Least-element Time-Stepping Methods for Simulation of Linear Networks With Ideal Switches

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***Abstract*—Linear networks with ideal switches have various applications in power converters, signal processing and control problems, which can be modeled by linear complementarity systems (LCSs). This note presents new results of the least element time-stepping method for simulation of linear networks with ideal switches for a class of LCSs. The method is efficient and stable, and can be easily implemented. The convergence results and preliminary numerical results show that the least-element time-stepping method is efficient for verifying accuracy of approximate solutions.**

***Index Terms*—linear network, ideal switch, circuit, time-stepping method, convergence, linear complementarity system.**

I. INTRODUCTION

LINEAR networks with ideal switches are widely used in various applications from power converters to signal processing. A modeling framework, linear complementarity systems (LCSs), consists of combinations of linear time-invariant dynamical systems and complementarity conditions [2], [5], [7], [9], [10], [11], [12], [17], [18], [23], [26], [27], [28], [29], [30]. The LCSs provide a powerful mathematical paradigm for the increasing number of engineering and economics problems that involve dynamics, inequalities and complementarity conditions [1], [2], [25].

An LCS is defined as follows

$$\begin{array}{l} \dot{x}(t) = Ax(t) + By(t) + f(t) \\ 0 \le y(t) \perp w(t) = Cx(t) + Dy(t) + g(t) \ge 0 \\ x(0) = x_0, \quad t \in [0,T], \end{array} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$, $f : R \to R^n$ and $g : R \to R^m$ are Lipschitz continuous. The orthogonality condition is called the complementarity condition, which means that one of these two nonnegative components $y_i(t)$ and $(Cx(t) + Dy(t) + g(t))_i$ must be zero.

A linear complementarity problem denoted by LCP(q, M) is to find a vector $z \in R^m$ such that

$$Mz + q \ge 0, \quad z \ge 0, \quad z \perp (Mz + q) = 0$$

for a given a vector $q \in R^m$ and a matrix $M \in R^{m\times m}$. We say that the LCP(q, M) is solvable if such a z exists. The set of all solutions of LCP(q, M) is denoted by SOL(q, M).

Simulation of the LCS (1) has been studied extensively in circuit theory [1], [2], [3], [4], [8], [15], [19], [21], [22], [24]. A popular numerical method for solving the LCS is the time-stepping method, which divides the time interval $[0, T]$ into N_h subintervals

$$0 = t_{h,0} < t_{h,1} < \cdots < t_{h,N_h} = T, \; t_{h,i+1} - t_{h,i} = h = T/N_h$$

for $i = 0, \ldots, N_h - 1$, and computes two finite families of vectors

$$\{x^{h,1}, x^{h,2}, \cdots, x^{h,N_h}\} \subset R^n, \{y^{h,1}, y^{h,2}, \cdots, y^{h,N_h}\} \subset R^m$$

by the recursion with $x^{h,0} = x^0 \in R^n$:

$$\begin{array}{rcl} x^{h,i+1} &=& x^{h,i} + h\left\{A\left[\theta x^{h,i} + (1-\theta)x^{h,i+1}\right]\right. \\ && \left. +By^{h,i+1} + f(t_{h,i+1})\right\}, \\ y^{h,i+1} &\in& \text{SOL}(Cx^{h,i+1} + g(t_{h,i+1}), D), \end{array} \tag{2}$$

where $\theta \in [0, 1]$ is a scalar.

The critical part in numerical implementation of the time-stepping method is to find a "good" solution $y^{h,i+1}$ in the solution set SOL$(Cx^{h,i+1} + g(t_{h,i+1}), D)$. In many cases, the solution set contains multiple solutions. Using an arbitrary solution can cause the numerical method unstable or make the linear complementary problem unsolvable in the next step. Therefore, the choice of a solution at each time t is essential to construct convergence methods in practical use for LCS (1).

The property of convergence plays an important role in many control problems including tracking, synchronization, observer design and output regulation [8], [17], [18].

Pang and Stewart [26] used the following conditions

$$\|x^{h,i+1}\|_2 \le c_1 + c_2\|x^0\|_2, \quad \|y^{h,i+1}\|_2 \le c_3 + c_4\|x^0\|_2 \tag{3}$$

with $c_j > 0$ for $j = 1, 2, 3, 4$ to guarantee that there is a sequence $\{h_\nu\}$ such that the continuous piecewise linear interpolant of $\{x^{h_\nu,i}\}$, and the piecewise constant interpolant of $\{y^{h_\nu,i}\}$ converge to a weak solution of (1).

To ensure the convergence of method (2), some references [26] assume that condition (3) holds for all solutions in the solution sets SOL$(Cx^{h,i} + g(t_{h,i}), D)$ for all $i = 0, \ldots, N_h$. This assumption is very strong, and can not be verified.

