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Joint Inspection and Inventory Control for Deteriorating Items with Random Maximum Lifetime

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Abstract

In this paper, we consider inventory control problems for deteriorating items. Six practical models are developed for an inventory system with deterioration rate depending on the maximum lifetime of items. Due to the physical nature of items, the maximum lifetime of items is assumed to be random over a time period. Items exceeding the maximum lifetime are regarded as scarp and no longer serviceable. Two replenishment policies: (i) quantity-based policy and (ii) time-based policy, and two inspection scenarios: (i) one inspection and (ii) continuous monitoring have been discussed in these models. Examples and sensitivity analysis are given for each model. The results indicate that it may be beneficial for the inventory holder to adopt the quantity-based policy and drop the plan of continuous monitoring if the cost of continuous monitoring is too high. Otherwise, performing continuous monitoring is able to increase the optimal long-run average profit. These results provide useful insights for inventory holder in making managerial decisions.

Keywords: Deteriorating items; Replenishment policy; Inspection policy; Random maximum lifetime; Inventory system.

1 Introduction

Products such as fresh food, chemicals and medicines may deteriorate when they are stored in warehouses. Electronic devices and fashion goods can also be considered as deteriorating items. These deteriorating items may lose their utility with time due to decay, damage or spoilage. Once their quality becomes unacceptable, they are no longer serviceable in the market. Deriving and obtaining an optimal inventory decision is a very complicated and difficult task in practice, but this is essential for decision makers in the industries. This study is motivated by an agricultural inventory control problem faced by a farmer who grows vegetables. The quality of harvested vegetables deteriorates during storage and they are sold with random shelf life. Due to the insufficiency of manpower, spoiled vegetables may not be able to screen out in time and therefore they may be sold to customers. Some studies on agricultural inventory control include Lodree and Uzochukwu [17] and Widodo et al. [31].

When the items stored in the inventory have deteriorated, it will be essential to screen them out from the good quality items since customers receiving deteriorated items may create damage to business reputation. In order to screen out deteriorated items, inspections can be preformed in the inventory. However, typical inventory models usually focus on screening out the deteriorated items, not much attention is paid to the situation that a proportion of deteriorated items may be sold to consumers together with serviceable items during a replenishment cycle. In this paper, we consider such kind of mixed sales with different inspection policies. Performing one inspection during the cycle is a possible choice. Another one is conducting continuous monitoring in the cycle. Continuous inspection is effective as it will screen out deteriorated item from inventory immediately, but it will be costly.

In reality, products have maximum lifetime due to their physical nature. It is also required by law in most countries that an expiry date has to be specified on certain products. Recently, researchers are paying more attention to “maximum lifetime” in modeling deteriorating inventory. This paper extends the model of Sarkar [20] along with random maximum lifetime. Sarkar proposed an Economic Order Quantity (EOQ) model for deteriorating items with maximum lifetime. In Sarkar’s model, the products deteriorate at a rate which depends on the maximum lifetime of the products. However, in practice it is not common for a product to have a deterministic lifetime. Various environmental factors such as temperature, humidity and air pressure may affect the maximum lifetime of the products. Thus the maximum lifetime of items may be uncertain and it may be difficult to confirm the exact time when the items would become deteriorated. In such cases, the assumption that products have deterministic lifetime is unrealistic, and we relax this assumption by considering products with random maximum lifetime.

We suppose that the deterioration rate of the inventory system depends on the maximum lifetime of items, and the maximum lifetime is assumed to be random. When an item stored in inventory exceeds its maximum lifetime, it is regarded as scarp and no longer serviceable. Two replenishment policies are provided to the inventory holder: (i) Quantity-based policy, which means that the inventory is replenished when all stocks are used; (ii) Time-based policy, which means that the inventory is replenished for every pre-determined period. During the replenishment cycle, two inspection scenarios can be considered: (i) one inspection during the cycle; (ii) continuous monitoring in the cycle. Six models have been proposed to discuss different cases. Examples and sensitivity analysis are conducted for each model.

1.1 Literature review

In the past fifty years, the exponential model based on the work of Ghare and Schrader [8] has been widely adopted in modeling the product decay in inventory. Readers may refer to the survey papers by Raafat [18], Goyal and Giri [11], Bakker et al. [2], Ghiami and Williams [9], Gilding [10], Janssen et al. [13], Taleizadeh [27] for research works on deteriorating inventory. The development of EOQ model with inventory deterioration grows rapidly since the work of Ghare and Schrader [8]. Recently, Ahmed et al. [1] proposed an inventory model with a general deterioration rate. They provided a general method which seems to be able to handle models with time varying deterioration rate. Dye [6] considered an inventory system with non-instantaneous deterioration and investigated the effect of preservation technology investment to the model. Yang et al. [35] extended the model of Dye [6] by considering a joint problem of trade credit and preservation technology investment.

Sarkar [20] proposed an EOQ model with time varying deterioration rate. The deterioration rate is an increasing function of time. When the lifetime of items reaches the maximum, which is assumed to be a constant, the deterioration rate becomes 1. This implies that all items in the inventory were deteriorated. Chen and Teng [3] extended Sarkar's model by considering the case when the retailer is offered a permissible delay in payment from its supplier. Wang et al. [30] extended Sarkar's model from the seller's point of view. The seller offers its buyer a trade credit period, which can boost the demands but also increase the default risk. Shah et al. [22] further extended the model by introducing two-level trade credits. The retailer passes credit period, which is received from its supplier, to its buyers. Wu et al. [33] considered a supply chain with supplier, retailer and customer in which the retailer offers a downstream partial trade credit to credit-risk customers. Other extensions of the Sarkar's model include Chen and Teng [4], Dye et al. [7], Sarkar et al. [21] and Wu and Chan [34].

Most of the research works on deteriorating items considered continuous monitoring, which means that deteriorated items are screened out from the inventory once they become deteriorated. Related discussions can be found in Chung and Wee [5], Lin et al. [14], Liao

[15], Teng and Chang [23] and Widyadana and Wee [32]. Tai et al. [28] proposed a model to formulate the process of mixed sales of deteriorated items and good items, and further studied the effect of inspection policies on optimal decisions for a deteriorating inventory system.

The remainder of this paper is organized as follows. Section 2 introduces the notations and assumptions adopted in the proposed models. Section 3 presents two models under quantity-based replenishment policy with and without an inspection during a replenishment cycle. Section 4 presents two models under time-based replenishment policy with and without an inspection during a replenishment cycle. Section 5 considers two models under continuous monitoring using different replenishment policies. Numerical examples and extensions are given in Sections 6 and 7, respectively. Finally, conclusions and future research issues are given in Section 8.

2 Notations and assumptions

In this section, we give the notations and assumptions adopted in the development of the models. The notations including decision variables (Q, τ and T) and other parameters are presented in Table 1.

Table 1: Notations

Q	the replenishment quantity (units)
T	the length of a replenishment cycle (days)
τ	the inspection time, where $\tau < T$ (days)
m	the maximum lifetime of a product item (days)
λ	demand rate (units per day)
p	sale price of a product item ($ per unit)
c	purchasing cost ($ per unit)
K	ordering cost ($ per order)
h	inventory holding cost ($ per unit per day)
c_b	lost sale penalty cost ($ per unit)
c_d	inspection cost ($ per unit)
s	salvage value of a product item ($ per unit)

The mathematical models proposed in this study have the following assumptions.

1. The maximum lifetime of a product item, denoted as m, is a positive random variable. For simplicity, the maximum lifetimes of all items in the same batch are assumed to be the same. This study assumes that m is uniformly distributed with probability

density function

$$f(m) = \begin{cases} \frac{1}{b-a}, & a < m < b; \\ 0, & \text{otherwise}, \end{cases}$$

where a and b ($b > a > 0$) are known constants.

2. Inventory holder is able to observe the items which have reached their maximum lifetime.

3. The deterioration rate of an inventory system depends on the time t and the products' maximum lifetime m. Here we follow the assumption as in Sarkar [20] that the deterioration rate is given by

$$\theta(t) = \frac{1}{1+m-t}, \quad 0 \le t \le m,$$

which has been widely used in the literature, such as Chen and Teng [4], Dye et al. [7], Teng et al. [26], Wang et al. [30], Wu et al. [33] and Wu and Chan [34]. The products begin to deteriorate at the beginning of the replenishment cycle. For example, fruit and vegetable, they begin to deteriorate once they are harvested.

4. Shortage is not allowed. Lost sales will incur a penalty cost to cover the lost of goodwill. Furthermore, customers receive full refund if the purchased item is deteriorated. Hence, the inventory holder receives revenue only when the sold item is serviceable.

5. The inspection is perfect, which means that it will correctly screen out deteriorated items. In Lee and Rosenblatt [16], they modeled a production system under the assumption that each inspection is perfect and then gave some discussions of imperfect inspection.

6. The replenishment lead time is negligible in this work. For future research, it would be interesting to study the effects of non-negligible lead-times on the decision variables, such as the replenishment quantity, the length of a replenishment cycle and the inspection time.

7. The long-run average profit can be determined by the renewal reward theorem, see, Ross [19], for more details. Similar methods can be found in Wahab and Jaber [29].

$$\text{Average profit} = \frac{\text{Expected Revenue} - \text{Expected Total Cost}}{\text{Expected Replenishment Cycle Length}}. \quad (1)$$

3 Quantity-based Replenishment Policy Models

Suppose that an inventory holder replenishes the inventory when all items in the inventory are sold or deteriorated. Two models, one has no inspection and another has one inspection during a replenishment cycle, are presented and discussed in this section. The timelines of the two models are shown in Figures 1 and 2.

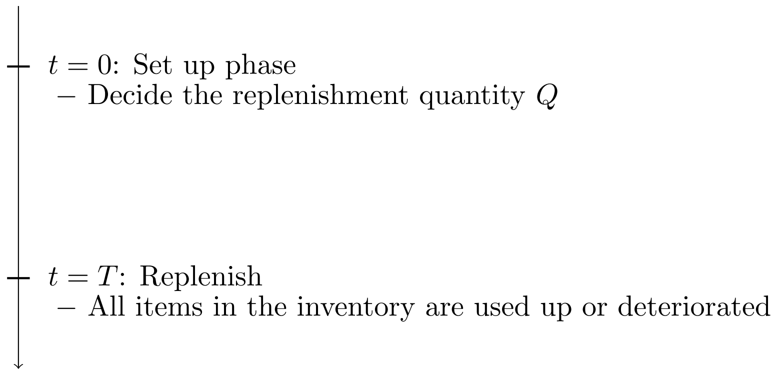


Figure 1: Timeline of the replenishment cycle (quantity-based) without inspection.

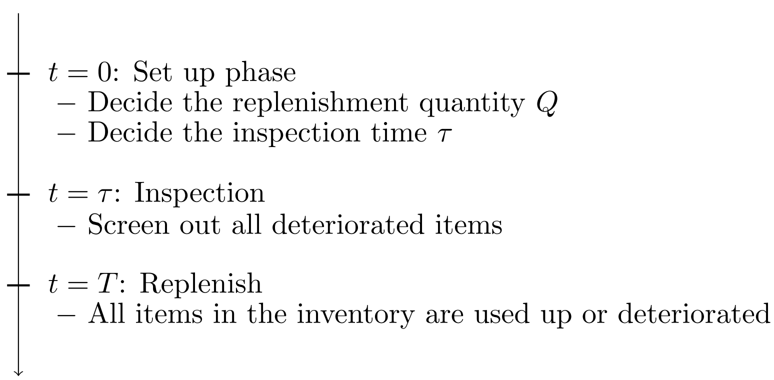


Figure 2: Timeline of the replenishment cycle (quantity-based) with one inspection.

