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The generalized numerical range of a set of matrices

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Abstract

For a given set of $n \times n$ matrices $\mathcal{F}$, we study the union of the C-numerical ranges of the matrices in the set $\mathcal{F}$, denoted by $W_C(\mathcal{F})$. We obtain basic algebraic and topological properties of $W_C(\mathcal{F})$, and show that there are connections between the geometric properties of $W_C(\mathcal{F})$ and the algebraic properties of C and the matrices in $\mathcal{F}$. Furthermore, we consider the starshapedness and convexity of the set $W_C(\mathcal{F})$. In particular, we show that if $\mathcal{F}$ is the convex hull of two matrices such that $W_C(A)$ and $W_C(B)$ are convex, then the set $W_C(\mathcal{F})$ is star-shaped. We also investigate the extensions of the results to the joint C-numerical range of an m-tuple of matrices.

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1 Introduction

Let M_n be the set of all $n \times n$ complex matrices. The numerical range of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\},$$

which is a useful tool for studying matrices and operators; for example, see [8, Chapter 22] and [9, Chapter 1]. In particular, there is an interesting interplay between the geometrical properties of $W(A)$ and the algebraic properties of A.

In this paper, for a nonempty set $\mathcal{F}$ of matrices in M_n, we consider

$$W(\mathcal{F}) = \cup\{W(A) : A \in \mathcal{F}\}.$$

We show that there are also interesting connections between the geometrical properties of $W(\mathcal{F})$ and the properties of the matrices in $\mathcal{F}$.

The study has motivations from different topics. We mention two of them in the following. The first one arises in the study of Crouzeix's conjecture asserting that for any $A \in M_n$,

$$\|f(A)\| \leq 2 \max\{|f(\mu)| : \mu \in W(A)\}$$

for any complex polynomial $f(z)$, where $\|A\|$ denotes the operator norm of A, see [7]. Instead of focusing on a single matrix $A \in M_n$, one may consider a complex convex set K and show that

$$\|f(A)\| \leq 2 \max\{|f(\mu)| : \mu \in K\}$$

whenever $W(A) \subseteq K$. One readily shows that this is equivalent to the Crouzeix's conjecture. In fact, one may focus on the case when K is a convex polygon (including interior) because $W(A)$ can always be approximated by convex polygons from inside or outside.

Another motivation comes from quantum information science. In quantum mechanics, a pure state in M_n is a rank one orthogonal projection of the form xx^* for some unit vector $x \in \mathbb{C}^n$, and a general state is a density matrix, which is a convex combination of pure states. For a measurement operator $A \in M_n$, which is usually Hermitian, the measurement of a state P is computed by $\langle A, P\rangle = \mathrm{tr}\,(AP)$. As a result,

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\} = \{\mathrm{tr}\,(Axx^*) : x \in \mathbb{C}^n, x^*x = 1\}$$

can be viewed as the set of all possible measurements on pure states for a given measurement operator A. By the convexity of the numerical range, we have

$$W(A) = \{\mathrm{tr}\,(Axx^*) : x \in \mathbb{C}^n, x^*x = 1\} = \{\mathrm{tr}\,(AP) : P \text{ is a general state}\}.$$

So, $W(A)$ actually contains all possible measurements on general states for a given measurement operator A. If we consider all possible measurements under a set $\mathcal{F}$ of measurement operators, then it is natural to study $W(\mathcal{F})$. In fact, if we know the set $W(\mathcal{F})$, we may deduce some properties about the measurement operators in $\mathcal{F}$. For example, one can show that

(a) $W(\mathcal{F}) = \{\mu\}$ if and only if $\mathcal{F} = \{\mu I\}$.
(b) $W(\mathcal{F}) \subseteq \mathbb{R}$ if and only if all matrices in $\mathcal{F}$ are Hermitian.
(c) $W(\mathcal{F}) \subseteq [0, \infty)$ if and only if all matrices in $\mathcal{F}$ are positive semi-definite.

It is worth pointing out that if $A \in M_n$ is not Hermitian, one may consider the Hermitian decomposition $A = A_1 + iA_2$ such that A_1, A_2 are Hermitian, and identify $W(A)$ as the joint numerical range of (A_1, A_2) defined by

$$W(A_1, A_2) = \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{R}^2.$$

One can study the joint numerical range of k-tuple of Hermitian matrices $(A_1, \ldots, A_k)$ defined by

$$W(A_1, \ldots, A_k) = \{(x^*A_1x, \ldots, x^*A_kx) : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{R}^k.$$

Accordingly, one may consider the joint numerical range $W(\mathcal{F}) \subseteq \mathbb{R}^k$ of a set $\mathcal{F}$ of k-tuple of Hermitian matrices in M_n.

It turns out that we can study $W(\mathcal{F})$ under a more general setting. For a matrix $C \in M_n$, the C-numerical range of $A \in M_n$ is defined by

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \text{ is unitary }\},$$

which has been studied by many researchers in connection to different topics; see [10] and its references. In particular, the C-numerical range is useful in the study of quantum control and quantum information; for example, see [16]. Denote by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for M_n. When $C = E_{11}$, $W_C(A)$ reduces to $W(A)$; if C is a rank m orthogonal projection, then $W_C(A)$ reduces to the mth-numerical range of A; see [8, 9, 12]. For a non-empty set $\mathcal{F}$ of matrices in M_n, we consider

$$W_C(\mathcal{F}) = \cup\{W_C(A) : A \in \mathcal{F}\}.$$

In Section 2, we study the connection between the geometric properties of $W_C(\mathcal{F})$ and the properties of the set $\mathcal{F}$. In Sections 3 and 4, we study conditions for $W_C(\mathcal{F})$ to be star-shaped or convex. In Section 5, we consider the joint C-numerical range of $(A_1, \ldots, A_k) \in M_n^k$ defined by

$$W_C(A_1, \ldots, A_k) = \{(\mathrm{tr}\,(CU^*A_1U), \ldots, \mathrm{tr}\,(CU^*A_kU)) : U \text{ is unitary}\} \subseteq \mathbb{C}^k.$$

If $C, A_1, \ldots, A_k$ are Hermitian matrices, then $W_C(A_1, \ldots, A_k) \subseteq \mathbb{R}^k$. One may see [10] for the background and references on the C-numerical range.

If $\mu \in \mathbb{C}$ and $C = \mu I$, then

$$W_C(A) = \{\mu \mathrm{tr}\, A\} \quad \text{and} \quad W_C(A_1, \ldots, A_k) = \{\mu(\mathrm{tr}\, A_1, \ldots, \mathrm{tr}\, A_k)\}.$$

We will always assume that C is not a scalar matrix to avoid trivial consideration.

For convenience of discussion, we always identify $\mathbb{C}$ with $\mathbb{R}^2$.

2 Basic properties of $W_C(A)$ and $W_C(\mathcal{F})$

We first list some basic results in the first two propositions below about the C-numerical range; see [4, 10, 13, 17, 18] and their references.

Proposition 2.1. *Let $C, A \in M_n$.*

(a) *For any unitary $U, V \in M_n$, $W_{V^*CV}(U^*AU) = W_C(A) = W_A(C)$.*

(b) *For any $\alpha, \beta \in \mathbb{C}$,*

$$W_C(\alpha A + \beta I) = \alpha W_C(A) + \beta \mathrm{tr}\, C = \{\alpha\mu + \beta \mathrm{tr}\, C : \mu \in W_C(A)\}.$$

(c) *The set $W_C(A)$ is compact.*

(d) *The set $W_C(A)$ is star-shaped with $(\mathrm{tr}\, C)(\mathrm{tr}\, A)/n$ as a star center, i.e.,*

$$t\mu + (1-t)(\mathrm{tr}\, C)(\mathrm{tr}\, A)/n \in W_C(A) \quad \text{for any } \mu \in W_C(A) \text{ and } t \in [0, 1].$$

(e) *The set $W_C(A)$ is convex if there is $\gamma \in \mathbb{C}$ such that $\tilde{C} = C - \gamma I_n$ satisfies any one of the following conditions.*

(e.1) *$\tilde{C}$ is rank one.*

(e.2) $\tilde{C}$ *is a multiple of a Hermitian matrix.*

(e.3) $\tilde{C}$ *is unitarily similar to a block matrix of the form* $(C_{ij})_{1\le i,j\le m}$ *such that* $C_{11},\dots,C_{mm}$ *are square matrices possibly with different sizes, and* $C_{ij}=0$ *if* $j\neq i+1$.

Researchers have extended the results on classical numerical range, and showed that there is interesting interplay between the geometry of $W_C(A)$ and the algebraic properties of C and A. A boundary point ν of $W_C(A)$ is a corner point if there is $d>0$ such that $W_C(A)\cap\{\mu\in\mathbb{C}:|\mu-\nu|\le d\}$ is contained in a pointed cone with ν as a vertex.

Proposition 2.2. *Let* $C=(c_{ij})\in M_n$ *be a non-scalar matrix in lower triangular form with diagonal entries* $c_1,\dots,c_n$. *Suppose* $A=(a_{ij})\in M_n$ *is also in lower triangular form with diagonal entries* $a_1,\dots,a_n$.

