SCIENCE CHINA
Information Sciences

• **RESEARCH PAPER** •

A Halanay-type inequality with delayed impulses and its applications

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Abstract This paper studies some properties of novel Halanay-type inequality containing impulses and delayed impulses simultaneously. Two concepts with respect to average impulsive gain are proposed to describe the hybrid impulsive strength and hybrid delayed impulsive strength. Then, using the obtained Halanay-type results, some sufficient conditions are derived to obtain the stability of linear systems with hybrid impulses. It is shown that the stability of impulsive systems is robust with respect to time-delayed impulses whose strength is relatively weak. While, if the magnitude of impulses is small, the time-delayed impulses can also promote the stability. Two examples are employed to show the validity of our theoretical results.

Keywords Halanay-type Inequality, Hybrid Impulses, Average Impulsive Gain, Linear Impulsive Systems

Citation Wang Y Q, Lu J Q, Lou Y J. A Halanay-type inequality with delayed impulses and its applications. Sci China Inf Sci, for review

1 INTRODUCTION

As well-know, the stability of dynamical systems has received many interests for its potential applications in many areas such as secure communication [1], image processing [2], and harmonic oscillation generation [3, 4]. Actually, there exist many stability phenomena in nature and man-made world, such as fireflies in the forest [5], distributed computing systems [6], routing messages in the internet [7], and so on [8–10]. Massive outstanding achievements have been obtained with respect to both theoretical analysis and applications of stability for dynamical systems.

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As a powerful tool studying dynamical behaviors of the differential systems, several differential inequalities, such as Halanay inequality [11], Hilger-type impulsive differential inequality [12] and Lieb-Thirring-type inequality [13,14], have received much attention from researchers [15]. All the inequalities have been proved to be effective in the investigation of stability problems for differential dynamical systems. One of the most common methods of theses is the Halanay inequality:

$$u'(t) \leqslant \alpha_1 u(t) + \alpha_2 [u(t)]_\tau, \quad -\alpha_1 > \alpha_2 > 0,$$

and the generalized Halanay inequality:

$$u'(t) \leqslant \alpha_1(t) u(t) + \alpha_2(t) [u(t)]_{\tau(t)},$$

where $-\alpha_1(t) > \alpha_2(t) > 0$, $\tau(t) < t$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$. A new generalized Halanay inequality without the requirement on uniform positiveness conditions and a novel concept of global generalized exponential stability are proposed in [16] to study the stability of nonlinear non-autonomous time-delayed systems. Song *et al*. in [17] introduced the discrete Halanay-type inequalities and applied the results to the stability problems for discrete neural networks with time-varying delays. He *et al*. extended the Halanay inequality to an integral inequality and presented several asymptotical stability conditions for the fractional-order differential systems in [18].

As one kind of hybrid characters, time-delayed impulses commonly appear in many different areas of real world, and they in general may cause oscillation, divergence, chaos, instability or other undesirable performances in the dynamical systems. Much efforts have been devoted to studying the stability and asymptotic behaviors of real-valued differential systems with delayed impulses, and many significant achievements have been reported, such as [19–21]. Recently, many impulsive delay differential inequalities have been established to study the stability problems for impulsive dynamical systems with delay. It was reported in [22] that global exponential stability of time delayed systems was achieved via an impulsive differential inequality with time-varying delays. Based on a novel extended Halanay-type differential inequality with delayed impulses, Wu *et al*. [23] investigated the exponential stability of recurrent delayed neural networks. Then, Yang *et al*. [24] extended this inequality to impulsive delayed differential equations in which the impulsive items had multiple time-varying delays, and they used the inequality to study the exponential stability of these equations. By utilizing the result, exponential synchronization problems of

TS fuzzy complex networks [24] and complex-valued complex networks with stochastic perturbations [25] were studied. Therefore, based on the above analysis, it is necessary and natural to investigate the effects of delayed impulses on differential systems.