To relax the assumption, Han, Tiwar, Camlibel and Pang [18] proposed the following method for the passive LCS of initial-value and boundary-value problems to a weak solution of the system:

IMPLICIT LEAST-NORM TIME-STEPPING METHOD (IL-NTS METHOD)

$$\begin{array}{rcl} x^{h,i+1} &=& x^{h,i} + h(Ax^{h,i+1} + By^{h,i+1} + f(t_{h,i+1})) \\ y^{h,i+1} &=& \text{argmin}\,\{\|v\|_2 \;\;|\;\; 0 \le v \;\;\perp \\ && \qquad Cx^{h,i+1} + g(t_{h,i+1}) + Dv \ge 0\}. \end{array} \tag{4}$$

This method sets $\theta = 0$ in the time-stepping method (2) and chooses the least-norm solution in the solution set of the linear complementarity problem at each step. An important contribution in [18] is to prove that the least-norm solution satisfies condition (3) under assumptions that D is positive semi-definite, the least-norm solutions exist for all $h > 0$ sufficiently small, and the system is passive.

The use of a least-norm solution in the time-stepping scheme is a constructive suggestion for solving passive LCSs. A natural question can be asked about why such a particular solution to the LCP (4) is needed. In section III, we show that choosing the least-norm solution is necessary for a passive LCS (1) in order to find a weak solution.

However, finding a least-norm solution in (4) is equivalent to finding a least-norm solution in the solution set SOL$(q^{h,i}, D^h)$, which is to solve the following quadratic program with linear complementarity constraints (QPLCC)

$$\begin{array}{ll} \text{minimize} & y^T y \\ \text{subject to} & 0 \le y \perp q^{h,i} + D^h y \ge 0, \end{array} \tag{5}$$

where

$$q^{h,i} = C(I - hA)^{-1}[x^{h,i} + hf(t_{h,i+1})] + g(t_{h,i+1})$$

and

$$D^h = D + hC(I - hA)^{-1}B.$$

The ILNTS method has many difficulties in numerical implementation and theoretical analysis. In particular, (5) is not a convex minimization problem and there is no feasible solution satisfying all inequalities strictly. The usual mathematical programming constraint qualification does not hold at any feasible solution.

In this note, we use the LCS to study a class of circuits and show existence of week solutions of the LCS and a special solution $(x^*(t), y^*(t))$ where x^* is Lipschitz continuously differentiable and y^* is Lipschitz continuous by using the least element solution. Moreover, we give a closed-form of the least element solution and error bounds of approximate solutions to the solution $(x^*(t), y^*(t))$. Using PSpice and existing numerical methods, we can not show the existence of the special solution $(x^*(t), y^*(t))$ or give error bounds of numerical solutions to the exact solution $(x^*(t), y^*(t))$. Preliminary numerical results in section V verify the convergence and accuracy of approximate solutions.

II. Preliminaries

In this note, $R^{n\times m}$ denotes the set of $n \times m$ matrices with real elements, R^n the set of $n \times 1$ vectors with real elements and e the suitable vector with all elements equal to 1. When two vectors v and w in R^n are orthogonal, i.e., $v^T w = 0$, we write $v \perp w$. The norm $\|v\|_2 = (\sum_{j=1}^n v_j^2)^{1/2}$ is the Euclidean norm and $\|v\|_\infty = \max_{1\le j\le m} |v_j|$. The transpose of A is denoted by A^T, A_{ij} for the (i, j) element of A and $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^m |A_{ij}|$. $M \in R^{m\times m}$ is called a Z-matrix if $M_{ij} \le 0$ $(i \ne j)$ for $i, j = 1, 2, \ldots, m$. M is called positive semi-definite if $x^T M x \ge 0$ for all $x \in R^m$.

Suppose that the solution set SOL(q, M) of the LCP(q, M) is nonempty. z^* is called *a least-element solution* in SOL(q, M) if for any z in SOL(q, M), $z^* \le z$ holds. z^* is called *a least-norm solution* in SOL(q, M) if for any z in SOL(q, M), $\|z^*\|_2 \le \|z\|_2$ holds.

It is well-known that SOL(q, M) contains a unique least-element solution if M is a Z-matrix and SOL(q, M) is nonempty, while SOL(q, M) is convex and contains a unique least-norm solution if M is positive semi-definite and SOL(q, M) is nonempty [14]. Obviously, a least-element solution is a least-norm solution. Additionally, it is known from [14] that if the LCP(q, M) has a least-element solution v^*, then v^* is the unique solution of

$$\begin{array}{ll} \text{minimize} & e^T v \\ \text{subject to} & 0 \le v \perp q + Mv \ge 0. \end{array} \tag{LCC}$$

Moreover, if M is a Z-matrix, then v^* is the unique solution of

$$\begin{array}{ll} \text{minimize} & e^T v \\ \text{subject to} & v \ge 0, \;\; q + Mv \ge 0. \end{array}$$

A weak solution of (1) is a pair of trajectories $(x(t), y(t))$ such that x is absolutely continuous and y is integrable on $[0, T]$ which satisfies

$$x(t) - x(s) = \int_s^t [Ax(\tau) + By(\tau) + f(\tau)]d\tau, \;\; 0 \le s \le t \le T \tag{6}$$

and

$$y(t) \in \text{SOL}(Cx(t) + g(t), D), \quad \text{almost all } t \in [0, T].$$

The passive LCS (1) was extensively studied in [9], [10], [11], [17], [20], [26]. The passive property of the LCS (1) is equivalent to that there exists a symmetric positive semidefinite matrix K such that

$$\begin{pmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{pmatrix}$$

is negative semi-definite [9], [10].