3.1 Model 1: No Inspection

In this model, we assume that there is no inspection during a replenishment cycle. Let $I(t)$ and $J(t)$ be the inventory levels of serviceable and deteriorated items at time t, respectively, where $t \in [0, T]$. At the beginning of a replenishment cycle, Q units of items are transported from the supplier to the inventory. In this paper, for simplicity of discussion, the inventory is subjected to a constant demand rate λ. The chance that a customer receiving a serviceable or a deteriorated item is naturally corresponding to the proportion of serviceable and deteriorated items in the inventory. The inventory levels of serviceable and deteriorated items in a replenishment cycle $[0, T]$ are governed by the following differential equations:

$$\begin{cases} I'(t) &= -\dfrac{I(t)}{I(t)+J(t)}\lambda - \theta(t)I(t), \quad 0 \le t \le T, \\ J'(t) &= \theta(t)I(t) - \dfrac{J(t)}{I(t)+J(t)}\lambda, \quad 0 \le t \le T. \end{cases} \tag{2}$$

Solving the system of differential equations (2) with the boundary conditions $I(0) = Q$, $J(0) = 0$, we have

$$\begin{cases} I(t) &= \dfrac{(Q-\lambda t)(1+m-t)}{1+m}, \quad 0 \le t \le T, \\ J(t) &= \dfrac{(Q-\lambda t)t}{1+m}, \quad 0 \le t \le T. \end{cases} \tag{3}$$

The replenishment cycle ends when one of the two cases occurs: (i) time t reaches m or (ii) all items in the inventory are used up, i.e., $t = Q/\lambda$, whichever comes first. Hence, the length of the replenishment cycle, which depends on m, is

$$T = \min\left\{\frac{Q}{\lambda}, m\right\}.$$

The number of serviceable items sold to customers in a replenishment cycle is

$$\int_0^T \frac{I(t)}{I(t)+J(t)}\lambda\, dt = \int_0^T \frac{(1+m-t)\lambda}{1+m}\, dt = \lambda\Big[T - \frac{T^2}{2(1+m)}\Big].$$

The revenue of the inventory holder is generated from the sales of serviceable items. Hence, the revenue per cycle is

$$TR = p\lambda\left(T - \frac{T^2}{2(1+m)}\right), \tag{4}$$

where p is the sale price of a product item. The total cost of the inventory holder consists of three parts: the ordering cost per cycle K, the purchasing cost cQ and the inventory holding cost per cycle $h(QT - \frac{\lambda T^2}{2})$. Hence the total cost is

$$TC = cQ + K + h\left(QT - \frac{\lambda T^2}{2}\right). \tag{5}$$

Based on Eq. (4) and Eq. (5), the long-run average profit can be determined by the renewal reward theorem:

$$\begin{aligned}\Pi_1(Q) &= \frac{E[TR]-E[TC]}{E[T]} \\ &= p\lambda - \frac{p\lambda}{2}\frac{E\Big[\frac{T^2}{1+m}\Big]}{E[T]} - \frac{cQ}{E[T]} - \frac{K}{E[T]} - hQ + \frac{h\lambda}{2}\frac{E[T^2]}{E[T]}. \end{aligned} \tag{6}$$

To obtain $E[T]$ and other expected values in terms of T, three cases: $Q \le a\lambda$, $a\lambda < Q < b\lambda$, and $Q \ge b\lambda$, are discussed in the follows.

Case 1: $Q \le a\lambda$. We have the following result in Lemma 1. The proof is presented in Appendix.

Lemma 1 *If* $Q \in (0, a\lambda]$, *denote*

$$Q^* = \begin{cases} q, & \text{if } q < a\lambda, \\ a\lambda, & \text{if } q \ge a\lambda, \end{cases}$$

where

$$q = \sqrt{\frac{2K\lambda}{\frac{p}{b-a}\ln\left(\frac{1+b}{1+a}\right) + h}},$$

and it has $\Pi_1'(q) = 0$. *Then we have*

$$\Pi_1(Q^*) = \max_{Q\in(0,a\lambda]}\{\Pi_1(Q)\}.$$

Case 2: $a\lambda < Q < b\lambda$. In this case, $m \le Q/\lambda$ with probability $\frac{Q/\lambda - a}{b-a}$ and $m > Q/\lambda$ with complementary probability. Thus

$$E[T] = \int_a^{Q/\lambda} m f(m)\,dm + \int_{Q/\lambda}^b \frac{Q}{\lambda} f(m)\,dm = \frac{2b\lambda Q - a^2\lambda^2 - Q^2}{2\lambda^2(b-a)},$$

$$E[T^2] = \int_a^{Q/\lambda} m^2 f(m)\,dm + \int_{Q/\lambda}^b \frac{Q^2}{\lambda^2} f(m)\,dm = \frac{3b\lambda Q^2 - a^3\lambda^3 - 2Q^3}{3\lambda^3(b-a)},$$

and

$$\begin{aligned} E\Big[\frac{T^2}{1+m}\Big] &= \int_a^{Q/\lambda} \frac{m^2}{1+m} f(m)\,dm + \int_{Q/\lambda}^b \frac{Q^2/\lambda^2}{1+m} f(m)\,dm \\ &= \frac{1}{b-a}\Big[\frac{(Q/\lambda-1)^2}{2} - \frac{(a-1)^2}{2} + \ln\left(\frac{1+Q/\lambda}{1+a}\right) \\ &\quad + \frac{Q^2}{\lambda^2}\ln\left(\frac{1+b}{1+Q/\lambda}\right)\Big]. \end{aligned}$$

Hence, substituting $E[T]$, $E[T^2]$ and $E[\frac{T^2}{1+m}]$ into Eq. (6), one has

$$\begin{aligned}\Pi_1(Q) &= p\lambda - \frac{p\lambda^3\Big[\frac{(Q/\lambda-1)^2}{2} - \frac{(a-1)^2}{2} + \ln\Big(\frac{1+Q/\lambda}{1+a}\Big) + \frac{Q^2}{\lambda^2}\ln\Big(\frac{1+b}{1+Q/\lambda}\Big)\Big]}{2b\lambda Q - a^2\lambda^2 - Q^2} \\ &\quad - \frac{2\lambda^2(b-a)(K+cQ)}{2b\lambda Q - a^2\lambda^2 - Q^2} - \frac{h(a^3\lambda^3 - 3a^2\lambda^2 Q + 3b\lambda Q^2 - Q^3)}{3(2b\lambda Q - a^2\lambda^2 - Q^2)}.\end{aligned}$$

We remark that since T is a continuous function in Q, $\Pi_1(Q)$ is also continuous in Q. By the extreme value theorem, $\Pi_1(Q)$ has a local maximum in $[a\lambda, b\lambda]$.

Case 3: $Q \geq b\lambda$. The following result in Lemma 2 is obtained, and the proof is shown in Appendix.

Lemma 2 *The long-run average profit $\Pi_1(Q)$ decreases as Q increases when $Q \in [b\lambda, \infty)$. That is, denote $Q^* = b\lambda$, and we have*

$$\Pi_1(Q^*) = \max_{Q\in[b\lambda,\infty)} \{\Pi_1(Q)\}.$$

Lemma 2 indicates that the replenishment quantity should be less than $b\lambda$ in case that there will be only deteriorated items left at the end of the cycle, which also means that the replenishment cycle should be shorter than the products' maximum lifetime.

3.2 Model 2: One Inspection

In this model, an inspection process is preformed in the replenishment cycle to screen out deteriorated items in the inventory. Suppose that the inspection is preformed at time $t = \tau$. The inspection is perfect in the sense that it will screen out all deteriorated items in the inventory at time $t = \tau$. To ensure that the inspection is preformed before the end of the replenishment cycle, we further assume that $\tau < T$. Hence, we restrict our choices of τ such that

$$\tau \in \mathcal{T} = \Big(0, \min\Big\{a, \frac{Q}{\lambda}\Big\}\Big),$$

so that we have $\tau < T$ with probability 1.

Let Q_1 be the inventory level after the inspection. By using the analogical argument as in the development of Eq. (3), the inventory levels of serviceable items in the periods $[0, \tau]$ and $(\tau, T]$ are

$$\begin{cases} I_1(t) &= \dfrac{(Q-\lambda t)(1+m-t)}{1+m}, \quad 0 \leq t \leq \tau, \\ I_2(t) &= \dfrac{(Q_1-\lambda(t-\tau))(1+m-(t-\tau))}{1+m}, \quad \tau < t \leq T. \end{cases} \tag{7}$$

Observing that $I_1(\tau) = Q_1$, hence we have

$$Q_1 = \frac{(Q-\lambda\tau)(1+m-\tau)}{1+m}. \tag{8}$$

Since

$$I_1(t) + J_1(t) = Q - \lambda t \quad \text{and} \quad I_2(t) + J_2(t) = Q_1 - \lambda(t - \tau),$$

the number of serviceable items sold to customers is

$$\begin{aligned} & \int_0^\tau \frac{I_1(t)}{I_1(t) + J_1(t)} \lambda \, dt + \int_\tau^T \frac{I_2(t)}{I_2(t) + J_2(t)} \lambda \, dt \\ = & \frac{\lambda}{1+m} \Big((1+m)T + \tau T - \tau^2 - \frac{T^2}{2} \Big). \end{aligned}$$

Since the revenue of the inventory holder is generated from the sales of serviceable items, the revenue per cycle is

$$TR = p\lambda \left(T + \tau \frac{T}{1+m} - \frac{\tau^2}{1+m} - \frac{T^2}{2(1+m)} \right). \tag{9}$$

The total cost of the inventory holder consists of four parts. The ordering cost per cycle is K, the purchasing cost is cQ and the holding cost per cycle is

$$\begin{aligned} & \frac{h(Q + Q - \lambda\tau)\tau}{2} + \frac{h[Q_1 + Q_1 - \lambda(T - \tau)](T - \tau)}{2} \\ = & h \Big[QT - \frac{\lambda T^2}{2} - (Q\tau - \lambda\tau^2) \frac{T}{1+m} + \frac{Q\tau^2 - \lambda\tau^3}{1+m} \Big]. \end{aligned}$$

Inspection is carried out at time τ and the inventory level at time τ is $Q - \lambda\tau$. The inspection cost per cycle is then given by $c_d(Q - \lambda\tau)$. Hence, the total cost is

$$\begin{aligned} TC = & cQ + K + c_d(Q - \lambda\tau) \\ & + h \left(QT - \frac{\lambda T^2}{2} - (Q\tau - \lambda\tau^2) \frac{T}{1+m} + \frac{Q\tau^2 - \lambda\tau^3}{1+m} \right). \end{aligned} \tag{10}$$

Based on Eq. (9) and Eq. (10), the long-run average profit can be determined by the renewal reward theorem:

$$\begin{aligned} & \Pi_2(Q, \tau) \\ = & \frac{E[TR] - E[TC]}{E[T]} \\ = & p\lambda - hQ + [p\lambda\tau + h(Q\tau - \lambda\tau^2)] \frac{E\Big[\frac{T}{1+m}\Big]}{E[T]} - [p\lambda\tau^2 + h(Q\tau^2 - \lambda\tau^3)] \frac{E\Big[\frac{1}{1+m}\Big]}{E[T]} \\ & - \frac{p\lambda}{2} \frac{E\Big[\frac{T^2}{1+m}\Big]}{E[T]} - \frac{cQ + K + c_d(Q - \lambda\tau)}{E[T]} + \frac{h\lambda}{2} \frac{E[T^2]}{E[T]}. \end{aligned} \tag{11}$$

The expectations involving T in Eq. (11) are calculated as follows. The replenishment cycle ends when (i) time t reaches m or (ii) all items in the inventory are used up. Hence, the length of the replenishment cycle, which depends on m, is

$$T = \begin{cases} t(m), & \text{if } m > t(m); \\ m, & \text{if } m \le t(m). \end{cases}$$

Here $t(m)$ is the time that all items in the inventory are used up, which is obtained according to Eq. (8),

$$t(m) = \frac{Q_1}{\lambda} + \tau = \frac{(Q - \lambda\tau)(1 + m - \tau)}{\lambda(1 + m)} + \tau.$$

The following proposition is introduced to simplify the calculations of expected values in terms of T. The proof is given in Appendix.

Proposition 1 *Let*

$$\begin{cases} m_+ = \dfrac{Q - \lambda + \sqrt{(Q + \lambda(1 - 2\tau))^2 + 4\tau\lambda^2}}{2\lambda}, \\ m_- = \dfrac{Q - \lambda - \sqrt{(Q + \lambda(1 - 2\tau))^2 + 4\tau\lambda^2}}{2\lambda}, \end{cases}$$

which are quantities independent of m *and let*

$$\mathcal{I} = \Big(\frac{Q - \sqrt{Q^2 - 4\lambda Q}}{2\lambda}, \frac{Q + \sqrt{Q^2 - 4\lambda Q}}{2\lambda}\Big).$$

If $Q > 4\lambda$ *and* $\tau \in \mathcal{T} \cap \mathcal{I}$*, then we have*

$$T = \begin{cases} t(m), & \text{if } m > m_+ \text{ or } 0 < m < m_-; \\ m, & \text{if } m_- \leq m \leq m_+. \end{cases}$$

Otherwise, we have

$$T = \begin{cases} t(m), & \text{if } m > m_+; \\ m, & \text{if } 0 < m \leq m_+. \end{cases}$$

The expectations involving T in the objective function $\Pi_2(Q, \tau)$ can then be computed using Proposition 1. For example, for the case of $Q \leq 4\lambda$, if the maximum lifetime of the items is greater than m_+, the replenishment cycle ends when all items in the inventory are used up. Otherwise the replenishment cycle ends at the time of the items reaching their maximum lifetime. In this case, the expected replenishment cycle length $E[T]$ is given by

$$E[T] = \int_0^{m_+} m f(m)\, dm + \int_{m_+}^{\infty} t(m) f(m)\, dm.$$

The remaining expectations can be computed in a similar way. Since a closed-form solution for maximizing Eq. (11) is generally not available, the problem can be solved by employing a numerical method.

