# A Hodge decomposition method for dynamic Ginzburg–Landau equations in nonsmooth domains — a second approach

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**Abstract.** In a general polygonal domain, possibly nonconvex and multi-connected (with holes), the time-dependent Ginzburg–Landau equation is reformulated into a new system of equations. The magnetic field $B := \nabla\times\mathbf{A}$ is introduced as an unknown solution in the new system, while the magnetic potential $\mathbf{A}$ is solved implicitly through its Hodge decomposition into divergence-free part, curl-free and harmonic parts, separately. Global well-posedness of the new system and its equivalence to the original problem are proved. A linearized and decoupled Galerkin finite element method is proposed for solving the new system. The convergence of numerical solutions is proved based on a compactness argument by utilizing the maximal $L^p$-regularity of the discretized equations. Compared with the Hodge decomposition method proposed in [27], the new method has the advantage of approximating the magnetic field $B$ directly and converging for initial conditions that are incompatible with the external magnetic field. Several numerical examples are provided to illustrate the efficiency of the proposed numerical method in both simply connected and multi-connected nonsmooth domains. We observe that even in simply connected domains, the new method is superior to the method in [27] for approximating the magnetic field.

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**Key words**: superconductivity, reentrant corner, singularity, multi-connected domain, finite element method, convergence, Hodge decomposition

## 1 Introduction

The time-dependent Ginzburg–Landau equation (TDGL) is widely used for numerical simulations of vortex dynamics of superconducting density and magnetic field for type-II superconductors [8,14,19,29]. In this model, the state of a superconductor is described by a complex-valued order parameter $\psi$, a real vector-valued magnetic potential $\mathbf{A}$, and a real scalar-valued electric potential $\phi$. In a two-dimensional domain, the TDGL can be written as (with non-dimensionalisation)

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi+i\eta\kappa\psi\phi=0, \tag{1.1}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\times(\nabla\times\mathbf{A})+\nabla\phi+\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=\nabla\times H, \tag{1.2}$$

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with the following notations of curl, divergence and gradient operators:

$$\nabla\times\mathbf{A}=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2},\qquad \nabla\cdot\mathbf{A}=\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2},$$
$$\nabla\times H=\left(\frac{\partial H}{\partial x_2},-\frac{\partial H}{\partial x_1}\right),\quad \nabla\psi=\left(\frac{\partial\psi}{\partial x_1},\frac{\partial\psi}{\partial x_2}\right).$$

The time-independent external magnetic field $H$ is given, $\eta$ and $\kappa$ are positive physical parameters, and $\overline{\psi}$ denotes the complex conjugate of $\psi$. The physically interesting quantities in this model are the magnetic field $B=\nabla\times\mathbf{A}$ and the superconductivity density $|\psi|^2$, which satisfies $0\le|\psi|^2\le 1$ and represents the superconducting state of a superconductor. In particular, $|\psi|^2=1$ indicates that the superconductor is in the superconducting state, and $|\psi|^2=0$ indicates the normal state. If the superconductor occupies a domain $\Omega$, then the following physical boundary conditions hold:

$$\left(\frac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{1.3}$$
$$\nabla\times\mathbf{A}=H \qquad \text{on } \partial\Omega, \tag{1.4}$$
$$\mathbf{A}\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$.

In addition to (1.1)-(1.2), one needs a gauge condition to determine the solution uniquely [1,12]. For example, the zero electric potential gauge $\phi=0$ and the Lorentz gauge $\phi=-\nabla\cdot\mathbf{A}$ are often used for numerical simulations [10,11,20,23,33,35,36]. The solutions under the different gauges are equivalent in producing the physical quantities $|\psi|^2$ and $B$ (see [12]). In this paper, we focus on the Lorentz gauge $\phi=-\nabla\cdot\mathbf{A}$, which reduces the TDGL to the following equations:

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi-i\eta\kappa\psi\nabla\cdot\mathbf{A}=0, \tag{1.6}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\times(\nabla\times\mathbf{A})-\nabla(\nabla\cdot\mathbf{A})+\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=\nabla\times H. \tag{1.7}$$

The above equations can be solved for given initial data

$$\psi(\cdot,0)=\psi_0,\quad \mathbf{A}(\cdot,0)=\mathbf{A}_0 \quad \text{in}\,\Omega. \tag{1.8}$$

Existence and uniqueness of solution for the system (1.3)-(1.8) has been proved in [12] for smooth domains. Numerical simulations and analysis in [10,11] also show that the finite element solutions of (1.3)-(1.8) converge to the PDE's solution if the computational domain $\Omega$ is smooth. Independently, numerical analysis of the TDGL in smooth or convex domains under the zero electric potential gauge $\phi=0$ was presented in [18,32,36] for different numerical methods.

As well as smooth or convex domains, numerical approximation of the TDGL in nonsmooth domains with reentrant corners is also important for physicists and engineers [3,13, 35]. In this case, the magnetic potential $\mathbf{A}$ may not be in $H^1(\Omega)\times H^1(\Omega)$, and the Galerkin finite element method (FEM) may yield spurious solutions [27]. To overcome this computational difficulty, the TDGL was reformulated into the following form in [27] (with proper boundary conditions):

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi-i\eta\kappa\psi\nabla\cdot\mathbf{A}=0, \tag{1.9}$$

$$\Delta p = -\nabla\times\left(\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]\right) \tag{1.10}$$

$$\Delta q = \nabla\cdot\left(\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]\right) \tag{1.11}$$

$$\frac{\partial u}{\partial t} - \Delta u = H - p, \tag{1.12}$$

$$\frac{\partial v}{\partial t} - \Delta v = -q, \tag{1.13}$$

where $p$ and $q$ are auxiliary variables, and $u$ and $v$ can be regarded as anti-derivatives of the magnetic potential $\mathbf{A}$ in the following sense:

$$\mathbf{A} = \nabla\times u + \nabla v. \tag{1.14}$$

In a nonconvex polygon, the unknown variables $\psi,p,q,u,v$ of the reformulated system (1.9)-(1.13) are expected to be in $H^1(\Omega)$. Hence, applying Galerkin FEMs to solve (1.9)-(1.13) is expected to yield correct solutions of the physical quantity $|\psi|$ (but may not for $B$). This has been proved in [28]. A Galerkin FEM for solving the TDGL under the zero electric potential gauge $\phi=0$ was presented in [21] recently. This method introduces $B=\nabla\times\mathbf{A}$ as an unknown solution while evaluates $\mathbf{A}$ by integrating the equation (1.2) using the explicit Euler method. Convergence of this method was proved in smooth domains [37] and remains open in nonsmooth domains.

Besides Galerkin FEMs, a mixed FEM was proposed in [20] for solving (1.6)-(1.8) directly in nonsmooth and nonconvex domains. Error estimates for this numerical method was proved in [22] for both $\psi$ and $B$ based on reasonable regularity assumptions on solutions, which requires the external magnetic field to be compatible with the initial data. For possibly incompatible initial data, the well-posedness of the TDGL was proved in [26,28] for general polygonal and polyhedral domains, and the convergence of numerical solutions was proved in [25] for an alternative mixed FEM by a compactness argument, without regularity assumptions on the solutions. Besides numerical methods for the TDGL under the Lorentz gauge, the discrete gauge invariant finite difference method introduced in [17] preserves the gauge invariant property in the discrete settings. Convergence of this numerical method was proved in rectangular domains.

Overall, in a nonconvex and nonsmooth domain, the standard Galerkin FEM for (1.6)-(1.8) often yields spurious solutions. The analysis of a Galerkin FEM in [28] for solving (1.9)-(1.13) only focused on the approximation of $|\psi|$ and is limited to simply connected domains due to the formula (1.14) used therein. The convergence of numerical solutions of the magnetic field $B$ in nonsmooth, nonconvex and possibly multi-connected domains was proved only for some mixed FEMs [22,25] but remains open for Galerkin FEMs, which are often preferred by physicists and engineers in numerical simulations.

In this paper, we develop a new method for solving the TDGL in a general polygonal domain, possibly nonconvex and multi-connected, by reformulating the TDGL into an equivalent system of PDEs based on the Hodge decomposition approach introduced by Brenner et. al. [7] for the Maxwell equations. The equivalent system is discretized by a linearized and decoupled Galerkin FEM, and the convergence of numerical solutions is proved. The main contributions of this paper include:

- In order to approximate the magnetic field, we introduce $B:=\nabla\times\mathbf{A}$ as an unknown solution in the new system, while $\mathbf{A}$ is solved implicitly through its Hodge decomposition. Compared with the method proposed in [27], the new method is superior in approximating the magnetic field $B$ (even in simply connected domains) and therefore is not a straightforward extension of the old method to multi-connected domains.

- The proposed new method makes it possible to prove convergence of numerical solutions with more general initial conditions that can be incompatible with the external magnetic field $H$ (this is the case of the numerical examples considered in [10,11,32]).
- Convergence in $C([0,T];L^2(\Omega))$ for the numerical solutions of the physical quantities $|\psi|$ and $B$ is proved based on the regularity assumptions on the initial data and external magnetic field, without any regularity assumptions on the solutions.
- The standard energy estimates are not sufficient for proving the compactness and convergence of the nonlinear terms. To overcome this difficulty, $\ell^p(H^1)$ estimates (Lemma 4.3) with $2\le p<4$ are established and utilized.

## 2 Hodge decomposition of the TDGL in multi-connected domains

### 2.1 Formal derivation

Let $\Omega$ be polygonal type domain (a domain with piecewise linear boundary) with $m$ holes, i.e., $\Omega=\Omega_0\backslash\big(\cup_{j=1}^m\Omega_j\big)$ with $\Gamma_j=\partial\Omega_j$, as shown in Figure 2.1. Under this setting, we have $\partial\Omega=\cup_{k=0}^m\Gamma_k$.

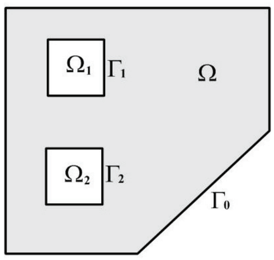



Figure 2.1: Illustration of the domain $\Omega$ (shadow part).