III. Necessity of the choice of the least-norm solution for ILNTS

Han, Tiwari, Camlibel and Pang [18] introduced the implicit least-norm time-stepping method (4) for the passive LCS, and showed the convergence of the method for initial-value and boundary-value problems to a weak solution of the system.

In this section, we show that choosing the least-norm solution is necessary for a passive LCS (1) in order to find a weak solution.

Let us consider the following LCS

$$\begin{array}{l} \dot{x}(t) = e^T y \\ 0 \le y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \\ \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} -c \\ 0 \end{pmatrix} \ge 0, \quad c > 0 \\ x(0) = 0. \end{array} \tag{7}$$

In this example, we see that

$$A = 0, \quad B = (1, 1), \;\; D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \;\; C = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$f(t) \equiv 0, \quad g(t) \equiv (-c, 0)^T.$$

Let $K = 1 \in R^{1\times 1}$. Then

$$\begin{pmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is negative semi-definite, which implies that the system (7) is passive [18]. The solution set of the LCP in (7) can be explicitly given as

$$\mathrm{SOL}\,(Cx(t)+g(t),D) = \begin{cases} \left\{ \begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix}, \quad \lambda_1 \geq c \right\}, & x(t)=0 \\ \begin{pmatrix} x(t) \\ c-x(t) \end{pmatrix}, & 0<x(t)<c \\ \begin{pmatrix} \frac{c}{2}(1+\mathrm{sgn}(c-x(t))) \\ 0 \end{pmatrix}, & x(t) \geq c, \end{cases} \tag{8}$$

where $\mathrm{sgn}(t) = \begin{cases} -1 & \text{if } t<0 \\ 1 & \text{if } t>0 \\ [-1,1] & \text{if } t=0 \end{cases}$ ([6]). Hence the system can be expressed as $x(0)=0$,

$$\dot{x}(t) \;=\; \begin{cases} \{\lambda_1, \quad \lambda_1 \geq c\}, & x(t)=0 \\ \frac{c}{2}\left[1+\mathrm{sgn}(c-x(t))\right], & x(t)>0. \end{cases} \tag{9}$$

Furthermore, from (8), the least-norm solution and the ODE corresponding to the least-norm solution respectively are

$$y\,(x(t)) = \begin{cases} \begin{pmatrix} x(t) \\ c-x(t) \end{pmatrix}, & 0 \leq x(t) < c \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x(t) \geq c \end{cases},$$

$$x(0)=0, \qquad \dot{x}(t) \;=\; \begin{cases} c, & 0 \leq x(t) < c \\ 0, & x(t) \geq c \end{cases}$$

with $\dot{x}(t)=c$ when $x(t)=0$ and $\dot{x}(t)=0$ when $x(t)>0$, which leads to a weak solution of (7)

$$x(t) = \begin{cases} ct, & 0 \leq t < 1 \\ c, & t \geq 1. \end{cases} \tag{10}$$

Moreover, from the definition of a weak solution (6) and the expression of $\mathrm{SOL}\,(Cx(t)+g(t),D)$, it follows that each weak solution of (9) satisfies

$$x(t)=ct,\; 0\leq t<1; \quad x(t)-x(1)=\int_1^t e^T y(\tau)d\tau = 0,\; t>1,$$

which, together with the continuity of a weak solution, derives that the above weak solution (10) is also unique.

In addition, from (9) and (10), we see that if $x(t)$ is a weak solution of (7) then

$$x(t)-x(1)=\lambda_2 t, \qquad \text{for} \quad t>1.$$

It is clear that for $\lambda_2 \in (0,c]$, the system (9) does not have a weak solution since $x(t)-x(1)=\lambda_2 t>0$ for $t>1$. However, if we choose the least-norm solution $\dot{x}(t)=c$ when $x(t)=0$ and $\dot{x}(t)=0$ when $x(t)>0$, the system (9) has a weak solution.

Hence the choice of the least-norm solution is necessary for defining a weak solution of (7).

IV. A Least-element time-stepping method for LCS

As mentioned in section I, solving

$$y^{h,i+1} \;\in\; \operatorname{argmin} \; \|y\|^2 \quad \text{subject to } y \in \mathrm{SOL}(q^{h,i},D^h)$$

by ILNTS method (4) may suffer from difficulty in numerical implementation and theoretical analysis.