It is worth noting that, the above literatures just discussed the robustness with delayed impulses, i.e. they normally regard the delayed impulses as a kind of instantaneous perturbations or some destabilizing sources. Do all the delayed impulses restrain the stability of systems? It's really not. In [26], Li *et al.* found that the time-delayed impulses may contribute to the stabilization of delayed systems by restricting the the impulse interval and impulsive gain. Subsequently, Yang *et al.* used this result to design a delayed impulsive distributed controller to investigate the exponential synchronization for nonlinear complex dynamical systems in [27]. However, few work considered dynamical systems with stabilizing impulses and stabilizing delayed impulses simultaneously. There is a natural question: under what circumstances would both the normal impulses and delayed impulses enhance the stability?

Motivated by the above analysis, this paper aims to establish a novel Halanay-type inequality with impulses and delayed impulses simultaneously, and use this inequality to study the exponentially stability of dynamical impulsive systems. The contributions of the paper are listed as follows. Firstly, this paper investigates the impact of the relationship between impulsive gain and delayed impulsive gain on the stability of impulsive systems. Secondly, in order to depict the hybrid impulsive gain and hybrid delayed impulsive gain, two novel concepts concerning the average impulsive gain are proposed. At last, we employ the obtained results to study the exponentially stability of linear impulsive systems. The remainder of this paper is organized as follows. The model description and some necessary preliminaries are formulated in Section 2. Section 3 studies the dynamical behaviors of the Halanay-type inequality and the applications of the obtained results in the linear impulsive systems are given in Section 4. In Section 5, two numerical examples are presented to show the effectiveness of the theoretical results.

$Notions$: The symbol D^+ represents the upper right-hand Dini derivative. Denote $\mathbb{R}^+$ as the set of positive real number. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are the n-dimensional Euclidean space and the set of $n \times m$ real matrices, respectively. Set $PC(U, V) = \{u : U \to V$ is continuous everywhere except at finite number of point t, at which $u(t^+)$, and $u(t^-)$ exist and $u(t^+) = u(t)\}$. I is the identity matrix with suitable order, and superscript 'T' denotes the transpose of a matrix or a vector. $|\cdot|$ represents the Euclidean norm. For a matrix P, $\lambda_{max}(P)$ and $\lambda_{min}(P)$ are respectively the maximum eigenvalue and minimum eigenvalue. Unless stated otherwise, a matrix in this paper has compatible dimensions.

2 Model description and preliminaries

This paper considers the following differential inequality with impulsive effects and delayed impulsive effects

$$\begin{cases} D^+u(t) \leqslant pu(t), \ t \neq t_k, \\ u(t_k) \leqslant b_k u(t_k^-) + d_k[u(t_k)^-]_{\tau^-}, \end{cases} \tag{1}$$

where $u \in PC(\mathbb{R}, \mathbb{R}^+)$, $p \in \mathbb{R}$, $\tau > 0$, $b_k > 0$, $d_k > 0$ and the impulsive sequence $\zeta = \{t_1, t_2, t_3, \ldots\}$. The impulsive sequence satisfies $\lim\limits_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > \tau$.

Remark 1. In the above inequality, p is not restricted to be a negative number, which is more refined and less conservative compared with existing Halanay-type inequalities. On the other hand, different time-dependent parameters b_k and d_k were adopted to draw the impulsive strength and delayed impulsive strength, which is more general and has a broader range of applications.

Definition 1 ([28] [29] Average Impulsive Interval)**.** Let $N_\zeta(t, t_0)$ be the number of impulsive times of the impulsive sequence ζ on the interval (t_0, t). The average impulsive interval T_α is defined as follows

$$T_\alpha = \lim_{t\to\infty} \frac{t - t_0}{N_\zeta(t, t_0)}.$$

In general, for dynamical systems, there exist two kinds of impulses, one is stabilizing impulse and another one is destabilizing impulse. The concept of average impulsive gain was proposed in [29] to describe the hybrid impulsive gain, which is also a pretty method to investigate the impulsive systems with two kinds of impulses. Motivated by [29], for the Halanay-type inequality (1), the following definition is proposed to draw the hybrid impulsive gain and hybrid delayed impulsive gain.