If the replenishment cycle ends at the time of the items reaching their maximum lifetime, that is $T = m$, the inventory holder has to replenish his stock in a short time in order to reduce the loss caused by the shortage. However, m is random and if the supplier is required to provide products without any prior notice, the purchasing cost and the ordering cost would be high. The total cost of the inventory holder would increase. Therefore, the inventory holder is suggested to conduct an inspection at time $\tau = Q/2\lambda$. Then the value of m_+ (m_-) could be minimized (maximised), and further the probability of $T = m$ would be reduced.

4 Time-based Replenishment Policy

Under the replenishment policy in last section, the inventory is replenished when all items in the inventory are sold or deteriorated. Therefore, the replenishment cycle length is uncertain. In this section, the replenishment cycle length T is one of the decision variables. The inventory holder determines the time to make an order for replenishment at the beginning of the sales. The timelines of the models with and without inspection are shown in Figures 3 and 4.

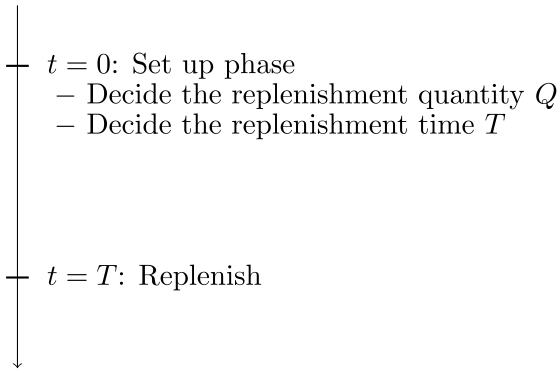


Figure 3: Timeline of the replenishment cycle (time-based) without inspection.

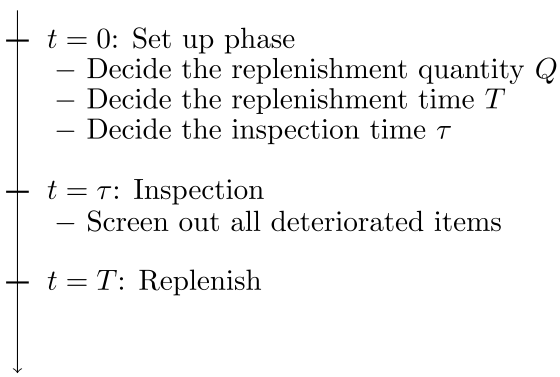


Figure 4: Timeline of the replenishment cycle (time-based) with one inspection.

4.1 Model 3: No Inspection

In this model, no inspection is performed during a replenishment cycle. Denote c_b as the shortage cost per item and s $(< p)$ as the salvage value of a serviceable item. The inventory level at time t is governed by a similar function as in Model 1, which is given by

$$\begin{cases} I(t) = \dfrac{(Q-\lambda t)(1+m-t)}{1+m}, & 0 \le t \le \min\left\{m, T, \dfrac{Q}{\lambda}\right\}, \\ J(t) = \dfrac{(Q-\lambda t)t}{1+m}, & 0 \le t \le \min\left\{m, T, \dfrac{Q}{\lambda}\right\}. \end{cases} \tag{12}$$

Since the replenishment cycle T is assumed to be decided by the inventory holder at the beginning of the inventory system, the long-run average profit is expressed as

$$\Pi_3(Q,T) = \frac{E[TR] - E[TC]}{T}. \tag{13}$$

First, some results relating to two cases: $T \geq Q/\lambda$ and $T \geq b$, are given in Lemmas 3 and 4. The proofs are shown in Appendix.

Lemma 3 *The long-run average profit* $\Pi_3(Q,T)$ *decreases as* T *increases when* $T \in [Q/\lambda, \infty)$. *That is, denote* $T^* = Q/\lambda$, *then we have*

$$\Pi_3(Q,T^*) = \max_{T\in[Q/\lambda,\infty)} \{\Pi_3(Q,T)\}.$$

Lemma 4 *The long-run average profit* $\Pi_3(Q,T)$ *decreases as* T *increases when* $T \in [b, \infty)$. *That is, denote* $T^* = b$, *then we have*

$$\Pi_3(Q,T^*) = \max_{T\in[b,\infty)} \{\Pi_3(Q,T)\}.$$

Lemma 3 and Lemma 4 imply that it is not beneficial for an inventory holder to set the replenishment cycle to be longer than the time of all items are used up or the maximum value of m. Therefore, we next investigate the long-run average profit with $T \leq Q/\lambda$ and $T \leq b$.

Denote that

$$T_0 = \min\Big\{m, \frac{Q}{\lambda}, T\Big\},$$

since $T \leq Q/\lambda$, then

$$T_0 = \begin{cases} T, & \text{if } m > T; \\ m, & \text{if } m \leq T, \end{cases}$$

which means that at time T_0, the replenishment cycle ends or all products become deteriorated. The inventory holder sells the products including serviceable and deteriorated items together to customers at a rate of λ before time T_0. After that, no serviceable item would be sold to customers. Therefore, the number of serviceable items sold to customers in a replenishment cycle is given by

$$\int_0^{T_0} \frac{I(t)}{I(t)+J(t)}\lambda\, dt = \int_0^{T_0} \frac{(1+m-t)\lambda}{1+m}\, dt = \lambda\left(T_0 - \frac{T_0^2}{2(1+m)}\right).$$

The sale price of one product is p, then the total revenue is

$$TR = p\lambda\Big(T_0 - \frac{T_0^2}{2(1+m)}\Big).$$

The total cost is composed by the ordering cost, the purchasing cost, the holding cost and the cost generated by shortage or salvage value. It is given by

$$TC = cQ + K + \begin{cases} h\dfrac{(2Q-\lambda T)T}{2} - sI(T), & \text{if } m > T; \\[2ex] h\dfrac{(2Q-\lambda m)m}{2} + c_b(T-m)\lambda, & \text{if } m \leq T. \end{cases}$$

Here $I(T)$ is the inventory level at the end of the replenishment cycle, which depends on m.

Based on the above expressions of TR and TC, the values of $E[TR]$ and $E[TC]$ can be determined as follows:

Case 1: $T \le a$. In this case, the replenishment cycle ends before the product items in the inventory reach their maximum lifetime, which means $T_0 = T$. The expected cost is

$$E[TC] = cQ + K + h\frac{(2Q - \lambda T)T}{2} - sE[I(T)], \tag{14}$$

where

$$E[I(T)] \;=\; (Q - \lambda T)\Big(1 - T \cdot E\Big[\frac{1}{1+m}\Big]\Big).$$

The expected revenue is

$$E[TR] \;=\; p\lambda \cdot E\Big[T_0 - \frac{T_0^2}{2(1+m)}\Big] = p\lambda\Big[T - \frac{T^2}{2(b-a)}\ln\Big(\frac{1+b}{1+a}\Big)\Big]. \tag{15}$$

Substituting Eq. (14) and Eq. (15) into Eq. (13), one has

$$\begin{aligned}\Pi_3(Q,T) &= \frac{E[TR] - E[TC]}{T} \\ &= p\lambda - \frac{p\lambda}{2(b-a)}\ln\Big(\frac{1+b}{1+a}\Big)T - (cQ+K)\frac{1}{T} - hQ + \frac{h\lambda}{2}T \\ &\quad + sQ\frac{1}{T} - \frac{sQ}{b-a}\ln\Big(\frac{1+b}{1+a}\Big) - s\lambda + \frac{s\lambda}{b-a}\ln\Big(\frac{1+b}{1+a}\Big)T.\end{aligned}$$

Then the following result in Lemma 5 is obtained, and the proof is given in Appendix.

Lemma 5 *Suppose $T \le a$ and set*

$$t_a = \sqrt{\frac{2(cQ - sQ + K)}{\frac{(p-2s)\lambda}{b-a}\ln\big(\frac{1+b}{1+a}\big) - h\lambda}}.$$

Denote

$$T^* = \begin{cases} t_a, & \text{if } t_a < \min\{a, Q/\lambda\}; \\ \min\{a, Q/\lambda\}, & \text{if } t_a \ge \min\{a, Q/\lambda\}, \end{cases}$$

then

$$\Pi_3(Q, T^*) = \max_{T \in (0,a]}\{\Pi_3(Q,T)\}.$$

Case 2: $a < T \le b$. Then $T_0 = m$ when $m \le T$ and $T_0 = T$ when $m > T$. The expected cost is

$$\begin{aligned}E[TC] &= cQ + K + \int_a^T \Big[h\frac{(2Q - \lambda m)m}{2} + c_b(T-m)\lambda\Big] f(m)\,dm \\ &\quad + \int_T^b \Big[h\frac{(2Q - \lambda T)T}{2} - sI(T)\Big] f(m)\,dm \\ &= cQ + K + \frac{1}{b-a}\Big[\frac{(hQ - c_b\lambda)(T^2 - a^2)}{2} - \frac{h\lambda(T^3 - a^3)}{6} + c_bT\lambda(T-a)\Big] \\ &\quad + \frac{1}{b-a}\Big[\frac{hT(2Q - \lambda T)(b-T)}{2} - s(Q - \lambda T)(b - T - T\ln\Big(\frac{1+b}{1+T}\Big))\Big].\end{aligned} \tag{16}$$

The expected revenue is

$$\begin{aligned} E[TR] &= p\lambda\Big\{\int_a^T \Big[m - \frac{m^2}{2(1+m)}\Big] f(m)\, dm + \int_T^b \Big[T - \frac{T^2}{2(1+m)}\Big] f(m)\, dm\Big\} \\ &= \frac{p\lambda}{2(b-a)}\Big[2Tb - T^2 - a^2 + \frac{(a-1)^2}{2} - \frac{(T-1)^2}{2} + \ln\Big(\frac{1+a}{1+T}\Big) \\ &\quad + T^2 \ln\Big(\frac{1+T}{1+b}\Big)\Big]. \end{aligned} \tag{17}$$

The long-run average profit is

$$\Pi_3(Q,T) = \frac{E[TR] - E[TC]}{T}.$$

Figure 3 indicates that the replenishment time T in Model 3 depends on the replenishment quantity Q since T is decided after Q is known. Hence, in order to maximize $\Pi_3(Q,T)$, we adopt the backward induction method, which means we need to find the optimal T first and then the optimal Q. We remark that even though the log parts in Eqs. (15) and (16) do not contain Q, and Q thus can be treated as a constant, we are not able to use the second derivative of $\Pi_3(Q,T)$ with respect to T to determine the concavity of $\Pi_3(Q,T)$. That is because $\Pi_3(Q,T)$ contains complicated logarithmic parts of T. Therefore, a closed-form solution is not available and we have to solve the problem by using numerical methods.

4.2 Model 4: One Inspection

In this model, an inspection process is performed to screen out deteriorated items in the inventory during the replenishment cycle. Similar to Model 2, suppose that the inspection is performed at time $t = \tau$ with $\tau \le \min\{a, T, \frac{Q}{\lambda}\}$ and all items in inventory are used up at time T_1,

$$T_1 = \tau + \frac{Q_1}{\lambda} = \frac{Q}{\lambda} - \Big(\frac{Q}{\lambda} - \tau\Big)\frac{\tau}{1+m}, \tag{18}$$

where Q_1 is illustrated in Eq. (8).

Here the expression of T_1 is same to $t(m)$ in Model 2. For the discussion about whether T_1 exceeds the product's maximum lifetime m, one can refer to Proposition 1. We then investigate whether T_1 exceeds the replenishment cycle T. Let

$$h(m) = T_1 - T = \frac{1}{1+m}\Big(\frac{Q}{\lambda} - T\Big)\Big(m + 1 - \tau\frac{Q - \tau\lambda}{Q - T\lambda}\Big), \tag{19}$$

then Lemma 6 is obtained without proof.

Lemma 6 *Denote T_1 as in Eq. (18) and $h(m)$ as in Eq. (19), then*

- *If $\dfrac{Q}{\lambda} = T$, then $T_1 < T$;*
- *If $\dfrac{Q}{\lambda} < T$, then $h(m) < 0$, which implies $T_1 < T$;*

- *If* $\dfrac{Q}{\lambda} > T$*, denote* $m_4 = \tau\dfrac{Q-\tau\lambda}{Q-T\lambda} - 1$*, then* $h(m) > 0$ *when* $m > m_4$*, which implies* $T_1 > T$*;* $h(m) < 0$ *when* $m < m_4$*, which implies* $T_1 < T$*.*

Together with the conditions defined in Proposition 1, we define the following 4 intervals for different values of Q, τ and T in Table 2.

Table 2: Intervals for the values of Q, τ and T.