In [13], a least-element time-stepping method was studied for D being a Z-matrix. Here we extend the least-element time-stepping method for D possibly not a Z-matrix with an assumption that $\mathrm{LCP}(q,D)$ has a least-element solution, which is efficient for solving linear networks with particular patterns: Least-Element LCS

$$\begin{aligned} &\dot{x}(t)=Ax(t)+By(x(t))+f(t) \\ &y(x(t))=\operatorname{argmin}\{e^T v \quad | \quad 0\leq v \quad \perp \\ &\qquad\qquad Cx(t)+g(t)+Dv \geq 0\} \\ &x(0)=x_0, \quad t\in[0,T], \end{aligned} \tag{11}$$

and the corresponding Implicit Least-Element Time-Stepping Method (ILETS method)

$$\begin{aligned} x^{h,i+1} &= x^{h,i}+h(Ax^{h,i+1}+By^{h,i+1}+f(t_{h,i+1})) \\ y^{h,i+1} &= \operatorname{argmin}\{e^T v \quad | \quad 0\leq v \\ &\qquad \perp Cx^{h,i+1}+g(t_{h,i+1})+Dv \geq 0\}. \end{aligned} \tag{12}$$

Many linear networks with ideal switches [1], [7], [8], [17], [18], [19], [20], [30] can be reformulated by LCS (1) as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = A\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + B\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + Eu(t)+h(t)$$

$$0 \leq \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \perp D\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \geq 0, \tag{13}$$

where $(x_1(t),x_2(t))^T \in R^n, y_i(t),w_i(t),q_i(t) \in R^m$, $A \in R^{n\times n}$, $B\in R^{n\times 2m}$, $C \in R^{n\times 2m}$, $E\in R^{n\times n}$, $I \in R^{m\times m}$ is the identity matrix,

$$D=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = C\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + g(t),$$

$u,h : [0,T] \to R^n$ and $g : [0,T] \to R^{2m}$ are Lipschitz continuous.

It is easy to verify that the LCP in (13) is solvable if and only if $q_2(t)\geq 0$. In the case that (13) is solvable, the least-element solution of the LCP in (13) can be given as follows: for components $i=1,2,\ldots,m$

$$\begin{pmatrix} (y_1(t))_i \\ (y_2(t))_i \end{pmatrix} = \begin{cases} (0,0)^T, & (q_1(t))_i \geq 0 \\ ((q_2(t))_i, -(q_1(t))_i)^T, & (q_1(t))_i < 0. \end{cases} \tag{14}$$

We also denote the solution (14) by $y(q(t))$.

Then using the form of the least-element solutions (14), ILETS method (12) can be easily implemented for LCS (13). Moreover, from (14) we see that the least-element solution function is Lipschitz continuous

$$\|y(q(t))-y(p(t))\|_\infty \leq \|q(t)-p(t)\|_\infty, \tag{15}$$

which, together with (11), (15) and the Lipschitz continuity of g, yields that the right-side of (13)

$$\dot{x}(t)=Ax(t)+By(x(t))+Eu(t)+h(t) \tag{16}$$

is Lipschitz continuous with respect to x.

Based on the Lipschitz conditions (15) and (16), we can establish the existence and uniqueness of solution of the system (11) and error bound of the ILETS method (12). See [13] for a proof.

Theorem *Suppose that for (13) there is $v \in R^{2m}$ such that*

$$Cx(0) + g(0) + Dv > 0, \qquad v > 0, \tag{17}$$

then the following statements hold.

(i) There are constants $T > 0$ and $\gamma > 0$ such that for $t \in [0, T]$ and $z \in \mathcal{B}(x_0, \gamma) = \{z : \|z - x_0\|_2 \le \gamma\}$

$$z \ge 0, \quad Cz + g(t) \ge 0. \tag{18}$$

(ii) The least-element LCS (11) has a unique solution $(x^*, y^*) \in C^1[0, T_0] \times C[0, T_0]$, where $T_0 = \min\{T, \dfrac{\gamma}{c_0 + \kappa\gamma}\}$, $c_0 = \|Ax_0 + By_0 + f\|_\infty$, $\kappa = \|A\|_\infty + \|B\|_\infty \|N\|_\infty$.

(iii) For the implicity least-element time-stepping method (12), we have the error bound

$$\|x^{h,i} - x(t_{h,i})\|_\infty \le O(h). \tag{19}$$

The ILETS (12) can also be solved by the following generalized Newton method to find $x^{h,i+1}$ at each step. Newton-type methods are popular for solving nonsmooth equation $F(u) = 0$, which is of superlinear convergence from any starting point in a small neighborhood of the solution if the generalized Jacobian $\partial F(u)$ is well-defined.

Define the function $H : R^n \to R^n$ by

$$\begin{array}{l} H(u) = x^{h,i} + h[Au + By(u) + f(t_{h,i+1})] \\ y(u) = \operatorname{argmin}\{ e^T v \quad | \quad v \in \mathrm{SOL}(q(u), D)\}, \end{array} \tag{20}$$

where

$$q(u) := Cu + g(t_{h,i+1}).$$

The generalized Newton method to calculate $x^{h,i+1}$ is defined as follows:

$$u^{k+1} = u^k - V_k^{-1} F(u^k), \quad u^0 = x^{h,i}, \tag{21}$$

where $F(u) = u - H(u)$ and

$$V_k = I - h[A - B(I - \Lambda_k + \Lambda_k D)^{-1} \Lambda_k C].$$

Here $\Lambda_k = \Lambda_{q(u^k)}$ is a $2m \times 2m$ diagonal matrix with

$$(\Lambda_k)_{ii} = \mathrm{sgn}(\max(0, y_i(u^k))) \tag{22}$$

From the definition of D and (22), we see that $I - \Lambda_k + \Lambda_k D$ is nonsingular since it has exact $2m$ nonzero entries with absolute values all ones on the different rows and columns. Then V_k^{-1} exists for sufficiently small step size h. Note that V_k belongs to the generalized Jacobian $\partial F(u^k)$ of F, that is, $V_k \in \partial F(u^k)$. In a similar proof to [13], the generalized Newton method (20) with the starting point $x^{h,i}$ converges to $x^{h,i+1}$ superlinearly if the step length h is sufficiently small.