Definition 2. The average impulsive gain and the average delayed impulsive gain of inequality (1) are respectively defined as follows

$$\mu_b = \lim_{t\to\infty} \frac{|b_1| + |b_2| + \cdots + |b_{N_\zeta(t,t_0)}|}{N_\zeta(t, t_0)}, \tag{2}$$

and

$$\mu_d = \lim_{t\to\infty} \frac{|d_1| + |d_2| + \cdots + |d_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}. \tag{3}$$

Lemma 1. For any two vectors x, $y \in \mathbb{R}^n$ and positive-definite matrix $Q \in \mathbb{R}^{n\times n}$, the following inequality holds

$$2xy \leqslant x^T Q x + y^T Q^{-1} y.$$

3 MAIN RESULTS

This section will propose some results concerning the Halanay-type inequality (1).

Theorem 1. For the impulsive inequality (1), suppose that there exists a positive constant M such that

$$b_k \geqslant M d_k e^{-p\tau}. \tag{4}$$

Then any solution $u(t)$ of inequality (1) satisfies

$$u(t) \leqslant (1 + \frac{1}{M})^{N_\zeta(t,t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} b_i e^{p(t-t_0)} u_0.$$

Proof. Let $u_0 = \sup\limits_{t_0-\tau\leqslant s\leqslant t_0} u(s)$. When $t \in [t_0, t_1)$, one has $u(t) \leqslant e^{p(t-t_0)}$ and

$$u(t_1) \leqslant b_1 u(t_1^-) + d_1[u(t_1^-)]_{\tau^-} \leqslant [b_1 e^{p(t_1-t_0)} + d_1 e^{p(t_1-\tau-t_0)}]u_0.$$

For $t \in [t_1, t_2)$, one has

$$u(t) \leqslant e^{p(t-t_1)} u(t_1) \leqslant [b_1 e^{p(t-t_0)} + d_1 e^{p(t-\tau-t_0)}]u_0$$

and

$$\begin{aligned} u(t_2) &\leqslant b_2 u(t_2^-) + d_2[u(t_2^-)]_{\tau^-} \\ &\leqslant b_1 b_2 e^{p(t_2-t_0)} + b_2 d_1 e^{p(t_2-\tau-t_0)} \\ &\quad + b_1 d_2 e^{p(t_2-\tau-t_0)} + d_1 d_2 e^{p(t_2-2\tau-t_0)} \end{aligned}$$

$$\leqslant (1+\frac{1}{M})^2 b_1 b_2 e^{p(t_2-t_0)} u_0.$$

When $t \in [t_2, t_3)$, it follows from (4) that

$$u(t) \leqslant e^{p(t-t_2)} u(t_2) \leqslant (1+\frac{1}{M})^2 b_1 b_2 e^{p(t-t_0)} u_0,$$

and

$$\begin{aligned} u(t_3) \leqslant\ & b_3 u(t_3^-) + d_3 [u(t_3^-)]_{\tau^-} \\ \leqslant\ & [b_1 b_2 b_3 e^{p(t_3-t_0)} + b_1 b_2 d_3 e^{p(t_3-\tau-t_0)} \\ & + b_1 b_3 d_2 e^{p(t_3-\tau-t_0)} + b_2 b_3 d_1 e^{p(t_3-\tau-t_0)} \\ & + b_1 d_2 d_3 e^{p(t_3-2\tau-t_0)} + b_2 d_1 d_3 e^{p(t_3-2\tau-t_0)} \\ & + b_3 d_1 d_2 e^{p(t_3-2\tau-t_0)} + d_1 d_2 d_3 e^{p(t_3-3\tau-t_0)}] u_0 \\ \leqslant\ & (1+\frac{1}{M})^3 b_1 b_2 b_3 e^{p(t_3-t_0)} u_0. \end{aligned}$$

By mathematical induction method, we can conclude that

$$u(t) \leqslant (1+\frac{1}{M})^{N_\zeta(t,t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} b_i e^{p(t-t_0)} u_0.$$

Hence, the proof is completed.