	$\frac{Q}{\lambda} \leqslant T$	$\frac{Q}{\lambda} > T$
Condition 1	$I_1 = (m_-, m_+) \cap (0, T) \cap (a, b)$,	$I_1 = (m_-, m_+) \cap (0, T) \cap (0, m_4) \cap (a, b)$,
$Q > 4\lambda$	$I_2 = \emptyset$,	$I_2 = (m_-, m_+) \cap (0, T) \cap (m_4, \infty) \cap (a, b)$,
and	$I_3 = ((0, m_-) \cup (m_+, \infty)) \cap (a, b)$,	$I_3 = ((0, m_-) \cup (m_+, \infty)) \cap (0, m_4) \cap (a, b)$,
$\tau \in \mathcal{T} \cap \mathcal{I}$	$I_4 = \emptyset$;	$I_4 = (T, \infty) \cap (m_4, \infty) \cap (a, b)$;
Condition 2	$I_1 = (0, m_+) \cap (0, T) \cap (a, b)$,	$I_1 = (0, m_+) \cap (0, T) \cap (0, m_4) \cap (a, b)$,
Otherwise	$I_2 = \emptyset$,	$I_2 = (0, m_+) \cap (0, T) \cap (m_4, \infty) \cap (a, b)$,
	$I_3 = (m_+, \infty) \cap (a, b)$,	$I_3 = (m_+, \infty) \cap (0, m_4) \cap (a, b)$,
	$I_4 = \emptyset$;	$I_4 = (T, \infty) \cap (m_4, \infty) \cap (a, b)$.

For different intervals, we get

(i) $m \in I_1$ is equivalent to $m < T_1 < T$. The net revenue, which equals to the total revenue minus the total cost, is given by

$$\begin{aligned} TR_1 - TC_1 &= \frac{p\lambda}{1+m}\Big((1+m)m + \tau m - \tau^2 - \frac{m^2}{2}\Big) - K - cQ - c_b\lambda(T-m) \\ &\quad -c_d(Q-\lambda\tau) - h\Big(\frac{(2Q-\lambda\tau)\tau}{2} + \frac{(2Q_1 - \lambda(m-\tau))(m-\tau)}{2}\Big). \end{aligned}$$

(ii) $m \in I_2$ is equivalent to $m < T < T_1$. The net revenue is given by

$$\begin{aligned} & TR_2 - TC_2 \\ = \; & \frac{p\lambda}{1+m}\Big((1+m)m + \tau m - \tau^2 - \frac{m^2}{2}\Big) - K - cQ - c_b\lambda(T-m) \\ & -c_d(Q-\lambda\tau) - h\Big(\frac{(2Q-\lambda\tau)\tau}{2} + \frac{(2Q_1 - \lambda(m-\tau))(m-\tau)}{2}\Big). \end{aligned}$$

(iii) $m \in I_3$ is equivalent to $T_1 < m$ and $T_1 < T$. The net revenue is given by

$$\begin{aligned} TR_3 - TC_3 &= \frac{p\lambda}{1+m}\Big((1+m)T_1 + \tau T_1 - \tau^2 - \frac{T_1^2}{2}\Big) - K - cQ - c_b(T-T_1)\lambda \\ &\quad -c_d(Q-\lambda\tau) - h\Big(\frac{(2Q-\lambda\tau)\tau}{2} + \frac{Q_1(T_1-\tau)}{2}\Big). \end{aligned}$$

(iv) $m \in I_4$ is equivalent to $T < m$ and $T < T_1$. The net revenue is given by

$$\begin{aligned} TR_4 - TC_4 &= \frac{p\lambda}{1+m}\Big((1+m)T + \tau T - \tau^2 - \frac{T^2}{2}\Big) - K - cQ - c_d(Q - \lambda\tau) \\ &\quad -h\Big(\frac{(2Q-\lambda\tau)\tau}{2} + \frac{[2Q_1 - \lambda(T-\tau)](T-\tau)}{2}\Big) \\ &\quad +s\Big[\frac{(Q_1 - \lambda(T-\tau))(1+m-(T-\tau))}{1+m}\Big]. \end{aligned}$$

Based on the four cases, the long-run average profit is

$$\Pi_4(Q,\tau,T) = \frac{E[TR]-E[TC]}{T} = \frac{\sum_{i=1}^{4}\int_{I_i}(TR_i - TC_i)f(m)\,dm}{T}. \quad (20)$$

Then some conclusions in Lemmas 7 and 8 are obtained and their proofs can be found in Appendix.

Lemma 7 *The long-run average profit $\Pi_4(Q,\tau,T)$ decreases as T increases when $T \in [Q/\lambda, \infty)$. That is, denote $T^* = Q/\lambda$, then we have*

$$\Pi_4(Q,\tau,T^*) = \max_{T\in[Q/\lambda,\infty)}\{\Pi_4(Q,\tau,T)\}.$$

Lemma 8 *The long-run average profit $\Pi_4(Q,\tau,T)$ decreases as T increases when $T \in [b, \infty)$. That is, denote $T^* = b$, then we have*

$$\Pi_4(Q,\tau,T^*) = \max_{T\in[b,\infty)}\{\Pi_4(Q,\tau,T)\}.$$

A closed-form solution for maximizing Eq. (20) is generally not available, but the problem can be solved by using numerical methods.

5 The Cases of Continuous Monitoring

Different to the models with one or no inspection in Sections 3 and 4, in this section, we suppose that the items are continuously monitored and deteriorated items are screened out instantaneously. The total cost of monitoring is denoted as M. Two models under different replenishment policies are presented, and the timelines are shown in Figures 5 and 6.

5.1 Model 5: Quantity-based Replenishment Policy Model

In this subsection, suppose that an inventory holder adopts quantity-based replenishment policy. The items in the inventory are continuously monitored such that deteriorated items are screened out instantaneously. The total monitoring cost is M. The inventory level of serviceable items in a replenishment cycle is governed by the following differential equation:

$$I'(t) = -\lambda - \theta(t)I(t), \quad 0 \le t \le T, \quad (21)$$

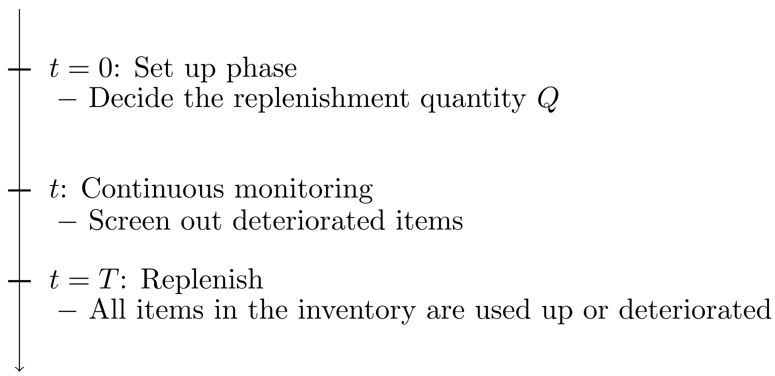


Figure 5: Timeline of the replenishment cycle (quantity-based) with continuous monitoring.

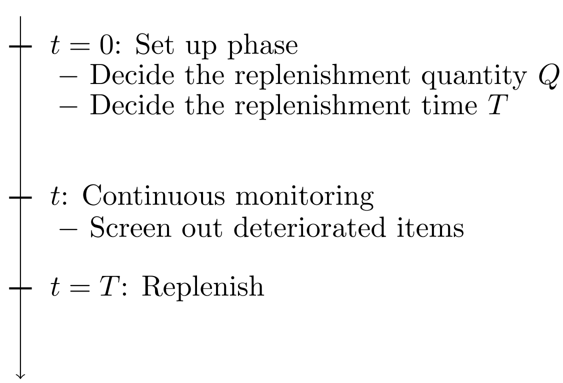


Figure 6: Timeline of the replenishment cycle (time-based) with continuous monitoring.

with boundary condition $I(0) = Q$. The replenishment cycle ends when time t reaches m or all serviceable items in the inventory are used up. Solving the differential equation (21), one obtains

$$I(t) = \lambda(1+m-t)\ln(1+m-t) + (1+m-t)\Big[\frac{Q}{1+m} - \lambda\ln(1+m)\Big], \quad 0 \le t \le T. \quad (22)$$

The holding cost per cycle is

$$\begin{aligned} h\int_0^T I(t)\,dt &= \frac{h}{2}\Big[\lambda(1+m)^2\ln(1+m) - T\Big(1+m-\frac{T}{2}\Big)\Big(\lambda - \frac{2Q}{1+m} + 2\lambda\ln(1+m)\Big) \\ &\quad -\lambda(1+m-T)^2\ln(1+m-T)\Big]. \end{aligned}$$

Again, the long-run average profit can be determined by the renewal reward theorem:

$$\begin{aligned} \Pi_5(Q) &= \frac{E[TR]-E[TC]}{E[T]} \\ &= p\lambda - \frac{cQ}{E[T]} - \frac{K}{E[T]} - \frac{M}{E[T]} - \frac{hE\Big[\int_0^T I(t)\,dt\Big]}{E[T]}. \end{aligned} \quad (23)$$

We next show how to obtain the expected replenishment length $E[T]$. The same argument can also be applied to find the expected value of any function of T. Denote t_0 as the time of all items are used up, which means $I(t_0) = 0$. Thus

$$t_0 = 1+m-(1+m)\exp\Big[-\frac{Q}{\lambda(1+m)}\Big]. \quad (24)$$

It is easy to see that $t_0 > 0$ when $Q > 0$. The replenishment cycle ends when (i) time t reaches m or (ii) all items in the inventory are used up. Hence, the length of the replenishment cycle, which depends on m, is

$$T = \begin{cases} t_0, & \text{if } m > t_0, \\ m, & \text{if } m \le t_0. \end{cases}$$

The positive root of the following function $g_5(m)$ is denoted by m_0,

$$g_5(m) = \ln(1+m) - \frac{Q}{\lambda(1+m)}.$$

Then we have the following lemma

Lemma 9 *The condition $t_0 < m$ is equivalent to $m > m_0$. In other words,*

$$t_0 < m \Leftrightarrow m > m_0 \quad \textit{and} \quad t_0 \ge m \Leftrightarrow m \le m_0.$$

To obtain $E[T]$, we consider the following three cases.

(i) $m_0 < a$. In this case, $m > m_0$ with probability 1. Hence, by Lemma 1, $t_0 < m$ with probability 1 and

$$E[T] = \int_a^b t_0 f(m)\,dm.$$

(ii) $a \leq m_0 < b$. In this case, $m \leq m_0$ with probability $\int_a^{m_0} f(m)\,dx$ and $m > m_0$ with complementary probability. Hence, by Lemma 1, $T = m$ with probability $\int_a^{m_0} f(m)\,dx$ and $T = t_0$ with complementary probability. Then

$$E[T] = \int_a^{m_0} mf(m)\,dm + \int_{m_0}^{b} t_0 f(m)\,dm.$$

(iii) $m_0 \geq b$. In this case, $m \leq m_0$ with probability 1. Hence, by Lemma 1, $t_0 \geq m$ with probability 1 and

$$E[T] = \int_a^{b} mf(m)\,dm.$$

It is clear that $\Pi_5(Q) \to -\infty$ when $Q \to 0$. When $Q \to +\infty$, by Eq. (24), $t_0 = 1 + m$, thus $T = m$ and $E[T] = E[m] = (a + b)/2$. The expected holding cost per cycle is

$$\frac{h}{2}\Big[\lambda(1+m)^2\ln(1+m) - m\Big(1+\frac{m}{2}\Big)\Big(\lambda - \frac{2Q}{1+m} + 2\lambda\ln(1+m)\Big)\Big],$$

which approaches $+\infty$ when $Q \to +\infty$. Hence, $\Pi_5(Q) \to -\infty$ when $Q \to \infty$. In conclusion, there exists a positive Q such that $\Pi_5(Q)$ is maximized. Again a closed form-solution for maximizing Eq. (23) is generally not available, the problem has to be solved by using numerical methods.

5.2 Model 6: Time-based Replenishment Policy Model

In this subsection, suppose that an inventory holder adopts time-based replenishment policy. The length of a replenishment cycle is denoted as T and it is fixed. Since the items are deteriorating at a random rate and deteriorated items are screened out instantaneously, the situations of shortages or surpluses may appear at the end of a replenishment cycle.

Denote $T_6 = \min\Big\{m, T, t_0\Big\}$. The inventory level at time t is governed by the same function as in Model 5:

$$\begin{aligned} I(t) &= \lambda(1+m-t)\ln(1+m-t) \\ &\quad +(1+m-t)\Big[\frac{Q}{1+m} - \lambda\ln(1+m)\Big], \quad 0 \leq t \leq T_6. \end{aligned}$$

The solution of $I(t) = 0$ is denoted as t_0, which is given in Eq. (24),

$$t_0 = 1 + m - (1+m)\exp\Big[-\frac{Q}{\lambda(1+m)}\Big].$$

If a replenishment cycle ends before both of times t_0 and m, then there is a surplus at the end of the cycle. On the other hand, if a replenishment cycle ends after time t_0, then there is a shortage at the end of the cycle. The total cost is then given by

$$TC = cQ + K + \begin{cases} h\displaystyle\int_0^{t_0} I(t)\,dt + c_b\lambda(T - t_0), & \text{if } T_6 = t_0; \\ h\displaystyle\int_0^{m} I(t)\,dt + c_b\lambda(T - m), & \text{if } T_6 = m; \\ h\displaystyle\int_0^{T} I(t)\,dt - sI(T), & \text{if } T_6 = T. \end{cases} \tag{25}$$

The long-run average profit can be determined by the renewal reward theorem:

$$\Pi_6(Q,T) = \frac{E[TR]-E[TC]}{T} = \frac{p\lambda E[T_6]-E[TC]}{T}. \tag{26}$$

Then the following results are presented in Lemmas 10 and 11. The proofs are given in Appendix.