(a) *If* $\mathrm{tr}\,(CA)=\sum_{j=1}^n c_ja_j$ *is a boundary point of* $W_C(A)$, *then* $a_{rs}=0$ *whenever* $c_r\neq c_s$ *and* $c_{pq}=0$ *whenever* $a_p\neq a_q$. *In particular, if* C *and* A *has distinct eigenvalues, then* C *and* A *will be in diagonal form, i.e.,* C *and* A *are normal.*

(b) *If* ν *is a corner point of* $W_C(A)$, *then there is a unitary* V *such that* V^*AV *is lower triangular form with diagonal entries* $a_{j_1},\dots,a_{j_n}$ *such that* $(j_1,\dots,j_n)$ *is a permutation of* $(1,\dots,n)$ *and*

$$\nu=\mathrm{tr}\,(CV^*AV)=\sum_{\ell=1}^n c_\ell a_{j_\ell}.$$

(c) *The set* $W_C(A)$ *is a singleton if and only if* A *is a scalar matrix.*

(d) *The set* $W_C(A)$ *is a non-degenerate line segment if and only if* C *and* A *are non-scalar normal matrices having collinear eigenvalues in* $\mathbb{C}$.

(e) *The set* $W_C(A)$ *is a convex polygon if and only if*

$$W_C(A)=\mathrm{conv}\{(c_1,\dots,c_n)P(a_1,\dots a_n)^t : P \text{ is a permutation matrix}\}.$$

Now, we extend the basic properties of $W_C(A)$ to $W_C(\mathcal{F})$. We always assume that C is not a scalar matrix and $\mathcal{F}$ is a non-empty subset of M_n.

Theorem 2.3. *Suppose* $C\in M_n$ *is non-scalar and* $\mathcal{F}\subseteq M_n$ *is non-empty.*

(a) *If* $U\in M_n$ *is unitary, then* $W_C(\mathcal{F})=W_C(U^*\mathcal{F}U)$, *where* $U^*\mathcal{F}U=\{U^*AU:A\in\mathcal{F}\}$.

(b) *For any* $\alpha,\beta\in\mathbb{C}$, *let* $\alpha\mathcal{F}+\beta I=\{\alpha A+\beta I:A\in\mathcal{F}\}$. *Then*

$$W_C(\alpha\mathcal{F}+\beta I)=\alpha W_C(\mathcal{F})+\beta\mathrm{tr}\,C=\{\alpha\mu+\beta\mathrm{tr}\,C:\mu\in W_C(\mathcal{F})\}.$$

(c) *If $\mathcal{F}$ is bounded, then so is $W_C(\mathcal{F})$.*

(d) *If $\mathcal{F}$ is connected, then so is $W_C(\mathcal{F})$.*

(e) *If $\mathcal{F}$ is compact, then so is $W_C(\mathcal{F})$.*

Proof. (a) and (b) can be verified readily.

(c) If $\mathcal{F}$ is bounded so that there is $R > 0$ such that for every $A \in M_n$ we have $\|A\| < R$, then

$$|\mathrm{tr}\,(CU^*AU)| \le n\|C\|\|A\| < n\|C\|R.$$

Thus, $W_C(\mathcal{F})$ is bounded.

(d) Note that for any $A \in M_n$, $W_C(A)$ is star-shaped with $(\mathrm{tr}\,C)(\mathrm{tr}\,A)/n$ as a star center. If $\mu_1 = \mathrm{tr}\,(CU^*A_1U)$ and $\mu_2 = \mathrm{tr}\,(CV^*A_2V)$ with $A_1, A_2 \in \mathcal{F}$ and U, V unitary, then there are a line segment with end points μ_1 and $(\mathrm{tr}\,C)(\mathrm{tr}\,A_1)/n$, and another line segment with end points μ_2 and $(\mathrm{tr}\,C)(\mathrm{tr}\,A_2)/n$. If $\mathcal{F}$ is connected, then so are the sets $\{\mathrm{tr}\,A : A \in \mathcal{F}\}$ and $\{(\mathrm{tr}\,A)(\mathrm{tr}\,C)/n : A \in \mathcal{F}\}$. Hence, there is a path joining μ_1 to $(\mathrm{tr}\,C)(\mathrm{tr}\,A_1)/n$, then to $(\mathrm{tr}\,C)(\mathrm{tr}\,A_2)/n$ and then to μ_2.

(e) Suppose $\mathcal{F}$ is compact. Then $\mathcal{F}$ is bounded and closed. By (c), $W_C(\mathcal{F})$ is also bounded. To show that $W_C(\mathcal{F})$ is closed, let $\{\mathrm{tr}\,(CU_k^*A_kU_k) : k = 1, 2, \ldots\}$ be a sequence in $W_C(\mathcal{F})$ converging to $\mu_0 \in \mathbb{C}$, where $A_k \in \mathcal{F}$ and U_k is unitary for each k. Since $\mathcal{F}$ is compact, there is a subsequence $\{A_{j_k} : k = 1, 2, \ldots\}$ of $\{A_k : k = 1, 2, \ldots\}$ converging to $A_0 \in \mathcal{F}$. We can further consider a subsequence $\{U_{j(\ell)} : \ell = 1, 2, \ldots\}$ of $\{U_{j_k} : k = 1, 2, \ldots\}$ converging to U_0. Thus

$$\{\mathrm{tr}\,(CU_{j(\ell)}^*A_{j(\ell)}U_{j(\ell)}) : k = 1, 2, \ldots\} \to \mathrm{tr}\,(CU_0^*A_0U_0) = \mu_0 \in W_C(\mathcal{F})\,.$$

Thus, $W_C(\mathcal{F})$ is closed. As a result, $W_C(\mathcal{F})$ is compact. ■

The following examples show that none of the converses of the assertions in Theorem 2.3 (c) – (e) is valid, and there are no implications between the conditions that "$\mathcal{F}$ is closed" and "$W_C(\mathcal{F})$" is closed.

Example 2.4. (a) Suppose $C \in M_n$ is non-scalar and has trace zero, and $\mathcal{F} = \{\mu I : \mu \in \mathbb{C}\}$. Then $W_C(\mathcal{F}) = \{0\}$ is bounded and compact, but $\mathcal{F}$ is not bounded.

(b) Suppose $C \in M_n$ is non-scalar, and $\mathcal{F} = \{A_0, A_1\}$ such that $A_0 = 0$ and $A_1 = xy^*$ for a pair of orthonormal vectors x, y. Then $W_C(A_0) = \{0\}$ and $W_C(\mathcal{F}) = W_C(A_1)$ is a circular disk center at the origin with radius

$$R = \max\{|u^*Cv| : \{u, v\} \text{ is an orthonormal set}\}.$$

Thus, $W_C(\mathcal{F})$ is connected, but $\mathcal{F}$ is not.

(c) Let $\mathcal{F} = \text{conv}\mathcal{G}$ with

$$\mathcal{G} = \{2E_{12}\} \cup \{\text{diag}(e^{ir}, e^{-ir}) : r \text{ is a rational number}\}.$$

Then $\mathcal{F}$ is not closed but $W(\mathcal{F}) = W(2E_{12})$ is closed.

(d) Let $\mathcal{F} = \left\{\text{diag}\left(0, x + \frac{i}{x}\right) : x > 0\right\} \cup \{\text{diag}(0,0)\}$. Then $\mathcal{F}$ is closed, but

$$W(\mathcal{F}) = \{x + iy : x,\ y > 0,\ xy \leq 1\} \cup \{0\}$$

is not closed.

Next, we consider the connection between the geometrical properties of $W_C(\mathcal{F})$ and the properties of C and $\mathcal{F}$. Note that for any subset $\mathcal{S}$ of $\mathbb{C}$, if $\text{tr}\, C \neq 0$ and $\mathcal{F} = \{\mu I/\text{tr}\, C : \mu \in \mathcal{S}\}$, then we have $W_C(\mathcal{F}) = \mathcal{S}$. Thus, the geometrical shape of $W_C(\mathcal{F})$ may be quite arbitrary. Also, if $C = \mu I$ is a scalar matrix, then $W_C(\mathcal{F}) = \{\mu \text{tr}\, A : A \in \mathcal{F}\}$. Again, $W_C(\mathcal{F})$ does not contain much information about the matrices in $\mathcal{F}$. Nevertheless, we have the following.

Theorem 2.5. *Suppose $C \in M_n$ is non-scalar, and $\mathcal{F} \subseteq M_n$ is non-empty. The following conditions hold.*

(a) *The set $W_C(\mathcal{F}) = \{\mu\}$ if and only if $\mathcal{F} = \{\nu I : \nu \text{tr}\, C = \mu\}$.*

(b) *The set $W_C(\mathcal{F})$ is a subset of a straight line L if and only if*

(i) $\mathcal{F} \subseteq \{\nu I : \nu \in \mathbb{C}, \nu \text{tr}\, C \in L\}$, *or*

(ii) *there are complex units $\alpha, \gamma \in \mathbb{C}$ such that*

$\gamma(C - (\text{tr}\, C)I/n)$ *and* $\alpha(A - (\text{tr}\, A)I/n)$ *are Hermitian for all* $A \in \mathcal{F}$,

and $\{(\text{tr}\, C)(\text{tr}\, A)/n : A \in \mathcal{F}\}$ *is collinear.*

(c) *The set $W_C(\mathcal{F})$ is a convex polygon if and only if $W_C(\mathcal{F}) = \text{conv}\{v_1, \ldots, v_m\}$ where each v_j is of the forms $(c_1, \ldots, c_n)(a_1, \ldots, a_n)^t$. Here $c_1, \ldots, c_n$ are eigenvalues of C, and $a_1, \ldots, a_n$ are eigenvalues of some $A_j \in \mathcal{F}$.*

Proof. Condition (a) follows from the fact that $W_C(A) = \{\mu\}$ if and only if $A = \nu I$ with $\nu \text{tr}\, C = \mu$.