Corollary 1. Suppose that there exists a positive constant M such that (4) holds. Then for any solution $u(t)$ of inequality (1) there exists a sufficient large $T \geqslant 0$ such that when $t \geqslant T$ we have

$$u(t) \leqslant e^{\eta_1 (t-t_0)} u_0,$$

where $\eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha} + p$, T_α is the average impulsive interval and μ_b is defined as (2).

Proof. Based on the mean value inequality one has

$$\begin{aligned} u(t) \leqslant\ & \prod_{i=1}^{N_\zeta(t,t_0)} (\frac{M}{M+1} b_i) e^{p(t-t_0)} u_0 \\ \leqslant\ & (\frac{M}{M+1})^{N_\zeta(t,t_0)} (\frac{|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)})^{N_\zeta(t,t_0)} e^{p(t-t_0)} u_0 \\ =\ & (\frac{M}{M+1} \cdot \frac{|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)})^{N_\zeta(t,t_0)} e^{p(t-t_0)} u_0 \\ =\ & e^{N_\zeta(t,t_0) \ln(\frac{\frac{M}{M+1}(|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})} e^{p(t-t_0)} u_0 \end{aligned}$$

$$= e^{\frac{A}{B}(t-t_0)}e^{p(t-t_0)}u_0,$$

where $A = \ln(\frac{\frac{M}{M+1}(|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})$ and $B = \frac{t-t_0}{N_\zeta(t,t_0)}$. For any η_1 satisfying $\eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha} + p$, there exists $T \geqslant 0$ such that when $t \geqslant T$ we have

$$u(t) \leqslant e^{(\frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha}+\eta_1-\eta+p)(t-t_0)}u_0$$

$$= e^{\eta_1(t-t_0)}u_0.$$

Theorem 2. Consider inequality (1), if there exists $\widetilde{M} \geqslant 0$ such that

$$d_k \geqslant \widetilde{M} d_k e^{p\tau}, \tag{5}$$

then any solution $u(t)$ of inequality (1) satisfies

$$u(t) \leqslant [(1+\frac{1}{\widetilde{M}})e^{-p\tau}]^{N_\zeta(t,t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} d_i e^{p(t-t_0)}u_0.$$

Proof. Let $u_0 = \sup_{t_0-\tau\leqslant s\leqslant t_0} u(s)$. When $t \in [t_0, t_1)$, one has $u(t) \leqslant e^{p(t-t_0)}$ and

$$u(t_1) \leqslant b_1 u(t_1^-) + d_1[u(t_1^-)]_{\tau^-} \leqslant [b_1 e^{p(t_1-t_0)} + d_1 e^{p(t_1-\tau-t_0)}]u_0.$$

For $t \in [t_1, t_2)$, combining (1) and (5), one can conclude that

$$u(t) \leqslant e^{p(t-t_1)}u(t_1) \leqslant [b_1 e^{p(t-t_0)} + d_1 e^{p(t-\tau-t_0)}]u_0$$

and

$$\begin{aligned} u(t_2) &\leqslant b_2 u(t_2^-) + d_2[u(t_2^-)]_{\tau^-} \\ &\leqslant b_1 b_2 e^{p(t_2-t_0)} + b_2 d_1 e^{p(t_2-\tau-t_0)} \\ &\quad + b_1 d_2 e^{p(t_2-\tau-t_0)} + d_1 d_2 e^{p(t_2-2\tau-t_0)} \\ &\leqslant (1+\frac{1}{\widetilde{M}})^2 d_1 d_2 e^{p(t_2-2\tau-t_0)}u_0. \end{aligned}$$

