., Chen, Y.W., and Yuan, J. (2003). On a dynamic preventive maintenance policy for a system under inspection. Reliability Engineering and System Safety, 80(1), 41–47.

Das Adhikary, D., Bose, G.K., Jana, D.K., Bose, D., and Mitra, S. (2016). Availability and cost-centered preventive maintenance scheduling of continuous operating series systems using multi-objective genetic algorithm: A case study. Quality Engineering, 28(3), 352–357.

Deloux, E., Castanier, B., and Bérenguer, C. (2009). Predictive maintenance policy for a gradually deteriorating system subject to stress. Reliability Engineering and System Safety, 94(2), 418–431.

Espino-Rodríguez, T.F., and Padrón-Robaina, V. (2006). A review of outsourcing from the resource-based view of the firm. International Journal of Management Reviews, 8(1), 49–70.

Gómez, J., Crespo, A., Moreu, P., Parra, C., and Díaz, V. G. (2009). Outsourcing maintenance in services providers. Safety, Reliability and Risk Analysis: Theory, Methods and Applications–Martorell et al.(eds), 829–837.

Grall, A., Dieulle, L., Bérenguer, C., and Roussignol, M. (2002). Continuous-time predictive-maintenance scheduling for a deteriorating system. IEEE Transactions on Reliability, 51(2), 141–150.

Gürler, U., Kaya, A. (2002). A maintenance policy for a system with multi-state components: an approximate solution. Reliability Engineering and System Safety, 76(2), 117–127.

Hajeeh, M.A. (2012). Availability of a system with different repair options. International Journal of Mathematics in Operational Research, 4(1), 41–55.

Hong, H.P., Zhou, W., Zhang, S., and Ye, W. (2014). Optimal condition-based maintenance decisions for systems with dependent stochastic degradation of components. Reliability Engineering and System Safety, 121, 276–288.

Huynh, K.T., Barros, A., Berenguer, C., and Castro, I.T. (2011). A periodic inspection and replacement policy for systems subject to competing failure modes due to degradation and traumatic events. Reliability Engineering and System Safety, 96(4), 497–508.

Jain, M., and Rani, S. (2013). Availability analysis for repairable system with warm standby, switching failure and reboot delay. International Journal of Mathematics in Operational Research, 5(1), 19–39.

Lehmann, A. (2009). Joint modeling of degradation and failure time data. Journal of Statistical Planning and Inference, 139(5), 1693–1706.

Lewandowski, M., and Scholz-Reiter, B. (2013). A Framework for Systematic Design and Operation of Condition-Based Maintenance Systems: Evidence from a Case Study of Fleet Management at Sea Ports. International Journal of Industrial Engineering: Theory, Applications And Practice, 20(1–2), 2–11.

Lin, D., and Makis, V. (2006). On-line parameter estimation for a partially observable system subject to random failure. Naval Research Logistics (NRL), 53(5), 477–483.

Lisnianski, A., Frenkel, I., Khvatskin, L., and Ding, Y. (2008). Maintenance contract assessment for aging systems. Quality and Reliability Engineering International, 24(5), 519–531.

Lugtigheid, D., Jardine, A.K., and Jiang, X. (2007). Optimizing the performance of a repairable system under a maintenance and repair contract. Quality and Reliability Engineering International, 23(8), 943–960.

Marseguerra, M., Zio, E., and Podofillini, L. (2002). Condition-based maintenance optimization by means of genetic algorithms and Monte Carlo simulation. Reliability Engineering and System Safety, 77(2), 151–165.

Martin, H.H. (1997). Contracting out maintenance and a plan for future research. Journal of Quality in Maintenance Engineering, 3(2), 81–90.

Park, S., Hong, K.S., and Lee, C. (2015). Optimal Maintenance Operations Using a RFID-based Monitoring System. International Journal of Industrial Engineering, 22(1), 117–125.

Park, C., and Padgett, W.J. (2005). Accelerated degradation models for failure based on geometric Brownian motion and gamma processes. Lifetime Data Analysis, 11(4), 511–527.

Sanchez-Silva, M., Klutke, G.A., and Rosowsky, D.V. (2011). Life-cycle performance of structures subject to multiple deterioration mechanisms. Structural Safety, 33(3), 206–217.