(b) Suppose $W_C(\mathcal{F}) \subseteq L$. If $\mathcal{F} \subseteq \{\mu I : \mu \in \mathbb{C}\}$, then clearly $\mathcal{F} \subseteq \{\nu I : \nu \text{tr}\, C \in L\}$. Let $\mathcal{F}$ contains a non-scalar matrix A. Then $W_C(A)$ must be a non-degenerate line segment contained in L. Therefore C, A are normal with collinear eigenvalues in $\mathbb{C}$, see [10, (7.3)]. There exist complex units $\alpha, \gamma \in \mathbb{C}$ such that $\gamma(C - (\text{tr}\, C)I/n)$ and $\alpha(A - (\text{tr}\, A)I/n)$ are Hermitian. If $B \in \mathcal{F}$ is a scalar matrix, then $\alpha(B - (\text{tr}\, B)I/n) = 0$ which is Hermitian. Now assume $B \in \mathcal{F}$ is non-scalar. Let $\tilde{C} = \gamma(C - (\text{tr}\, C)I/n)$. Then

$$W_{\tilde{C}}(\alpha(B - (\text{tr}\, B)I/n)) = \gamma\alpha W_C(B) + \mu_B \subseteq \{\gamma\alpha z + \mu_B : z \in L\},$$

for some constant $\mu_B \in \mathbb{C}$, and

$$W_{\tilde{C}}(\alpha(A - (\operatorname{tr} A)I/n)) = \gamma\alpha W_C(A) + \mu_A \subseteq \{\gamma\alpha z + \mu_A : z \in L\},$$

for some constant $\mu_A \in \mathbb{C}$. Hence $W_{\tilde{C}}(\alpha(B - (\operatorname{tr} B)I/n))$ is a subset of a line segment parallel to $W_{\tilde{C}}(\alpha(A - (\operatorname{tr} A)I/n)) \subseteq \mathbb{R}$. As $0 \in W_{\tilde{C}}(\alpha(B - (\operatorname{tr} B)I/n))$, we have $W_{\tilde{C}}(\alpha(B - (\operatorname{tr} B)I/n)) \subseteq \mathbb{R}$. Therefore, $\alpha(B - (\operatorname{tr} B)I/n)$ is Hermitian. The last assertion follows from $\{(\operatorname{tr} C)(\operatorname{tr} A)/n : A \in \mathcal{F}\} \subseteq W_C(\mathcal{F}) \subseteq L$. The sufficiency can be verified readily.

(c) Suppose $W_C(\mathcal{F}) = \operatorname{conv}\{v_1, \ldots, v_m\}$ is a convex polygon. Then for every v_j, there is $A_j \in \mathcal{F}$ such that $v_j = \operatorname{tr}(CU_j^* A_j U_j) \in W_C(A_j)$ for some unitary $U_j \in M_n$. Since $W_C(A_j) \subseteq W_C(\mathcal{F})$, we see that v_j is a vertex point of $W_C(A_j)$. It follows that v_j has the form $(c_1, \ldots, c_n)(a_1, \ldots, a_n)^t$, where $c_1, \ldots, c_n$ are eigenvalues of C and $a_1, \ldots, a_n$ are eigenvalues of A_j arranged in some suitable order. The converse of the assertion is clear. ■

3 Star-shapedness and Convexity

In this section, we study the star-shapedness and convexity of $W_C(\mathcal{F})$. If $\mathcal{F}$ is not connected, then $W_C(\mathcal{F})$ may not be connected so that $W_C(\mathcal{F})$ is not star-shaped or convex. One might hope that if $\mathcal{F}$ is star-shaped or convex, then $W_C(\mathcal{F})$ will inherit the properties. However, the following examples show that $W_C(\mathcal{F})$ may fail to be convex (star-shaped, resp.) even if $\mathcal{F}$ is convex (star-shaped, resp.).

Example 3.1. *Let $C = E_{11}$, $A = \operatorname{diag}(1+i, 1-i)$ and $\mathcal{F} = \operatorname{conv}\{A, -A\}$. Then*

$$W_C(\mathcal{F}) = W(\mathcal{F}) = \bigcup_{t\in[0,1]} W(tA + (1-t)(-A)) = \bigcup_{s\in[-1,1]} sW(A).$$

As $W(A) = \operatorname{conv}\{1+i, 1-i\}$, we have $W_C(\mathcal{F}) = \operatorname{conv}\{0, 1+i, 1-i\} \cup \operatorname{conv}\{0, -1-i, -1+i\}$ which is not convex, see Figure 1.

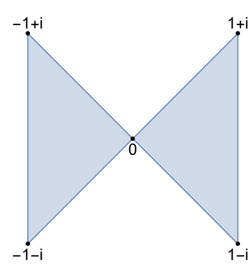

Figure 1

Example 3.2. *Let $C = E_{11}$, $A = \operatorname{diag}(1+i, 1-i)$, $\mathcal{F}_1 = \operatorname{conv}\{A, -A\}$ and $\mathcal{F}_2 = \operatorname{conv}\{A, -A+4I\}$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is star-shaped with star-center A. Since $tA + (1-t)(-A+4I) = (1-2t)(2I-A) + 2I$, we have*

$$W(\mathcal{F}_2) = \bigcup_{t\in[0,1]} W((1-2t)(-A+2I) + 2I) = \bigcup_{s\in[-1,1]} sW(-A+2I) + 2.$$

Note that $W(-A+2I) = \text{conv}\{1+i, 1-i\} = W(A)$. *Then* $W(\mathcal{F}_2) = W(\mathcal{F}_1) + 2$ *and*

$$W_C(\mathcal{F}) = W(\mathcal{F}_1 \cup \mathcal{F}_2) = W(\mathcal{F}_1) \cup W(\mathcal{F}_2)$$

equals $\text{conv}\{0, -1+i, -1-i\} \cup \text{conv}\{0, 2, 1-i, 1+i\} \cup \text{conv}\{2, 3+i, 3-i\}$ *which is not star-shaped, see Figure 2.*

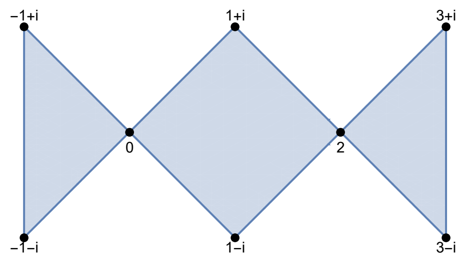

Figure 2

Notice that in Example 3.1, $\mathcal{F}$ is convex and $W_C(\mathcal{F})$ is star-shaped with star-center at the origin. One may ask if $W_C(\mathcal{F})$ is always star-shaped for a convex set $\mathcal{F}$. We will study this question in the following, and pay special attention on the case where $\mathcal{F} = \text{conv}\{A_1, \ldots, A_m\}$ for some $A_1, \ldots, A_m \in M_n$.

Denote by $S_C(A)$ the set of all star-centers of $W_C(A)$ and $\mathcal{U}_n$ the group of all $n \times n$ unitary matrices. We begin with the following result showing that $W_C(\mathcal{F})$ is star-shaped if C or $\mathcal{F}$ satisfies some special properties.

Proposition 3.3. *Suppose* $C \in M_n$ *and* $\mathcal{F}$ *is a convex matrix set.*

(a) *If* $\mathcal{F}$ *contains a scalar matrix* μI*, then* $W_C(\mathcal{F})$ *is star-shaped with* $\mu \text{tr}\, C$ *as a star center.*

(b) *Suppose the intersection of all (or any three of)* $S_C(A)$ *with* $A \in \mathcal{F}$ *is nonempty. Then* $W_C(\mathcal{F})$ *is star-shaped with* μ *as a star center for any* $\mu \in \cap\{S_C(A) : A \in \mathcal{F}\}$.

(c) *If* $\text{tr}\, C = 0$*, then* $W_C(\mathcal{F})$ *is star-shaped with 0 as a star-center.*

(d) *If all matrices in* $\mathcal{F}$ *have the same trace* ν*, then* $W_C(\mathcal{F})$ *is star-shaped with* $\nu \text{tr}\, C$ *as a star-center.*

Proof. (a) Suppose $\mathcal{F}$ contains a scalar matrix μI and $B \in \mathcal{F}$. Then

$$\text{conv}\{\mu \text{tr}\, C, W_C(B)\} \subseteq W_C(\text{conv}\{\mu I, B\}) \subseteq W_C(\mathcal{F}).$$

The result follows.

(b) Suppose $\mu \in \cap\{S_C(A) : A \in \mathcal{F}\}$. Then for any $\nu \in W_C(\mathcal{F})$, there is $B \in \mathcal{F}$ such that $\nu \in W_C(B)$. As $\mu \in S_C(B)$, the line segment joining μ and ν will lie in $W_C(B) \subseteq W_C(\mathcal{F})$. Thus, $W_C(\mathcal{F})$ is star-shaped with μ as a star center.

If $S_C(A_0) \cap S_C(A_1) \cap S_C(A_2) \neq \emptyset$ for any $A_0, A_1, A_2 \in \mathcal{F}$, then $\cap\{S_C(A) : A \in \mathcal{F}\} \neq \emptyset$ by Helly's Theorem. So, the result follows from the preceding paragraph.

(c) Note that $\frac{1}{n}(\operatorname{tr} C)(\operatorname{tr} A) \in S_C(A)$ for any $C, A \in M_n$, see [4] or Proposition 2.1 (d). If $\operatorname{tr} C = 0$, then $0 \in \bigcap\{S_C(A) : A \in \mathcal{F}\}$ is a star-center of $W_C(\mathcal{F})$ by (b).