Similarly, for $t \in [t_2, t_3)$, one obtains

$$u(t) \leqslant e^{p(t-t_2)}u(t_2) \leqslant (1+\frac{1}{\widetilde{M}})^2 d_1 d_2 e^{p(t-2\tau-t_0)}u_0,$$

and

$$u(t_3) \leqslant b_3 u(t_3^-) + d_3[u(t_3^-)]_{\tau^-}$$

$$\begin{aligned}
&\leqslant [b_1b_2b_3e^{p(t_3-t_0)} + b_1b_2d_3e^{p(t_3-\tau-t_0)} \\
&\quad +b_1b_3d_2e^{p(t_3-\tau-t_0)} + b_2b_3d_1e^{p(t_3-\tau-t_0)} \\
&\quad +b_1d_2d_3e^{p(t_3-2\tau-t_0)} + b_2d_1d_3e^{p(t_3-2\tau-t_0)} \\
&\quad +b_3d_1d_2e^{p(t_3-2\tau-t_0)} + d_1d_2d_3e^{p(t_3-3\tau-t_0)}]u_0 \\
&\leqslant (1+\frac{1}{\widetilde{M}})^3 d_1d_2d_3e^{p(t_3-3\tau-t_0)}u_0.
\end{aligned}$$

According to mathematical induction method, we can conclude that

$$u(t) \leqslant [(1+\frac{1}{\widetilde{M}})e^{-p\tau}]^{N_\varsigma(t,t_0)} \prod_{i=1}^{N_\varsigma(t,t_0)} d_i e^{p(t-t_0)}u_0.$$

The proof if completed here.

Corollary 2. Suppose that there exists $\widetilde{M} \geqslant 0$ such that (5) holds. Then for any solution $u(t)$ of inequality (1), there exists $T \geqslant 0$ such that when $t \geqslant T$, we have

$$u(t) \leqslant e^{\eta_1(t-t_0)}u_0,$$

where $\eta_1 > \eta = \frac{\ln\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}\mu_d}{T_\alpha} + p$, T_α is the average impulsive interval and μ_d is defined as (3).

Proof. It follows from the mean value inequality that

$$\begin{aligned}
u(t) &\leqslant \prod_{i=1}^{N_\varsigma(t,t_0)} (\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}d_i)e^{p(t-t_0)}u_0 \\
&\leqslant (\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau})^{N_\varsigma(t,t_0)}(\frac{|d_1|+|d_2|+\cdots+|d_{N_\varsigma(t,t_0)}|}{N_\varsigma(t,t_0)})^{N_\varsigma(t,t_0)}e^{p(t-t_0)}u_0 \\
&= e^{N_\varsigma(t,t_0)\ln(\frac{\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}(|d_1|+|d_2|+\cdots+|d_{N_\varsigma(t,t_0)}|)}{N_\varsigma(t,t_0)})}e^{p(t-t_0)}u_0 \\
&= e^{\frac{\widetilde{A}}{\widetilde{B}}(t-t_0)}e^{p(t-t_0)}u_0,
\end{aligned}$$

where $\widetilde{A} = \ln(\frac{\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}(|d_1|+|d_2|+\cdots+|d_{N_\varsigma(t,t_0)}|)}{N_\varsigma(t,t_0)})$ and $\widetilde{B} = \frac{t-t_0}{N_\varsigma(t,t_0)}$. For any η_1 satisfies $\eta_1 > \eta = \frac{\ln\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}\mu_d}{T_\alpha} + p$, there exists $T \geqslant 0$ such that when $t \geqslant T$ we have

$$\begin{aligned}
u(t) &\leqslant e^{(\frac{\ln\frac{\widetilde{M}+1}{\widetilde{M}}e^{-p\tau}\mu_d}{T_\alpha}+\eta_1-\eta+p)(t-t_0)}u_0 \\
&= e^{\eta_1(t-t_0)}u_0.
\end{aligned}$$

Remark 2. In the real world, the effect of impulses and delayed impulses on the stability of dynamical systems is terrific complex. Impulses that promote the stability of dynamical systems are the same that suppress the stability. Before [26], the delayed impulses in almost all of the existing results were regarded as destabilizing effect. For example, in order to guarantee stability, the delayed impulsive gain in [25] should be kept as small as possible. Although Li *et al*. proved that the delayed impulses may facilitate the stability, the considering dynamical system only contains the delayed impulses without delay-free impulses. This paper considered the systems with hybrid impulses and hybrid delayed impulses simultaneously. By comparing their parameters, the influences of impulses and delayed impulses on the stability were discussed.

