Unified Stability Criteria of Switched Systems with Limiting Average Dwell Time

Yaqi Wang, Jianquan Lu, Yijun Lou

***Abstract*—In this paper, the stability problems of a class of switched systems with limiting average dwell time are concerned. The common average dwell time (ADT) is improved to a form of limit, and the limiting average dwell time even can be infinite. Different from previous results, in order to take full advantage of stabilizing switchings, switching-dependent switched parameters are firstly used to describe the relationship of two consecutive activated switchings. Then, unified stability criteria of switched systems with limiting average dwell time are established, whatever the subsystems are stable or not, which are less conservative comparing with the existing results. Additionally, the unified stability criteria of switched systems including continuous-time and discrete-time cases are firstly derived. Finally, the validity and effectiveness of our results are elucidated by numerical examples.**

***Index Terms*—Limiting Average Dwell Time, Stabilizing Switching, Switched System, Unified Stability Criteria.**

I. Introduction

The environment of real-world dynamical networks always suffered from the external disturbances, which potentially lead to the sudden changes of states and can be modeled as switched system [1], [2]. The switched systems consist of a finite number of subsystems and a logical rule regulating the switching behavior among them called switching signal. During the past decades, many researchers paid their attention on switched systems due to its extensive applications in power electronics [3], [4], formation control systems [5], networked control systems [6], and so on.

The stability and stabilization of dynamical systems are of great importance in many areas. Moreover, the switching signals always have a strong impact on the dynamical behaviors of switched systems [7], which always caused undesire results. Therefore, it is necessary and difficult to study how the switching signals affect the stability of dynamical systems. Although there exist many valid control methods in general dynamical systems, it is still quite hard to achieve the stability and stabilization of switched systems. The reason is that the switched systems may be unstable in some special switching signals even if all the subsystems are stable. For overcoming this obstacle, a lot of attempts were made. Using the state transformation method to decompose each subsystem into stable and unstable portions, Li *et al*. investigated the stabilization of a class of switched systems in [8]. In [9], the state-feedback path-wise switching law was presented to study switching stabilization problems for discrete-time switched linear systems. Moreover, Sun further used the switching control strategy to prove that any stabilizable switched linear system is also robust against switching perturbations [10]. For the switched systems in which each subsystem is completely observable, Wu *et al*. [11] constructed an observer-driven path-wise switching law to investigate the exponential stability of switched linear systems and obtained a weaker version of the separation principle. And, more remarkable, LaSalle's invariance principle and Lyapunov method, as two of most important tools in the stability analysis of common dynamical systems, were extended to the switched systems. It was shown in [12] that one extension of LaSalle's invariance principle for global asymptotic stability of switched linear systems was proposed. Morevoer, a class of weak common linear copositive Lyapunov functions were constructed to investigate the stabilization problem of positive switched linear systems with disturbances via a time-dependent switching law [13].

However, in the above literatures, the upper bound and lower bound of two consecutive switching signals' interval were restricted, which led the results to be conservative. In order to describe the frequency of switching signals, the dwell time (DT) [14] and the average dwell time (ADT) [15] were proposed. Actually, the ADT switchings are more flexible and less conservative than the dwell time (DT) switchings, which has attracted more attention [16], [17], [18]. In [19], Zhang *et al*. used the concept of ADT to study the stability and l_2-gain problems for a class of discrete-time switched systems where the Lyapunov-type functions were allowed to increase during the running time of activated subsystems. Then, the asynchronously switched control problem of switched systems with ADT was investigated in [20], where the switchings between the candidate controllers and subsystem models were asynchronous. Meanwhile, the ADT switchings method was used to investigated the stabilization and stability problems of many special dynamical systems, such as positive linear systems [21] and time-delayed Markov jump systems [22] and so on. Soon afterwards, Zhao *et al*. presented more general concept named mode-dependent average dwell time (MDADT) [23]. The notion of MDADT describes the being activated frequency of each subsystem, i.e. every subsystem has its own ADT. Furthermore, even if all the subsystems may be unstable, the stabilization problems for a class of slowly switched linear systems can be obtained by using the property of MDADT [24].

Similar with the switched system, another discontinuous dynamical system, the impulsive system is used to model the

This work was supported by the National Natural Science Foundation of China under Grant Nos. 61573102 and 61673110, the Natural Science Foundation of Jiangsu Province of China under Grant No. BK20170019, and the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence under Grant No. BM2017002.

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system whose states suffer abrupt change at some constants. Therefore, the control methods of these two systems can learn from each other. Inspired by the concept of ADT, Lu *et al*. proposed a notion named average impulsive interval (AII) in impulsive control systems, which defined the impulses' frequency [25]. This notion was used to obtain a unified synchronization construction of impulsive dynamical networks no matter the impulses are synchronizing impulses or desynchronizing impulses. Afterwards, much attention was attracted by this results [26], [27], [28]. In [29], the consensus of second-order multi-agent system was achieved by impulsive controller with AII. Since the above results only considered two kinds of impulses separately, Wang *et al*. proposed a novel concept called average impulsive gain (AIG) to describe the condition that both synchronizing impulses and desynchronizing impulses occur simultaneously [30]. They also extended the AII to the form of limit, which is more general. Then, pinning impulsive synchronization of Lur'e systems was studied based on limiting AII and AIG [31]. Similarly, for switched systems, the switchings between different subsystems also can be divided into two types, stabilizing switchings and destabilizing switchings [1], [32], [33]. The former can maintain loop stability and enhance system performance, while the later may restrain the systems stability. However, the most of existing results did not consider this problem. They always used the conditions $V_i(x(t)) \leq \mu V_j(x(t))$, with $\mu > 1$ to describe the relationship between two consecutive activated subsystems. But this relationship can not reflect that the switching is stabilizing switching or not. They always omitted the effect of stabilizing switchings.