V. Numerical examples

All the numerical results in this section are carried out by the proposed ILETS method by using Matlab R2012a on a desktop (2.8 GB RAM, 2 Core2 (32 bit) processors at 2.80 GHz) with Windows XP operating system.

To show the reliability of the least-element LCS approach (11) and the efficiency of the ILETS method (12), we consider the circuit system in [18] (see Fig. 1)

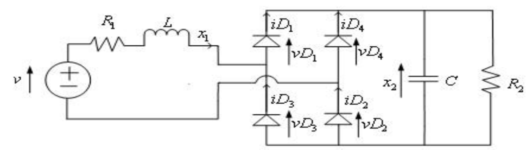

Fig. 1. The circuit system in [18].

By extracting the ideal diodes and using Kirchoff laws, this system can be reformulated by LCS (1) as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2 C} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -\frac{1}{L} & \frac{1}{L} & 0 & 0 \\ 0 & 0 & \frac{1}{C} & \frac{1}{C} \end{pmatrix} \begin{pmatrix} v_{D_3} \\ v_{D_2} \\ i_{D_1} \\ i_{D_4} \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} v$$

$$0 \le \begin{pmatrix} v_{D_3} \\ v_{D_2} \\ i_{D_1} \\ i_{D_4} \end{pmatrix} \perp \begin{pmatrix} i_{D_3} \\ i_{D_2} \\ v_{D_1} \\ v_{D_4} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{D_3} \\ v_{D_2} \\ i_{D_1} \\ i_{D_4} \end{pmatrix} \ge 0.$$

Here x_1 is the current through the inductor L, x_2 is the voltage across the capacitor C, (v_{D_i}, i_{D_i}) is the voltage-current pair associated to the ith diode, and v is a DC voltage source.

The corresponding least-element solution of the LCP is

$$\begin{pmatrix} v_{D_3} \\ v_{D_2} \\ i_{D_1} \\ i_{D_4} \end{pmatrix} = \begin{cases} (x_2(t), 0, x_1(t), 0)^T, & x_2(t) \ge 0, x_1(t) > 0 \\ (0, 0, 0, 0)^T, & x_2(t) \ge 0, x_1(t) = 0 \\ (0, x_2(t), 0, -x_1(t))^T, & x_2(t) \ge 0, x_1(t) < 0. \end{cases}$$

The corresponding least-element solution ODE system is

$$\begin{cases} \begin{cases} \dot{x_1}(t) = -\dfrac{R_1}{L} x_1(t) - \dfrac{1}{L} x_2(t) + \dfrac{v}{L}, \\ \dot{x_2}(t) = \dfrac{1}{C} x_1(t) - \dfrac{1}{R_2 C} x_2(t), \end{cases} & x_2(t) \ge 0, x_1(t) > 0; \\ \begin{cases} \dot{x_1}(t) = -\dfrac{R_1}{L} x_1(t) + \dfrac{v}{L}, \\ \dot{x_2}(t) = -\dfrac{1}{R_2 C} x_2(t), \end{cases} & x_2(t) \ge 0, \quad x_1(t) = 0; \\ \begin{cases} \dot{x_1}(t) = -\dfrac{R_1}{L} x_1(t) + \dfrac{1}{L} x_2(t) + \dfrac{v}{L}, \\ \dot{x_2}(t) = -\dfrac{1}{C} x_1(t) - \dfrac{1}{R_2 C} x_2(t). \end{cases} & x_2(t) \ge 0, x_1(t) < 0. \end{cases}$$

Now, we consider the implementation of ILETS (12) together with (14). Let $e = 5$V, $R_1 = 0.1\Omega$, $L = 0.2$mH, $R_2 = 20\Omega$

and $C = 40\mu$F. To compare the ILETS method with PSpice, we also simulate the system in Fig. 1 by PSpice. Fig. 2, Fig. 3 and Table 1 illustrate the ILETS method for the LCS with initial $x(0) = (1, 2)^T$, compared with the PSpice simulation results. In Table 1 the differences are calculated by

$$\Delta x_j = \max\{|x_j^{h,i} - x_j^{PS}(t_{h,i})|\}_{i=1}^{N_h},\ j = 1, 2,$$

where $x^{PS}(t)$ denotes the PSpice simulation solution with step size $h = 0.1\mu s$. We can see from Table 1 and Fig. 3 that the solutions generated from the ILETS method are close to the solutions of PSpice. For the same step size, the ILETS method requires less CPU time than PSpice. It is interesting to see that the difference between $x^{h,i}$ and x^{PS} becomes smaller when the step size h decreases.