Lemma 10 *The long-run average profit $\Pi_6(Q,T)$ decreases as T increases when $T \in [Q/\lambda, \infty)$. That is, denote $T^* = Q/\lambda$, then we have*

$$\Pi_6(Q,T^*) = \max_{T\in[Q/\lambda,\infty)}\{\Pi_6(Q,T)\}.$$

Lemma 11 *The long-run average profit $\Pi_6(Q,T)$ decreases as T increases when $T \in [b, \infty)$. That is, denote $T^* = b$, then we have*

$$\Pi_6(Q,T^*) = \max_{T\in[b,\infty)}\{\Pi_6(Q,T)\}.$$

Lemma 10 and Lemma 11 imply that it is not beneficial for the inventory holder to set the replenishment cycle T to be greater than Q/λ and b. Thus we consider the situations that $T \le Q/\lambda$ and $T \le b$.

To calculate $E[T_6]$ and $E[TC]$ in Eq. (26), we discuss the value of T_6. Denote

$$g_6(m) = T - t_0 = T - (1+m)\Big[1-\exp\big(-\frac{Q}{\lambda(1+m)}\big)\Big].$$

It is not difficult to show that $g_6(m)$ is strictly decreasing in m. Therefore, we have

(i) $g_6(a) > 0$ and $g_6(b) \ge 0$. In this case, $T \ge t_0$ with probability 1. The values of T_6 corresponding to different cases are shown in Table 3. For example, if $m_0 < a$, then $m > m_0$ with probability 1. Hence, by Lemma 1, $t_0 < m$ with probability 1. Thus $T_6 = t_0$. Then $E[T_6]$ and $E[TC]$ in Eq. (26) can be determined. For example, if $a \le m_0 < b$, refer to Table 3 and Eq. (25), we have

$$\begin{aligned} E[T_6] &= \int_a^{m_0} mf(m)\,dm + \int_{m_0}^{b} t_0 f(m)\,dm \\ &= \frac{1}{b-a}\Big(\frac{m_0^2-a^2}{2} + t_0(b-m_0)\Big), \end{aligned}$$

and

$$\begin{aligned} E[TC] &= cQ + K + \int_a^{m_0}\Big(h\int_0^m I(t)\,dt + c_b\lambda(T-m)\Big)f(m)\,dm \\ &\quad + \int_{m_0}^{b}\Big(h\int_0^{t_0} I(t)\,dt + c_b\lambda(T-t_0)\Big)f(m)\,dm. \end{aligned}$$

(ii) $g_6(a) > 0$ and $g_6(b) < 0$. In this case, there exists $m_6 \in (a,b)$ such that $g_6(m_6) = 0$. The values of T_6 are summarized in Table 4. Similar to case (i), $E[T_6]$ and $E[TC]$ in Eq. (26) can be determined according to Table 4.

Table 3: T_6 in case (i).

m_0	$(0,a)$	$[a,b)$		$[b,\infty)$
m	(a,b)	(a,m_0)	$[m_0,b)$	(a,b)
T_6	t_0	m	t_0	m

Table 4: T_6 in case (ii).

m_0	$(0,a)$		$[a,m_6)$			(m_6,b)	
T	$(0,b]$		$[0,b]$			$(0,m_6)$	
m	(a,m_6)	$[m_6,b)$	(a,m_0)	$[m_0,m_6)$	$[m_6,b)$	(a,m_6)	$[m_6,b)$
T_6	t_0	T	m	t_0	T	m	T

m_0	(m_6,b)				$[b,\infty)$			
T	$[m_6,m_0)$		$[m_0,b]$		$(0,m_6)$		$[m_6,b]$	
m	(a,T)	$[T,b]$	(a,m_0)	$[m_0,b)$	(a,m_6)	$[m_6,b)$	(a,T)	$[T,b)$
T_6	m	T	m	T	m	T	m	T

(iii) $g_6(a) \leq 0$ and $g_6(b) < 0$. In this case, $T < t_0$ with probability 1. The values of T are summarized in Table 5. Similar to the case (i), $E[T_6]$ and $E[TC]$ in Eq. (26) can be determined according to Table 5.

Table 5: T_6 in case (iii).

m_0	$(0,a)$	$[a,b)$					$[b,\infty)$	
T	$(0,b]$	$(0,a)$	$[a,m_0)$		$[m_0,b]$		$(0,a)$	$[a,b]$
m	(a,b)	(a,b)	(a,T)	$[T,b)$	(a,m_0)	$[m_0,b)$	(a,T)	(a,b)
T_6	T	T	m	T	m	T	m	T

We remark that $[b,b)$ is the empty set. A closed-form solution for maximizing Eq. (26) is generally not available, we have to solve the problem by using numerical methods.

6 Numerical Examples

In this section, numerical examples are given to illustrate the models in this work. Parameters adopted in the examples are summarized in Table 6. Here Q_i^*, τ_i^* and T_i^* represent the optimal replenishment quantity, inspection time and length of a replenishment cycle in

Model i, respectively, $i = 1, 2, \ldots, 6$.

Table 6: Data of parameters

a	b	λ	K	p	c
10 days	60 days	50 units/day	4000 \$	20 \$/unit	4 \$/unit
h	s	c_d	c_b	M	
0.001 \$/unit/day	2 \$/unit	0.01 \$/unit	1 \$/unit	1000 \$	

6.1 A Comparison of the Six Models

The optimal long-run average profit in each model is shown in Table 7. A larger b means a greater range of m. We compare the results of the six models and get some conclusions as follows.

First, $\Pi_1(Q_1^*) > \Pi_3(Q_3^*, T_3^*)$, $\Pi_2(Q_2^*, \tau_2^*) > \Pi_4(Q_4^*, \tau_4^*, T_4^*)$ and $\Pi_5(Q_5^*) > \Pi_6(Q_6^*, T_6^*)$ imply that fixing the length of the replenishment cycle would decrease the long-run average profit. That is because if the inventory holder sets the length of T at the beginning of a replenishment cycle, the cost caused by the lost sale penalty or the lost due to the processing of the surplus products at time T may lower the total profit of the inventory holder. Hence, it is not beneficial for the inventory holder to fix the length of T.

Second, it shows that $\Pi_1(Q_1^*) < \Pi_2(Q_2^*, \tau_2^*)$ and $\Pi_3(Q_3^*, T_3^*) < \Pi_4(Q_4^*, \tau_4^*, T_4^*)$ at each value of b. This indicates that performing an inspection process in the inventory system would lead to an increase in the long-run average profit.

Specially, it shows that the optimal long-run average profit in Model 2 is greater than those in Models 1, 3, 4 and 6 at each value of b. Then, it seems that Model 2 has an advantage over Models 1, 3, 4 and 6.

Furthermore, if continuous monitoring is considered in the inventory system, it is found that $\Pi_5(Q_5^*)$ is greater than $\Pi_2(Q_2^*, \tau_2^*)$ when b is large. This suggests that Model 5 would have an advantage over Model 2 as the range of m increases. However, Table 10 indicates that the optimal long-run average profit in Model 5 decreases as the cost of conducting continuous monitoring M increases. This implies that if M is too high such that the optimal long-run average profit in Model 5 is less than that in Model 2, then the inventory holder is suggested to employ Model 2. Hence, as the range of m increases, there must exist a threshold value of M denoted as M_0 such that if $M \leqslant (>) M_0$ then the inventory holder would get more (less) profit under Model 5 than that under Model 2. Therefore, based on the market information, the inventory holder would find the value of M_0 and then decide to whether to employ Model 2 or Model 5. This is consistent with the actual situation in the market. For example, since conducting continuous monitoring is expensive, the management would inspect the inventory (fruit and vegetables) at their scheduled time intervals and screen out any deteriorating product. This would reduce the management cost, and the product quality (such as apples and potatoes) could be guaranteed.

Table 7: Optimal value of the long-run average profit in each model.

	$\Pi_1(Q_1^*)$	$\Pi_2(Q_2^*, \tau_2^*)$	$\Pi_3(Q_3^*, T_3^*)$	$\Pi_4(Q_4^*, \tau_4^*, T_4^*)$	$\Pi_5(Q_5^*)$	$\Pi_6(Q_6^*, T_6^*)$
b=15	25.05	167.01	25.05	154.04	116.67	100.81
b=20	80.54	214.75	79.95	199.95	204.46	175.95
b=30	158.02	283.26	154.89	259.81	298.78	251.57
b=40	211.55	329.86	206.57	301.63	358.28	300.06
b=50	252.26	364.05	245.50	333.16	400.87	335.55
b=60	284.41	391.47	276.27	358.30	433.21	362.63

6.2 Sensitivity Analysis

The sensitivity analysis is performed by changing each of the parameters by -30%, -15%, 15%, and 30%. We adopt the method that one parameter is changed at a time and the remaining parameters are kept constant.

The optimal long-run average profits for varying parameters are shown in Tables 8, 9 and 10. It is found that $\Pi_1(Q_1^*)$, $\Pi_2(Q_2^*, \tau_2^*)$, $\Pi_3(Q_3^*, T_3^*)$, $\Pi_4(Q_4^*, \tau_4^*, T_4^*)$, $\Pi_5(Q_5^*)$ and $\Pi_6(Q_6^*, T_6^*)$ increase as λ and p increase, but decrease as the other parameters (except s) increase. That is because the market demand rate λ and the sale price p affect the incomes during the replenishment cycle while other parameters except s affect the inventory costs. Here $\Pi_4(Q_4^*, \tau_4^*, T_4^*)$ slightly increases as s increases since the loss due to the processing of the surplus products would be lower when the salvage value of a product is higher. In particular, $\Pi_3(Q_3^*, T_3^*)$ remains unchanged when s varies, which indicates that there is no serviceable items at time T_3^*. Thus it implies that $T_3^* \geq Q_3^*/\lambda$. Considering Lemma 3, T_3^* should be Q_3^*/λ in these examples.

Furthermore, Tables 8, 9 and 10 show that the optimal long-run average profit is sensitive to the changes in parameters λ, K, p, c and M, but moderately sensitive to the changes in other parameters. Thus, reducing the costs which include the ordering cost, the purchasing cost and the continuous monitoring cost will result in a significant saving.

Next, we discuss the impacts of the parameters λ, K, p, c and M on the optimal replenishment quantity, the optimal inspection time and the optimal length of the replenishment cycle.

Figures 7, 8 and 9 show that the optimal replenishment quantity in each model presents an increasing trend as the parameters λ and K increase, and it presents a decreasing trend as p and c increase. Thus the inventory holder has to increase the quantity of products when λ and K increase, and reduce the quantity of products when p and c go up. Recall that the optimal long-run average profit increases as p increases. And the optimal length of the replenishment cycle decreases as p increases by referring to Table 12. These suggest that when the sale price increases, the inventory holder can enhance the long-run average profit by optimizing the replenishment quantity of products and the length of the replenishment cycle. That is the inventory holder can reduce Q, which shortens the length of the replenishment cycle, or reduce T directly. Figure 9 indicates that if the inventory holder adopts Model 5 or Model 6, he needs to slightly increase the replenishment quantity of products as the total cost M increases.