(d) The assumption implies that $\nu \operatorname{tr} C$ is the common star-center of $W_C(A)$ for all $A \in \mathcal{F}$. Thus, the result follows from (b). ■

Lemma 3.4. *Let $C, A, B \in M_n$ and $\mathcal{F} = \operatorname{conv}\{A, B\}$. If $\mu \in S_C(A) \cap S_C(B)$, then $W_C(\mathcal{F})$ is star-shaped with star-center μ.*

Proof. Let $\zeta \in W_C(\mathcal{F})$. There are $V \in \mathcal{U}_n$ and $0 \leq t \leq 1$ such that $\zeta = \operatorname{tr}(CV^*(tA + (1-t)B)V)$. It suffices to show $\operatorname{conv}\{\mu, \operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV)\} \subseteq W_C(\mathcal{F})$. For any $a, b, c \in \mathbb{C}$, we let $\triangle(a, b, c) = \operatorname{conv}\{a, b\} \cup \operatorname{conv}\{b, c\} \cup \operatorname{conv}\{a, c\}$, i.e., the triangle (without the interior) with the vertices a, b, c. Let $U_A \in \mathcal{U}_n$ such that $\operatorname{tr}(CU_A^*AU_A) = \mu$. As $\mu \in S_C(A) \cap S_C(B)$ we have

$$\operatorname{conv}\{\operatorname{tr}(CV^*AV), \mu\} \cup \operatorname{conv}\{\operatorname{tr}(CV^*BV), \mu\} \subseteq W_C(\mathcal{F}).$$

Moreover we have

$$\operatorname{conv}\{\operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV)\} = \{\operatorname{tr}(CV^*(tA + (1-t)B)V) : 0 \leq t \leq 1\} \subseteq W_C(\mathcal{F}).$$

Hence $\triangle(\operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV), \mu) \subseteq W_C(\mathcal{F})$. We shall show that

$$\operatorname{conv}\{\operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV), \mu\} \subseteq W_C(\mathcal{F}). \tag{3.1}$$

If $\triangle(\operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV), \mu)$ is a line segment or a point, then eq. (3.1) holds clearly. Now assume that $\triangle(\operatorname{tr}(CV^*AV), \operatorname{tr}(CV^*BV), \mu)$ is non-degenerate. As $\mathcal{U}_n$ is path-connected, we define a continuous function $f : [0, 1] \to \mathcal{U}_n$ with $f(0) = V$ and $f(1) = U_A$. For $0 \leq t \leq 1$, let $V_A(t) = \operatorname{tr}(Cf(t)^*Af(t))$ and $V_B(t) = \operatorname{tr}(Cf(t)^*Bf(t))$. Note that for any $t \in [0, 1]$, we have

$$\triangle(t) = \triangle(V_A(t), V_B(t), \mu) \subseteq W_C(\mathcal{F}).$$

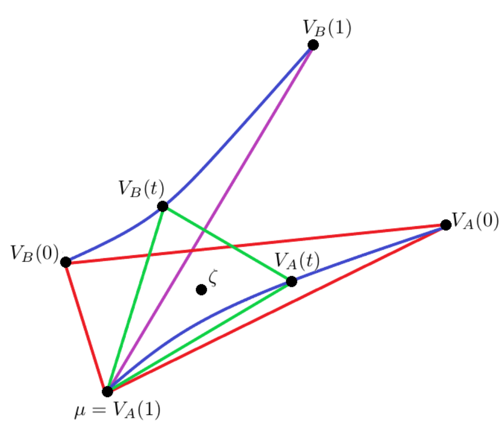

Figure 3: $\triangle(0)$ and $\triangle(t)$ are triangles in the red and green, respectively. In particular, $\triangle(1)$ dengenerates to a line segment in purple.

For any $\zeta \in \mathrm{conv}\triangle(0)$, see Figure 3, let

$$t_0 = \max\{t : \zeta \in \mathrm{conv}\triangle(s) \text{ for all } 0 \leq s \leq t\}\,.$$

Since $\triangle(1)$ degenerates, by continuity of f, we have $\zeta \in \triangle(t_0) \subseteq W_C(\mathcal{F})$. Hence the result follows. ∎

One can extend Lemma 3.4 to a more general situation.

Theorem 3.5. *Let $C \in M_n$ and $\mathcal{G}$ be a (finite or infinite) family of matrices in M_n. If $\mu \in \bigcap_{A\in\mathcal{G}} S_C(A)$, then $W_C(\mathcal{F})$ is star-shaped with star-center μ for $\mathcal{F} = \mathrm{conv}\mathcal{G}$.*

Proof. By Lemma 3.4, the result holds if $\mathcal{G}$ has 2 elements. Suppose $|\mathcal{G}| \geq 3$ and $\mu \in \bigcap_{A\in\mathcal{G}} S_c(A)$. Let $\zeta \in W_C(\mathcal{F})$. Then there exist $A_1, ..., A_m \in \mathcal{G}$, $t_1, ..., t_m > 0$ with $t_1 + \cdots + t_m = 1$ and $U \in \mathcal{U}_n$ such that $\zeta = \mathrm{tr}\,(CU^*(t_1A_1 + \cdots + t_mA_m)U)$. Let $\zeta_i = \mathrm{tr}\,(CU^*A_iU)$, $i = 1, ..., m$. Then $\zeta \in \mathrm{conv}\{\zeta_1, ..., \zeta_m\}$. The half line from μ through ζ intersects a line segment joining some ζ_i, ζ_j with $1 \leq i \leq j \leq m$ such that $\zeta \in \mathrm{conv}\{\mu, \zeta_i, \zeta_j\}$. By Lemma 3.4, we have $\mathrm{conv}\{\mu, \zeta_i, \zeta_j\} \subseteq W_C(\mathrm{conv}\{A_i, A_j\}) \subseteq W_C(\mathcal{F})$. ∎

Note that if $W_C(A)$ is convex, then $W_C(A) = S_C(A)$. So, if $W_C(A)$ is convex for every $A \in \mathcal{G}$, and if $\mu \in \cap_{A\in\mathcal{G}} W_C(A)$, then μ is a star-center of $W_C(\mathrm{conv}(\mathcal{G}))$. Checking the condition that $W_C(A)$ is convex for every $A \in \mathcal{G}$ may not be easy. On the other hand, if $C \in M_n$ is such that $W_C(A)$ is always convex, then one can skip the checking process, and we have the following.

Corollary 3.6. *Suppose $C \in M_n$ satisfies any one of the conditions* (e.1)—(e.3) *in Proposition 2.1* (e). *If $\mathcal{G} \subseteq M_n$ and $\mu \in \cap_{A\in\mathcal{G}} W_C(A)$, then μ is a star-center of $W_C(\mathrm{conv}\mathcal{G})$.*

Next, we show that if $W_C(A)$ and $W_C(B)$ are convex, then $W_C(\mathcal{F})$ is star-shaped for $\mathcal{F} = \mathrm{conv}\{A, B\}$ even when $W_C(A) \cap W_C(B) = \emptyset$.

Theorem 3.7. *Let $C \in M_n$. Suppose $A, B \in M_n$ such that $W_C(A)$ and $W_C(B)$ are convex sets with empty intersection. Let $\mathcal{F} = \mathrm{conv}\{A, B\}$.*

If $W_C(A) \cup W_C(B)$ lies on a line, then $W_C(\mathcal{F}) = \mathrm{conv}\{W_C(A), W_C(B)\}$ is convex.

Otherwise, there are two non-parallel lines L_1 and L_2 intersecting at μ such that for each $j = 1, 2$, L_j is a common supporting line of $W_C(A)$ and $W_C(B)$ separating the two convex sets (i.e., $W_C(A)$ and $W_C(B)$ lying on opposite closed half spaces determined by L_j); the set $W_C(\mathcal{F})$ is star-shaped with star-center μ, see Figure 4.

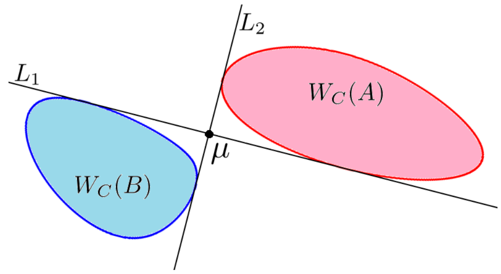

Figure 4

Proof. Assume $W_C(A) \cup W_C(B)$ lies on a line. As $\mathcal{F}$ is connected, by Theorem 2.3, $W_C(\mathcal{F})$ is connected. We have

$$W_C(A) \cup W_C(B) \quad \subseteq \quad W_C(\mathcal{F}) \subseteq \text{conv}\{W_C(A), W_C(B)\}.$$

Therefore, $W_C(\mathcal{F}) = \text{conv}\{W_C(A), W_C(B)\}$ which is convex.

Otherwise, we may assume that $W_C(A)$ lies on the left open half plane and $W_C(B)$ lies on the right open half plane. For $t \in [-\pi, \pi]$, let $L(t)$ be the support line of $W_C(A)$ with (outward pointing) normal $n(t) = (\cos t, \sin t)$. Let $P(t) = L(t) \cap W_C(A)$. Therefore, for each $t \in [-\pi, \pi]$ and $p \in P(t)$, we have $n(t) \cdot (w - p) \leq 0$ for all $w \in W_C(A)$. Let $q(t) = \min\{n(t) \cdot (w - p) : w \in W_C(B), p \in P(t)\}$. Then we have $q(0) > 0$ and $q(-\pi) = q(\pi) < 0$. Hence, there exist $-\pi < t_1 < 0 < t_2 < \pi$ such that $q(t_1) = q(t_2) = 0$. Then $L_j = L(t_j)$ for $j = 1, 2$ will satisfy the requirement.

We claim that $L(t_1)$ and $L(t_2)$ in (a) cannot be parallel. Otherwise, we must have $t_1 = t_2 + \pi$. Therefore, $n(t_1) = -n(t_2)$ and $W_C(A), W_C(B) \subset L_1 = L_2$, is a contradiction. Since L_1 and L_2 are not parallel, they intersect at a point μ.