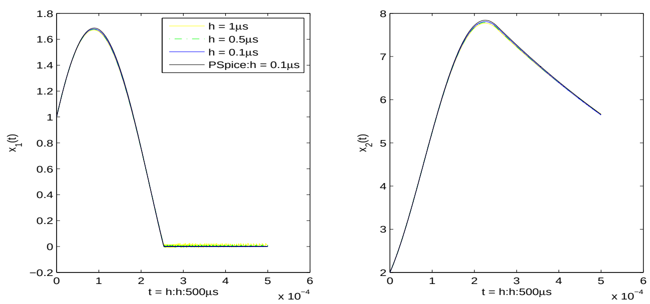

Fig. 2. Approximations by the ILETS (12) together with (14) for the LCS derived from Fig. 1.

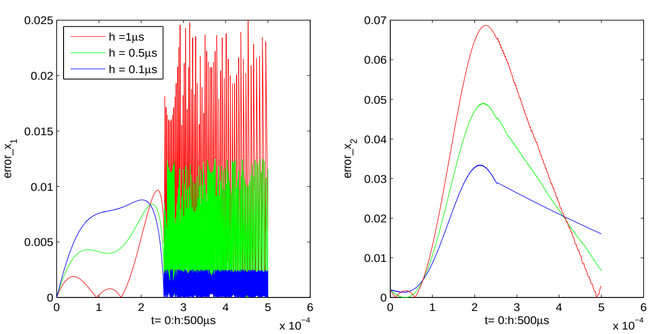

Fig. 3. Differences between solutions in Fig. 2 and solution of PSpice for the circuit system in Fig. 1

Table 1. Comparison of the simulation results of Fig.1 by ILETS and PSpice

h	$1\mu s$	$0.5\mu s$	$0.1\mu s$
PSpice simulation time	$0.42s$	$0.17s$	$0.22s$
ILETS simulation time	$0.030s$	$0.046s$	$0.203s$
Δx_1	$0.0250A$	$0.0125A$	$0.0088A$
Δx_2	$0.0687V$	$0.0490V$	$0.0335V$

It is also worth noting that PSpice can not provide the existence of the solution of the LCS. Chen and Wang [12] proposed an algorithm to estimate the computational error bound of the time stepping method when applied to solve the LCS. Using the LCS we can show the existence of the solution of the system. Moreover, we can provide error bounds of numerical solution $x^{h,j}$ generated from the ILETS with D being a P-matrix. Although in our example M is not a P-matrix, by (15) we could get the Lipschitz constant $\beta_M = 1$ in Lemma 2.1 in [12] and apply the algorithm to this example. Figure 4 shows the computational error bounds for this example. Due to the large norm of matrix A and B, we only estimate the error bound at the first $200\mu s$ and choose the step size as $h = 1 \times 10^{-9}s$ and $2 \times 10^{-10}s$. In [12] it proved that the error bound could be improved when the step size h decreases with the order $O(h)$, which also coincides with the result in Fig. 4.

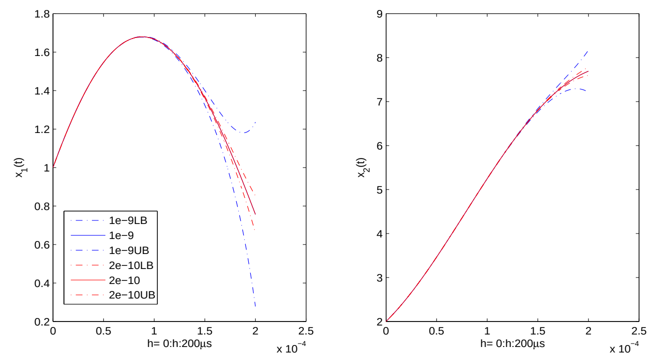

Fig. 4. Computational error bound of the ILETS (12) for the LCS derived from Fig. 1.

The next example on the three-phase converter in Fig. 5 mentioned in [30] also shows the efficiency (see Table 2) of the ILETS (12): which can be reformulated by LCS with $f \equiv Eu$, $g \equiv 0$,

$$A = -\frac{R}{L}\begin{bmatrix} \frac{L}{RR_CC} & -\frac{L}{RC} & 0 & 0 \\ \frac{L}{RL_f} & \frac{LR_f}{RL_f} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = -\frac{1}{3L}\begin{bmatrix} \frac{3L}{C} & \frac{3L}{C} & \frac{3L}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \end{bmatrix},$$

$$E = \frac{1}{3L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{3L}{L_f} & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad I \in R^{3\times 3},$$

and $x = (x_C, x_L, x_r, x_s)^T$, and $u = (e, e_r, e_s, e_t)$.

Set $R = 1\Omega$, $L = 100$mH, $R_f = 2\Omega$, $L_f = 10$mH, $C = 10$mF, $R_C = 10k\Omega$, $e = 300$V, $f_e = 48$Hz, $e_r = 100\sin(2\pi f_e t)$, $e_r = 100\sin(2\pi f_e t - 2\pi/3)$ and $e_r = 100\sin(2\pi f_e t - 4\pi/3)$. Figs 6-7 show the results computed by the ILETS with respect to different initial vectors, which implies the approximations vary smaller as the time-step h decreases. Especially, the computation of x_C and x_L vary very

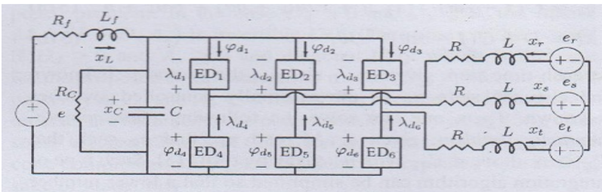

Fig. 5. Three-phase power converter with input filter [30].

small by different hs in Figs. 6-7. For more details on the type of electronic devices used for that converter see [30].