Sheu, S.H., Lin, Y.B., and Liao, G.L. (2006). Optimum policies for a system with general imperfect maintenance. Reliability Engineering and System Safety, 91(3), 362–369.

Tai, A.H., Chan, L.Y., Zhou, Y., Liao, H., and Elsayed, E.A. (2009). Condition based maintenance of periodically inspected systems. In Proceedings of the World Congress on Engineering, 2, 1256–1261.

Tai, A.H., and Chan, L.Y. (2010). Maintenance models for a continuously degrading system. Computers and industrial engineering, 58(4), 578–583.

van Noortwijk, J.M. (2009). A survey of the application of gamma processes in maintenance. Reliability Engineering and System Safety, 94(1), 2–21.

van Noortwijk, J.M., van der Weide, J.A., Kallen, M.J., and Pandey, M.D. (2007). Gamma processes and peaks-over-threshold distributions for time-dependent reliability. Reliability Engineering and System Safety, 92(12), 1651–1658.

Wang, W. (2010). A model for maintenance service contract design, negotiation and optimization. European Journal of Operational Research, 201(1), 239–246.

Yusuf, I., Suleiman, K., Bashir, Y., and Ali, U. A. (2014). Stochastic modelling and analysis of a repairable 2-out-of-4 system. International Journal of Mathematics in Operational Research, 6(1), 29–40.

Yusuf, I. (2015). Availability modelling and evaluation of a repairable system subject to minor deterioration under imperfect repairs. International Journal of Mathematics in Operational Research, 7(1), 42–51.

Zhu, Y., Elsayed, E.A., Liao, H., and Chan, L.Y. (2010). Availability optimization of systems subject to competing risk. European Journal of Operational Research, 202(3), 781–788.

Appendix

Proof of Proposition 1. If $M''(t) > 0$, then for any $\tau_1 \in [(n-1)T, nT]$ and $\tau_2 \in [nT, (n+1)T]$, we have

$$M'(\tau_2) > M'(\tau_1).$$

By the mean value theorem,

$$\frac{M\big((n+1)T\big) - M(nT)}{T} > \frac{M(nT) - M\big((n-1)T\big)}{T}.$$

Hence for any positive integer n and $\theta \in [0,1]$, we have

$$\begin{aligned} Y^{(n+1)}(T) - Y^{(n)}(T) &= \big[M\big((n+1)T\big) - \theta M(nT)\big] - \big[M(nT) - \theta M\big((n-1)T\big)\big] \\ &> \big[M\big((n+1)T\big) - M(nT)\big] - \big[M(nT) - M\big((n-1)T\big)\big] \\ &> 0. \end{aligned}$$

□

Proof of Lemma 2. Let $T_B^{(n)}$ be the time to failure starting from the completion of the nth inspection with $Z^{(n)}(T) = z$. Then

$$\begin{aligned} G_z^{(n)}(\tau) &= \Pr(T_B^{(n)} < \tau) \\ &= \Pr(X(n(T+T_I)+\tau) > B) \\ &= \Pr(z + Z(\tau) + Y(n(T+T_I)+\tau) > B) \\ &= \Pr(Z(\tau) > B - Y(n(T+T_I)+\tau) - z) \\ &= 1 - F_\tau(B - Y(n(T+T_I)+\tau) - z). \end{aligned}$$

□

Proof of Lemma 3.

$$\begin{aligned} \int_0^u xg(x)\,dx &= \int_0^u \int_0^x g(x)\,ds\,dx = \int_0^u \int_s^u g(x)\,dx\,ds \\ &= \int_0^u (G(u) - G(s))\,ds = uG(u) - H(u). \end{aligned}$$