Now, we show that μ is a star-center of $W_C(\mathcal{F})$. We may apply a suitable affine transform to A and B, and assume that $\mu = 0$, L_1 and L_2 are the x-axis and y-axis respectively. We may further assume that $W_C(A)$ lies in the first quadrant and $W_C(B)$ lies in the third quadrant. For any $\zeta \in W_C(\mathcal{F})$, there are $V \in \mathcal{U}_n$ and $0 \leq t \leq 1$ such that $\zeta = \text{tr}\,(CV^*(tA + (1-t)B)V)$. Denote $\zeta_A = \text{tr}\,(CV^*AV)$ and $\zeta_B = \text{tr}\,(CV^*BV)$. We claim that $\text{conv}\{\zeta_A, \zeta_B, 0\} \subseteq W_C(\mathcal{F})$. Once the claim holds, we have $\{s\zeta + (1-s) \cdot 0 : 0 \leq s \leq 1\} \subseteq \text{conv}\{\zeta_A, \zeta_B, 0\} \subseteq W_C(\mathcal{F})$. Then the star-shapedness of $W_C(\mathcal{F})$ follows.

We now show the claim. We denote by $\overline{\zeta_1\zeta_2}$ the line segment with end points $\zeta_1, \zeta_2 \in \mathbb{C}$. By symmetry, we may assume that the line segment $\overline{\zeta_A\zeta_B}$ intersects the y-axis at $(0, b)$ with $b \geq 0$. The situation is depicted in Figure 5a and Figure 5b.

We shall first show that $\text{conv}\{\zeta_B, bi, 0\} \subseteq W_C(\mathcal{F})$. Let y_B be a point in the intersection of $W_C(B)$ and the y-axis, $U_B \in \mathcal{U}_n$ be such that $y_B = \text{tr}\,(CU_B^*BU_B)$. Let $y_A = \text{tr}\,(CU_B^*AU_B)$. Then by the convexity of $W_C(A)$, $W_C(B)$ and $\mathcal{F}$, we have

$$Q((\zeta_A, y_A); (\zeta_B, y_B)) := \overline{\zeta_A y_A} \cup \overline{\zeta_B y_B} \cup \overline{\zeta_A \zeta_B} \cup \overline{y_A y_B} \subseteq W_C(\mathcal{F}).$$

(a) $Q((\zeta_A, y_A); (\zeta_B, y_B))$ is a quadrilateral.

(b) $Q((\zeta_A, y_A); (\zeta_B, y_B))$ is a union of two triangle.

Figure 5

Note that $Q((\zeta_A, y_A); (\zeta_B, y_B))$ is either a quadrilateral or a union of two triangle, see Figure 5a and Figure 5b. In both case, $\triangle(\zeta_B, bi, 0)$ lies inside the region determined by the closed curve $Q((\zeta_A, y_A); (\zeta_B, y_B))$. Now consider a continuous function $f : [0,1] \to \mathcal{U}_n$ with $f(0) = V$ and $f(1) = U_B$. Then for $t \in [0,1]$, let $\zeta'_A(t) = \operatorname{tr}(Cf(t)^* Af(t))$ and $\zeta'_B(t) = \operatorname{tr}(Cf(t)^* Bf(t))$, we have

$$Q((\zeta'_A(t), y_A); (\zeta'_B(t), y_B)) := \overline{\zeta'_A(t) y_A} \cup \overline{\zeta'_B(t) y_B} \cup \overline{\zeta'_A(t)\zeta'_B(t)} \cup \overline{y_A y_B} \subseteq W_C(\mathcal{F}).$$

Note that $Q((\zeta'_A(1), y_A); (\zeta'_B(1), y_B))$ degenerates, by continuity of f, for any point ζ enclosed by the closed curve $Q((\zeta_A, y_A); (\zeta_B, y_B))$, there exists $0 \le t_0 \le 1$ such that $\zeta \in Q((\zeta'_A(t_0), y_A); (\zeta'_B(t_0), y_B)) \subseteq W_C(\mathcal{F})$. Hence, $\operatorname{conv}\{\zeta_B, bi, 0\} \subseteq W_C(\mathcal{F})$. By symmetry, one can show that $\operatorname{conv}\{\zeta_A, a, 0\} \subseteq W_C(\mathcal{F})$ where a is the intersection point of $\overline{\zeta_A\zeta_B}$ and x-axis. Hence, $\operatorname{conv}\{\zeta_A, \zeta_B, 0\} = \operatorname{conv}\{\zeta_A, a, 0\} \cup \operatorname{conv}\{\zeta_B, bi, 0\} \subseteq W_C(\mathcal{F})$. The claim follows. ∎

4 Additional results on star-shapedness and convexity

It is not easy to extend Theorem 3.7. The following example shows that if $W_C(A)$ and $W_C(B)$ are not convex, then $W_C(\mathcal{F})$ may not be star-shaped for $\mathcal{F} = \operatorname{conv}\{A, B\}$ even when $W_C(A) \cap W_C(B) \neq \emptyset$.

Example 4.1. *Let $w = e^{2\pi i/3}$ and $C = \operatorname{diag}(1, w, w^2)$. Suppose $A = C - \frac{1}{6}I$, $B = e^{\pi i/3}C + \frac{1}{6}I$ and $\mathcal{F} = \operatorname{conv}\{A, B\}$. Then $W_{C+I}(\mathcal{F})$ is not star-shaped.*

To prove our claim, for $0 \le t \le 1$, let

$$A(t) = tA + (1-t)B = (t + (1-t)e^{\pi i/3})C + \frac{1-2t}{6}I\,.$$

$$\begin{aligned} W_{C+I}(A(t)) &= W_{C+I}\left((t + (1-t)e^{\pi i/3})C + \frac{1-2t}{6}I\right) \\ &= (t + (1-t)e^{\pi i/3})W_C(C) + \left(t + (1-t)e^{\pi i/3} + \frac{1-2t}{6}\right)\operatorname{tr} C + \frac{1-2t}{2} \\ &= (t + (1-t)e^{\pi i/3})W_C(C) + \frac{1-2t}{2}. \end{aligned}$$

Therefore,

$$W_{C+I}(\mathcal{F}) = \bigcup\{(t + (1-t)e^{\pi i/3})W_C\,(C) + \frac{1-2t}{2} : 0 \le t \le 1\}\,.$$

By [14], $W_C\,(C)$ is star-shaped with the origin as the unique star-center. Moreover $W_C(C) = e^{2\pi i/3} W_C(C)$ and its boundary is given by

$$\Gamma = \{2e^{i\theta} + e^{-2i\theta} : -\pi \le \theta \le \pi\}.$$

Hence, $W_{C+I}(A(t))$ is star-shaped with $\frac{1-2t}{2}$ as the unique star-center and its boundary is given by

$$\Gamma(t) = \{(t + (1-t)e^{\pi i/3})\left(2e^{i\theta} + e^{-2i\theta}\right) : -\pi \le \theta \le \pi\}. \tag{4.2}$$

For $0 \le t \le 1$ and $-\pi \le \theta \le \pi$, define

$$\begin{aligned} f(\theta, t) &= (t + (1-t)e^{\pi i/3})(2e^{i\theta} + e^{-2i\theta}) + \frac{1-2t}{2} \\ &= \frac{1+t}{2}(2\cos\theta + \cos 2\theta) - \frac{1-t}{2}\sqrt{3}(2\sin\theta - \sin 2\theta) + \frac{1-2t}{2} \\ &\quad + i\left(\frac{1-t}{2}\sqrt{3}(2\cos\theta + \cos 2\theta) + \frac{1+t}{2}(2\sin\theta - \sin 2\theta)\right). \end{aligned}$$

Using this description, we can plot $W_{C+I}(\mathcal{F})$ as follows.

First consider the plot of $\Gamma(0)$ and $\Gamma(1)$, which are the boundaries of $W_{C+I}(B)$ and $W_{C+I}(A)$ respectively, see Figure 6.

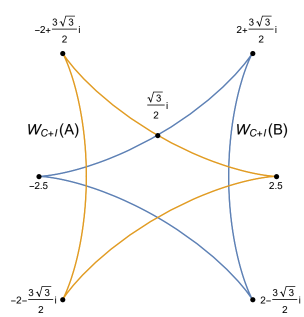

Figure 6: $W_{C+I}(A) \cup W_{C+I}(B)$

The "vertices" of $\Gamma(0)$ and $\Gamma(1)$ are given by

$$\begin{aligned} f\left(-\tfrac{2\pi}{3}, 0\right) &= 2 - \tfrac{3\sqrt{3}}{2}i \quad & f(0,0) &= 2 + \tfrac{3\sqrt{3}}{2}i \quad & f\left(\tfrac{2\pi}{3}, 0\right) &= -\tfrac{5}{2} \\ f\left(-\tfrac{2\pi}{3}, 1\right) &= -2 - \tfrac{3\sqrt{3}}{2}i \quad & f(0,1) &= \tfrac{5}{2} \quad & f\left(\tfrac{2\pi}{3}, 1\right) &= -2 + \tfrac{3\sqrt{3}}{2}i\,. \end{aligned}$$

As t increases from 0 to 1, $\Gamma(t)$ changes from $W_{C+I}(B)$ to $W_{C+I}(A)$, For $\theta = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}$, the vertex $f(\theta, t)$ moves along the line segment from $f(\theta, 0)$ to $f(\theta, 1)$, see Figure 7.