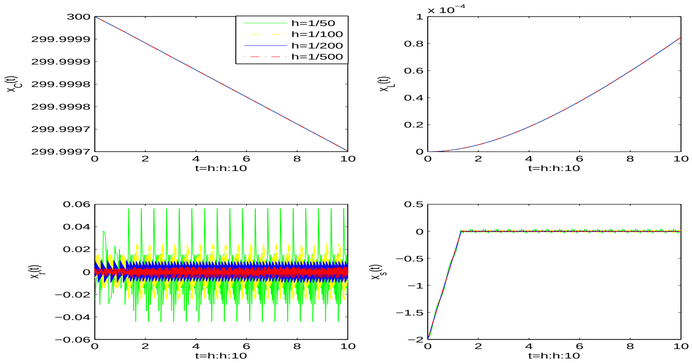

Fig. 6. Approximations by the ILETS (12) together with (14) for the LCS derived from Fig. 4 with $x(0) = (300, 0, 0, -2)^T$.

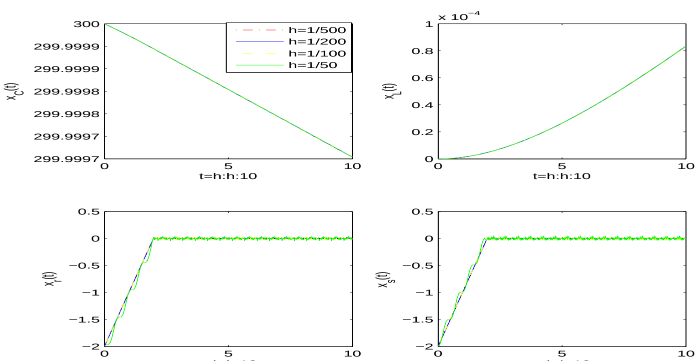

Fig. 7. Approximations by the ILETS (12) together with (14) for the LCS derived from Fig. 4 with $x(0) = (300, 0, -2, -2)^T$.

Next we illustrate the generalized Newton method to evaluate the LCS derived from Fig. 1 and present differences by the ILETS (12) together with (14) in Fig. 8. Here, Table 2 shows the total cpu time for Fig. 8 by the ILETS (12) together with the generalized Newton method for the LCS derived from Fig. 1 by setting stopping criteria with tolerance 10^{-14}, where it takes at most 2 iterations for the generalized Newton method at each step and the elapsed time for each iteration is $0.000162s$.

***Table 2**. The total cpu time of the simulation results of Fig.8*

h	$0.1\mu s$	$0.5\mu s$	$1\mu s$
simulation time	$1.7838s$	$3.5465s$	17.6628

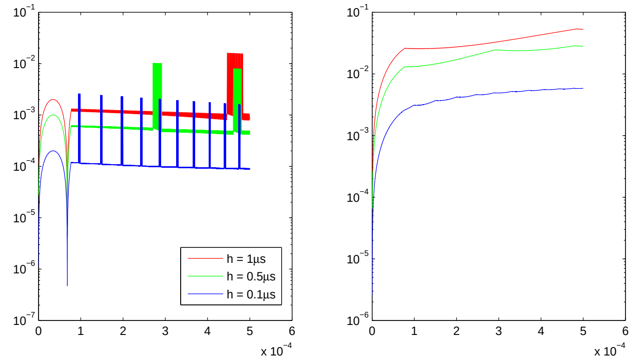

Fig. 8. Absolute errors on the ILETS (12) for the LCS derived from Fig. 1 by using the generalized Newton method and (14): $x(0) = (1, 2)^T$.

Remark. Han et al. [18] showed the cpu time for solving the LCS derived from Fig. 1 by ILNTS (4) on a high-end desktop (2.0 GB RAM, 2 Core2 (64 bit) processors at 2.40 GHz) running Ubuntu 9.04 (i386). One just used the Lemke method [14] to find a solution of the LCP subproblems without any norm-minimization, which costs about 15 seconds for 2^{14} steps. The other with the norm-minimization step included, costs about 85 seconds.

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The authors would like to thank the associate editor and the reviewers for their insightful comments and constructive suggestions to improve this note. This work is supported partly by Hong Kong Research Grant Council grant, partly by NSF of China (No.11371376) and partly by upported by Mathematics and Interdisciplinary Sciences Project, Central South University.