Remark 3. Zhang *et al*. in [25] pointed out that the delayed impulsive gain is the smaller the better, which is consistent with Theorem 1 and Corollary 1. What's more, our results present the explicit relationship between impulsive parameters and delayed impulsive parameters. On the other hand, in view of the stabilizing delayed impulses, Theorem 2 and Corollary 2 studied the stability of impulsive systems when the delayed impulses played the main roles.

4 Applications

By using the obtained results in Section 3, we will establish some stability criteria for the following linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t),\ t \neq t_k, \\ x(t_k) = C_k x(t_k^-) + D_k x(t_k^- - \tau_k), \\ x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0]. \end{cases} \tag{6}$$

where A, C_k, $D_k \in \mathbb{R}^{n\times n}$ are constant matrices, $0 < \tau_k < \tau$, $x(t) \in \mathbb{R}^n$ is state variable, and $\varphi(t) \in C([t_0 - \tau, t_0], \mathbb{R}^n)$ is initial function. The impulsive sequence $\zeta = \{t_1, t_2, t_3, \cdots\}$ is same as those in the impulsive differential inequality (1).

Theorem 3. System (6) is global exponentially stable if there exist positive-definite matrices $P \in \mathbb{R}^{n\times n}$ and constants $\beta \in \mathbb{R}$, $b_k > 0$, $d_k > 0$, such that (2), (4) and the following and LMIs hold:

$$PA + A^T P - \beta P < 0, \tag{7}$$

$$\begin{pmatrix} C_k^T P C_k - b_k P & C_k^T P D_k \\ D_k^T P C_k & D_k^T P D_k - d_k P \end{pmatrix} < 0, \tag{8}$$

where

$$0 > \eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha} + \beta.$$

Proof. Let $V(t) = x^T(t)Px(t)$. Taking the derivative of $V(t)$ along the trajectories of the system (6), one has

$$D^+V(t) = x^T(t)PAx(t) + x^T(t)A^TPx(t).$$

According to (7), we can get

$$D^+V(t) \leqslant \beta V(t).$$

Furthermore, it follows from (8) that

$$\begin{aligned} V(t_k) &= x^T(t_k^-)C_k^TPC_kx(t_k^-) + x^T(t_k^-)C_k^TPD_kx(t_k^- - \tau_k) \\ &\quad + x^T(t_k^- - \tau_k)D_k^TPC_kx(t_k^-) + x^T(t_k^- - \tau_k)D_k^TPD_kx(t_k^- - \tau_k) \\ &\leqslant b_kx^T(t_k^-)Px(t_k^-) + d_kx^T(t_k^- - \tau_k)Px(t_k^- - \tau_k) \\ &= b_kV(t_k^-) + d_kV(t_k^- - \tau_k). \end{aligned}$$

Then, by Corollary 1, we have

$$V(t) \leqslant V_0 e^{\eta_1(t-t_0)},$$

where $V_0 = \sup\limits_{t_0-\tau\leqslant s\leqslant t_0} V(s)$. It follows from the definition of $V(t)$ that

$$\|x(t)\|^2 \leqslant \lambda_{max}(P)/\lambda_{min}(P)\|\varphi\|^2 e^{\eta(t-t_0)}. \tag{9}$$

Hence, we can deduce that the system (6) is exponentially stable from (9).

Similarly, we can obtain the following result from Corollary 2. The proof is omitted here.