Table 8: Sensitivity analysis of average profit in Model 1 and Model 2

$\Pi_1(Q_1^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	133.19	206.94	364.72	447.63
K	364.34	322.34	249.28	216.75
p	69.47	175.53	395.47	508.65
c	346.19	315.30	253.52	222.63
h	284.54	284.48	284.35	284.28
$\Pi_2(Q_2^*, \tau_2^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	214.72	301.56	482.93	575.98
K	460.14	424.28	360.15	330.82
p	138.24	264.25	519.62	648.50
c	465.49	428.29	354.65	317.95
h	391.64	391.55	391.38	391.30
c_d	391.59	391.53	391.41	391.35

6.3 Managerial Implications

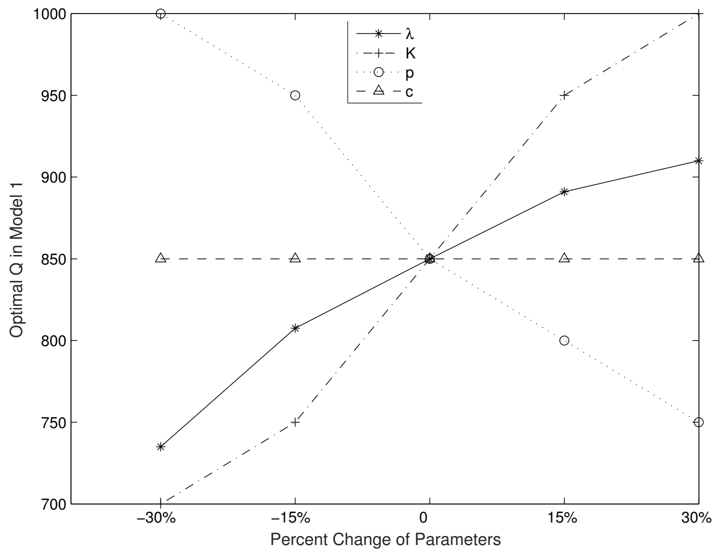
Some helpful managerial insights are provided to help inventory holder in making smart decisions. Table 11 and Table 12 provide managerial insights to help inventory holder in making decisions related to the optimal inspection time and the optimal length of replenishment cycle. Table 11 suggests the inventory holder who adopts Model 2 or Model 4 to conduct the inspection earlier when the parameters λ, p and c become higher, while delay

Table 9: Sensitivity analysis of average profit in Model 3 and Model 4

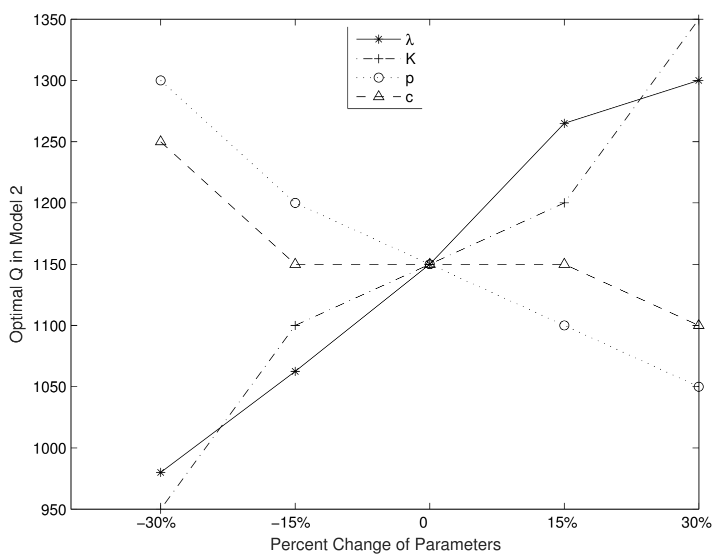
$\Pi_3(Q_3^*, T_3^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	124.51	198.26	357.83	441.96
K	361.27	316.75	239.48	205.29
p	63.73	167.98	387.97	502.16
c	336.28	306.28	246.28	216.28
h	276.39	276.33	276.21	276.16
s	276.27	276.27	276.27	276.27
c_b	276.61	276.44	276.11	275.94
$\Pi_4(Q_4^*, \tau_4^*, T_4^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	188.55	271.78	446.85	537.67
K	431.87	393.17	325.31	294.42
p	118.60	236.68	482.22	607.66
c	428.70	393.31	323.29	289.03
h	358.45	358.38	358.23	358.15
s	357.68	357.99	358.61	358.92
c_d	358.42	358.36	358.24	358.18
c_b	359.09	358.70	357.90	357.51

Table 10: Sensitivity analysis of average profit in Model 5 and Model 6

$\Pi_5(Q_5^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	234.90	333.17	534.64	637.16
K	492.51	462.31	404.99	377.50
p	133.21	283.21	583.21	733.21
c	534.63	483.05	384.75	337.37
h	433.44	433.32	433.09	432.97
M	447.64	440.40	426.07	419.00
$\Pi_6(Q_6^*, T_6^*)$ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	186.89	273.06	455.14	549.14
K	424.98	393.03	334.08	307.17
p	100.94	230.08	498.09	634.59
c	451.17	406.05	320.62	280.12
h	362.81	362.72	362.53	362.44
M	377.55	370.05	355.48	348.34
s	361.67	362.15	363.22	363.89
c_b	364.28	363.45	361.83	361.11

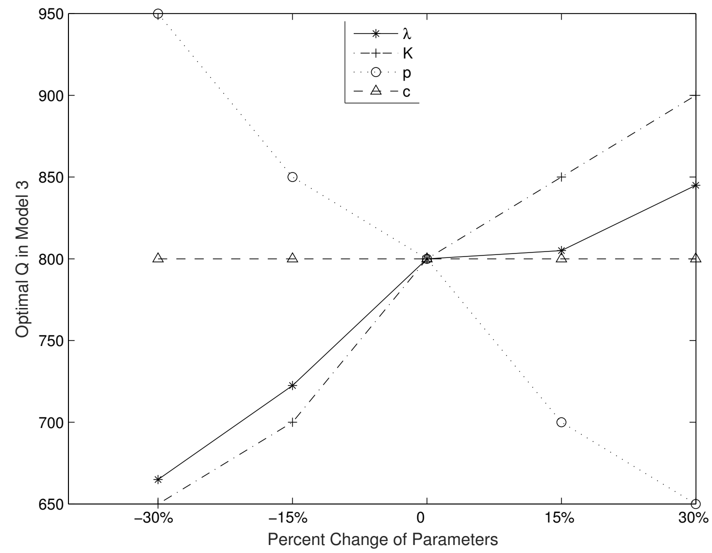


(a)

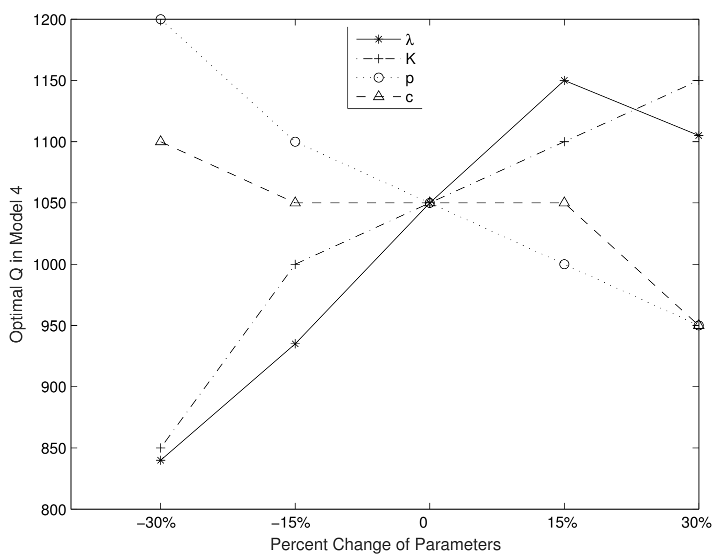


(b)

Figure 7: (a): Optimal replenishment quantity for varying parameters in Model 1, (b): Optimal replenishment quantity for varying parameters in Model 2.

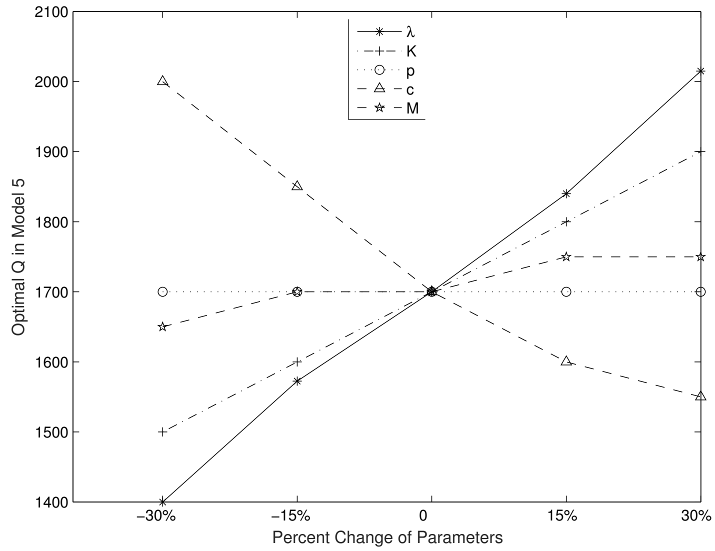


(a)

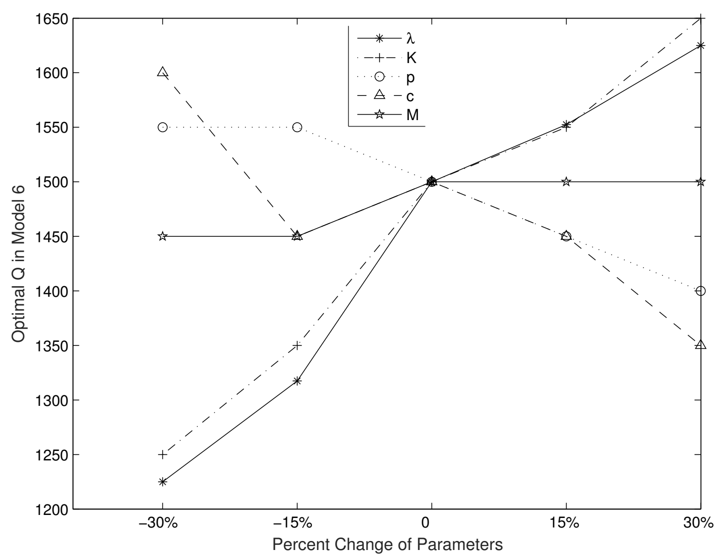


(b)

Figure 8: (a): Optimal replenishment quantity for varying parameters in Model 3, (b): Optimal replenishment quantity for varying parameters in Model 4.



(a)



(b)

Figure 9: (a): Optimal replenishment quantity for varying parameters in Model 5, (b): Optimal replenishment quantity for varying parameters in Model 6.

Table 11: Sensitivity analysis of the optimal inspection time in Model 2 and Model 4

Inspection time τ_2^* (Change / Parameter)	-30% changed	-15% changed	15% changed	30% changed
λ	9	8	8	7
K	7	8	8	9
p	8	8	8	8
c	9	8	8	8
Inspection time τ_4^* (Change / Parameter)	-30% changed	-15% changed	15% changed	30% changed
λ	8	7	7	6
K	6	7	7	7
p	8	7	7	7
c	7	7	7	6

the inspection time when K increases. Table 12 suggests the inventory holder chooses Model 3, Model 4 or Model 6 to set a shorter length of replenishment when the parameters λ, p and c become larger, while set a longer length of the replenishment when K becomes larger. Moreover, the inventory holder in Model 6 has to increase the length of the replenishment when M increases.

Previous analyses show that it may be benefit for the inventory holder to adopt the quantity-based policy, since the optimal long-run average profit may decrease if choosing the time-based policy. The inventory holder is suggested to drop the plan of continuous monitoring if the cost of the continuous monitoring is too high. Otherwise, performing continuous monitoring is able to increase the optimal long-run average profit.

7 Extensions

In this section, we discuss the extensions of our model from two aspects: one is with time-dependent demand and the other one is with other distributions of the maximum lifetime.

7.1 Discussion on λ

Demand has been always one of the most influential factors in the decisions relating to inventory and production activities. Hence, this section extends our model into the case with time dependent demand (e.g., Hung [12], Teng et al. [24] and Teng [25]). Here $\lambda(t)$ is denoted as the demand rate and it is assumed to be any nonnegetive function. Then

Table 12: Sensitivity analysis of the optimal length of the replenishment cycle in Models 3, 4 and 6.

Cycle T_3^* \ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	19	17	14	13
K	13	14	17	18
p	19	17	14	13
c	16	16	16	16
Cycle T_4^* \ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	21	19	17	15
K	15	17	19	20
p	21	19	17	16
c	18	18	18	17
Cycle T_6^* \ Change / Parameter	-30% changed	-15% changed	15% changed	30% changed
λ	24	22	19	18
K	18	19	22	23
p	23	22	20	19
c	21	21	21	20
M	20	20	21	21

the inventory levels of serviceable and deteriorated items in a replenishment cycle $[0,T]$ are governed by the following differential equations:

$$\begin{cases} I'(t) &= -\dfrac{I(t)}{I(t)+J(t)}\lambda(t) - \theta(t)I(t), \quad 0 \le t \le T, \\ J'(t) &= \theta(t)I(t) - \dfrac{J(t)}{I(t)+J(t)}\lambda(t), \quad 0 \le t \le T,. \end{cases} \tag{27}$$

Solving the system of differential equations (27) with the boundary conditions $I(0) = Q$, $J(0) = 0$, we have

$$\begin{cases} I(t) &= \dfrac{\Big(Q - \displaystyle\int_0^t \lambda(x)dx\Big)\Big(1+m-t\Big)}{1+m}, \quad 0 \le t \le T, \\ J(t) &= \dfrac{\Big(Q - \displaystyle\int_0^t \lambda(x)dx\Big)t}{1+m}, \quad 0 \le t \le T. \end{cases} \tag{28}$$

The above functions of $I(t)$ and $J(t)$ can be applied to calculate the number of serviceable items sold the customers, the revenue per cycle and the total cost. Hence Models 1, 2, 3 and 4 can be extended to the situation with time dependent demand for future research.