The Hodge decomposition says that the vector field $\mathbf{A}$ can be decomposed into its divergence-free part, curl-free part and harmonic part (see Appendix A), i.e.

$$\mathbf{A}=\nabla\times u+\nabla v+\sum_{j=1}^m\alpha_j\nabla\times\varphi_j. \tag{2.1}$$

where $u$ and $v$ are solutions of the elliptic equations

$$\begin{cases}-\Delta u=\nabla\times\mathbf{A} & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}\qquad\text{and}\qquad\begin{cases}\Delta v=\nabla\cdot\mathbf{A} & \text{in }\Omega,\\ \nabla v\cdot\mathbf{n}=0 & \text{on }\partial\Omega,\end{cases} \tag{2.2}$$

and $\varphi_j$ is solution of

$$\begin{cases}-\Delta\varphi_j=0 & \text{in }\Omega,\\ \varphi_j=\delta_{jk} & \text{on }\Gamma_k,\quad k=0,1,\cdots,m,\end{cases} \tag{2.3}$$

with $\delta_{jk}$ denoting the Kronecker symbol. Note that $u$ is zero on the boundary, and $\varphi$ has piecewise constant boundary conditions, i.e., constant on each $\Gamma_j$. This implies $\mathbf{n}\cdot\nabla\times u=\mathbf{n}\cdot\nabla\times\varphi_j=0$ on $\partial\Omega$, consistent with the boundary condition $\mathbf{A}\cdot\mathbf{n}=0$ on $\partial\Omega$.

Let $B=\nabla\times\mathbf{A}$. Then the curl of (1.7) yields

$$\frac{\partial B}{\partial t}-\Delta(B-H)+\nabla\times\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=0, \tag{2.4}$$

with the Dirichlet boundary condition $B=H$ on $\partial\Omega$, and the divergence of (1.7) gives

$$\frac{\partial\phi}{\partial t}-\Delta\phi-\nabla\cdot\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=0. \tag{2.5}$$

Since $\nabla\times(B-H)\cdot\mathbf{n}$ coincides with the tangential derivative of $B-H$ on the boundary $\partial\Omega$ where $B-H=0$, it follows that $\nabla\times(B-H)\cdot\mathbf{n}=0$ on $\partial\Omega$. In view of the boundary conditions (1.3) and (1.5), the inner product of the equation (1.7) with the normal vector $\mathbf{n}$ on the boundary yields

$$\nabla\phi\cdot\mathbf{n}=-\frac{\partial(\mathbf{A}\cdot\mathbf{n})}{\partial t}-(\nabla\times B)\cdot\mathbf{n}-\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]\cdot\mathbf{n}=0 \quad \text{on } \partial\Omega. \tag{2.6}$$

It remains to determine the coefficients $\alpha_j$ in (2.1). To this end, we substitute (2.1) into (1.7) and obtain

$$\nabla\times\left(\frac{\partial u}{\partial t}-\Delta u-H\right)+\nabla\left(\frac{\partial v}{\partial t}-\Delta v\right)+\sum_{j=1}^{m}\alpha_j'(t)\nabla\times\varphi_j=-\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]. \tag{2.7}$$

Then integrating this equation against $\nabla\times\varphi_k$ and using integration by parts (since $\mathbf{n}\cdot\nabla\times\varphi_j=0$ on $\partial\Omega$, the integration by parts does not yield boundary terms), we obtain

$$\sum_{j=1}^{m}M_{kj}\alpha_j'(t)=-\left(\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right],\nabla\times\varphi_k\right), \tag{2.8}$$

where $M_{kj}:=(\nabla\times\varphi_j,\nabla\times\varphi_k)$ is a positive definite $m\times m$ matrix [7] ($m$ denoting the number of holes inside the domain $\Omega$).

To summarize, the TDGL system (1.6)-(1.8) can be reformulated into the following system of equations:

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi+i\eta\kappa\psi\phi=0, \tag{2.9}$$

$$\frac{\partial B}{\partial t}-\Delta(B-H)+\nabla\times\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=0, \tag{2.10}$$

$$\frac{\partial\phi}{\partial t}-\Delta\phi-\nabla\cdot\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=0, \tag{2.11}$$

$$\Delta u=-B \tag{2.12}$$

$$\Delta v=-\phi \tag{2.13}$$

$$\sum_{j=1}^{m}M_{kj}\alpha_j'(t)=-\left(\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right],\nabla\times\varphi_k\right), \quad k=1,\ldots,m, \tag{2.14}$$

where $M_{kj}=(\nabla\times\varphi_j,\nabla\times\varphi_k)$ and $\mathbf{A}$ is given by (2.1), expressed in terms of $u$, $v$ and $\alpha_j$. The corresponding boundary conditions are

$$\nabla\psi\cdot\mathbf{n}=0, \quad B=H, \quad \nabla\phi\cdot\mathbf{n}=u=\nabla v\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{2.15}$$

and the initial conditions are given by

$$\psi(\cdot,0)=\psi_0, \quad B(\cdot,0)=\nabla\times\mathbf{A}_0, \quad \phi(\cdot,0)=-\nabla\cdot\mathbf{A}_0. \tag{2.16}$$

## 2.2 Well-posedness and equivalence to the original problem

For $1<q<\infty$ and $s\in\mathbb{R}$, we denote by $W^{s,q}$, $L^q$ and $L^\infty$ the conventional Sobolev spaces of real-valued functions defined on $\Omega$ (see [2]). The notation $W_0^{s,q}$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{s,q}$, and $\widetilde{W}^{s,q}$ denotes the space of functions in $W^{s,q}$ whose zero extension to $\mathbb{R}^2$ is in $W^{s,q}(\mathbb{R}^2)$. For simplicity of notation, we denote $H^s=W^{s,2}$, $H_0^s=W_0^{s,2}$ and $\widetilde{H}^s=\widetilde{W}^{s,2}$ for $s\in\mathbb{R}$. Then $\widetilde{H}^{-s}$ coincides with the dual spaces of $H^s$ and $\widetilde{H}^s=H_0^s$ for $s\neq\frac{1}{2},\ \frac{3}{2},\ \dots$ (cf. [31, Theorems 3.30 and 3.33]).

The complex-valued extensions of these function spaces are denoted by $\mathcal{W}^{s,q}$, $\mathcal{L}^q$, $\mathcal{W}_0^{s,q}$, $\widetilde{\mathcal{W}}^{s,q}$, $\mathcal{H}^s$, $\mathcal{H}_0^s$, and $\widetilde{\mathcal{H}}^s$, respectively.

For a Banach space $X$ and a nonnegative integer $k$, we define $W^{k,p}(0,T;X)$ to be the space of functions $f:(0,T)\to X$ equipped with the following norm:

$$\|f\|_{W^{k,p}(0,T;X)}=\left(\int_0^T\sum_{\ell=0}^k\|\partial_t^\ell f(\cdot,t)\|_X^p\mathrm{d}t\right)^{\frac{1}{p}}, \tag{2.17}$$

with the conventional notation $L^p(0,T;X)=W^{0,p}(0,T;X)$. The notation $C([0,T];X)$ denotes the space of functions which are continuous in the time direction, taking values in the Banach space $X$, equipped with the norm

$$\|f\|_{C([0,T];X)}=\max_{t\in[0,T]}\|f(\cdot,t)\|_X. \tag{2.18}$$

Analogous to (2.17), for a sequence $f^n\in X$, $n=1,\dots,N$, we define a time-discrete norm:

$$\|(f^n)_{n=1}^N\|_{\ell^p(X)}:=\left(\sum_{n=1}^N\tau\|(f^n)_{n=1}^N\|_X^p\right)^{\frac{1}{p}},\quad 1\le p\le\infty. \tag{2.19}$$

The well-posedness of the new system (2.9)-(2.16) and its equivalence with the original problem (1.3)-(1.8) is given in the following theorem.

**Theorem 2.1.** *If the initial data and external magnetic field satisfy*

$$\psi_0\in\mathcal{H}^1,\quad \mathbf{A}_0\in\mathbf{L}^4,\quad B_0=\nabla\times\mathbf{A}_0\in H^1,\quad \phi_0=-\nabla\cdot\mathbf{A}_0\in H^1,\quad H\in H^1, \tag{2.20}$$

*and $\alpha_j(0)\in\mathbb{R}$, $j=1,\dots,m$, are determined by*

$$\sum_{j=1}^m(\nabla\times\varphi_j,\nabla\times\varphi_k)\alpha_j(0)=(\mathbf{A}_0,\nabla\times\varphi_k),\quad k=1,\dots,m,$$

*then we have the following results:*

*(i) there exists a weak solution of the reformulated system* (2.9)-(2.16) *with following the regularity:*

$$\begin{aligned}
&\psi\in C([0,T];\mathcal{L}^2)\cap L^\infty(0,T;\mathcal{H}^1)\cap L^2(0,T;\mathcal{W}^{1,4}),\quad \partial_t\psi\in L^2(0,T;\mathcal{L}^2),\\
&B-H\in C([0,T];L^2)\cap L^2(0,T;H_0^1),\quad \partial_tB\in L^2(0,T;H^{-1}),\\
&\phi\in C([0,T];L^2)\cap L^2(0,T;H^1),\quad \partial_t\phi\in L^2(0,T;(H^1)'),\\
&u\in C([0,T];W_0^{1,4}),\quad v\in C([0,T];W^{1,4}),\quad \partial_tu\in L^2(0,T;H_0^1),\\
&\partial_tv\in L^2(0,T;H^1),\quad \alpha_j\in W^{1,\infty}(0,T),\quad \mathbf{A}\in C([0,T];\mathbf{L}^4).
\end{aligned} \tag{2.21}$$

*(ii) the weak solution of* (2.9)-(2.16) *with the regularity* (2.21) *is unique.*

*(iii) the unique weak solution of* (2.9)-(2.16) *coincides with the unique weak solution of* (1.3)-(1.8) *with the regularity*

$$\begin{aligned}
&\psi \in C([0,T];\mathcal{L}^2)\cap L^\infty(0,T;\mathcal{H}^1)\cap L^2(0,T;\mathcal{W}^{1,4}), \quad \partial_t\psi \in L^2(0,T;\mathcal{L}^2),\\
&\mathbf{A}\in C([0,T];\mathbf{L}^4), \quad \partial_t\mathbf{A}\in L^2(0,T;\mathbf{L}^2),\\
&\nabla\times\mathbf{A}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1), \quad \nabla\cdot\mathbf{A}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1).
\end{aligned} \tag{2.22}$$

The proof of Theorem 2.1 is presented in Sections 4.5 and 4.6.

**Remark 2.1.** Since we have not assumed the compatibility condition $B_0 = H$ at the initial time $t=0$, it follows that $B-H$ is not in $C([0,T];H_0^1)$.