References

[1] V. Acary, O. Bonnefon and B. Brogliato, *Nonsmooth Modelling and Simulation for Switched Circuits*, LNEE 69, Springer Verlag, 2010.
[2] V. Acary, O. Bonnefon and B. Brogliato, *Time-stepping numerical simulation of switched circuits with the nonsmooth dynamical systems approach*, IEEE Trans. Computer-Aided Design Integr. Circ. Syst., 29(2010), 1042-1055.
[3] D. Bedrosian and J. Vlach, *Time-domain analysis of networks with internally controlled switches*, IEEE Trans. Cirsuit Syst. I, 39(1992), 199-212.
[4] W. Bokhoven, *Piecewise Linear Modeling and Analysis*, Deventer, the Netherland, Kluwer, 1981.
[5] B. Brogliato, A. Daniilidis, C. Lemarchal and V. Acary, *On the equivalence between complementarity systems, projected systems and unilateral differential inclusions*, Systems and Control Letters, 55(2006), 45-51
[6] B. Brogliato and D. Goeleven, *Existence, uniqueness of solutions and stability of nonsmooth multivalued Lur'e dynamical systems*, J. Convex Analysis, 20(2013), 881-900.
[7] M.K. Camlibel, *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*, Ph.D. thesis, Center for Economic Research, Tilburg University, The Netherlands, 2001.
[8] M.K. Camlibel, W.P.M.H. Heemels and J.M. Schumacher, *Consistency of a time-stepping method for a class of piecewise-linear networks*, IEEE Trans. Cirsuit Syst. I, 49(2002), 349-357.
[9] M.K. Camlibel, W.P.M.H. Heemels and J.M. Schumacher, *On linear passive complementarity systems, European J. Control*, 8(2002), 220-237.
[10] M.K. Camlibel, W.P.M.H. Heemels, A.J. van der Schaft and J.M. Schumacher, *Switched networks and complementarity*, IEEE Trans. Circuits Syst. I, 50(2003), 1036-1046.
[11] W.P.M.H. Heemels, M.K. Camlibel, J.M. Schumacher and B. Brogliato, *Observer-based control of linear complementarity systems*, Int. J. Robust Nonlin. Contr., 21(10)(2011), 1193-1218.

[12] X. Chen and Z. Wang, *Computational error bounds for differential linear variational inequality*, IMA J. Numer. Anal., 32(2012), 957-982.

[13] X. Chen and S. Xiang, *Newton iterations in implicit time-stepping method for differential linear complementarity systems*, Math. Program. Ser. A, 138(2013), 579-606.

[14] R.W. Cottle, J.-S. Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[15] J. van Eijndhoven, *Solving the linear complementarity problem in circuit simulation*, SIAM J. Control Optim., 24(1986), 1050-1062.

[16] M.C. Ferris and J.S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev., 39(1997), 669-713

[17] R. Frasca, M.K. Camlibel, I.C. Goknar, L. Iannelli and F. Vasca, *Linear passive networks with ideal switches: Consistent initial conditions and state discontinuities*, IEEE Trans. Cirsuit Syst. I, 57(2010), 3138-3151.

[18] L. Han, A. Tiwari, M.K. Camlibel and J.-S. Pang, *Convergence of time-stepping methods for passive and extended linear complementarity systems*, SIAM J. Numer. Anal., 47(2009), 3768-3796.

[19] W.P.M.H. Heemels, M.K, Camlibel and J. Schumacher *On the dynamic analysis of piecewise linear networks*, IEEE Trans. Cirsuit Syst. I, 49(2002), 315-327.

[20] W.P.M.H. Heemels, M.K. Camlibel, A. van der Schaft and J Schumacher, *Modelling, well-posedness, and stability of switched electrical networks*, In O. Maler and A. Pnueli (ed.), Hybrid Systems: Computation and Control, LNCS 2623, 249-266, Springer, Berlin, 2003.

[21] W.P.M.H. Heemels and B. Brogliato, *The complementarity class of hybrid dynamical systems*, European J. Control, 9(2003), 322-360.

[22] D. Leenaerts, *On linear dynamic complementarity systems*, IEEE Trans. Cirsuit Syst. I, 46(1999), 1022-1026.

[23] Y. Lootsma, A.J. van der Schaft and M.K. Camlibel, *Uniqueness of solutions of relay systems*, Automatic, 35(1999), 467-478.

[24] A. Massarini, U. Reggiani and K. Kazimierzuk, *Analysis of networks with ideal switches by state equations*, IEEE Trans. Cirsuit Syst. I, 44(1997), 692-697.

[25] J.-S. Pang, *Three modeling paradigms in mathematical programming*, Math. Program. Ser., B, 125(2010), 297-323.

[26] J.-S. Pang and D.E. Stewart, *Differential variational inequalities*, Math. Program. Ser. A, 113(2008), 345-424.

[27] A.J. van der Schaft and J.M. Schumacher, *The complementarity-slackness class of hybrid systems*, Math. Contr. Signals Syst., 9(1996), 266-301.

[28] A.J. van der Schaft and J.M. Schumacher, *Complementarity modeling of hybrid systems*, IEEE Trans. Automatic Control, 43(1998), 483- 490.

[29] J.M. Schumacher, *Complementarity systems in optimization*, Math. Program. Ser. B, 101(2004), 263-296.

[30] F. Vasca, L. Iannelli, M.K. Camlibel and R. Frasca, *A new perspective for modeling power electronics converters: complementarity framework*, IEEE Trans. Power Electron., 24(2009), 456-468.