If the items in the inventory are continuously monitored, the inventory level of serviceable items with time dependent demand is written as

$$I'(t) = -\lambda(t) - \theta(t)I(t), \quad 0 \le t \le T, \tag{29}$$

with boundary condition $I(0) = Q$. Solving the differential equation of Eq. (29), one has

$$I(t) = \frac{1+m-t}{1+m}\Big(\int_0^t \lambda(s)\left(\frac{1+m}{1+m-s}\right) ds + Q\Big), \quad 0 \le t \le T. \tag{30}$$

Furthermore, Models 5 and 6 with time dependent demand $\lambda(t)$ can be investigated based on the function of $I(t)$ in Eq. (30). More useful managerial insights can be provided for the inventory holder to deal with different market situations.

7.2 Discussion on m

This work aims to maximize the long-run average profit which is determined by Eq. (1). Then based on Eq. (1), we need to get the expected revenue, the expected total cost and the expected replenishment cycle length for each model. It is found that the calculations of these expected values will be affected significantly by the properties of m. However, since the maximum lifetime of the items m is random and the expected parts involving m and T are not simple, it is not easy to solve the optimization problem for each model. Even though we assume that m is uniformly distributed, it is still very hard to find closed-form solutions for the models in the work. For example, we do not have a closed-form solution for maximizing Eq. (11), which is the long-run average profit in Model 2. Hence, it is suggested to solve these optimization problems by using numerical methods. In practice, m may follow other distributions, for example, normal distribution. The probability density function $f(m)$ would be different for each possible situation. Then the method of calculating the expected revenue, the expected total cost and the expected replenishment cycle length for each model would remain the same. While the results of these expected values and the optimal strategy in each model would change with $f(m)$. Therefore, the models in this study can be readily extended to the cases of other distributions. Some more interesting results can be obtained under different $f(m)$.

8 Conclusions

This study is motivated by an agricultural inventory control problem in which products are assumed to have random maximum lifetime. Items in the inventory are deteriorating with a deterioration rate which is assumed to depend on the maximum lifetime. Once the item stored in inventory exceeds the maximum lifetime, it is regarded as scarp and no longer serviceable. We develop six models for the inventory system by considering two replenishment policies and two inspection scenarios. Two replenishment policies include: (i) quantity-based policy, which means that the inventory is replenished when all stock are used up and (ii) time-based policy, which means that the inventory is replenished for every pre-determined period. Two inspection scenarios include: (i) one inspection during each cycle and (ii) continuous monitoring in the cycle. Each of the models aims to optimize the long-run average profit. Finally, examples and sensitivity analysis are conducted for each model. The results show that the parameters λ, K, p, c and M have significant effects on the long-run average profit. Some useful insights are provided to the management for making managerial decisions. For example, it is beneficial for the inventory holder to adopt the quantity-based policy. If the cost of the continuous monitoring is too high, the inventory holder is suggested not to employ the plan of continuous monitoring, otherwise, it is beneficial to choose continuous inspection.

One limitation of this study is that it can only handle the case when the inventory holder is aware of the situation that the products reach their maximum lifetime. Under such case, a replenishment can be preformed. We also assume that the inspection process is performed perfectly in the sense that it will correctly screen out the deteriorated items. For future research, one may consider imperfect inspection with errors. Another research direction is to consider an extended inventory model for multiple items with different maximum lifetimes. Moreover, this work could be further extended to consider the case of non-negligible lead-time. In this situation, two important factors including transportation cost and the length of lead time would be included in the function of the average profit. Then the decision variables such as Q, T and τ would be dependent on these two factors.

9 Appendix

9.1 Proof of Lemma 1

Suppose that $Q \leq a\lambda$. Then $m > Q/\lambda$ with probability 1, which implies that $T = \min\Big\{Q/\lambda, m\Big\} = Q/\lambda$. Then

$$E[T] = \frac{Q}{\lambda}, \quad E[T^2] = \frac{Q^2}{\lambda^2}, \tag{31}$$

and

$$E\Big[\frac{T^2}{1+m}\Big] = \frac{Q^2}{\lambda^2}\int_a^b \frac{1}{(1+m)} f(m)\, dm = \frac{Q^2}{\lambda^2(b-a)} \ln\Big(\frac{1+b}{1+a}\Big). \tag{32}$$

Hence, substituting Eq. (31) and Eq. (32) into Eq. (6), one has

$$\Pi_1(Q) = p\lambda - \frac{pQ}{2(b-a)} \ln\Big(\frac{1+b}{1+a}\Big) - c\lambda - \frac{K\lambda}{Q} - \frac{hQ}{2}.$$

Since $\Pi_1''(Q) = -\dfrac{2K\lambda}{Q^3} < 0$ for any $Q > 0$, $\Pi_1(Q)$ is a concave function of Q. Denote

$$q = \sqrt{\frac{2K\lambda}{\frac{p}{b-a}\ln\big(\frac{1+b}{1+a}\big) + h}},$$

and it has $\Pi_1'(q) = 0$. The optimal value of Q relating to the maximum average profit denoted by Q^* is

$$Q^* = \begin{cases} q, & \text{if } q < a\lambda, \\ a\lambda, & \text{if } q \geq a\lambda. \end{cases}$$

This completes the proof.

9.2 Proof of Lemma 2

Suppose that $Q \geq b\lambda$. Then $m \leq Q/\lambda$ with probability 1, which implies that

$$T = \min\Big\{Q/\lambda, m\Big\} = m.$$

Then

$$E[T] = E[m] = \frac{a+b}{2}, \quad E[T^2] = E[m^2] = \frac{a^2+ab+b^2}{3},$$

and

$$\begin{aligned} E\Big[\frac{T^2}{1+m}\Big] &= \int_a^b \frac{m^2}{(1+m)} \frac{1}{(b-a)}\, dm \\ &= \frac{1}{b-a}\Big[\frac{(b-1)^2}{2} - \frac{(a-1)^2}{2} + \ln\Big(\frac{1+b}{1+a}\Big)\Big]. \end{aligned}$$

Hence, substituting the above equations into Eq. (6), one has

$$\Pi_1(Q) = -\frac{2cQ}{a+b} - hQ + C_2,$$

where C_2 is a constant and independent of Q. It is easy to see that $\Pi_1(Q)$ is monotonic decreasing and $\Pi_1(Q)$ is maximized at $Q^* = b\lambda$. This completes the proof.

9.3 Proof of Proposition 1

Denote $g(m) = t(m) - m$. Hence $T = m$ when $g(m)$ is positive and $T = t(m)$ otherwise. Rewrite $g(m)$ as

$$\begin{aligned} g(m) &= t(m) - m = \frac{Q}{\lambda} - \Big(\frac{Q}{\lambda} - \tau\Big)\frac{\tau}{1+m} - m \\ &= \frac{1}{1+m}\Big[- m^2 + \Big(\frac{Q}{\lambda} - 1\Big)m + \Big(\tau - \frac{Q}{2\lambda}\Big)^2 + \frac{Q}{4\lambda}\Big(4 - \frac{Q}{\lambda}\Big)\Big]. \end{aligned}$$

Consider the discriminant of the quadratic polynomial in the brackets:

$$\begin{aligned} \Delta &= \Big(\frac{Q}{\lambda} - 1\Big)^2 + 4\Big[\Big(\tau - \frac{Q}{2\lambda}\Big)^2 + \frac{Q}{4\lambda}\Big(4 - \frac{Q}{\lambda}\Big)\Big] \\ &\geq \Big(\frac{Q}{\lambda} - 1\Big)^2 + \frac{Q}{\lambda}\Big(4 - \frac{Q}{\lambda}\Big) \\ &= \frac{2Q}{\lambda} + 1 > 0. \end{aligned}$$

Hence there are two roots for $g(m) = 0$, and they are

$$m_+ = \frac{Q - \lambda + \sqrt{(Q + \lambda(1 - 2\tau))^2 + 4\tau\lambda^2}}{2\lambda},$$

which is always positive, and

$$m_- = \frac{Q - \lambda - \sqrt{(Q + \lambda(1 - 2\tau))^2 + 4\tau\lambda^2}}{2\lambda}.$$

Also

$$g'(m) = \Big(\frac{Q}{\lambda} - \tau\Big)\frac{\tau}{(1+m)^2} - 1,$$

and

$$g''(m) = -\Big(\frac{Q}{\lambda} - \tau\Big)\frac{2\tau}{(1+m)^3} < 0.$$

Hence $g(m)$ is a concave function for positive m. Now

$$g(0) = \Big(\tau - \frac{Q}{2\lambda}\Big)^2 + \frac{Q}{4\lambda}\Big(4 - \frac{Q}{\lambda}\Big).$$

Consider 3 cases:

1. $Q \leq 4\lambda$. Then $g(0) \geq 0$. Hence $g(m) \geq 0$ when $0 < m \leq m_+$ and $g(m) < 0$ when $m > m_+$.
2. $Q > 4\lambda$ and $\tau \in \left(0, \frac{Q - \sqrt{Q^2 - 4\lambda Q}}{2\lambda}\right] \cup \left[\frac{Q + \sqrt{Q^2 - 4\lambda Q}}{2\lambda}, \min\left\{a, \frac{Q}{\lambda}\right\}\right)$. Then $g(0) \geq 0$. Hence $g(m) > 0$ when $0 < m \leq m_+$ and $g(m) < 0$ when $m > m_+$.
3. $Q > 4\lambda$ and $\tau \in \left(\frac{Q - \sqrt{Q^2 - 4\lambda Q}}{2\lambda}, \frac{Q + \sqrt{Q^2 - 4\lambda Q}}{2\lambda}\right)$. Then $g(0) < 0$. Hence $g(m) \geq 0$ when $m_- \leq m \leq m_+$ and $g(m) < 0$ when $m > m_+$ or $0 < m < m_-$.

This completes the proof.

9.4 Proof of Lemma 3

Assume that the replenishment quantity Q is given. Suppose that $T \in [Q/\lambda, \infty)$, thus the number of serviceable items sold to customers in a replenishment cycle is

$$\int_0^{T_0} \frac{I(t)}{I(t)+J(t)} \lambda \, dt = \int_0^{T_0} \frac{(1+m-t)\lambda}{1+m} dt = \lambda \Big[T_0 - \frac{T_0^2}{2(1+m)} \Big],$$

where

$$T_0 = \begin{cases} Q/\lambda, & \text{if } m > Q/\lambda, \\ m, & \text{if } m \le Q/\lambda. \end{cases}$$

The total revenue is

$$TR = p\lambda \Big(T_0 - \frac{T_0^2}{2(1+m)} \Big),$$

and the total cost is

$$TC = cQ + K + \begin{cases} \frac{hQ^2}{2\lambda} + c_b(T - \frac{Q}{\lambda})\lambda, & \text{if } m > Q/\lambda; \\ \frac{h}{2}(2Q - \lambda m)m + c_b(T-m)\lambda, & \text{if } m \le Q/\lambda. \end{cases}$$

The expected revenue and cost can be computed as follows.

1. If $Q/\lambda < a$, the expected cost is

$$E[TC] = cQ + K + \frac{hQ^2}{2\lambda} + c_b(T - \frac{Q}{\lambda})\lambda,$$

and the expected revenue is

$$\begin{aligned} E[TR] &= p\lambda \cdot E\Big[T_0 - \frac{T_0^2}{2(1+m)} \Big] = p\lambda \cdot E\Big[\frac{Q}{\lambda} - \frac{Q^2}{2\lambda^2(1+m)} \Big] \\ &= \frac{p}{b-a} \Big[Q(b-a) - \frac{Q^2}{2\lambda} \ln\left(\frac{1+b}{1+a}\right) \Big]. \end{aligned}$$

The long-run average profit is

$$\Pi_3(Q,T) = \frac{E[TR] - E[TC]}{T} = \frac{C_{31}}{T} - c_b\lambda,$$

Where $C_{31} > 0$ and C_{31} is independent of T. Thus $\Pi_3(Q,T)$ decreases in T.

2. If $Q/\lambda \geqslant b$, the expected cost is

$$\begin{aligned} E[TC] &= cQ + K + hE\Big[\frac{(2Q - \lambda m)m}{2} \Big] + c_b E[(T-m)\lambda] \\ &= cQ + K + \frac{h}{2}\Big[Q(a+b) - \frac{\lambda}{3}(a^2+ab+b^2) \Big] + c_b\lambda(T - \frac{a+b}{2}), \end{aligned}$$

and the expected revenue is

$$\begin{aligned} E[TR] &= p\lambda \cdot E\Big[T_0 - \frac{T_0^2}{2(1+m)} \Big] = p\lambda \cdot E\Big[m - \frac{m^2}{2(1+m)} \Big] \\ &= \frac{p\lambda}{4(b-a)} \Big[(b-a)(b+a+2) - 2\ln\left(\frac{1+b}{1+a}\right) \Big]. \end{aligned}$$

Similarly, the long-run average profit is given by

$$\Pi_3(Q,T) \quad = \quad \frac{E[TR]-E[TC]}{T} = \frac{C_{32}}{T} - c_b\lambda,$$

where $C_{32} > 0$ and C_{32} is independent of T. Thus $\Pi_3(Q,T)$ decreases in T.