# 3 A linearized and decoupled Galerkin finite element method

## 3.1 Time discretization

In this subsection, we propose a linearized and decoupled time-stepping scheme for solving the new system (2.9)-(2.16) introduced in this paper.

Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of the time interval $[0,T]$ and define $\tau=T/N$. For any given

$$(\psi^{n-1},B^{n-1},\phi^{n-1},u^{n-1},v^{n-1},\alpha^{n-1}),$$

we define

$$\mathbf{A}^{n-1}=\nabla\times u^{n-1}+\nabla v^{n-1}+\sum_{j=1}^{m}\alpha_j^{n-1}\nabla\times\varphi_j \tag{3.1}$$

and solve $(\psi^n,B^n,\phi^n,u^n,v^n,\alpha^n)$ from the following decoupled linear equations:

$$\eta\frac{\psi^n-\psi^{n-1}}{\tau}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}^{n-1}\right)^2\psi^n+(|\psi^{n-1}|^2-1)\psi^n+i\eta\kappa\psi^n\phi^{n-1}=0, \tag{3.2}$$

$$\frac{B^n-B^{n-1}}{\tau}-\Delta(B^n-H)+\nabla\times\mathrm{Re}\left[\overline{\psi}^{n-1}\left(\frac{i}{\kappa}\nabla+\mathbf{A}^{n-1}\right)\psi^n\right]=0, \tag{3.3}$$

$$\frac{\phi^n-\phi^{n-1}}{\tau}-\Delta\phi^n-\nabla\cdot\mathrm{Re}\left[\overline{\psi}^{n-1}\left(\frac{i}{\kappa}\nabla+\mathbf{A}^{n-1}\right)\psi^n\right]=0, \tag{3.4}$$

$$\Delta u^n=-B^n \tag{3.5}$$

$$\Delta v^n=-\phi^n \tag{3.6}$$

$$\sum_{j=1}^{m}M_{kj}\frac{\alpha_j^n-\alpha_j^{n-1}}{\tau}=-\left(\mathrm{Re}\left[\overline{\psi}^{n-1}\left(\frac{i}{\kappa}\nabla+\mathbf{A}^{n-1}\right)\psi^n\right],\nabla\times\varphi_k\right), \tag{3.7}$$

with the boundary conditions

$$\nabla\psi^n\cdot\mathbf{n}=0, \quad B^n=H, \quad \nabla\phi^n\cdot\mathbf{n}=u^n=\nabla v^n\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{3.8}$$

and the initial conditions

$$\psi^0=\psi_0, \quad B^0=\nabla\times\mathbf{A}_0, \quad \phi^0=-\nabla\cdot\mathbf{A}_0. \tag{3.9}$$

The initial data $u^0$ and $v^0$ can be calculated by

$$\left\{\begin{array}{ll} -\Delta u^0=\nabla\times\mathbf{A}_0 & \text{in } \Omega,\\ u^0=0 & \text{on } \partial\Omega,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} \Delta v^0=\nabla\cdot\mathbf{A}_0 & \text{in } \Omega,\\ \nabla v^0\cdot\mathbf{n}=0 & \text{on } \partial\Omega,\end{array}\right. \tag{3.10}$$

respectively.

### 3.2 Full discretization

Let $\Omega$ be triangulated quasi-uniformly and denote the complex-valued $L^2$ inner product on $\Omega$ by

$$(f,g) := \int_\Omega f(x)\overline{g(x)}\mathrm{d}x.$$

Let $\mathcal{V}_h$ be the space of complex-valued globally continuous piecewise polynomials of degree $r\geq 1$ defined on $\Omega$. Let $V_h$ be the subspace of $\mathcal{V}_h$ consisting of real-valued functions, and let $\mathring{V}_h$ be the subspace of $V_h$ consisting of functions which are zero on $\partial\Omega$.

For any given $k=0,1,\cdots,m$, let $\Gamma_{k,h}$ be the piecewise linear approximation of $\Gamma_k$ subject to the triangulation. For $j=1,\ldots,m$, we let $\varphi_{h,j}\in V_h$ be the finite element solution of

$$(\nabla\varphi_{h,j},\nabla\chi_h)=0, \quad \forall\chi_h\in\mathring{V}_h, \tag{3.11}$$

with the Dirichlet boundary condition

$$\varphi_{h,j}=\delta_{jk} \quad \text{on}\,\Gamma_{k,h},\, k=0,1,\cdots,m. \tag{3.12}$$

The functions $\varphi_{h,j}$, $j=1,\ldots,m$, are finite element approximations of the harmonic functions $\varphi_j$, $j=1,\ldots,m$, and the $m\times m$ matrix

$$M_{h,kj}=(\nabla\times\varphi_{h,j},\nabla\times\varphi_{h,k}) \tag{3.13}$$

is positive definite [7].

At the initial time step, we choose $\psi_h^0=I_h\psi_h$, the Lagrange interpolation of $\psi_0$, and set $B_h^0=\nabla\times\mathbf{A}_0$, $\phi_h^0=-\nabla\cdot\mathbf{A}_0$. We solve $u_h^0\in\mathring{V}_h$, $v_h^0\in V_h$ and $\alpha_{h,j}^0\in\mathbb{R}$, $j=1,\cdots,m$, from the following equations:

$$(\nabla u_h^0,\nabla\xi_h)=\big(\mathbf{A}_0,\nabla\times\xi_h\big), \qquad \forall\,\xi_h\in\mathring{V}_h, \tag{3.14}$$

$$(\nabla v_h^0,\nabla\zeta_h)=\big(\mathbf{A}_0,\nabla\zeta_h\big), \qquad \forall\,\zeta_h\in V_h, \tag{3.15}$$

$$\sum_{j=1}^m M_{h,kj}\alpha_{h,j}^0=(\mathbf{A}_0,\nabla\times\varphi_{h,k}), \qquad k=1,\cdots,m. \tag{3.16}$$

where the normalization condition $\int_\Omega v_h^0\mathrm{d}x=0$ is imposed for the uniqueness of the solution of (3.15).

For $1\leq n\leq N$ and given $\psi_h^{n-1}$, $B^{n-1}$, $\phi^{n-1}$, $u_h^{n-1}$, $v_h^{n-1}$, $\alpha_{h,j}^{n-1}$, we define

$$\mathbf{A}_h^{n-1}=\nabla\times u_h^{n-1}+\nabla v_h^{n-1}+\sum_{j=1}^m\alpha_{h,j}^{n-1}\nabla\times\varphi_{h,j}, \tag{3.17}$$

and solve $\psi_h^n\in\mathcal{V}_h$ from the equation

$$\begin{aligned}\Big(\eta\frac{\psi_h^n-\psi_h^{n-1}}{\tau},\omega_h\Big)+\Big(\Big(\frac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\Big),\Big(\frac{i}{\kappa}\nabla\omega_h+\mathbf{A}_h^{n-1}\omega_h\Big)\Big)&\\ +\big((|\Upsilon(\psi_h^{n-1})|^2-1)\psi_h^n,\omega_h\big)+\big(i\eta\kappa\phi_h^{n-1}\psi_h^n,\omega_h\big)=0, \qquad \forall\omega_h\in\mathcal{V}_h,&\end{aligned} \tag{3.18}$$

where $\Upsilon:\mathbb{C}\to\mathbb{C}$ is a cut-off function, defined by

$$\Upsilon(z)=z/\max(|z|,1), \quad \forall\,z\in\mathbb{C}, \tag{3.19}$$

which is Lipschitz continuous and satisfies that $|\Upsilon(z)|\leq 1$ for all $z\in\mathbb{C}$.

Next, we define

$$\mathbf{F}_h^n = \mathrm{Re}\left[\Upsilon(\overline{\psi}_h^{n-1})\left(\frac{i}{\kappa}\nabla\psi_h^n + \mathbf{A}_h^{n-1}\psi_h^n\right)\right] \tag{3.20}$$

and solve $B_h^n \in V_h$, $u_h^n \in \mathring{V}_h$ and $\phi_h^n, v_h^n \in V_h$ from the equations

$$\left(\frac{B_h^n - B_h^{n-1}}{\tau}, \omega_h\right) + \left(\nabla(B_h^n - H), \nabla\omega_h\right) = -(\mathbf{F}_h^n, \nabla\times\omega_h) \qquad \forall\omega_h \in \mathring{V}_h, \tag{3.21}$$

$$\left(\frac{\phi_h^n - \phi_h^{n-1}}{\tau}, \chi_h\right) + \left(\nabla\phi_h^n, \nabla\chi_h\right) = -(\mathbf{F}_h^n, \nabla\chi_h), \qquad \forall\chi_h \in V_h, \tag{3.22}$$

$$(\nabla u_h^n, \nabla\xi_h) = \left(B_h^n, \xi_h\right), \qquad \forall\xi_h \in \mathring{V}_h, \tag{3.23}$$

$$(\nabla v_h^n, \nabla\zeta_h) = \left(\phi_h^n, \zeta_h\right), \qquad \forall\zeta_h \in V_h, \tag{3.24}$$

$$\sum_{j=1}^m M_{h,kj}\frac{\alpha_{h,j}^n - \alpha_{h,j}^{n-1}}{\tau} = -(\mathbf{F}_h^n, \nabla\times\varphi_{h,k}), \quad k = 1,\dots,m, \tag{3.25}$$

with the boundary condition $B_h^n = H$ on $\partial\Omega$ and the normalization condition $\int_\Omega v_h^n \mathrm{d}x = 0$ for the uniqueness of the solution of (3.15). After solving $u_h^n$, $v_h^n$ and $\alpha_{h,j}^n$ from the equations above, the magnetic potential $\mathbf{A}_h^n$ can be computed by using the formula

$$\mathbf{A}_h^n = \nabla\times u_h^n + \nabla v_h^n + \sum_{j=1}^m \alpha_{h,j}^n \nabla\times\varphi_{h,j}. \tag{3.26}$$

**Remark 3.1.** In (3.18) and (3.20), we have truncated the order parameter $\psi_h^{n-1}$ by using a cut-off function $\Upsilon$. This truncation is consistent with PDEs' solution $\psi$, which satisfies $0 \le |\psi| \le 1$. In the next section, we will see that the truncation helps us to derive basic energy estimates for the numerical solutions, which are fundamental for further proving the convergence of the numerical solutions.