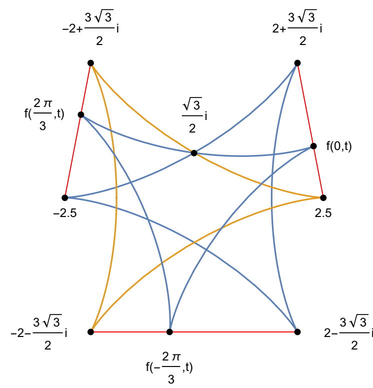

Figure 7: $W_{C+I}(\mathcal{F})$

Therefore, we see that $W_{C+I}(\mathcal{F})$ is the union of $W_{C+I}(A)$, $W_{C+I}(B)$ and three triangles, $\triangle\left(0.5, 2.5, 2+\frac{3\sqrt{3}}{2}i\right)$, $\triangle\left(-0.5, -2.5, -2+\frac{3\sqrt{3}}{2}i\right)$, $\triangle\left(\frac{\sqrt{3}}{2}i, 2-\frac{3\sqrt{3}}{2}i, -2-\frac{3\sqrt{3}}{2}i\right)$, see Figure 8.

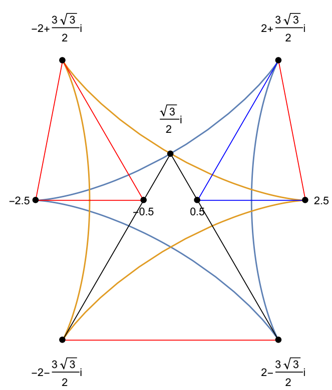

Figure 8: $W_{C+I}(\mathcal{F})$

Suppose $W_{C+I}(\mathcal{F})$ is star-shaped. By symmetry, $W_{C+I}(\mathcal{F})$ has a star-center c on the imaginary axis. The line segment joining c and $f(0,0)$ must lie below the tangent line to $\Gamma(0)$ at $f(0,0) = 2+\frac{3\sqrt{3}}{2}i$. By direct calculation, this tangent line is the line through 0.5 and $f(0,0)$. Similarly, The line segment joining c and $f(1,0)$ must lie above the tangent line to $\Gamma(1)$ at $f(1,0) = 2.5$. By direct calculation, this tangent line is the line through 0.5 and $f(1,0)$. Since these two tangent lines intersect at 0.5, no c on the imaginary axis will satisfy the above conditions.

In the following, we present an example of $C, A_1, A_2, A_3 \in M_n$ such that $W_C(A_1)$, $W_C(A_2)$, $W_C(A_3)$ are convex and $W_C(A_1) \cap W_C(A_2) \cap W_C(A_3) = \emptyset$, but $W_C(\mathcal{F})$ is not star-shaped for $\mathcal{F} = \text{conv}\{A_1, A_2, A_3\}$. In particular, we choose $C = E_{11}$ so that $W_C(\mathcal{F}) = W(\mathcal{F})$.

To describe our example, we first show that the set $W(\mathcal{F})$ is connected to the concept of product numerical range arising in the study of quantum information theory, [15].

Let $A \in M_m \otimes M_n = M_m(M_n)$, the product numerical range of A is

$$W^{\otimes}(A) = \{(u \otimes v)^* A (u \otimes v) : u \in \mathbb{C}^m,\ v \in \mathbb{C}^n\}.$$

We note that $W^{\otimes}(A)$ depends on order of the factors M_m and M_n in the representation of $M_{mn} = M_m \otimes M_n$.

Theorem 4.2. *Let $A_1, \ldots, A_m \in M_n$ and $\mathcal{F} = \mathrm{conv}(\{A_1, \ldots, A_m\})$. Then*

$$W(\mathcal{F}) = W^{\otimes}(\oplus_{i=1}^m A_i).$$

Proof. Let $\mu \in W(\mathcal{F})$ Then there exist $t_1, \ldots, t_m \geq 0$ with $\sum_{i=1}^m t_i = 1$ and $v \in \mathbb{C}^n$ such that $\mu = v^* \left(\sum_{i=1}^m t_i A_i\right) v$. Let $u = (\sqrt{t_1}, \ldots, \sqrt{t_m})^t$. Then $\mu = (u \otimes v)^* (\oplus_{i=1}^m A_i)(u \otimes v) \in W^{\otimes}(\oplus_{i=1}^m A_i)$.

Conversely, suppose $\mu \in W^{\otimes}(\oplus_{i=1}^m A_i)$. Then there exist unit vectors $u \in \mathbb{C}^m,\ v \in \mathbb{C}^n$ such that $\mu = (u \otimes v)^* (\oplus_{i=1}^m A_i)(u \otimes v)$. Let $u = (u_1, \ldots, u_m)^t$. Set $t_i = |u_i|^2$ for $i = 1, \ldots, m$. Then $\mu = v^* \left(\sum_{i=1}^m t_i A_i\right) v \in W(\mathcal{F})$. ■

Now, we describe the follow example showing that $W(\mathcal{F})$ is not necessarily star-shaped for $\mathcal{F} = \mathrm{conv}\{A_1, A_2, A_3\}$ if $W(A_1) \cap W(A_2) \cap W(A_3) = \emptyset$.

Example 4.3. Suppose $A = \mathrm{diag}\left(e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}, 0.95 e^{i\frac{\pi}{4}}\right)$. Let $A_1 = e^{i\frac{\pi}{3}} A,\ A_2 = e^{-i\frac{\pi}{3}} A$ and $A_3 = 0.95 e^{i\frac{\pi}{4}} A$. Then by Theorem 4.2, $W(\mathrm{conv}\{A_1, A_2, A_3\})$ is equal to

$$W^{\otimes}(\oplus_{i=1}^m A_i) = W^{\otimes}(A \otimes A) = W(A) \cdot W(A) = \{\mu_1 \mu_2 : \mu_1, \mu_2 \in W(A)\}.$$

By the result in [11, Example 3.1], $W(A) \cdot W(A)$ is not star-shaped.

We note that in this example, $W(A_i) \cap W(A_j) \neq \emptyset$ for all i, j, but $W(A_1) \cap W(A_2) \cap W(A_3) = \emptyset$. Therefore, in some sense, the condition in Theorem 3.5 is optimal.

While Theorem 3.5 and Theorem 3.7 provide some sufficient conditions for the star-shapedness of $W_C(\mathcal{F})$, it is a challenging problem to determine whether $W_C(\mathcal{F})$ is star-shaped or not for a given $C \in M_n$ and $\mathcal{F} \subseteq M_n$.

Proposition 4.4. *Suppose $\mathcal{F} = \mathrm{conv}\{A_1, \ldots, A_m\} \subseteq M_n$. For each unit vector $x \in \mathbb{C}^n$, let $\mathcal{A}_x = \mathrm{diag}(x^* A_1 x, \ldots, x^* A_m x)$, and let $\hat{\mathcal{F}} = \{\mathcal{A}_x : x \in \mathbb{C}^n,\ x^* x = 1\}$. Then*

$$W(\mathcal{F}) = W(\hat{\mathcal{F}}) = W(\mathrm{conv}\hat{\mathcal{F}}).$$

Proof. Let $\mu \in W(\mathcal{F})$. Then there exist $t_1, \ldots, t_m \geq 0$, $t_1 + \cdots + t_m = 1$ and a unit vector $x \in \mathbb{C}^n$ such that $\mu = x^*(t_1 A_1 + \cdots + t_m A_m)x = t_1 x^* A_1 x + \cdots + t_m x^* A_m x \in W(\hat{\mathcal{F}})$. Therefore, $W(\mathcal{F}) \subseteq W(\hat{\mathcal{F}})$.

Clearly, $W(\hat{\mathcal{F}}) \subseteq W(\text{conv}\hat{\mathcal{F}})$. Finally, we show that $W(\text{conv}\hat{\mathcal{F}}) \subseteq W(\mathcal{F})$. To see this, let $\mu \in W(\text{conv}\hat{\mathcal{F}})$, i.e.,

$$\mu = y^* \left(\sum_{j=1}^{r} t_j D_j \right) y$$

for a unit vector $y = (y_1, \ldots, y_m)^t$, some $D_j = \text{diag}(x_j^* A_1 x_j, \ldots, x_j^* A_m x_j) \in \hat{\mathcal{F}}$ with $j = 1, \ldots, r$ and $t_1, \ldots, t_r > 0,\ t_1 + \cdots + t_r = 1$. Thus,

$$\begin{aligned} \mu &= \sum_{\ell=1}^{m} |y_\ell|^2 \sum_{j=1}^{r} t_j x_j^* A_\ell x_j = \sum_{j=1}^{r} t_j \left(x_j^* \left(\sum_{\ell=1}^{m} |y_\ell|^2 A_\ell \right) x_j \right) \\ &\in \text{conv}(W(A_0)) = W(A_0) \subseteq W(\mathcal{F}) \end{aligned}$$

where $A_0 = \sum_{\ell=1}^{m} |y_\ell|^2 A_\ell \in \mathcal{F}$. ■

Example 4.5. Suppose that $A_1 = \text{diag}\left(1, e^{i\frac{\pi}{3}}\right)$, $A_2 = \text{diag}\left(e^{i\frac{2\pi}{3}}, e^{i\pi}\right)$, $A_2 = \text{diag}\left(e^{i\frac{4\pi}{3}}, e^{i\frac{5\pi}{3}}\right)$ and $\mathcal{F} = \text{conv}\{A_1, A_2, A_3\}$. Then $W(A_1) \cap W(A_2) \cap W(A_3) = \emptyset$. So the condition in Theorem 3.5 is not satisfied. However, by direct computation, we have

$$\hat{\mathcal{F}} = \left\{ \text{diag}\left(t + (1-t)e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}(t + (1-t)e^{i\frac{\pi}{3}}), e^{i\frac{4\pi}{3}}(t + (1-t)e^{i\frac{\pi}{3}}) \right) : 0 \le t \le 1 \right\},$$

where $\hat{\mathcal{F}}$ is defined as in Proposition 4.4. Therefore, $0 \in \cap_{A \in \hat{\mathcal{F}}} W(A)$. Hence, by Proposition 4.4 and Theorem 3.5, $W(\mathcal{F}) = W(\hat{\mathcal{F}})$ is star-shaped.