3. If $a \leqslant Q/\lambda < b$, the expected cost is

$$\begin{aligned}
E[TC] &= cQ + K + \int_a^{Q/\lambda} \Big[h\frac{(2Q-\lambda m)m}{2} + c_b(T-m)\lambda\Big] f(m)\, dm \\
&\quad + \int_{Q/\lambda}^{b} \Big[h\frac{Q^2}{2\lambda} + c_b(T-\frac{Q}{\lambda})\lambda\Big] f(m)\, dm \\
&= cQ + K + \frac{1}{b-a}\Big[\frac{1}{2}(hQ - c_b\lambda)(\frac{Q^2}{\lambda^2} - a^2) - \frac{1}{6}h\lambda(\frac{Q^3}{\lambda^3} - a^3) + c_bT\lambda(\frac{Q}{\lambda} - a)\Big] \\
&\quad + \frac{1}{b-a}\Big[\frac{hQ^2}{2\lambda} + c_b\lambda(T - \frac{Q}{\lambda})\Big](b - \frac{Q}{\lambda}),
\end{aligned}$$

and the expected revenue is

$$\begin{aligned}
E[TR] &= p\lambda \cdot E\Big[T_0 - \frac{T_0^2}{2(1+m)}\Big] \\
&= p\lambda\Big\{\int_a^{Q/\lambda} \Big[m - \frac{m^2}{2(1+m)}\Big] f(m)\, dm + \int_{Q/\lambda}^{b} \Big[\frac{Q}{\lambda} - \frac{Q^2}{2\lambda^2(1+m)}\Big] f(m)\, dm\Big\} \\
&= \frac{p\lambda}{2(b-a)}\Big[\frac{2bQ}{\lambda} - \frac{Q^2}{\lambda^2} - a^2 + \frac{(a-1)^2}{2} - \frac{(Q/\lambda-1)^2}{2} + \ln\Big(\frac{1+a}{1+Q/\lambda}\Big) \\
&\qquad + \frac{Q^2}{\lambda^2}\ln\Big(\frac{1+Q/\lambda}{1+b}\Big)\Big].
\end{aligned}$$

Similarly, the long-run average profit is

$$\Pi_3(Q,T) \quad = \quad \frac{E[TR]-E[TC]}{T} = \frac{C_{33}}{T} - c_b\lambda,$$

where $C_{33} > 0$ and C_{33} is independent of T. Thus $\Pi_3(Q,T)$ decreases in T.

Thus, the above three cases show that $\Pi_3(Q,T)$ decreases in T when Q is given and $T \in [Q/\lambda, \infty)$. This completes the proof.

9.5 Proof of Lemma 4

Suppose that $T \in [b, \infty)$. The value of $\Pi_3(Q,T)$ is discussed as follows.

1. $Q/\lambda > b$. Since $T \geq b$, we consider two intervals $T \in [b, Q/\lambda)$ and $T \in [Q/\lambda, \infty)$.

   (a) If $T \in [b, Q/\lambda)$, then $m < T < Q/\lambda$. Hence, $E[I(T)] = 0$. The expected cost is

$$\begin{aligned}
E[TC] &= cQ + K + hE\Big[\frac{(2Q-\lambda m)m}{2}\Big] + c_bE[(T-m)\lambda] \\
&= cQ + K + \frac{h}{2}\Big[Q(a+b) - \frac{\lambda}{3}(a^2+ab+b^2)\Big] + c_b\lambda(T - \frac{a+b}{2}).
\end{aligned}$$

The expected revenue is

$$
\begin{aligned}
E[TR] &= p\lambda \cdot E\Big[T_0 - \frac{T_0^2}{2(1+m)}\Big] = p\lambda \cdot E\Big[m - \frac{m^2}{2(1+m)}\Big] \\
&= p\lambda\Big\{\frac{a+b}{2} - \frac{1}{2(b-a)}\Big[\frac{(b-1)^2}{2} - \frac{(a-1)^2}{2} + \ln\Big(\frac{1+b}{1+a}\Big)\Big]\Big\}.
\end{aligned}
$$

The long-run average profit is

$$
\Pi_3(Q,T) = \frac{E[TR]-E[TC]}{T} = \frac{C_4}{T} - c_b\lambda.
$$

where C_4 is positive and independent with T.

$$
\partial\Pi_3(Q,T)/\partial T = -\frac{C_4}{T^2} < 0.
$$

$\Pi_3(Q,T)$ decreases in T when $T \in [b, Q/\lambda)$.

(b) If $T \in [Q/\lambda, \infty)$, refer to Lemma 3, $\Pi_3(Q,T)$ decreases in T.

$\Pi_3(Q,T)$ is a continuous function of T. Therefore, the above cases (a) and (b) imply that if $Q/\lambda > b$, $\Pi_3(Q,T)$ decreases in T when $T \in [b,\infty)$.

2. $Q/\lambda \le b$. Since $T \ge b$, then $Q/\lambda \le b \le T$. Refer to Lemma 3, $\Pi_3(Q,T)$ decreases in T when $T \in [b,\infty)$.

The above two cases show that $\Pi_3(Q,T)$ decreases in T when $T \in [b,\infty)$ whatever $Q/\lambda \le b$ or $Q/\lambda > b$. This completes the proof.

9.6 Proof of Lemma 5

Let

$$
t_a = \sqrt{\frac{2(cQ - sQ + K)}{\frac{(p-2s)\lambda}{b-a}\ln\left(\frac{1+b}{1+a}\right) - h\lambda}}.
$$

It is easy to verify that

$$
\frac{\partial\Pi_3(Q,t_a)}{\partial T} = 0.
$$

Since

$$
\frac{\partial^2\Pi_3(Q,T)}{\partial T^2} = -\frac{2(cQ-sQ+K)}{T^3} < 0,
$$

as $T > 0$ and $c > s$, the optimal value of T, which is denoted by T^*, is t_a if $t_a < \min\{a, Q/\lambda\}$; otherwise, $T^* = \min\{a, Q/\lambda\}$ as the function $\Pi_3(Q,T)$ is increasing in $(0, \min\{a, Q/\lambda\}]$. Hence, we have

$$
T^* = \begin{cases} t_a, & \text{if } t_a < \min\{a, Q/\lambda\}; \\ \min\{a, Q/\lambda\}, & \text{if } t_a \ge \min\{a, Q/\lambda\}. \end{cases}
$$

This completes the proof.

9.7 Proof of Lemma 7

Suppose that $T \geq Q/\lambda$ and T_1 is given in Eq. (18). Since $Q/\lambda > T_1$, then $T > T_1$.
Case 1. $Q/\lambda \leq T$ and condition (i) in Table 2 is satisfied. When $m \in I_1$, then $m < T_1 < T$, which implies that $I_1 \subseteq (0, T_1) \subset (0, T)$. Denote

$$I_1 = (m_-, m_+) \cap (0, T) \cap (a, b) = (i_1, i_2).$$

Here i_1 and i_2 must be independent of T. Thus

$$\begin{aligned} \Pi_4(Q,\tau,T) &= \frac{E[TR]-E[TC]}{T} \\ &= \frac{\sum_{i=1}^{4} \int_{I_i} (TR_i - TC_i) f(m)\, dm}{T} \\ &= \frac{\int_{I_1} (TR_1 - TC_1) f(m)\, dm}{T} + \frac{\int_{I_3} (TR_3 - TC_3) f(m)\, dm}{T} \\ &= \frac{C_7}{T} - c_b \lambda \left(\frac{|I_1| + |I_3|}{b-a} \right), \end{aligned}$$

where $|I_i|$ means the length of the interval I_i. C_7 and $|I_i|$ are positive and independent with T. Thus, in this case, $\Pi_4(Q,\tau,T)$ decreases as T increases. Similar results can be obtained for the other case: $Q/\lambda \leq T$ with condition (ii) in Table 2.

Therefore, $\Pi_4(Q,\tau,T)$ decreases as T increases when $T \in [Q/\lambda, \infty)$, and

$$\Pi_4(Q,\tau,b) = \max_{T \in [Q/\lambda, \infty)} \{\Pi_4(Q,\tau,T)\}.$$

This completes the proof.

9.8 Proof of Lemma 8

Suppose that $T \geq b$. For Case 1, $Q/\lambda \leq T$ and condition (i) in Table 2 is satisfied. Since $T \geq b$, I_1 and I_3 given in Table 2 are independent with T.

$$\begin{aligned} \Pi_4(Q,\tau,T) &= \frac{E[TR]-E[TC]}{T} = \frac{1}{T} \sum_{i=1}^{4} \int_{I_i} (TR_i - TC_i) f(m)\, dm \\ &= \frac{1}{T} \left(\int_{I_1} (TR_1 - TC_1) f(m)\, dm + \int_{I_3} (TR_3 - TC_3) f(m)\, dm \right) \\ &= \frac{C_8}{T} - c_b \lambda, \end{aligned}$$

where C_8 is positive and independent of T. Thus, in this case, $\Pi_4(Q,\tau,T)$ decreases as T increases. Similar results can be obtained for other three cases: $Q/\lambda \leq T$ with condition (ii), $Q/\lambda > T$ with condition (i), and $Q/\lambda > T$ with condition (ii). Therefore, $\Pi_4(Q,\tau,T)$ decreases as T increases when $T \in [b, \infty)$, and

$$\Pi_4(Q,\tau,b) = \max_{T \in [b, \infty)} \{\Pi_4(Q,\tau,T)\}.$$

This completes the proof.

9.9 Proof of Lemma 9

Differentiate $g_5(m)$ we get

$$g_5'(m) = \frac{1}{1+m} + \frac{Q}{\lambda(1+m)^2},$$

which is positive for any $m > 0$. We also have $g_5(0) = -\dfrac{Q}{\lambda}$ and $g_5(m) \to +\infty$ as $m \to +\infty$. Hence, there is a unique and positive m_0 such that $g_5(m_0) = 0$. Then

$$\begin{aligned} t_0 - m < 0 \quad &\Leftrightarrow \quad 1 - (1+m)\exp\Big[-\frac{Q}{\lambda(1+m)}\Big] < 0 \\ &\Leftrightarrow \quad \exp\Big[-\frac{Q}{\lambda(1+m)}\Big] > \frac{1}{1+m} \\ &\Leftrightarrow \quad -\frac{Q}{\lambda(1+m)} > \ln\Big(\frac{1}{1+m}\Big) \\ &\Leftrightarrow \quad \ln(1+m) - \frac{Q}{\lambda(1+m)} > 0 \\ &\Leftrightarrow \quad m > m_0. \end{aligned}$$

9.10 Proof of Lemma 10

Since the deteriorated items are screened out instantaneously, it has $t_0 < Q/\lambda$. Thus if $T \in [Q/\lambda, \infty)$, then $T_6 = \min\{m, T, t_0\} = \min\{m, t_0\}$ which is independent of T. Therefore $E[T_6]$ would be independent of T.

$$TC = cQ + K + \begin{cases} h\displaystyle\int_0^{t_0} I(t)\,dt + c_b\lambda(T - t_0), & \text{if } T_6 = t_0; \\ h\displaystyle\int_0^{m} I(t)\,dt + c_b\lambda(T - m), & \text{if } T_6 = m. \end{cases}$$

Thus $E[TC] = C_9 + c_b\lambda T$, where C_9 would be independent of T.

Then

$$\Pi_6(Q,T) \quad = \quad \frac{p\lambda E[T_6] - E[TC]}{T} = \frac{C_{10}}{T} - c_b\lambda,$$

where C_{10} is positive and independent of T. Here $\Pi_6(Q,T)$ decreases as T increases, which means that

$$\Pi_6(Q,b) = \max_{T\in[Q/\lambda,\infty)}\{\Pi_6(Q,T)\}.$$

This completes the proof.

9.11 Proof of Lemma 11

Suppose that $T \in [b, \infty)$, then $T_6 = \min\{m, T, t_0\} = \min\{m, t_0\}$ which is independent with T. Thus $E[T_6]$ would be independent with T.

$$TC = cQ + K + \begin{cases} h\displaystyle\int_0^{t_0} I(t)\,dt + c_b\lambda(T - t_0), & \text{if } T_6 = t_0; \\ h\displaystyle\int_0^{m} I(t)\,dt + c_b\lambda(T - m), & \text{if } T_6 = m. \end{cases}$$

Thus $E[TC] = C_{11} + c_b\lambda T$, where C_{11} would be independent of T.

Then

$$\Pi_6(Q,T) \quad = \quad \frac{p\lambda E[T_6] - E[TC]}{T} = \frac{C_{12}}{T} - c_b\lambda,$$

where C_{12} is positive and independent of T. Here $\Pi_6(Q,T)$ decreases as T increases, which means that

$$\Pi_6(Q,b) = \max_{T\in[b,\infty)}\{\Pi_6(Q,T)\}.$$

This completes the proof.

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