□

Proof of Proposition 4. We give the proof for the case when $n = 2, \ldots, N$. For the case when $n = 1$ the proof can be deduced in the same way with some minor modification and hence is skipped here. Denote the time to system failure starting from the completion of the $(n-1)$th inspection by $T_B^{(n-1)}$. First the system has already failed before the nth inspection means that

$$X^{(n)}(T) > B \Leftrightarrow T_B^{(n-1)} < T + T_I.$$

Then

$$\begin{aligned} &E\big[\overline{T_B^{(n-1)}}\big] \\ =&E[T_B^{(n-1)} | (X^{(n)}(T) \geq B) \cap (X^{(n-1)}(T) < L)] \\ =&E[T_B^{(n-1)} | (T_B^{(n-1)} < T + T_I) \cap (Z^{(n-1)}(T) < L - Y^{(n-1)}(T))] \\ =&\int_0^{L-Y^{(n-1)}(T)} E[T_B^{(n-1)} | (T_B^{(n-1)} < T + T_I) \cap (Z^{(n-1)}(T) = z)] \frac{f_{(n-1)(T+T_I)}(z)}{F_{(n-1)(T+T_I)}(L - Y^{(n-1)}(T))}\,dz. \end{aligned}$$

Follows from Lemma 3, the conditional expectation in the above expression is

$$\begin{aligned}
&E[T_B^{(n-1)}|(T_B^{(n-1)} < T+T_I) \cap (Z^{(n-1)}(T)=z)] \\
=&\int_0^{T+T_I} \tau g_z^{(n-1)}(\tau|T_B^{(n-1)} < T+T_I)\, d\tau \\
=&\int_0^{T+T_I} \tau \frac{g_z^{(n-1)}(\tau)}{G_z^{(n-1)}(T+T_I)}\, d\tau \\
=&T+T_I - \frac{H_z^{(n-1)}(T+T_I)}{G_z^{(n-1)}(T+T_I)}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
E\big[\overline{T_B^{(n-1)}}\big] =& \int_0^{L-Y^{(n-1)}(T)} \Big[T+T_I - \frac{H_z^{(n-1)}(T+T_I)}{G_z^{(n-1)}(T+T_I)}\Big] \frac{f_{(n-1)(T+T_I)}(z)}{F_{(n-1)(T+T_I)}(L-Y^{(n-1)}(T))}\, dz \\
=&T+T_I - \int_0^{L-Y^{(n-1)}(T)} \frac{H_z^{(n-1)}(T+T_I)}{G_z^{(n-1)}(T+T_I)} \cdot \frac{f_{(n-1)(T+T_I)}(z)}{F_{(n-1)(T+T_I)}(L-Y^{(n-1)}(T))}\, dz.
\end{aligned}$$

□

Proof of Lemma 6. The derivation of $\phi^{(1)}(z)$ is trivial. For $n = 2, \ldots, N$, the pdf of $Z^{(n-1)}(T)$ given that $Z^{(n-1)}(T) < L - Y^{(n-1)}(T)$ is

$$\frac{f_{(n-1)(T+T_I)}(u)}{F_{(n-1)(T+T_I)}(L-Y^{(n-1)}(T))}.$$

Since the increment of Z from time $(n-1)(T+T_I)$ to $n(T+T_I)$ is independent of $Z^{(n-1)}(T)$, the pdf of $Z^{(n)}(T)$ given that $Z^{(n-1)}(T) < L - Y^{(n-1)}(T)$ is

$$\int_0^{L-Y^{(n-1)}(T)} \frac{f_{T+T_I}(z-u) \cdot f_{(n-1)(T+T_I)}(u)}{F_{(n-1)(T+T_I)}(L-Y^{(n-1)}(T))}\, du. \tag{9}$$

Hence, the pdf of $Z^{(n)}(T)$ under the condition that $L - Y^{(n)}(T) \le Z^{(n)}(T) < B - Y^{(n)}(T)$ and $Z^{(n-1)}(T) < L - Y^{(n-1)}(T)$, $\phi^{(n)}(z)$, can be obtained by normalizing (9) such that $\int_{L-Y^{(n)}(T)}^{B-Y^{(n)}(T)} \phi^{(n)}(z)\, dz = 1$. □

Proof of Proposition 7. We give the proof for the case when $n = 2, \ldots, N$. For the case when $n = 1$ the proof can be deduced in the same way with some minor modification and hence is skipped here. Denote the system uptime starting from the completion of the nth inspection by $T_U^{(n)}$.