**Remark 3.2.** In (3.18), we have kept $\psi_h^n$ to be implicit in the two terms

$$(|\Upsilon(\psi_h^{n-1})|^2 - 1)\psi_h^n \quad \text{and} \quad i\eta\kappa\phi_h^{n-1}\psi_h^n.$$

By this scheme, it is easy to obtain an energy estimate for $\psi_h^n$ by substituting $\omega_h = \psi_h^n$ into (3.18) and considering the real part of the result; see (4.5).

### 3.3 Convergence of the numerical solutions

Let $\psi_{h,\tau}$, $B_{h,\tau}$, $\phi_{h,\tau}$, $u_{h,\tau}$, $v_{h,\tau}$, $\alpha_{h,\tau,j}$ and $\mathbf{A}_{h,\tau}$ be piecewise linear functions in time, defined by

$$\omega_{h,\tau}(t) = \frac{t_n - t}{\tau}\omega_h^{n-1} + \frac{t - t_{n-1}}{\tau}\omega_h^n, \qquad \text{for } t \in [t_{n-1}, t_n],\ n = 1,2,\dots,N. \tag{3.27}$$

The definition above and the expression (3.26) imply

$$\mathbf{A}_{h,\tau} = \nabla\times u_{h,\tau} + \nabla v_{h,\tau} + \sum_{j=1}^m \alpha_{h,\tau,j}\nabla\times\varphi_{h,j}. \tag{3.28}$$

Then we have the following result on the convergence of the numerical solutions.

**Theorem 3.1.** *If the initial data and external magnetic field satisfy* (2.20)*, then we have*

*(i) the discrete system of linear equations given by* (3.18) *and* (3.21)-(3.25) *admits a unique numerical solution when* $\tau < \eta$,

*(ii) the numerical solution converges to the unique weak solution of the PDE problem* (2.9)-(2.16) *as* $\tau, h \to 0$, *in the following sense:*

$$\begin{aligned}
&\psi_{h,\tau} \to \psi && \textit{strongly in } C([0,T];\mathcal{L}^2),\\
&B_{h,\tau} \to B \textit{ and } \phi_{h,\tau} \to \phi && \textit{strongly in } C([0,T];L^2),\\
&u_{h,\tau} \to u \textit{ and } v_{h,\tau} \to v && \textit{strongly in } C([0,T];H^1),\\
&\alpha_{h,\tau,j} \to \alpha_j && \textit{strongly in } C([0,T]),\\
&\mathbf{A}_{h,\tau} \to \mathbf{A} && \textit{strongly in } C([0,T];\mathbf{L}^4).
\end{aligned} \tag{3.29}$$

# 4 Proof of Theorem 2.1 and Theorem 3.1

## 4.1 Proof of Theorem 3.1 (i)

In this subsection, we show that when $\tau < \eta$, for any given $\psi_h^{n-1}$, $B_h^{n-1}$, $\phi_h^{n-1}$, $u_h^{n-1}$, $v_h^{n-1}$ and $\alpha_{h,j}^{n-1}$, with $\mathbf{A}_h^{n-1}$ given by (3.17), the proposed numerical scheme (3.18)-(3.25) admits a unique finite element solution $\psi_h^n$, $B_h^n$, $\phi_h^n$, $u_h^n$, $v_h^n$, $\alpha_{h,j}^n$.

In fact, for the given $\psi_h^{n-1}$ and $\mathbf{A}_h^{n-1}$, the inhomogeneous linear system (3.18) has a unique solution if and only if the corresponding homogeneous linear system

$$\begin{aligned}
&\left(\frac{\eta}{\tau}\psi_h^n, \omega_h\right) + \left(\left(\frac{i}{\kappa}\nabla\psi_h^n + \mathbf{A}_h^{n-1}\psi_h^n\right), \left(\frac{i}{\kappa}\nabla\omega_h + \mathbf{A}_h^{n-1}\omega_h\right)\right)\\
&\qquad + \left((|\Upsilon(\psi_h^{n-1})|^2 - 1)\psi_h^n, \omega_h\right) + \left(i\eta\kappa\phi_h^{n-1}\psi_h^n, \omega_h\right) = 0, \qquad \forall \omega_h \in \mathcal{V}_h,
\end{aligned}$$

does not have non-zero solution. In fact, by substituting $\omega_h = \psi_h^n$ into the equation above and considering the real part of the result, and using the fact

$$\mathrm{Re}\left(i\eta\kappa\phi_h^{n-1}\psi_h^n, \psi_h^n\right) = \mathrm{Re}\left(i\eta\kappa\phi_h^{n-1}|\psi_h^n|^2, 1\right) = 0,$$

we obtain

$$\frac{\eta}{\tau}\|\psi_h^n\|_{\mathcal{L}^2}^2 + \left\|\left(\frac{i}{\kappa}\nabla\psi_h^n + \mathbf{A}_h^{n-1}\psi_h^n\right)\right\|_{\mathcal{L}^2}^2 \le \|\psi_h^n\|_{\mathcal{L}^2}^2.$$

When $\tau < \eta$, the last inequality implies $\|\psi_h^n\|_{\mathcal{L}^2} = 0$. Hence, the corresponding homogeneous linear system only has the zero solution, which implies that the inhomogeneous linear system (3.18) has a unique solution.

Then, for the given $\mathbf{F}_h^n$, it is clear that the equations (3.21) and (3.22) have unique finite element solutions $B_h^n$ and $\phi_h^n$, respectively. Similarly, for the given $B_h^n$ and $\phi_h^n$, the equations (3.23) and (3.24) have unique finite element solutions $u_h^n$ and $v_h^n$, respectively.

Since the matrix $M_{h,kj}$ is positive definite (for any given mesh), it follows that (3.25) has a unique solution $\alpha_{h,j}^n$.

## 4.2 Energy estimates

In this subsection, we derive basic energy estimates for the numerical solutions, uniformly with respect to the time step size and spatial mesh size as $\tau, h \to 0$. These basic energy estimates are needed in the next subsection to derive further estimates for proving compactness and convergence of the numerical solutions. For simplicity of notation, we denote by $C$ a generic positive constant which may be different at each occurrence, but is independent of the time-step size $\tau$ and spatial mesh size $h$.

The following discrete Sobolev embedding inequalities was proved in [28, Lemma 5.1] and are needed in this subsection.

**Lemma 4.1.** *Let $\Delta_h^D$ and $\Delta_h^N$ denote the discrete Laplacian associated with the Dirichlet and Neumann boundary conditions, respectively, defined by*

$$\begin{aligned} &(\Delta_h^D\theta_h,\xi_h)=-(\nabla\theta_h,\nabla\xi_h), && \forall\theta_h,\xi_h\in\mathring{V}_h,\\ &(\Delta_h^N\vartheta_h,\zeta_h)=-(\nabla\vartheta_h,\nabla\zeta_h), && \forall\vartheta_h,\zeta_h\in V_h. \end{aligned}$$

*Then there exists a constant $q>4$, which depends on the domain $\Omega$, such that*

$$\begin{aligned} &\|\theta_h\|_{W^{1,q}}\le C\|\Delta_h^D\theta_h\|_{L^2}, && \forall\theta_h\in\mathring{V}_h,\\ &\|\vartheta_h\|_{W^{1,q}}\le C\|\Delta_h^N\vartheta_h\|_{L^2}, && \forall\vartheta_h\in V_h \text{ satisfying } \int_\Omega\vartheta_h\mathrm{d}x=0. \end{aligned}$$

The regularity of the continuous and discrete harmonic functions $\varphi_j$ and $\varphi_{h,j}$ are presented in the following lemma.

**Lemma 4.2.** *There exists a constant $q>4$ such that the solutions of* (2.3) *and* (3.11) *satisfy the following estimates:*

$$\|\varphi_j\|_{W^{1,q}}+\|\varphi_{h,j}\|_{W^{1,q}}\le C, \tag{4.1}$$

$$\|\varphi_{h,j}-\varphi_j\|_{H^1}\le Ch^{\min(1,\pi/\omega)}, \tag{4.2}$$

*where $\omega$ dentoes the maximal interior angle of the corners of the domain $\Omega$.*

*Proof.* For any given $j$, there exists a sufficiently smooth function (the constant 1 times a smooth cut-off function) $\psi$ such that

$$\psi=\delta_{jk}\quad \text{on } \Gamma_k,\quad k=0,1,\cdots,m. \tag{4.3}$$

Then

$$\begin{cases} -\Delta(\varphi_j-\psi)=\Delta\psi & \text{in } \Omega,\\ \varphi_j-\psi=0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

The problem has the following regularity (cf. [9]):

$$\|\varphi_j-\psi\|_{H^{1+\frac{\pi}{\omega}-\epsilon}}\le C\|\Delta\psi\|_{H^{-1+\frac{\pi}{\omega}-\epsilon}}\le C,$$

where $\epsilon\in(0,\frac{\pi}{\omega})$ can be arbitrary. Therefore, by choosing sufficiently small $\epsilon$ we have $\varphi_j\in H^{1+\frac{\pi}{\omega}-\epsilon}\hookrightarrow W^{1,q}$ for some $q>4$. Since $\varphi_{h,j}$ is the Galerkin finite element approximation of $\varphi_j$, it follows that (see [9] and [7])

$$\|\varphi_{h,j}-I_h\varphi_j\|_{H^1}\le Ch^{\min(1,\frac{\pi}{\omega})},$$

where $I_h$ is the Lagrange interpolation onto the finite element space $V_h$. By the inverse inequality,

$$\|\varphi_{h,j}-I_h\varphi_j\|_{W^{1,q}}\le Ch^{\frac{2}{q}-1}\|\varphi_{h,j}-I_h\varphi_j\|_{H^1}\le Ch^{\min(\frac{2}{q},\frac{2}{q}-1+\frac{\pi}{\omega})}.$$

Since $\omega\in(0,2\pi)$, there exists $q>4$ such that $\frac{2}{q}-1+\frac{\pi}{\omega}>0$. Therefore the inequality above implies $\|\varphi_{h,j}\|_{W^{1,q}}\le C$. This completes the proof of Lemma 4.2.

The main result of this subsection is the following proposition.