Motivated by the above discussions, this paper presents unified stability criteria of switched systems under limiting ADT switchings, in which the switchings may be helpful for the stability. The main contributions of this paper are as follows. Firstly, we improve the common average dwell time to a form of limit, i.e. the limiting average dwell time, and the limiting ADT even can be equal to infinite. Secondly, in order to take full advantage of stabilizing switching, switching-dependent parameters are used to depict the relationship between two consecutive activated subsystems, which is more practical than the existing work. At last, unified stability criteria for continuous-time switched systems and discrete-time switched systems are derived. The remainder of this paper is organized as following. Section II provides some definitions and lemmas which will be used in the derivation of the main results, and reviews the existing some stability criteria of switched systems with common ADT switching. The unified stability criteria of switched systems are derived in Section III, and then we further address the stability problems of linear switched systems both in continuous-time and discrete-time context. In Section IV, numerical examples are provided to show the validity of our results.

*Notation.*Denote $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional Euclidean space and the $n\times m$ real matrices, respectively. I is the identity matrix with suitable order, and $|\cdot|$ denotes the Euclidean norm. Superscript 'T' denotes the transpose of a matrix or a vector. We let $\lambda_{min}(A)$ and $\lambda_{max}(A)$ respectively denote the smallest and the largest eigenvalues of matrix A. A function $f:[0,+\infty)\to[0,+\infty)$ belong to class $\mathscr{K}_\infty$ if f is continuous, strictly increasing and $f(0)=0$.

II. Preliminaries and Model Description

Consider the classical switched systems given by

$$\delta x(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1}$$

where $x(t)\in\mathbb{R}^m$ is state vector, and $u(t)$ is control input vector. The symbol δ denotes respectively the derivative operator in continuous-time context, i.e. $\delta x(t) = \frac{dx(t)}{dt}$, and the shift forward operator in the discrete-time case, i.e. $\delta x(t) = x(t+1)$. The switching signal $\sigma(t)$ is a piecewise constant function and is continuity from the right, which takes its values in a finite set $S=\{1,\dots,N\}$. N is the number of subsystems. The switching sequence satisfies $0<t_1<t_2<\cdots$. We said that the $\sigma(t)th$ subsystem is activated when $t\in[t_i,t_{i+1})$, hence the trajectory $x(t)$ of system (1) is the trajectory of the $\sigma(t)th$ subsystem. Then $(A_{\sigma(t)},B_{\sigma(t)}) = (A_i,B_i)$ denotes the ith subsystem or ith model of (1). As commonly assumed in the literature, the state vector $x(t)$ is everywhere continuous even in the switching instants.

In general, the switching signal has the following ADT property which is proposed to describe how seldom or/and how often the switchings occur.

Definition 1: [15] For a switching signal σ and any $t_0\leq t_1<t_2$, $N_\sigma(t_1,t_2)$ represents the switching numbers of $\sigma(t)$ in the interval $[t_1,t_2)$. If there exist constants $N_0\geq 1$ and $\tau_a>0$, such that

$$N_\sigma(t_1,t_2) \leq N_0 + \frac{t_2-t_1}{\tau_a} \tag{2}$$

holds, then τ_a and N_0 are called the average dwell time and chatter bound, respectively.

Motivated by the concept of ADT, Lu *et al*. presented a novel concept named average impulsive interval (AII) in impulsive system [25]. Recently, they extend AII to a form of limit [30]. Inspired by [25] and [30], we propose the following concept.

Definition 2: For a switching signal σ, $N_\sigma(t_0,t)$ denotes the switching numbers of $\sigma(t)$ in the interval $[t_0,t)$. If there exists a positive constant τ_a such that

$$\tau_a = \lim_{t\to\infty}\frac{t-t_0}{N_\sigma(t_0,t)} \tag{3}$$

holds, then τ_a is called the limiting averaged well time of switching signal σ.

Remark 1: The classical ADT can be deduced by the Definition 2. In fact, it follows from $\tau_a = \lim_{t\to\infty}\frac{t-t_0}{N_\sigma(t_0,t)}$ that there exists N_0 such that (2) holds for any $t_1<t_2$. Hence the limiting ADT in this paper improves the classical ADT in [15]. In other hand, τ_a in [15] must be a positive constant and must be finite, while in the Definition 2, it may be equal to infinite. For example, if $N_\sigma(t_0,t) = \lceil t-t_0\rceil$ then $\tau_a=1$, if $N_\sigma(t_0,t) = ln(t-t_0)$ then $\tau_a=+\infty$, and if $N_\sigma(t_0,t) = exp(t-t_0)$ then $\tau_a=0$. For two special cases, $\tau_a=0$ means the switchings occur very frequently, and when the switchings become more and more sparse, τ_a tends to $+\infty$, see in Fig. 4.