$$\begin{aligned}
&E\big[\overline{T_U^{(n)}}\big] \\
=&E[T_U^{(n)}|(L \le X^{(n)}(T) < B) \cap (X^{(n-1)}(T) < L)] \\
=&E[T_U^{(n)}|(L-Y^{(n)}(T) \le Z^{(n)}(T) < B - Y^{(n)}(T)) \cap (Z^{(n-1)}(T) < L - Y^{(n-1)}(T))] \\
=&\int_{L-Y^{(n)}(T)}^{B-Y^{(n)}(T)} E[T_U^{(n)}|Z^{(n)}(T) = z]\phi^{(n)}(z)\, dz.
\end{aligned}$$

The proof is completed by showing the expectation

$$E[T_U^{(n)}|Z^{(n)}(T)=z] = \int_0^{T_W} \tau g_z^{(n)}(\tau)\, d\tau + \int_{T_W}^{\infty} T_W g_z^{(n)}(\tau)\, d\tau$$
$$= T_W - H_z^{(n)}(T_W),$$

where the last equality follows from Lemma 3. □

n	1	2	3	4	5
$P_U(n)$	0.9945	0.7400	0.1728	0.0093	0.0001
$P_C(n)$	0.0003	0.0725	0.2947	0.0546	0.0007
$P_P(n)$	0.0052	0.1820	0.2725	0.1089	0.0085

Table 1 The probabilities of different actions after the nth inspection when $T = 1$ and $L = 8$.

n	1	2	3	4	5	$E\big[\overline{T_B^{(N)}}\big]$
$E\big[\overline{T_B^{(n-1)}}\big]$	1.3532	1.3013	1.2177	1.1946	1.2064	0.4047

Table 2 The expected values $E\big[\overline{T_B^{(n-1)}}\big]$ and $E\big[\overline{T_B^{(N)}}\big]$.

n	1	2	3	4	5
$E\big[\overline{T_U^{(n)}}\big]$	0.8573	0.9139	0.9542	1.0095	1.0954
$E\big[\overline{T_D^{(n)}}\big]$	6.0000	6.0000	6.0000	6.0000	6.0000

Table 3 The expected values $E\big[\overline{T_U^{(n)}}\big]$ and $E\big[\overline{T_D^{(n)}}\big]$.

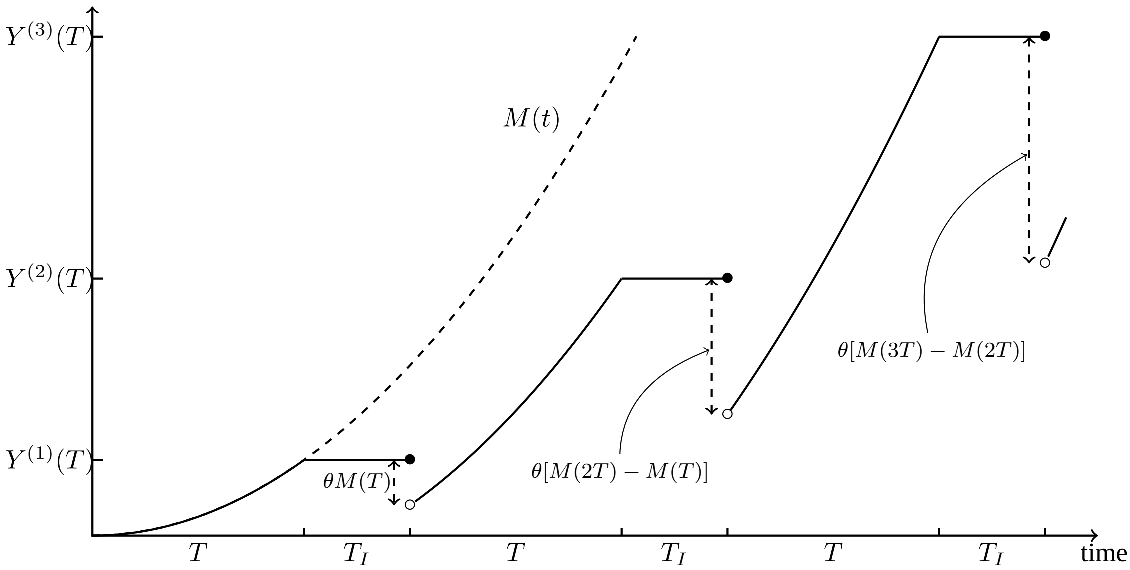

Figure 1 The internal degradation level of a system.