Recall that $W_C(\mathcal{F})$ may fail to be convex even if $\mathcal{F}$ is convex and $W_C(A)$ is convex for all $A \in \mathcal{F}$. In the following, we give a necessary and sufficient condition for $W_C(\mathcal{F})$ to be convex. First, it is easy to see that $W_C(A) \subseteq W_C(\mathcal{F})$ for every $A \in \mathcal{F}$. Thus, $\text{conv}\{\cup_{A \in \mathcal{F}} W_C(A)\}$ is the smallest convex set containing $W_C(\mathcal{F})$. As a result, we have the following observation.

Proposition 4.6. *Let $C \in M_n$ and $\mathcal{F} \subseteq M_n$. Then $W_C(\mathcal{F})$ is convex if and only if*

$$W_C(\mathcal{F}) = \text{conv}\left\{ \bigcup_{A \in \mathcal{F}} W_C(A) \right\}.$$

Proposition 4.7. *Suppose $C \in M_n$, $\mathcal{G} \subseteq M_n$ and $\mathcal{F} = \text{conv}\mathcal{G}$ with $|\mathcal{G}| \ge 3$. Then for every $\mu \in W_C(\mathcal{F})$, there are matrices $A_1, A_2, A_3 \in \mathcal{G}$ and a unitary matrix U such that $\mu \in \text{conv}\{\text{tr}\,(CU^*A_1U), \text{tr}\,(CU^*A_2U), \text{tr}\,(CU^*A_3U)\}$. If μ is a boundary point of $W_C(\mathcal{F})$, then there are $B_1, B_2 \in \mathcal{G}$ and a unitary V such that $\mu \in \text{conv}\{\text{tr}\,(CV^*B_1V), \text{tr}\,(CV^*B_2V)\}$.*

Proof. For any $\mu \in W_C(\mathcal{F})$, we have $\mu = t_1 \text{tr}\,(CU^*A_1U) + \cdots + t_r \text{tr}\,(CU^*A_rU)$ for some unitary matrix U and $A_1, \ldots, A_r \in \mathcal{G}$ with $t_1, \ldots, t_r > 0$ summing up to 1. So, $\mu \in \text{conv}\{\text{tr}\,(CU^*A_1U), \ldots, \text{tr}\,(CU^*A_rU)\}$. Thus, μ lies in the convex hull of no more than three of the points in $\{\text{tr}\,(CU^*A_1U), \ldots, \text{tr}\,(CU^*A_rU)\}$. In case μ is a boundary point of $W_C(\mathcal{F})$, then μ must be a boundary point of the set $\text{conv}\{\text{tr}\,(CU^*A_1U), \ldots, \text{tr}\,(CU^*A_rU)\}$. Thus, μ list in the convex hull of no more than two points in the set $\{\text{tr}\,(CU^*A_1U), \ldots, \text{tr}\,(CU^*A_rU)\}$. The assertion follows. ∎

Theorem 4.8. *Suppose $C \in M_n$, $\mathcal{G} \subseteq M_n$ is compact and $\mathcal{F} = \text{conv}(\mathcal{G})$. The following are equivalent.*

(a) *$W_C(\mathcal{F})$ is convex.*

(b) *$W_C(\mathcal{F})$ is simply connected and every boundary point $\mu \in \text{conv}W_C(\mathcal{F})$ has the form $\text{tr}\,(CU^*(tA + (1-t)B)U)$ for some unitary matrix U, $t \in [0, 1]$ and $A, B \in \mathcal{G}$.*

Proof. Suppose $W_C(\mathcal{F})$ is convex. Then it is clearly simply connected. Now, $\text{conv}W_C(\mathcal{F})$ and $W_C(\mathcal{F})$ have the same boundary. By Proposition 4.7, every boundary point of $W_C(\mathcal{F})$ is a convex combination of $\text{tr}\,(CU^*AU), \text{tr}\,(CU^*BU)$ with $A, B \in \mathcal{G}$.

Conversely, suppose (b) holds. We only need to show that each boundary point μ of $\text{conv}(W_C(\mathcal{F}))$ lies in $W_C(\mathcal{F})$, which is true by Proposition 4.7 and assumption (b). ∎

5 Extension to the joint C-numerical range

Let $A = A_1 + iA_2 \in M_n$, where $A_1, A_2 \in M_n$ are Hermitian matrices. Then $W(A)$ can be identified as the joint numerical range of (A_1, A_2) defined by

$$W(A_1, A_2) = \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\}.$$

One may consider whether our results can be extended to the joint numerical range of an m-tuple of Hermitian matrices $(A_1, \ldots, A_m)$ defined by

$$W(A_1, \ldots, A_m) = \{(x^*A_1x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

Some of the results on classical numerical range are not valid for the joint numerical range. For instance, the joint numerical range of three matrices may not be convex if $n = 2$. For $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have

$$W(A_1, A_2, A_3) = \{(a, b, c) : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\}.$$

The following is known, see [1, 12].

1. Suppose $A_1, A_2, A_3 \in M_2$ are Hermitian matrices such that $\{I_2, A_1, A_2, A_3\}$ is linearly independent. Then $W(A_1, A_2, A_3)$ is an ellipsoid without interior in $\mathbb{R}^3$.

2. If $n \geq 3$ and $A_1, A_2, A_3 \in M_n$ are Hermitian matrices, then $W(A_1, A_2, A_3)$ is convex.

3. Suppose $A_1, A_2, A_3 \in M_n$ such that $\{I_2, A_1, A_2, A_3\}$ is linearly independent. Then there is $A_4 \in M_n$ such that $W(A_1, A_2, A_3, A_4)$ is not convex.

There has been study of topological and geometrical properties of $W(A_1, \ldots, A_m)$. Researchers also consider the joint C-numerical range of $(A_1, \ldots, A_m)$ defined by

$$W_C(A_1, \ldots, A_m) = \{(\mathrm{tr}\,(CU^*A_1U), \ldots, \mathrm{tr}\,(CU^*A_mU)) : U \text{ unitary}\} \tag{5.3}$$

for a Hermitian matrix C, for example [5, 6]. In particular, it is known that $W_C(A_1, A_2, A_3)$ is convex for any Hermitian matrices $C, A_1, A_2, A_3 \in M_n$ if $n \geq 3$, see [2]. Of course, one can also consider the C-numerical range of $(A_1, \ldots, A_m)$ for general matrices $C, A_1, \ldots, A_m \in M_n$ defined as in Equation (5.3).

Denote by $\mathbf{A} = (A_1, \ldots, A_m)$ an m-tuple of matrices in M_n. Let $\mathbf{F}$ be a non-empty subset of M_n^m. We consider

$$W_C(\mathbf{F}) = \bigcup\{W_C(\mathbf{A}) : \mathbf{A} \in \mathbf{F}\}.$$

Evidently, when $\mathbf{F} = \{\mathbf{A}\}$, then $W_C(\mathbf{F}) = W_C(\mathbf{A})$.

5.1 Basic results

We begin with the following results.

Proposition 5.1. *Let $C \in M_n$ be non-scalar, and let $\mathbf{F}$ be a non-empty subset of M_n^m.*

1. *For any unitary $U, V \in M_n$, we have $W_C(\mathbf{F}) = W_{V^*CV}(U^*\mathbf{F}U)$, where*

$$U^*\mathbf{F}U = \{(U^*A_1U, \ldots, U^*A_mU) : (A_1, \ldots, A_m) \in \mathbf{F}\}.$$

2. *Let $\gamma_1, \gamma_2 \in \mathbb{C}$ with $\gamma_1 \neq 0$. If $\hat{C} = \gamma_1 C + \gamma_2 I_n$, then for any $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$,*

$$W_{\hat{C}}(\mathbf{A}) = \{\gamma_1(a_1, \ldots, a_m) + \gamma_2(\mathrm{tr}\,A_1, \ldots, \mathrm{tr}\,A_m) : (a_1, \ldots, a_m) \in W_C(\mathbf{A})\}.$$

3. *For any $T = (t_{ij}) \in M_m$ and $f = (f_1, \ldots, f_m)^t$, we can define an affine map R on $\mathbb{C}^m$ by $v \mapsto Tv + f$, and extend the affine map to M_n^m by mapping $\mathbf{A} = (A_1, \ldots, A_m)$ to $\mathbf{B} = (B_1, \ldots, B_m)$ with $B_i = \sum_{j=1}^n t_{ij}A_j + f_iI_n$. Then*

$$W_C(\mathbf{B}) = \{(b_1, \ldots, b_m) : (b_1, \ldots, b_m)^t = T(a_1, \ldots, a_n)^t + (\mathrm{tr}\,C)(f_1, \ldots, f_m)^t, (a_1, \ldots, a_m) \in W_C(\mathbf{A})\}.$$

Consequently,

$$R(W_C(\mathbf{F})) = W_C(R(\mathbf{F})).$$

4. *The linear span of $\{A_j - (\mathrm{tr}\,A_j)/n : j = 1, \ldots, m\}$ has dimension k if and only if $W_C(\mathbf{F}) \subseteq \mathbf{V} + f$ for a k-dimensional subspace $\mathbf{V} \subseteq \mathbb{C}^m$ and a vector $f \in \mathbb{C}^m$. In particular, $W_C(\mathbf{F})$ is a singleton $\{(\nu_1, \ldots, \nu_n)\}$ if and only if $\mathcal{F} = \{(\mu_1 I, \ldots, \mu_m I)\}$ with $(\mathrm{tr}\,C)(\mu_1, \ldots, \mu_m) = (\nu_1, \ldots, \nu_n)$.*

Note that if $C \in M_n$ is Hermitian, then $W_C(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ can be identified as $W_C(X_1, Y_1, X_2, Y_2, \ldots, X_m, Y_m) \subseteq \mathbb{R}^{2m}$, where $X_j = (A_j + A_j^*)/2$ and $Y_j = (A_j - A_j^*)/(2i)$. One can obtain a "real version" of Proposition 5.1 using real scalars γ_1, γ_2, real matrix T, real vector f, etc.