Next, we give the following stabilty definition of system (1) for later development, in which for discrete-time case replace time t by k.

Definition 3: The equilibrium $x=0$ of system (1) is globally uniformly exponentially stable (GUES) under certain switching signal $\sigma(t)$ if for $u(t)=0$ (or $u(k)=0$) and initial condition $x(t_0)=x_0$ (or $x(k_0=x_0)$), there exist constants $M>0$, $\eta>0$ (or $0<\iota<1$) such that the solution of system satisfies $\|x(t)\| \leq Me^{-\eta(t-t_0)}\|x_0\|,\ t\geq t_0$ (or $\|x(k)\| \leq M\iota^{(k-k_0)}\|x_0\|$, $k\geq k_0$).

In order to obtain GUES under switching signals, we will adopt the state feedback control protocol. Let $u(t)=K_{\sigma(t)}x(t)$ (or $u(k)=K_{\sigma(k)}x(k)$), where $K_{\sigma(t)}=K_s, \sigma(t)=s$ (or $K_{\sigma(k)}=K_s, \sigma(k)=s\in S$) is the control gain matrix which will be determined in the later.

There exist many results studying the stability problems for the switching systems. The following lemmas are provided for latter use.

Lemma 1: [15] For the switched system $\dot{x}(t)=f_{\sigma(t)}(x(t))$, $\sigma(t)\in S$, let $\rho>0$, $\mu>1$ be given constants. Suppose that there exist continuously differentiable functions $V_{\sigma(t)}:\mathbb{R}^n\to\mathbb{R}$ and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s\in S$

$$\beta_1(x_t)\leq V_s(x_t)\leq\beta_2(x_t), \tag{4}$$

$$\dot{V}_i(x_t)\leq-\rho V_s(x_t), \tag{5}$$

and for any $(\sigma(t_s)=i,\sigma(t_s^-)=j)\in S\times S$ one has

$$V_i(x(t_s))\leq\mu V_j(x(t_s)), \tag{6}$$

then the system is globally uniformly asymptotically stable (GUAS) for any switching signal with ADT τ_a satisfying

$$\tau_a>\tau_a^*=\frac{\ln\mu}{\rho}. \tag{7}$$

Lemma 2: [19] Consider the switched system $x(k+1)=f_{\sigma(k)}(x(k))$, $\sigma(k)\in S$, let $1>\gamma>0$, $\mu>1$ be given constants. The system is GUAS for any switching signal with ADT τ_a satisfying

$$\tau_a>\tau_a^*=\frac{\ln\mu}{\ln(\gamma)}, \tag{8}$$

if there exist continuously differentiable functions $V_{\sigma(k)}:\mathbb{R}^n\to\mathbb{R}$ and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s\in S$

$$\beta_1(x_k)\leq V_s(x_k)\leq\beta_2(x_k), \tag{9}$$

$$\dot{V}_i(x_k)\leq-\rho V_s(x_k), \tag{10}$$

and for any $(\sigma(k_s)=i,\sigma(k_s^-)=j)\in S\times S$ one has

$$V_i(x(k_s))\leq\mu V_j(x(k_s)). \tag{11}$$

III. Main Results

A. Continuous-time systems

This subsection will discuss the stability of continuous-time switched system under the limiting ADT. Firstly, we study the case of $\tau_a<\infty$.

Lemma 3: Consider the continuous-time system $\dot{x}(t)=f_{\sigma(t)}(x(t))$, where switching signal $\sigma(t)\in S$ has a limiting average dwell time τ_a. Suppose that there exist continuously differentiable functions $V_{\sigma(t)}:\mathbb{R}^n\to\mathbb{R}$, constant $\rho\in\mathbb{R}$ and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s\in S$

$$\beta_1(x_t)\leq V_s(x_t)\leq\beta_2(x_t), \tag{12}$$

$$\dot{V}_s(x_t)\leq\rho V_s(x_t), \tag{13}$$

and for any $(\sigma(t_s)=i,\sigma(t_s^-)=j)\in S\times S$ we has

$$V_i(x(t_s))\leq\mu_s V_j(x(t_s)), \tag{14}$$

where ρ is a constant, $\mu_k>0$ $(k=1,2,\cdots)$ satisfying

$$\mu=\lim_{t\to\infty}\frac{\mu_1+\mu_2+\cdots\mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)}, \tag{15}$$

then the system is globally uniformly asymptotically stable for any switching signal with limiting ADT τ_a satisfying

$$\gamma_1:=\rho+\frac{\ln\mu}{\tau_a}<0. \tag{16}$$

Proof: For $t\in[t_k,t_{k+1})$, according to (13) we have

$$V_{\sigma(t)}(x(t))\leq e^{\rho(t-t_k)}V_{\sigma(t_k)}(x(t_k)).$$

It follows from (14) that

$$\begin{aligned}V_{\sigma(t)}(x(t)) &\leq e^{\rho(t-t_k)}V_{\sigma(t_k)}(x(t_k))\\ &\leq e^{\rho(t-t_k)}\mu_k V_{\sigma(t_k^-)}(x(t_k))\\ &\leq e^{\rho(t-t_{k-1})}\mu_k V_{\sigma(t_{k-1})}(x(t_{k-1}))\\ &\cdots\\ &\leq e^{\rho(t-t_0)}\mu_k\cdots\mu_0 V_{\sigma(t_0)}(x(t_0)).\end{aligned}$$