**Proposition 4.1.** There exists a positive constant $h_0$, depending on the domain $\Omega$, such that when $\tau \le \eta/4$ and $h \le h_0$ the numerical solutions given by (3.18) and (3.21)-(3.25) satisfy

$$\max_{1\le n\le N}\left(\|\psi_h^n\|_{L^2}+\|B_h^n\|_{L^2}+\|\phi_h^n\|_{L^2}+\|\mathbf{A}_h^n\|_{L^4}+\|u_h^n\|_{W^{1,4}}+\|v_h^n\|_{W^{1,4}}+\sum_{j=1}^m|\alpha_{h,j}^n|\right)$$
$$+\sum_{n=1}^N\tau\left(\|\nabla\psi_h^n\|_{L^2}^2+\|\nabla B_h^n\|_{L^2}^2+\|\nabla\phi_h^n\|_{L^2}^2+\|\mathbf{F}_h^n\|_{L^2}^2\right)\le C.$$

*Proof.* First, by substituting $\omega_h=\psi_h^n$ into (3.18) and considering the real part of the result, we obtain

$$\frac{\eta}{2}\frac{\|\psi_h^n\|_{L^2}^2-\|\psi_h^{n-1}\|_{L^2}^2}{\tau}+\left\|\left(\frac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\right)\right\|_{L^2}^2\le\|\psi_h^n\|_{L^2}^2. \tag{4.5}$$

By choosing a step size $\tau\le\eta/4$, the last inequality implies (via discrete Grönwall's inequality)

$$\max_{1\le n\le N}\|\psi_h^n\|_{L^2}^2+\sum_{n=1}^N\tau\left\|\left(\frac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\right)\right\|_{L^2}^2\le C. \tag{4.6}$$

Second, we note that the cut-off function introduced in (3.20) guarantees

$$\|\mathbf{F}_h^n\|_{L^2}\le\left\|\left(\frac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\right)\right\|_{L^2}. \tag{4.7}$$

With the help of the inequality above, by substituting $\omega_h=B_h^n-H$ and $\chi_h=-\phi_h^n$ into (3.21)-(3.22), we obtain the following energy inequalities:

$$\frac{\|B_h^n-H\|_{L^2}^2-\|B_h^{n-1}-H\|_{L^2}^2}{2\tau}+\|\nabla(B_h^n-H)\|_{L^2}^2\le\frac{1}{2}\|\mathbf{F}_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla(B_h^n-H)\|_{L^2}^2,$$
$$\frac{\|\phi_h^n\|_{L^2}^2-\|\phi_h^{n-1}\|_{L^2}^2}{2\tau}+\|\nabla\phi_h^n\|_{L^2}^2\le\frac{1}{2}\|\mathbf{F}_h^n\|_{L^2}^2+\frac{1}{2}\|\nabla\phi_h^n\|_{L^2}^2$$

which together with (4.6)-(4.7) yield

$$\max_{1\le n\le N}\left(\|B_h^n-H\|_{L^2}^2+\|\phi_h^n\|_{L^2}^2\right)+\sum_{n=1}^N\tau\left(\|\nabla(B_h^n-H)\|_{L^2}^2+\|\nabla\phi_h^n\|_{L^2}^2\right)\le C \tag{4.8}$$

Since (3.23)-(3.24) imply $\Delta_h^D u_h=-B_h^n$ and $\Delta_h^N v_h=-\phi_h^n$ (see the definitions in Lemma 4.1), the last inequality gives

$$\max_{1\le n\le N}\left(\|\Delta_h^D u_h^n\|_{L^2}+\|\Delta_h^N v_h^n\|_{L^2}\right)=\max_{1\le n\le N}\left(\|B_h^n\|_{L^2}+\|\phi_h^n\|_{L^2}\right)\le C. \tag{4.9}$$

Next, Lemma 4.2 implies that the matrices $M_{h,kj}=(\nabla\times\varphi_{h,j},\nabla\times\varphi_{h,k})$ and $M_{kj}=(\nabla\times\varphi_j,\nabla\times\varphi_k)$ satisfy

$$|M_{h,kj}-M_{kj}|\le Ch^{\min(1,\pi/\omega)}.$$

Since the matrix $M_{kj}$ is invertible, there exists a positive constant $h_0$, depending on the domain $\Omega$, such that when $h\le h_0$ the matrix $M_{h,kj}$ has an inverse satisfying

$$|M_{h,kj}^{-1}|\le C|M_{kj}^{-1}|\le C. \tag{4.10}$$

Correspondingly, from (3.25) we see that

$$\left|\frac{\alpha_{h,j}^n-\alpha_{h,j}^{n-1}}{\tau}\right|=\left|\sum_{k=1}^m M_{h,jk}^{-1}(\mathbf{F}_h^n,\nabla\times\varphi_{h,k})\right|\le C\|\mathbf{F}_h^n\|_{L^2}\|\nabla\times\varphi_{h,k}\|_{L^2}\le C\|\mathbf{F}_h^n\|_{L^2},$$

which implies

$$\begin{aligned}|\alpha_{h,j}^n|&=\Big|\alpha_{h,j}^0+\textstyle\sum_{j=1}^n\tau\frac{\alpha_{h,j}^n-\alpha_{h,j}^{n-1}}{\tau}\Big|\le|\alpha_{h,j}^0|+C\textstyle\sum_{j=1}^n\tau\|\mathbf{F}_h^n\|_{L^2}\\&\le|\alpha_{h,j}^0|+CT^{\frac12}\big(\textstyle\sum_{j=1}^n\tau\|\mathbf{F}_h^n\|_{L^2}^2\big)^{\frac12}\qquad\text{use Hölder's inequality}\\&\le C,\qquad\qquad\text{use (4.6)-(4.7).}\end{aligned}\tag{4.11}$$

Finally, (4.9) and Lemma 4.1 imply

$$\max_{1\le n\le N}(\|u_h^n\|_{W^{1,4}}+\|v_h^n\|_{W^{1,4}})\le C\max_{1\le n\le N}(\|\Delta_h^D u_h^n\|_{L^2}+\|\Delta_h^N v_h^n\|_{L^2})\le C,\tag{4.12}$$

and by using the expression (3.26) we obtain

$$\max_{1\le n\le N}\|\mathbf{A}_h^n\|_{L^4}\le C\max_{1\le n\le N}\left(\|\nabla u_h^n\|_{L^4}+\|\nabla v_h^n\|_{L^4}+\sum_{j=1}^m|\alpha_j^n|\|\nabla\times\varphi_{h,j}\|_{L^4}\right)\le C.\tag{4.13}$$

With the inequality above, we have

$$\|\mathbf{A}_h^{n-1}\psi_h^n\|_{L^2}\le C\|\mathbf{A}_h^{n-1}\|_{L^4}\|\psi_h^n\|_{L^4}\le C\|\psi_h^n\|_{L^4}\le C\epsilon^{-1}\|\psi_h^n\|_{L^2}+\epsilon\|\nabla\psi_h^n\|_{L^2}.$$

As a consequence, we have

$$\begin{aligned}\|\nabla\psi_h^n\|_{L^2}&\le\big\|\tfrac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\big\|_{L^2}+\|\mathbf{A}_h^{n-1}\psi_h^n\|_{L^2}\\&\le\big\|\tfrac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\big\|_{L^2}+C\epsilon^{-1}\|\psi_h^n\|_{L^2}+\epsilon\|\nabla\psi_h^n\|_{L^2},\end{aligned}$$

which further reduces to (by choosing a sufficiently small $\epsilon$)

$$\|\nabla\psi_h^n\|_{L^2}\le\big\|\tfrac{i}{\kappa}\nabla\psi_h^n+\mathbf{A}_h^{n-1}\psi_h^n\big\|_{L^2}+C\|\psi_h^n\|_{L^2}.\tag{4.14}$$

The inequality above, together with (4.6), implies

$$\max_{1\le n\le N}\|\psi_h^n\|_{L^2}^2+\sum_{n=1}^N\tau\|\nabla\psi_h^n\|_{L^2}^2\le C.\tag{4.15}$$

The proof of Proposition 4.1 is complete.

## 4.3 Further estimates

In this section, we present further estimates for the numerical solutions based on the energy estimates derived in the last subsection. To this end, we need the following result on the maximal $\ell^p$-regularity of finite element solutions (a proof is presented in Appendix B).

**Lemma 4.3.** *The solutions of the discretized parabolic equations* (3.21) *and* (3.22) *satisfy*

$$\begin{aligned}&\left\|\left(\frac{B_h^n-B_h^{n-1}}{\tau}\right)_{n=1}^N\right\|_{\ell^p(H^{-1})}+\big\|\big(B_h^n-H\big)_{n=1}^N\big\|_{\ell^p(H^1)}\\&\le C\big\|\big(\mathbf{F}_h^n\big)_{n=1}^N\big\|_{\ell^p(L^2)}+C\|B_h^0-H\|_{(H^{-1},H_0^1)_{1-\frac1p,p}},\end{aligned}\tag{4.16}$$

$$\begin{aligned}&\left\|\left(\frac{\phi_h^n-\phi_h^{n-1}}{\tau}\right)_{n=1}^N\right\|_{\ell^p(\widetilde{H}^{-1})}+\big\|\big(\phi_h^n\big)_{n=1}^N\big\|_{\ell^p(H^1)}\\&\le C\big\|\big(\mathbf{F}_h^n\big)_{n=1}^N\big\|_{\ell^p(L^2)}+C\|\phi_h^0\|_{(\widetilde{H}^{-1},H^1)_{1-\frac1p,p}}.\end{aligned}\tag{4.17}$$

The main result of this subsection is the following proposition.