Theorem 4. System (6) is globally exponentially stable if there exist positive-definite matrices $P \in \mathbb{R}^{n\times n}$ and constants $\beta_1 \in \mathbb{R}$, $b_k > 0$, $d_k > 0$, such that (3), (5) and the following and LMIs hold:

$$PA + A^TP - \beta_1 P < 0, \tag{10}$$

$$\begin{pmatrix} C_k^T P C_k - b_k P & C_k^T P D_k \\ D_k^T P C_k & D_k^T P D_k - d_k P \end{pmatrix} < 0, \tag{11}$$

where

$$0 > \eta_1 > \eta = \frac{\ln \frac{M}{M+1} e^{-\beta_1 \tau} \mu_d}{T_\alpha} + p. \tag{12}$$

5 Numerical Simulations

This section will present two examples to show the validity of our results.

Example 1. Consider the following linear system

$$\begin{cases} \dot{x}(t) = Ax(t), \ t \neq t_k, \\ x(t_k) = C_k x(t_k^-) + D_k x(t_k^- - \tau_k), \end{cases} \tag{13}$$

where $A = \begin{pmatrix} 1 & 1 \\ -2 & 0.7 \end{pmatrix}$. It follows from the fact $Re(\lambda_A) > 0$ that this linear system is unstable without impulses. According to Theorem 3, we will design an impulsive controller to stabilize the system. Let the time decay $\tau_k = \tau = 0.3$, $T_\alpha = 0.25$, $M = 2$ and $\alpha = 0.3$. The impulsive sequence is shown in Figure 1. The matrices C_k can be chosen from set $\{0.2I, 0.4I, 0.3I\}$ and $D_k \in \{0.3I, 0.2I, 0.2I\}$ with same probability. Let impulsive parameters $b_k \in \{0.3, 0.5, 0.4\}$ and delayed impulsive parameters $d_k \in \{0.16, 0.27, 0.21\}$, then $\mu_b = 0.4$ and $\mu_d = 0.213$. Using LMI toolbox of MATLAB, we can find

$$P = \begin{pmatrix} 26.0614 & 0.7118 \\ 0.7118 & 25.672 \end{pmatrix}$$

which satisfys (7) and (8). It follows from Theorem 3 that the system (13) is stable under impulsive controller. The corresponding trajectory of $x(t)$ is depicted in Figure 2.

Example 2. This example considers the linear system (6) with $A = \begin{pmatrix} 0.2 & 0.1 \\ 0.06 & 0.04 \end{pmatrix}$. By simple calculation, A is not Hurwize, hence the system is unstable without impulses. Let $C_k \in \{0.3I, 0.1I, 0.15I\}$

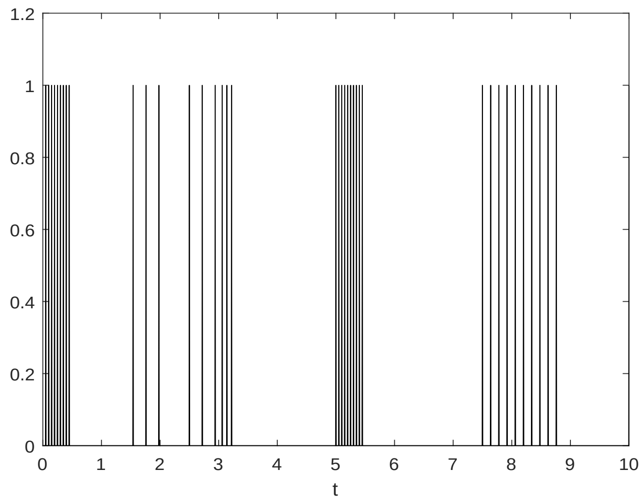

Figure 1 The impulsive sequence with average impulsive interval $T_\alpha = 0.25$. This sequence is repeated.