Then, based on mean value inequality, we have

$$\begin{aligned}V_{\sigma(t)}(x(t)) &\leq (\frac{\mu_1+\cdots+\mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)})^{N_\sigma(t_0,t)}e^{\rho(t-t_0)}V_0\\ &= e^{N_\sigma(t_0,t)\ln\frac{\mu_1+\cdots+\mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)}}e^{\rho(t-t_0)}V_0\\ &= e^{\omega(t-t_0)}e^{\rho(t-t_0)}V_0,\end{aligned}$$

where $V_0=V_{\sigma(t_0)}(x(t_0))$, $\omega=\ln\frac{\mu_1+\cdots+\mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)}/\frac{t-t_0}{N_\sigma(t_0,t)}$. Then it follows from (15), (16) and Definition 2 that there exist a sufficiently large scalar $T>0$ and a positive constant γ satisfying $\gamma+\gamma_1<0$ such that for any $t\geq T$ one has

$$\begin{aligned}V_{\sigma(t)}(x(t)) &\leq e^{(\frac{\ln\mu}{\tau_a}-\gamma_1-\gamma+\rho)(t-t_0)}V_0.\\ &\leq e^{-\gamma(t-t_0)}V_0,\end{aligned}$$

hence the switching system can achieve the globally uniformly asymptotically stable. ■

In the above lemma, μ is proposed to quantize the average switched parameters. We can reduce this condition to the following form

$$\mu_0-\frac{t-t_0}{\mu_a}\leq N_\sigma(t,t_0)\leq\mu_0+\frac{t-t_0}{\mu_a},$$

where $\mu_0>0$ is a constant. The corresponding proof is omitted here.

In many real-world engineering systems, switching signals will fade away as time goes on. However, no result considered this phenomenon and all the existing results did not give the corresponding description. In this paper, we can used limiting ADT $\tau_a = \infty$ to describe this case. Considered the case that the switches occur infinitely but more and more sparse, we study the stability of switching system with limiting ADT $\tau_a = \infty$.

Lemma 4: For the continuous-time system $\dot{x}(t) = f_{\sigma(t)}(x(t))$, the switching signal $\sigma(t) \in S$ has limiting average dwell time $\tau_a = \infty$. If there exist continuously differentiable functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$ and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s \in S$

$$\beta_1(x_t) \le V_s(x_t) \le \beta_2(x_t), \tag{17}$$

$$\dot{V}_s(x_t) \le -\alpha V_s(x_t), \tag{18}$$

and for any $(\sigma(t_s) = i, \sigma(t_s^-) = j) \in S \times S$ one has

$$V_i(x(t_s)) \le \mu_s V_j(x(t_s)), \tag{19}$$

where α is a positive constant, and $\mu_k > 0$ $k = 1, 2, \cdots$ satisfying

$$\mu = \lim_{t\to\infty} \frac{\mu_1 + \mu_2 + \cdots \mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)}, \tag{20}$$

then the system is globally uniformly asymptotically stable.

Proof: Similar to the proof of Lemma 3, we can get

$$V_{\sigma(t)}(x(t)) \le e^{\omega(t-t_0)} e^{-\alpha(t-t_0)} V_0,$$

where $\omega = \ln \frac{\mu_1+\cdots+\mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)} / \frac{t-t_0}{N_\sigma(t_0,t)}$.

When $\mu \le 1$, one has

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\le \left(\frac{\mu_1 + \cdots + \mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)}\right)^{N_\sigma(t_0,t)} e^{-\alpha(t-t_0)} V_0 \\ &\le e^{-\alpha(t-t_0)} V_0. \end{aligned}$$

It follows from $\alpha > 0$ that the system achieves globally uniformly asymptotically stable.

If $\mu > 1$, then

$$\lim_{t\to\infty} \ln \frac{\mu_1 + \cdots + \mu_{N_\sigma(t_0,t)}}{N_\sigma(t_0,t)} / \frac{t-t_0}{N_\sigma(t_0,t)} = 0.$$

Hence, there exist $\alpha_1 < \alpha$ and $T > 0$ such that for any $t > T$

$$V_{\sigma(t)}(x(t)) \le e^{-\alpha_1(t-t_0)} V_0.$$

Therefore, no matter $\mu < 1$ or $\mu > 1$, the system can achieve globally uniformly asymptotically stable. ■

Based on the above results, we next study the stability of the switching system (1) with limiting ADT τ_a.