**Proposition 4.2.** The numerical solutions given by (3.18) and (3.21)-(3.25) satisfy

$$\begin{aligned}&\max_{1\le n\le N}\left(\|\psi_h^n\|_{H^1}+\sum_{j=1}^m\left|\frac{\alpha_{h,j}^n-\alpha_{h,j}^{n-1}}{\tau}\right|\right)+\sum_{n=1}^N\tau\left(\left\|\frac{\psi_h^n-\psi_h^{n-1}}{\tau}\right\|_{L^2}^2+\|\Delta_h^N\psi_h^n\|_{L^2}^2\right)\\&+\|(B^n)_{n=1}^N\|_{\ell^p(H^1)}+\|(\phi^n)_{n=1}^N\|_{\ell^p(H^1)}\\&+\left\|\left(\frac{B^n-B^{n-1}}{\tau}\right)_{n=1}^N\right\|_{\ell^p(H^{-1})}+\left\|\left(\frac{\phi^n-\phi^{n-1}}{\tau}\right)_{n=1}^N\right\|_{\ell^p(\widetilde{H}^{-1})}\le C_p,\quad \forall 2\le p<4.\end{aligned}$$

*Proof.* Let $P_h$ denote the $L^2$ projection onto the finite element space $\mathcal{V}_h$, and define the operator $\nabla_h\cdot:L^2\times L^2\to\mathcal{V}_h$ by

$$(\nabla_h\cdot\mathbf{g},\omega_h)=-(\mathbf{g},\nabla\omega_h),\quad \forall\mathbf{g}\in L^2\times L^2\text{ and }\omega_h\in\mathcal{V}_h. \tag{4.18}$$

Then (3.18) can be rewritten as the following form:

$$\eta\frac{\psi_h^n-\psi_h^{n-1}}{\tau}-\frac{1}{\kappa^2}\Delta_h^N\psi_h^n=g^n, \tag{4.19}$$

with

$$g^n=-\frac{i}{\kappa}P_h[\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1}]-\frac{i}{\kappa}\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})-P_h\Big[|\mathbf{A}_h^{n-1}|^2\psi_h^n+(|\Upsilon(\psi_h^{n-1})|^2-1)\psi_h^n+i\eta\kappa\phi_h^{n-1}\psi_h^n\Big].$$

Integrating this equation against $-\Delta_h^N\psi_h^n$ yields

$$\eta\frac{\|\nabla\psi_h^n\|_{L^2}^2-\|\nabla\psi_h^{n-1}\|_{L^2}^2}{2\tau}+\frac{1}{\kappa^2}\|\Delta_h^N\psi_h^n\|_{L^2}^2\le\kappa^2\|g^n\|_{L^2}^2+\frac{1}{4\kappa^2}\|\Delta_h^N\psi_h^n\|_{L^2}^2, \tag{4.20}$$

with

$$\begin{aligned}\|g^n\|_{L^2}^2&\le C\|\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1}\|_{L^2}^2+C\|\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})\|_{L^2}^2\\&\quad+\||\mathbf{A}_h^{n-1}|^2\psi_h^n+(|\Upsilon(\psi_h^{n-1})|^2-1)\psi_h^n+i\eta\kappa\phi_h^{n-1}\psi_h^n\|_{L^2}^2\\&\le C\|\nabla\psi_h^n\|_{L^4}^2\|\mathbf{A}_h^{n-1}\|_{L^4}^2+C\|\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})\|_{L^2}^2\\&\quad+C\|\mathbf{A}_h^{n-1}\|_{L^4}^4\|\psi_h^n\|_{L^\infty}^2+\|\psi_h^n\|_{L^2}^2+C\|\phi_h^{n-1}\|_{L^2}^2\|\psi_h^n\|_{L^\infty}^2\\&\le C\big(\|\nabla\psi_h^n\|_{L^4}^2+\|\psi_h^n\|_{L^\infty}^2\big)+C\|\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})\|_{L^2}^2.\end{aligned} \tag{4.21}$$

where we have used Proposition 4.1 in the last inequality.

To estimate the term $\|\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})\|_{L^2}^2$ in the right-hand side of (4.21), we use a duality argument below. Let $\mathcal{K}_h$ denote the set of triangles in the triangulation of the domain $\Omega$, let $\mathcal{E}_h$ denote the set of edges in the triangulation, and let $[\omega_h]$ denote the jump of a function $\omega_h$ on an edge $e\in\mathcal{E}_h$. Due to the continuity of $u_h^n$ and $\varphi_{h,j}^n$ in $\Omega$, we have $[(\nabla\times u_h^n)\cdot\mathbf{n}]=[(\nabla\times\varphi_{h,j})\cdot\mathbf{n}]=0$ on each edge $e\in\mathcal{E}_h$, and so for any $\omega_h\in\mathcal{V}_h$

$$\begin{aligned}&(\mathbf{A}_h^n,\nabla(\overline{\psi}_h^n\omega_h))\\&=(\nabla\times u_h^n,\nabla(\overline{\psi}_h^n\omega_h))+(\nabla v_h^n,\nabla(\overline{\psi}_h^n\omega_h))+\sum_{j=1}^m\alpha_{h,j}^n(\nabla\times\varphi_{h,j},\nabla(\overline{\psi}_h^n\omega_h))=(\nabla v_h^n,\nabla(\overline{\psi}_h^n\omega_h)).\end{aligned} \tag{4.22}$$

By the definition of the discrete divergence in (4.18), for any $\omega_h\in\mathcal{V}_h$ we have

$$(\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1}),\omega_h)=-(\psi_h^n\mathbf{A}_h^{n-1},\nabla\omega_h)$$

$$
\begin{aligned}
&= (\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1},\omega_h) - (\mathbf{A}_h^{n-1},\nabla(\overline{\psi}_h^n\omega_h))\\
&= (\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1},\omega_h) - (\nabla v_h^{n-1},\nabla(\overline{\psi}_h^n\omega_h)) \quad \text{(here we use (4.22))}\\
&= (\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1},\omega_h) - (\nabla v_h^{n-1},\nabla(\overline{\psi}_h^n\omega_h - P_h(\overline{\psi}_h^n\omega_h))) - (\nabla v_h^{n-1},\nabla P_h(\overline{\psi}_h^n\omega_h))\\
&= (\nabla\psi_h^n\cdot\mathbf{A}_h^{n-1},\omega_h) - (\nabla v_h^{n-1},\nabla(\overline{\psi}_h^n\omega_h - P_h(\overline{\psi}_h^n\omega_h))) - (\phi_h^{n-1},P_h(\overline{\psi}_h^n\omega_h)) \quad \text{(use (3.24))}\\
&\le \|\nabla\psi_h^n\|_{L^4}\|\mathbf{A}_h^{n-1}\|_{L^4}\|\omega_h\|_{L^2} + \|\nabla v_h^{n-1}\|_{L^4}\|\nabla(\overline{\psi}_h^n\omega_h - P_h(\overline{\psi}_h^n\omega_h))\|_{L^{4/3}}\\
&\quad + \|\phi_h^{n-1}\|_{L^2}\|P_h(\overline{\psi}_h^n\omega_h)\|_{L^2}\\
&\le C\|\nabla\psi_h^n\|_{L^4}\|\omega_h\|_{L^2} + \|\nabla(\overline{\psi}_h^n\omega_h - P_h(\overline{\psi}_h^n\omega_h))\|_{L^{4/3}} + C\|\psi_h^n\|_{L^\infty}\|\omega_h\|_{L^2}
\end{aligned} \tag{4.23}
$$

where we have used Proposition 4.1 and (4.12) in the last inequality. Let $I_h$ denotes the Lagrange interpolation operator onto the finite element space. Since the $L^2$ projection is bounded with respect to the $W^{1,4/3}$ norm and the finite element functions $\omega_h$ and $\psi_h^n$ are piecewise linear, it follows that

$$
\begin{aligned}
\|\nabla(\overline{\psi}_h^n\omega_h - P_h(\overline{\psi}_h^n\omega_h))\|_{L^{4/3}} &\le C\|\nabla(\overline{\psi}_h^n\omega_h - I_h(\overline{\psi}_h^n\omega_h))\|_{L^{4/3}} \le Ch\sum_{K\in\mathcal{K}_h}\|\nabla^2(\overline{\psi}_h^n\omega_h)\|_{L^{4/3}(K)}\\
&\le Ch\sum_{i,j=1}^{2}\|\partial_i\overline{\psi}_h^n\partial_j\omega_h\|_{L^{4/3}} \le Ch\sum_{i,j=1}^{2}\|\partial_i\psi_h^n\|_{L^4}\|\partial_j\omega_h\|_{L^2}\\
&\le C\|\nabla\psi_h^n\|_{L^4}\|\omega_h\|_{L^2}. \quad \text{(use the inverse inequality here)}
\end{aligned}
$$

Substituting the last inequality into (4.23) yields which implies (via duality)

$$
\|\nabla_h\cdot(\psi_h^n\mathbf{A}_h^{n-1})\|_{L^2} \le C(\|\nabla\psi_h^n\|_{L^4} + \|\psi_h^n\|_{L^\infty}). \tag{4.24}
$$

Then, by substituting the last inequality into (4.21), we obtain

$$
\begin{aligned}
\|g^n\|_{L^2}^2 &\le C\big(\|\nabla\psi_h^n\|_{L^4}^2 + \|\psi_h^n\|_{L^\infty}^2\big)\\
&\le C\big(\|\nabla\psi_h^n\|_{L^4}^2 + \|\psi_h^n\|_{L^4}^2\big) \qquad \text{(Sobolev embedding } W^{1,4}\hookrightarrow L^\infty)\\
&\le C\big(\|\nabla\psi_h^n\|_{L^2}^2 + \|\psi_h^n\|_{L^2}^2\big)^{\frac{q-4}{2(q-2)}}\big(\|\nabla\psi_h^n\|_{L^q}^2 + \|\psi_h^n\|_{L^q}^2\big)^{\frac{q}{2(q-2)}} \quad \text{(for any } q>4)\\
&\le C\big(\|\nabla\psi_h^n\|_{L^2}^2 + \|\psi_h^n\|_{L^2}^2\big)^{\frac{q-4}{2(q-2)}}\|\Delta_h\psi_h^n\|_{L^2}^{\frac{q}{q-2}} \quad \text{(use Lemma 4.1)}\\
&\le C_\epsilon\big(\|\psi_h^n\|_{L^2}^2 + \|\nabla\psi_h^n\|_{L^2}^2\big) + \epsilon\|\Delta_h\psi_h^n\|_{L^2}^2,
\end{aligned} \tag{4.25}
$$

which together with (4.20) implies (by choosing a small $\epsilon$)

$$
\eta\frac{\|\nabla\psi_h^n\|_{L^2}^2 - \|\nabla\psi_h^{n-1}\|_{L^2}^2}{2\tau} + \frac{1}{2\kappa^2}\|\Delta_h^N\psi_h^n\|_{L^2}^2 \le C\big(\|\psi_h^n\|_{L^2}^2 + \|\nabla\psi_h^n\|_{L^2}^2\big). \tag{4.26}
$$

Summing up the last inequality for $n=1,\dots,N$ and using (4.15) yield

$$
\begin{aligned}
&\max_{1\le n\le N}\|\nabla\psi_h^n\|_{L^2}^2 + \sum_{n=1}^{N}\tau\|\Delta_h^N\psi_h^n\|_{L^2}^2\\
&\le C\|\nabla\psi_h^0\|_{L^2}^2 + C\sum_{n=1}^{N}\tau(\|\psi_h^n\|_{L^2}^2 + \|\nabla\psi_h^n\|_{L^2}^2) \le C.
\end{aligned} \tag{4.27}
$$