and $D_k \in \{0.8I, 0.4I, 0.3I\}$. The impulsive sequence is same as Example 1. Let the impulsive delays $\tau_k = \tau = 0.2$ and $\widetilde{M} = 2$. The impulsive parameters are chosen as $b_k \in \{0.18, 0.14, 0.23\}$ and delayed impulsive parameters are chosen as $d_k \in \{0.4, 0.3, 0.35\}$. Then we can obtain $\mu_b = 0.183$, $\mu_d = 0.35$ and $p = 0.3$ which satisfy (5) and (12). By utilizing the LMI toolbox of MATLAB, we can find

$$P = \begin{pmatrix} 0.3903 & 0.0947 \\ 0.0947 & 0.2723 \end{pmatrix}$$

satisfying (10) and (11). Based on the Theorem 4, we can see that this system is stable under the delayed impulses, and the trajectory is shown in Figure 3.

6 CONCLUSION

This paper studied a novel Halanay-type inequality with hybrid impulses and hybrid delayed impulses. Two concepts concerning average impulsive gain were proposed to describe the impulsive gain and delayed impulsive gain. For the purpose of investigating the effect of impulses and delayed impulses on the stability, the comparison between the impulsive parameters and the delayed impulsive parameters were employed. Stabilizing impulses and stabilizing delayed impulses were respectively analyzed to study the stability problem for impulsive dynamical systems. According the obtained Halanay-type inequality,

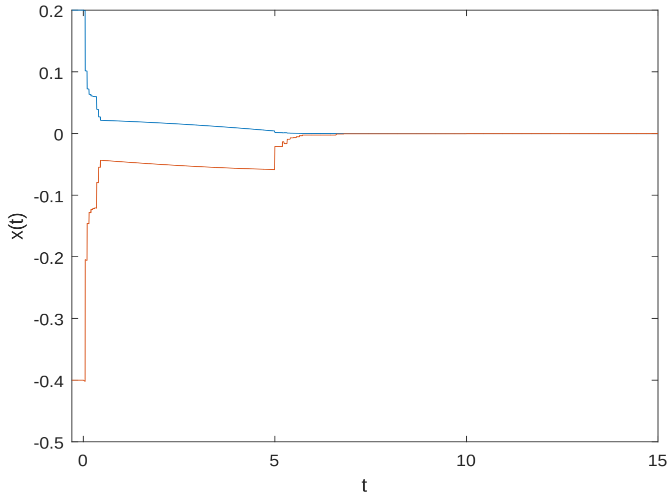

Figure 2 The trajectory of system (13) in Example 1.

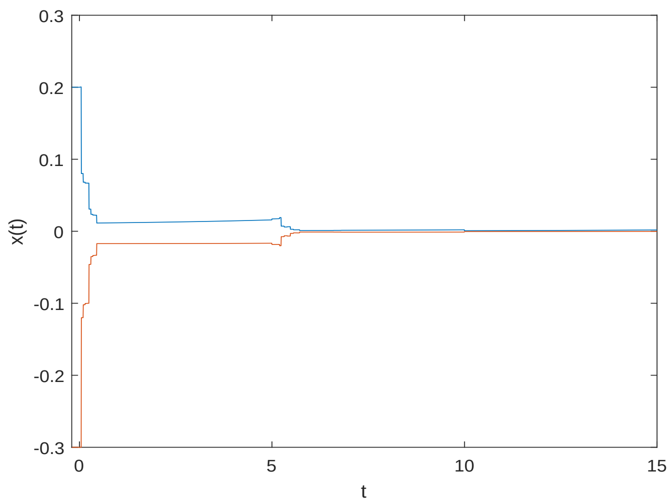

Figure 3 The trajectory of system (13) in Example 2.

some sufficient conditions were constructed to achieve the exponential stability of linear systems.

Acknowledgements This work was supported by the National Natural Science Foundation of China under Grant No. 61573102, the Natural Science Foundation of Jiangsu Province of China under Grant no. BK20170019, and Jiangsu Provincial Key Laboratory of Networked Collective Intelligence under Grant No. BM2017002.

Conflict of interest The authors declare that they have no conflict of interest.

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