Theorem 1: Consider the continuous-time switched linear system (1) under state-feedback controller $u(t) = K_{\sigma(t)}x(t)$ with limiting ADT $\tau_a < \infty$. ρ, γ_1, μ and μ_k $k = 1, 2, \cdots$ are given constants. If there exist positive-definite matrices P_s and R_s, $s \in S$ such that (15), (16) and the following matrix inequalities hold:

$$A_s P_s + B_s R_s + P_s A_s^T + R_s^T B_s^T - \rho P_s \le 0, \tag{21}$$

$$P_i \le \mu_k P_j, \tag{22}$$

where $(i = \sigma(t_k), j = \sigma(t_{k+1}))$, then there exists a set of controllers such that the switched system (1) is GUES. Moreover, if (21) and (22) have a solution, the admissible controller can be given by

$$K_s = R_s P_s^{-1}. \tag{23}$$

Proof: Construct a Lyapunov-type function in the quadratic form $V_s(x(t)) = x^T(t) Q_s x(t)$, $s = \sigma(t) \in S$, where $Q_s = P_s^{-1}$ is a positive definite matrix. Taking the derivative of $V_{\sigma(t)}(x(t))$ along the trajectory of the switched system (1), from (21) we have

$$\dot{V}_s(t) = x^T(t)[Q_s(A_s + B_s K_s) + (A_s + B_s K_s)^T Q_s] x(t),$$

where $\sigma(t) = s \in S$. It follow from $K_s = R_s P_s^{-1}$ and $Q_s = P_s^{-1}$ that

$$\dot{V}_s(t) = x^T(t)[A_s^T Q_s + Q_s A_s + Q_s R_s^T B_s^T Q_s + Q_s B_s R_s Q_s] x(t).$$

Then, based on (21) one has

$$\dot{V}_s(t) \le \rho V_s(x(t)), \sigma(t) = s \in S.$$

On other hand, according to (22) we can obtain

$$V_i(x(t_s)) \le \mu_s V_j(x(t_s)),$$

where $(\sigma(t_s) = i, \sigma(t_s^-) = j) \in S \times S$. Thus, by Lemma 3, the switched system (1) is GUAS with limiting ADT τ_a if (21) and (22) hold. In particular, for any $\varepsilon > 0$, let $\gamma = \rho + \frac{\ln \mu}{\tau_a} - \varepsilon$, $\eta = \frac{1}{2}\gamma$ and $M = \max_{s \in S}\{\frac{\lambda_{max}(Q_s)}{\lambda_{min}(Q_s)}\}$, then there exists a sufficient large scalar $T > 0$ such that the system (1) satisfies

$$\|x(t)\| \le M e^{-\eta(t-t_0)} \|x_0\|, t > T.$$

Thus the switched system (1) is GUES. ■

Now, we study the stability of switched system (1) with limiting ADT $\tau_a = \infty$.

Theorem 2: Consider the continuous-time switched linear system (1) under state-feedback controller $u(t) = K_{\sigma(t)}x(t)$ with limiting ADT $\tau_a = \infty$. Let $\rho < 0$, μ and μ_k $k = 1, 2, \cdots$ be given constants. The switched system (1) is GUES if there exist positive-definition matrices P_s and matrices R_s, $s \in S$ such that (15), (21) and (22) hold. Moreover, if (21) and (22) have a solution, the admissible controller can be given by

$$K_s = R_s P_s^{-1}. \tag{24}$$

The proof of Theorem 2 is omitted here. For the condition of $u(t) = 0$, we have the following corollaries.

Corollary 1: Consider the switched linear system (1) with $u(t) = 0$ and limiting ADT $\tau_a < \infty$. ρ, γ_1, μ and μ_k ($k = 1, 2, \cdots$) satisfying (15) and (16) are given constants. If there exist positive definite matrices P_s, $s \in S$ such that

$$A_s P_s + P_s A_s^T - \rho P_s \le 0, \tag{25}$$

and for any $(i = \sigma(t_k), j = \sigma(t_{k+1}))$

$$P_i \le \mu_k P_j, \tag{26}$$

then the continuous-time switched system (1) is GUES.

Corollary 2: Consider the switched linear system (1) with $u(t) = 0$ and limiting ADT $\tau_a = \infty$. Let $\rho < 0$, μ and μ_k

$(k=1,2,\cdots)$ satisfying (15) be given constants. The switched system (1) is GUES if there exist positive-definition matrices P_s, $s\in S$ such that (25) and (26) hold.

Remark 2: In general, there are two kinds of switchings: one can suppress the stability of switching systems called stabilizing switching and another is adverse called destabilizing switching. For (14), if $\mu_s<1$, then the switching from ith subsystem to jth subsystem enhance the system stability, and the switching from jth subsystem to ith subsystem is inverse at t_k. For example, in multi-agent networks, switching from the a strong connected graph to another which is non-connectivity will go gainst the system stability. What's more, in [1], Morse presented an bilinear system to show that, the system can achieve asymptotic stability without chattering under a switching signal, which cannot be locally asymptotically stabilized with any smooth control.

Remark 3: The profitable switching signals paly a very important role in the stability analysis of switched systems and should not be ignored. However, most of existing studies, such as [20], [23], always adopt the condition (6) to describe the relationship of two successive switching signal, which cannot reflect the impact of stability switchings. In this paper, different parameters μ_k of condition (14) are used in different switching instants. In this way, the positive roles of the stabilizing switchings can be fully utilized. If $\mu_k<1$, then this switching signal is benefit for stable. Hence, our results improve the existing results from this aspect.