Then, by using (4.25) and the last inequality, from (4.19) we can also derive

$$
\sum_{n=1}^{N}\tau\left\|\frac{\psi_h^n - \psi_h^{n-1}}{\tau}\right\|_{L^2}^2 \le C\sum_{n=1}^{N}\tau\big(\|\Delta_h^N\psi_h^n\|_{L^2}^2 + \|g^n\|_{L^2}^2\big) \le C. \tag{4.28}
$$

By using (4.7), (4.13) and (4.27) we have

$$\|\mathbf{F}_h^n\|_{L^2} \leq \big\| \big( \tfrac{i}{\kappa} \nabla \psi_h^n + \mathbf{A}_h^{n-1} \psi_h^n \big) \big\|_{L^2} \leq C \|\nabla \psi_h^n\|_{L^2} + C \|\mathbf{A}_h^{n-1}\|_{L^4} \|\psi_h^n\|_{L^4} \leq C. \tag{4.29}$$

Hence, by applying Lemma 4.3 to (3.21) and (3.22), respectively, we have

$$\begin{aligned}
&\big(\textstyle\sum_{n=1}^N \tau \big\| \big(\frac{B_h^n - B_h^{n-1}}{\tau}\big)_{n=1}^N \big\|_{H^{-1}}^p \big)^{\frac{1}{p}} + \big(\textstyle\sum_{n=1}^N \tau \big\| \big(B_h^n - H\big)_{n=1}^N \big\|_{H^1}^p \big)^{\frac{1}{p}} \\
&\leq C \big(\textstyle\sum_{n=1}^N \tau \big\| \big(\mathbf{F}_h^n\big)_{n=1}^N \big\|_{L^2}^p \big)^{\frac{1}{p}} + C \|B_h^0 - H\|_{(H^{-1}(\Omega), H_0^1(\Omega))_{1-1/p,p}} \\
&\leq C \big(\textstyle\sum_{n=1}^N \tau \big\| \big(\mathbf{F}_h^n\big)_{n=1}^N \big\|_{L^2}^p \big)^{\frac{1}{p}} + C \|B_h^0 - H\|_{H^{1-2/p}} \text{ (see Appendix C and note that } \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq p < 4 \text{ implies } 1 - 2/p < 1/2) \\
&\leq C \big(\textstyle\sum_{n=1}^N \tau \big\| \big(\mathbf{F}_h^n\big)_{n=1}^N \big\|_{L^2}^p \big)^{\frac{1}{p}} + C \|B_h^0 - H\|_{H^1} \\
&\leq C
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&\big(\textstyle\sum_{n=1}^N \tau \big\| \big(\frac{\phi_h^n - \phi_h^{n-1}}{\tau}\big)_{n=1}^N \big\|_{\widetilde{H}^{-1}}^p \big)^{\frac{1}{p}} + \big(\textstyle\sum_{n=1}^N \tau \big\| \big(\phi_h^n\big)_{n=1}^N \big\|_{H^1}^p \big)^{\frac{1}{p}} \\
&\leq C \big(\textstyle\sum_{n=1}^N \tau \big\| \big(\mathbf{F}_h^n\big)_{n=1}^N \big\|_{L^2}^p \big)^{\frac{1}{p}} + C \|\phi_h^0\|_{(\widetilde{H}^{-1}, H^1)_{1-\frac{1}{p},p}} \leq C.
\end{aligned} \tag{4.31}$$

From (3.25) and (4.29) we derive

$$\Big| \frac{\alpha_{h,j}^n - \alpha_{h,j}^{n-1}}{\tau} \Big| = \Big| \textstyle\sum_{k=1}^m M_{h,jk}^{-1} (\mathbf{F}_h^n, \nabla \times \varphi_{h,k}) \Big| \leq C \|\mathbf{F}_h^n\|_{L^2} \|\nabla \times \varphi_{h,k}\|_{L^2} \leq C. \tag{4.32}$$

The proof of Proposition 4.2 is completed.

## 4.4 Compactness of the finite element solutions

Lemma 4.1, Proposition 4.1 and Proposition 4.2 imply the following estimates for some $q > 4$ and arbitrary $2 < p < 4$:

$$\|\psi_{h,\tau}\|_{H^1(0,T;L^2)} + \|\psi_{h,\tau}\|_{L^\infty(0,T;H^1)} + \|\psi_{h,\tau}\|_{L^2(0,T;W^{1,q})} \leq C, \tag{4.33}$$

$$\|B_{h,\tau} - H\|_{W^{1,p}(0,T;H^{-1})} + \|B_{h,\tau} - H\|_{L^p(0,T;H_0^1)} \leq C, \tag{4.34}$$

$$\|\phi_{h,\tau}\|_{W^{1,p}(0,T;(H^1)')} + \|\phi_{h,\tau}\|_{L^p(0,T;H^1)} + |\alpha_{h,\tau,j}| + \left| \frac{\mathrm{d}\alpha_{h,\tau,j}}{\mathrm{d}t} \right| \leq C. \tag{4.35}$$

Since $\mathcal{H}^1$ is compactly embedded into $\mathcal{L}^q$ for all $2 \leq q < \infty$ and such $\mathcal{L}^q$ is continuously embedded into $L^2$, the Aubin–Lions lemma [6, Theorem II.5.16 (ii)] implies that $L^\infty(0,T;\mathcal{H}^1) \cap H^1(0,T;\mathcal{L}^2)$ is compactly embedded into $C([0,T];\mathcal{L}^q)$ for all $2 \leq q < \infty$. Hence, the inequality (4.33) implies that $\psi_{h,\tau}$, $h,\tau > 0$, are compact in $C([0,T];\mathcal{L}^q)$ for all $2 \leq q < \infty$. As a result, for any sequence $(h_m,\tau_m) \to (0,0)$ there exists a subsequence, also denoted by $(h_m,\tau_m)$ for the simplicity of the notations, such that

$$\begin{aligned}
&\partial_t \psi_{h_m,\tau_m} \to \partial_t \psi && \text{weakly in } L^2(0,T;\mathcal{L}^2), && (4.36)\\
&\psi_{h_m,\tau_m} \to \psi && \text{weakly}^* \text{ in } L^\infty(0,T;\mathcal{H}^1)\ , && (4.37)\\
&\psi_{h_m,\tau_m} \to \psi && \text{weakly in } L^2(0,T;\mathcal{W}^{1,q}) \text{ for some } q > 4, && (4.38)\\
&\psi_{h_m,\tau_m} \to \psi && \text{strongly in } C([0,T];\mathcal{L}^q) \text{ for arbitrary } 1 < q < \infty, && (4.39)
\end{aligned}$$

for some function $\psi$.

Similarly, for any $2 \le p < 4$ (required in Proposition 4.2) the following Sobolev embedding holds:

$$\begin{aligned} &W^{1,p}(0,T;H^{-1}) \cap L^p(0,T;H_0^1) \\ &\hookrightarrow C([0,T];(H^{-1},H_0^1)_{1-1/p,p}) \cap W^{1,p}(0,T;H^{-1}) \qquad \text{see [30, Proposition 1.2.10].} \end{aligned} \tag{4.40}$$

Since $(H^{-1},H_0^1)_{1-1/p,p}$ is compactly embedded into $(H^{-1},H_0^1)_{0,2}=L^2$ (cf. [4, Corollary 3.8.2]) and $L^2$ is continuously embedded into $H^{-1}$, the Aubin–Lions lemma [6, Theorem II.5.16 (ii)] implies that

$$W^{1,p}(0,T;H^{-1}) \cap L^p(0,T;H_0^1) \text{ is compactly embedded into } C([0,T];L^2).$$

Hence, for any sequence $(h_m,\tau_m) \to (0,0)$, (4.34) there exists a subsequence, also denoted by $(h_m,\tau_m)$ for the simplicity of the notations, such that

$$\partial_t B_{h_m,\tau_m} \to \partial_t B \qquad \text{weakly in } L^p(0,T;H^{-1}) \text{ for } 2 \le p < 4, \tag{4.41}$$

$$B_{h_m,\tau_m} - H \to B - H \qquad \text{weakly in } L^p(0,T;H_0^1) \text{ for } 2 \le p < 4, \tag{4.42}$$

$$B_{h_m,\tau_m} \to B \qquad \text{strongly in } C([0,T];L^2), \tag{4.43}$$

for some function $B$. The same argument yields the existence of a subsequence, also denoted by $(h_m,\tau_m)$ for the simplicity of the notations, such that

$$\partial_t \phi_{h_m,\tau_m} \to \partial_t \phi \qquad \text{weakly in } L^p(0,T;\widetilde{H}^{-1}) \text{ for } 2 \le p < 4, \tag{4.44}$$

$$\phi_{h_m,\tau_m} \to \phi \qquad \text{weakly in } L^p(0,T;H^1) \text{ for } 2 \le p < 4, \tag{4.45}$$

$$\phi_{h_m,\tau_m} \to \phi \qquad \text{strongly in } C([0,T];L^2), \tag{4.46}$$

for some function $\phi$.

Since $u_h^n$ and $v_h^n$ are determined by $B_h^n$ and $\phi_h^n$ in a linear way, satisfying the estimates (4.9) and (4.12), it follows that the convergence of $B_{h_m,\tau_m}$ and $\phi_{h_m,\tau_m}$ in (4.43) and (4.46) immediately imply the convergence of $u_{h_m,\tau_m}$ and $v_{h_m,\tau_m}$, i.e.,

$$u_{h_m,\tau_m} \to u \text{ and } v_{h_m,\tau_m} \to v \qquad \text{strongly in } C([0,T];W^{1,4}), \tag{4.47}$$

for some functions $u$ and $v$.

Finally, (4.35) implies the existence of a subsequence, also denoted by $(h_m,\tau_m)$ for the simplicity of the notations, such that

$$\partial_t \alpha_{h_m,\tau_m,j} \to \partial_t \alpha_j \qquad \text{weakly}^* \text{ in } L^\infty(0,T), \tag{4.48}$$

$$\alpha_{h_m,\tau_m,j} \to \alpha_j \qquad \text{strongly in } C([0,T]), \tag{4.49}$$

for some function $\alpha_j \in W^{1,\infty}(0,T)$. In view of (3.28), the results (4.47)-(4.49) imply

$$\mathbf{A}_{h_m,\tau_m} \to \mathbf{A} \qquad \text{strongly in } C([0,T];L^4), \tag{4.50}$$

for $\mathbf{A} = \nabla \times u + \nabla v + \sum_{j=1}^m \alpha_j \nabla \times \varphi_j$.