One can prove the following when $W_C(\mathbf{F})$ is a polyhedral set in $\mathbb{C}^m$, i.e., a convex combination of a finite set of vertices.

Proposition 5.2. *Suppose $C \in M_n$ is non-scalar, and $\mathbf{F}$ is a non-empty set of M_n^m. If $W_C(\mathbf{F})$ is polyhedral, then every vertex has the form $(\mu_1, \ldots, \mu_m)$ with $\mu_j = \operatorname{tr} V^*CVU^*A_jU$, where $(A_1, \ldots, A_m) \in \mathbf{F}$, $U, V \in M_n$ are unitary so that $V^*CV, U^*A_1U, \ldots, U^*A_mU$ are in lower triangular matrices with diagonal entries*

$$c_1, \ldots, c_n, a_1(1), \ldots, a_n(1), \ldots, a_1(m), \ldots, a_n(m)$$

and $\mu_j = \sum_{\ell=1}^n c_\ell a_\ell(j)$. Furthermore, if $c_1, \ldots, c_n$ are distinct, then

$$V^*CV, U^*A_1U, \ldots, U^*A_mU$$

are diagonal matrices.

If C has distinct eigenvalues, and if $W_C(\mathbf{A})$ has a conical point $(\mu_1, \ldots, \mu_m)$ on the boundary, i.e., there is a pointed cone $K \subseteq \mathbb{C}^m \equiv \mathbb{R}^{2m}$ with vertex $(\mu_1, \ldots, \mu_m)$ such that

$$W_C(\mathbf{A}) \cap \{(\mu_1 + \nu_1, \ldots, \mu_m + \nu_m) : |\nu_j| \le \varepsilon\} \subseteq K$$

for some sufficiently small $\varepsilon > 0$, then $\{C, A_1, \ldots, A_m\}$ is a commuting family of normal matrices, and $\operatorname{conv} W_C(\mathbf{A})$ is a polyhedral set, see [3].

Note that if there is $\mathbf{B} \in \mathbf{F}$ such that $W_C(\mathbf{B})$ is a singleton, a subset of a straight line, or a convex polygon, then we can apply the results on $W_C(\mathbf{B})$ to deduce that C and $\mathbf{B}$ has special structure. Then we can deduce results on $W_C(\mathbf{F})$.

We can obtain some topological properties $W_C(\mathbf{F})$.

Proposition 5.3. *Let $C \in M_n$ be non-scalar, and $\mathbf{F} \subseteq M_n^m$ be a nonempty set.*

1. *If $\mathcal{F}$ is bounded, then so is $W_C(\mathbf{F})$.*

2. *If $\mathcal{F}$ is connected, then so is $W_C(\mathbf{F})$.*

3. *If $\mathcal{F}$ is compact, then so is $W_C(\mathbf{F})$.*

Proof. The proof of the boundedness and compactness are similar to those for $W_C(\mathcal{F})$ in Section 2. The prove connectedness, for any $\mathbf{A}, \mathbf{B} \in \mathbf{F}$ and unitary $U_0, U_1 \in M_n$, there is a path joining U_t with $t \in [0, 1]$ joining U_0 and U_t, and hence there is a path joining $(\operatorname{tr} U_0^*CU_0A_1, \ldots, \operatorname{tr} U_0^*CU_0A_m)$ to $(\operatorname{tr} U_1^*CU_1A_1, \ldots, \operatorname{tr} U_1^*CU_1A_m)$, which is connected to $(\operatorname{tr} U_1^*CU_1B_1, \ldots, \operatorname{tr} U_1^*CU_1B_m)$. ∎

5.2 Star-shapedness and convexity

Here we consider whether $W(\mathbf{F})$ is star-shaped or convex. It is known that if $C \in M_n$ is Hermitian, then $W_C(A_1, A_2, A_3)$ is convex for any Hermitian $A_1, A_2, A_3 \in M_n$ with $n \geq 3$, see [2]. One may wonder whether Theorem 3.5 admits an extension to this setting. The following example shows that the answer is negative.

Example 5.4. *Let*

$$\mathbf{A}_t = \left(\begin{pmatrix} t & & \\ & t & \\ & & 1-t \end{pmatrix}, \begin{pmatrix} 1-t & & \\ & t & \\ & & 1-t \end{pmatrix} \begin{pmatrix} 0 & & \\ & t & \\ & & 1-t \end{pmatrix} \right), \qquad t \in [0,1].$$

Notice that $\operatorname{conv}\{\mathbf{A}_0, \mathbf{A}_1\} = \{\mathbf{A}_t : 0 \leq t \leq 1\}$*. Moreover,* $(0,0,0) \in W(\mathbf{A}_0) \cap W(\mathbf{A}_1)$*, but* $(0,0,0)$ *is not a star-center of* $W(\operatorname{conv}\{\mathbf{A}_0, \mathbf{A}_1\})$ *as* $(1/2, 1/2, 0) \in W(\mathbf{A}_{1/2})$*, however,* $\operatorname{conv}\{(0,0,0), (1/2,1/2,0)\} \not\subseteq W(\operatorname{conv}\{\mathbf{A}_0, \mathbf{A}_1\})$*. Nevertheless,* $(1/2,1/2,1/2) \in W(\mathbf{A}_t)$ *for all* $t \in [0,1]$*, and hence* $(1/2,1/2,1/2)$ *is a star-center of* $W(\operatorname{conv}\{\mathbf{A}_0, \mathbf{A}_1\})$*. Actually, one can show that* $W(\mathbf{A}_t) = \operatorname{conv}\{(t, 1-t, 0), (t,t,t), (1-t, 1-t, 1-t)\}$*.*

Here is an example showing that $W(\mathbf{A})$ may not be star-shaped in general.

Example 5.5. *Let* $\mathbf{A} = (A_1, A_2, A_3)$ *with* $A_1 = \operatorname{diag}(0,1,0)$*,* $A_2 = \operatorname{diag}(1,0,-1)$*,* $A_3 = I_3$ *and let* $\mathbf{B} = (B_1, B_2, B_3)$ *with* $B_1 = \operatorname{diag}(1,0,0)$*,* $B_2 = \operatorname{diag}(0,-1,1)$*,* $B_3 = 0_3$*. If* $\mathbf{F} = \operatorname{conv}\{\mathbf{A}, \mathbf{B}\}$*, then* $W(\mathbf{F})$ *is the union of the triangular disk with vertices*

$$(1-t, t, t), (t, t-1, t), (0, 1-2t, t).$$

Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ *be given by* $g((a,b,c)) = (a, -b, 1-c)$*. We have*

$$\begin{aligned} g((1-t,t,t)) &= (1-t, -t, 1-t) &&= (1-t, (1-t)-1, 1-t) \\ g((t,t-1,t)) &= (t, 1-t, 1-t) &&= (1-(1-t), (1-t), (1-t)) \\ g((0,1-2t,t)) &= (0, 2t-1, 1-t) &&= (0, 1-2(1-t), (1-t)) \end{aligned}$$

Therefore, $g(W(\mathbf{F})) = W(\mathbf{F})$*. Moreover, we claim that* $W(\mathbf{F})$ *is not star-shaped.*

Suppose the contrary that $W(\mathbf{F})$ is star-shaped with (a, b, t_0) be a star-center. Then $(a, -b, 1-t_0)$ is also a star-center. As the set of all star-center of a star-shaped set is convex. We may now assume without loss of generality that $b = 0$ and $t_0 = 1/2$. Assume now $a > 0$. By assumption, for all $0 \leq t \leq 1$, $t(a, 0, 1/2) + (1-t)(1,0,0) = (1 - t(1-a), 0, t/2) \in W(\mathbf{F})$. By direct computation, given $0 \leq t \leq 1$, we have

$$\max\{\alpha \in \mathbb{R} : (\alpha, 0, t) \in W(\mathbf{F})\} = t^2 + (1-t)^2 = 1 - 2t + 2t^2.$$

However for sufficient small $t > 0$

$$1 - t(1-a) > 1 - t(1 - t^2/2) > (t/2)^2 + (1 - t/2)^2,$$

which contradicts that $(1-t(1-a), 0, t/2) \in W(\mathbf{F})$. Therefore, we have $a=0$. However $(0,0,1/2)$ is not a star-center as

$$\frac{1}{3}(0,0,1/2)+\frac{2}{3}(0,0,1)=\left(0,0,\frac{5}{6}\right) \notin \operatorname{conv}\left\{\left(\frac{1}{6},\frac{5}{6},\frac{5}{6}\right),\left(\frac{5}{6},-\frac{1}{6},\frac{5}{6}\right),\left(0,-\frac{2}{3},\frac{5}{6}\right)\right\}.$$

Therefore, $W(\mathbf{F})$ is not star-shaped.

It is challenging to determine conditions on $\mathbf{F}$ so that $W(\mathbf{F})$ is star-shaped.

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