Remark 4: This paper proposes a unified stability criterion of switching system $\dot{x}(t)=f_{\sigma(t)}(x(t))$, which has never been derived. Notice that, compared with Lemma 1, ρ does not need to be negative, and μ is not required to satisfy $\mu>1$ or $\mu<1$. When $\rho<0$, i.e. all the subsystems are stable, μ_k can be lager than 1, while when $\rho>0$, there exists $\mu_k<1$ such that (16) holds, i.e. the effect of stabilizing switching is stronger.

Remark 5: It follows from (6) that the most previous work only used a parameter μ to describe the relationship of two consecutive activated subsystems, which is less practical. In [23], the authors proposed the mode-dependent parameters μ_p. In this paper, we extended it to different parameter μ_k which depended on the switching signal, such as inequality (14). Comparing with the existing work, the switching-dependent parameters improve the mutual parameters and model-dependent parameters in previous literatures and reduce the conservativeness.

B. Discrete-time systems

Lemma 5: Consider the discrete-time system $x(k+1)=f_{\sigma(k)}(x(k))$, where switching signal $\sigma(k)\in S$ has a limiting average dwell time τ_a. If there exist continuously differentiable functions $V_{\sigma(k)}:\mathbb{R}^n\to\mathbb{R}$, constant $\hat{\theta}>-1$ and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s\in S$

$$\beta_1(x(k))\le V_s(x(k))\le\beta_2(x(k)), \tag{27}$$

$$\triangle V_i(x(k))\le\hat{\theta}V_s(x(k)), \tag{28}$$

hold and for any $(\sigma(k_s)=i,\sigma(k_s^-)=j)\in S\times S$ one has

$$V_i(x(k_s))\le\mu_s V_j(x(k_s)), \tag{29}$$

where ρ is a constant, $\mu_k>0$ $k=1,2,\cdots$ satisfying

$$\mu=\lim_{k\to\infty}\frac{\mu_1+\mu_2+\cdots\mu_{N_\sigma(k_0,k)}}{N_\sigma(k_0,k)}, \tag{30}$$

then the system is globally uniformly asymptotically stable for any switching signal with ADT

$$\vartheta_1:=\ln\theta+\frac{\ln\mu}{\tau_a}<0, \tag{31}$$

where $\theta=1+\hat{\theta}$.

Proof: Based on (28) we have

$$V_{\sigma(k)}(x(k))\le\theta^{(k-k_s)}V_{\sigma(k_s)}(x(k_s)),\ k\in[k_s,k_{s+1}).$$

By (29), we can obtain

$$\begin{aligned}V_{\sigma(k)}(x(k)) &\le \theta^{(k-k_s)}\mu_s V_{\sigma(k_s^-)}(x(k_s))\\ &\le \theta^{(k-k_{s-1})}\mu_s V_{\sigma(k_{s-1})}(x(k_{s-1}))\\ &\quad \cdots\\ &\le \theta^{(k-k_0)}\mu_s\cdots\mu_0 V_{\sigma(k_0)}(x(k_0))\\ &= e^{\ln\theta(k-k_0)}\mu_s\cdots\mu_0 V_0,\end{aligned}$$

where $V_0=V_{\sigma(k_0)}(x(k_0))$.

Then, by means of mean value inequality, we have

$$\begin{aligned}V_{\sigma(k)}(x(k)) &\le (\frac{\mu_1+\cdots+\mu_{N_\sigma(k_0,k)}}{N_\sigma(k_0,k)})^{N_\sigma(k_0,k)}e^{\ln\theta(k-k_0)}V_0\\ &= e^{N_\sigma(k_0,k)\ln\frac{\mu_1+\cdots+\mu_{N_\sigma(k_0,k)}}{N_\sigma(k_0,k)}}e^{\ln\theta(k-k_0)}V_0\\ &= e^{\omega(k-k_0)}e^{\ln\theta(k-k_0)}V_0,\end{aligned}$$

where $\omega=\ln\frac{\mu_1+\cdots+\mu_{N_\sigma(k_0,k)}}{N_\sigma(k_0,k)}/\frac{k-k_0}{N_\sigma(k_0,k)}$. Then it follows from (30), (31) and Definition 2 that there exist a sufficiently large scalar $T>0$ and a positive constant ϑ satisfying $\vartheta+\vartheta_1<0$ such that for any $k\ge T$ one has

$$\begin{aligned}V_{\sigma(k)}(x(k)) &\le e^{(\frac{\ln\mu}{\tau_a}-\vartheta_1-\vartheta+\ln\theta)(k-k_0)}V_0\\ &\le e^{-\vartheta(k-k_0)}V_0,\end{aligned}$$

hence the switching system achieves globally uniformly asymptotically stable. ■

Next, we will consider the case of $\tau_a=\infty$ for discrete-time systems. According to Lemma 4 and Lemma 5, the proof of the following result can be omitted.

Lemma 6: Consider the discrete-time system $x(k+1)=f_{\sigma(k)}(x(k))$, where switching signal $\sigma(k)\in S$ has a limiting average dwell time $\tau_a=\infty$. If there exist continuously differentiable functions $V_{\sigma(k)}:\mathbb{R}^n\to\mathbb{R}$, a positive constant β, and two class $\mathscr{K}_\infty$ functions β_1 and β_2 such that for any $s\in S$

$$\beta_1(x(k))\le V_s(x(k))\le\beta_2(x(k)), \tag{32}$$

$$\dot{V}_i(x(k))\le-\beta V_s(x(k)), \tag{33}$$

and for any $(\sigma(k_s)=i,\sigma(k_s^-)=j)\in S\times S$ one has

$$V_i(x(k_s))\le\mu_s V_j(x(k_s)), \tag{34}$$

where $\mu_k > 0$ $(k = 1, 2, \cdots)$ satisfying

$$\mu = \lim_{k\to\infty} \frac{\mu_1 + \mu_2 + \cdots \mu_{N_\sigma(k_0,k)}}{N_\sigma(k_0,k)}, \tag{35}$$

then the system is globally uniformly asymptotically stable.

Using Lemma 5 and Lemma 6 for the linear switched system (1), we can obtain the following results.

Theorem 3: Consider the linear switched discrete-time system (1) under state-feedback controller $u(k) = K_{\sigma(k)}x(k)$ with limiting ADT $\tau_a < \infty$. Suppose that θ, ϑ_1, μ and μ_i $i = 1, 2, \cdots$ are given constants which satisfy (30) and (31). If there exist positive-definition matrices P_s and matrices R_s, $s \in S$ such that

$$A_s P_s + B_s R_s + P_s A_s^T + R_s^T B_s^T - \rho P_s \leq 0, \tag{36}$$

and for any $(i = \sigma(k_s), j = \sigma(k_{s+1}))$

$$P_i \leq \mu_s P_j, \tag{37}$$

then there exist a set of controllers such that the switched system (1) is GUES. Besides, if (21) and (22) have a solution, the admissible controller can be given by

$$K_s = R_s P_s^{-1}. \tag{38}$$

Theorem 4: Consider the linear switched discrete-time system (1) under state-feedback controller $u(k) = K_{\sigma(k)}x(k)$ with limiting ADT $\tau_a = \infty$. Let $\beta < 0$, μ and μ_k $k = 1, 2, \cdots$ satisfying (30) be given constants. If there exist matrices P_s and R_s, $s \in S$ such that (36) and (37) hold, then there exist a set of controllers such that the switched system (1) is GUES. Additionally, if (36) and (37) have a solution, the admissible controller can be given by

$$K_s = R_s P_s^{-1}. \tag{39}$$

For the condition of $u(t) = 0$ in discrete-time systems, we have the following results.

Corollary 3: Consider the linear switched discrete-time system (1) with $u(k) = 0$ and limiting ADT $\tau_a < \infty$. Suppose that θ, ϑ_1, μ and μ_i $(i = 1, 2, \cdots)$ are given constants which satisfy (30) and (31). If there exist positive-definition matrices P_s, $s \in S$ such that

$$A_s P_s + P_s A_s^T - \rho P_s \leq 0, \tag{40}$$

and for any $(i = \sigma(k_s), j = \sigma(k_{s+1}))$

$$P_i \leq \mu_s P_j, \tag{41}$$

then the switched system (1) is GUES.

Corollary 4: Consider the linear switched discrete-time system (1) with $u(k) = 0$ and limiting ADT $\tau_a = \infty$. Let $\beta < 0$, μ and μ_k $k = 1, 2, \cdots$ satisfying (30) be given constants. The switched system (1) is GUES if there exist positive definition matrices P_s, $s \in S$ such that (40) and (41) hold.

IV. Numerical Results

Example 1: Consider the switched system (1) consisting of the following three subsystems:

$$A_1 = \begin{pmatrix} 3.5 & 2.1 \\ 0 & 5 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1.5 & 2 \\ 2.01 & 1.7 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & -0.5 \\ 1.2 & -1.7 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} -0.2 \\ 0.1 \end{pmatrix}, B_2 = \begin{pmatrix} 0.17 \\ 0.1 \end{pmatrix}, B_3 = \begin{pmatrix} 0.11 \\ 0.23 \end{pmatrix}$$

In order to show the valid of our results, let the Lyapunov-type functions be $V_s(t) = x^T(t)Q_s x(t)$, $s \in S = \{1, 2, 3\}$, where Q_s will be determined later. Consider the condition of limiting average dwell time $\tau_a = 0.7$, μ_{ij} are given positive constants satisfying $V_i(x(t)) \leq \mu_{ij}V_j(x(t))$, $i, j = 1, 2, 3$. $U = (\mu_{ij})$ is defined as follows

$$U = (\mu_{ij}) = \begin{pmatrix} 0 & 1.6 & 1.2 \\ 0.62 & 0 & 0.6 \\ 0.7 & 3 & 0 \end{pmatrix}.$$

It is assume that the probabilities of the switchings between any two subsystems are same. Fig. 1 shows the switching signals. By calculating, we can get $\mu = 1.29$. For previous studies [19], they must restrict that $\mu = \max_{i,j}\{\mu_{ij}\} = 3$ and according to [23], one can get the mode-dependent parameters are $\mu_1 = 1.6$, $\mu_2 = 0.62$ and $\mu_3 = 3$. From the above comparison, our results are more flexible and less conservative. Based on Theorem 1, choose matrices P_i and Q_i, $i \in S$ satisfying (16), (21) and (22). Let $u(t) = K_{\sigma(t)}x(t)$ and $K_s = R_s P_s^{-1}$, then we can get

$$K_1 = \begin{pmatrix} 63.98 & 53.25 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} -85.81 & -45.28 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} -33.47 & -67.81 \end{pmatrix}.$$

Then, the condition (21) and (22) are satisfied. Hence, it follows from Theorem 1 that the stability of switched system with limiting ADT $\tau_a = 0.7$ can be realized. The trajectory of $x(t)$ can be see in Fig. 2.

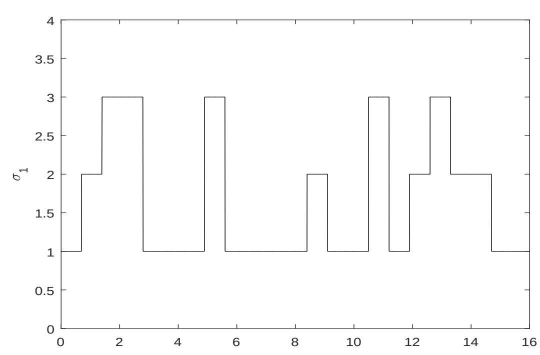

Fig. 1. The switching signal with $\tau_a = 0.7$ and $\mu = 1.29$. The following is repeated this sequence.

Example 2: Consider the case that limiting average dwell time is infinite, *i.e.* $\tau_a = \infty$. The corresponding switching signal is shown in Fig. 3, where $N_{\sigma(t)}(t_0, t) = \sqrt[3]{t - t_0}$. The model and the subsystems are same as Example 1. Let $\mu_{ij} = 2$ for any $i, j = 1, 2, 3$, then by LMI tools in MATLAB, we can find matrices P_i and R_i satisfying (21) and (22). Based on Theorem

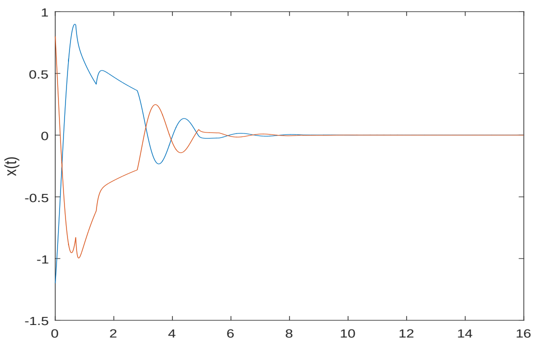

Fig. 2. The state trajectory of closed-loop system (1) in Example 1.

2, it follows that the stability of switched linear system 1 with limiting ADT $\tau_a = \infty$ is obtained. Fig. 4 shows the trajectory of $x(t)$.

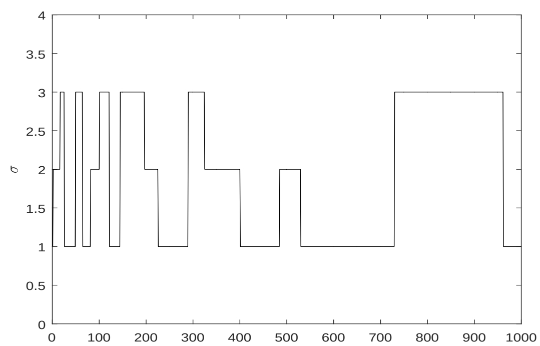

Fig. 3. The switching signal with $\tau_a = \infty$ and $N_{\sigma(t)}(t_0, t) = \sqrt[3]{t - t_0}$ in Example 2.

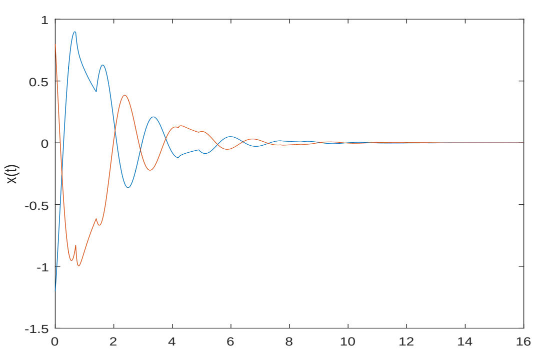

Fig. 4. The state trajectory of closed-loop system (1) in Example 2.

V. Conclusion

This paper firstly proposed a novel concept of limiting average dwell time inspired by average impulsive interval. Moreover, the limiting ADT can be infinite, which extended the concept of ADT. On the other hand, considering the fact that some switchings are helpful for the stability of switched systems, switching-dependent parameters were adopted to describe the relationship of two consecutive activated switchings. Some sufficient conditions were derived to ensure the stability of linear switching systems both in continuous-time and discrete-time circumstances.

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