The convergence of (4.36)-(4.50) imply that the piecewise constant functions $\psi_{h,\tau}^\pm$, $B^\pm$, $\phi^\pm$, $u^\pm$, $v^\pm$, $\alpha^\pm$, $\mathbf{A}^\pm$, defined by

$$\omega_{h,\tau}^+ := \omega_h^n \qquad \forall t \in (t_{n-1},t_n], \quad n=1,\dots,N, \tag{4.51}$$

$$\omega_{h,\tau}^- := \omega_h^{n-1} \qquad \forall t \in (t_{n-1},t_n], \quad n=1,\dots,N, \tag{4.52}$$

satisfy

$$\psi^{\pm}_{h_m,\tau_m} \to \psi \qquad \text{weakly* in } L^\infty(0,T;\mathcal{H}^1)\,, \tag{4.53}$$

$$\psi^{\pm}_{h_m,\tau_m} \to \psi \qquad \text{weakly in } L^2(0,T;\mathcal{W}^{1,q}) \text{ for some } q>4, \tag{4.54}$$

$$\psi^{\pm}_{h_m,\tau_m} \to \psi \qquad \text{strongly in } L^\infty(0,T;\mathcal{L}^q),\ \forall 1<q<\infty, \tag{4.55}$$

$$B^{\pm}_{h_m,\tau_m} \to B \text{ and } \phi^{\pm}_{h_m,\tau_m} \to \phi \qquad \text{weakly in } L^p(0,T;H^1) \text{ for } 2\le p<4, \tag{4.56}$$

$$B^{\pm}_{h_m,\tau_m} \to B \text{ and } \phi^{\pm}_{h_m,\tau_m} \to \phi \qquad \text{strongly in } L^\infty(0,T;L^2), \tag{4.57}$$

$$u^{\pm}_{h_m,\tau_m} \to u \text{ and } v^{\pm}_{h_m,\tau_m} \to v \qquad \text{strongly in } L^\infty(0,T;W^{1,4}), \tag{4.58}$$

$$\alpha^{\pm}_{h_m,\tau_m,j} \to \alpha_j \qquad \text{strongly in } L^\infty(0,T), \tag{4.59}$$

$$\mathbf{A}^{\pm}_{h_m,\tau_m} \to \mathbf{A} \qquad \text{strongly in } L^\infty(0,T;L^4). \tag{4.60}$$

Let $\mathbf{F}^+_{h,\tau} := \mathrm{Re}\big[\Upsilon(\overline{\psi}^-_{h,\tau})\big(\frac{i}{\kappa}\nabla\psi^+_{h,\tau} + \mathbf{A}^-_{h,\tau}\psi^+_{h,\tau}\big)\big]$. For the sequence $(h,\tau)=(h_m,\tau_m)\to(0,0)$, the convergence results (4.53)-(4.60) imply

$$\tfrac{i}{\kappa}\nabla\psi^+_{h,\tau} + \mathbf{A}^-_{h,\tau}\psi^+_{h,\tau} \to \big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\psi \qquad \text{weakly in } L^2(0,T;L^4), \tag{4.61}$$

$$\mathbf{A}^-_{h,\tau}\cdot\big(\tfrac{i}{\kappa}\nabla\psi^+_{h,\tau} + \mathbf{A}^-_{h,\tau}\psi^+_{h,\tau}\big) \to \mathbf{A}\cdot\big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\psi \qquad \text{weakly in } L^2(0,T;L^2), \tag{4.62}$$

$$\mathbf{F}^+_{h_m,\tau_m} \to \Upsilon(\overline{\psi})\big(\tfrac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\big) \qquad \text{weakly in } L^2(0,T;L^2), \tag{4.63}$$

$$\phi^-_{h_m,\tau_m}\psi^+_{h_m,\tau_m} \to \phi\psi \qquad \text{weakly in } L^2(0,T;L^2), \tag{4.64}$$

$$(|\Upsilon(\psi^-_{h,\tau})|^2-1)\psi^+_{h,\tau} \to (|\Upsilon(\psi)|^2-1)\psi \qquad \text{strongly in } L^2(0,T;L^2). \tag{4.65}$$

Moreover, from (4.39), (4.43), (4.46), (4.49) and (4.50) we see that the following initial conditions are satisfied:

$$\psi(\cdot,0)=\psi_0, \quad B(\cdot,0)=B_0, \quad \phi(\cdot,0)=\phi_0, \quad \alpha(\cdot,0)=\alpha_0, \quad \mathbf{A}(\cdot,0)=\mathbf{A}_0 \tag{4.66}$$

## 4.5 Proof of Theorem 2.1 (i)

First, for any given $\varphi\in L^2(0,T;\mathcal{H}^1)$, we choose finite element functions $\varphi_{h,\tau}\in L^2(0,T;\mathcal{V}_h)$ which converge to $\varphi$ strongly in $L^2(0,T;\mathcal{H}^1)$ as $h\to 0$. Then the equation (3.18) implies

$$\begin{aligned}&\int_0^T\big[(\eta\partial_t\psi_{h,\tau},\varphi_{h,\tau}) + (i\eta\kappa\phi^-_{h,\tau}\psi^+_{h,\tau},\varphi_{h,\tau})\big]\mathrm{d}t\\ &+\int_0^T\big[\big(\tfrac{i}{\kappa}\nabla\psi^+_{h,\tau} + \mathbf{A}^-_{h,\tau}\psi^+_{h,\tau}, \tfrac{i}{\kappa}\nabla\varphi_{h,\tau} + \mathbf{A}^-_{h,\tau}\varphi_{h,\tau}\big) + ((|\Upsilon(\psi^-_{h,\tau})|^2-1)\psi^+_{h,\tau},\varphi_{h,\tau})\big]\mathrm{d}t = 0.\end{aligned}$$

Let $h=h_m\to 0$ and $\tau=\tau_m\to 0$ in the equation above and use (4.36) and (4.61)-(4.65). We obtain

$$\begin{aligned}&\int_0^T\big[(\eta\partial_t\psi,\varphi) + (i\eta\kappa\phi\psi,\varphi) + \big(\big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\psi, \big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\varphi\big)\big]\mathrm{d}t\\ &+\int_0^T ((|\Upsilon(\psi)|^2)-1)\psi,\varphi)\mathrm{d}t = 0,\end{aligned} \tag{4.67}$$

which holds for any given $\varphi\in L^2(0,T;\mathcal{H}^1)$. The following lemma implies $|\psi|\le 1$, which can be proved in the same way as [28, Lemma 3.3].

**Lemma 4.4.** *For any given $\mathbf{A}\in L^\infty(0,T;\mathbf{L}^4)$, $\phi\in L^\infty(0,T;L^2)$ and $|\psi_0|\le 1$, the weak formulation (4.67) has a unique solution $\psi\in L^2(0,T;\mathcal{H}^1)\cap H^1(0,T;\widetilde{\mathcal{H}}^{-1})$ under the initial condition $\psi(\cdot,0)=\psi_0$. The solution satisfies $|\psi|\le 1$ a.e. in $\Omega\times(0,T)$.*

Since Lemma 4.4 implies $|\psi|\le 1$ a.e. in $\Omega\times(0,T)$, it follows that $\Upsilon(\psi)=\psi$. Hence, (4.67) implies

$$\int_0^T\big[(\eta\partial_t\psi,\varphi) + (i\eta\kappa\phi\psi,\varphi) + \big(\big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\psi, \big(\tfrac{i}{\kappa}\nabla + \mathbf{A}\big)\varphi\big)\big]\mathrm{d}t$$

$$+\int_0^T((|\psi|^2-1)\psi,\varphi)\mathrm{d}t=0, \qquad \forall\varphi\in L^2(0,T;\mathcal{H}^1). \tag{4.68}$$

In the same way, one can prove the following identities:

$$\int_0^T\left[(\partial_t B,\omega)+(\nabla(B-H),\nabla\omega)\right]\mathrm{d}t=-\int_0^T\left(\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\right],\nabla\times\omega\right)\mathrm{d}t \tag{4.69}$$

$$\int_0^T\left[(\partial_t\phi,\chi)+(\nabla\phi,\nabla\chi)\right]\mathrm{d}t=-\int_0^T\left(\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\right],\nabla\chi\right)\mathrm{d}t \tag{4.70}$$

$$\int_0^T(\nabla u,\nabla\xi)\mathrm{d}t=\int_0^T(B,\xi)\mathrm{d}t, \tag{4.71}$$

$$\int_0^T(\nabla v,\nabla\zeta)\mathrm{d}t=\int_0^T(\phi,\zeta)\mathrm{d}t, \tag{4.72}$$

$$\sum_{j=1}^m M_{kj}\alpha_j'(t)=-\left(\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\right],\nabla\times\varphi_k\right), \quad k=1,\dots,m, \tag{4.73}$$

which hold for all $\omega,\xi\in L^2(0,T;H_0^1)$ and $\chi,\zeta\in L^2(0,T;H^1)$. This proves that the functions $\psi$, $B$, $\phi$, $u$, $v$, $\alpha$ and $\mathbf{A}$ appearing in (4.36)-(4.50) form a weak solution of (2.9)-(2.16), with the regularity (2.21). This proves Theorem 2.1 (i).

### 4.6 Proof of Theorem 2.1 (ii)-(iii)

Differentiating (2.1) yields

$$\begin{aligned}
&\nabla\times\mathbf{A}=-\Delta u=B\in L^2(0,T;H^1),\\
&\nabla\cdot\mathbf{A}=\Delta v=-\phi\in L^2(0,T;H^1),\\
&\partial_t\mathbf{A}=\nabla\times\partial_t u+\nabla\partial_t v+\textstyle\sum_{j=1}^m\alpha_j'(t)\nabla\times\varphi_j\in L^2(0,T;\mathbf{L}^2).
\end{aligned} \tag{4.74}$$

Hence, the function

$$\mathbf{f}:=\partial_t\mathbf{A}+\nabla\times(\nabla\times\mathbf{A})-\nabla(\nabla\cdot\mathbf{A})+\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]-\nabla\times H$$

is well defined in $L^2(0,T;\mathbf{L}^2)$. In view of (2.10) and (2.11), we derive the following equality in the sense of distributions:

$$\begin{aligned}
\nabla\times\mathbf{f}&=\partial_t B-\Delta B+\nabla\times\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]+\Delta H=0,\\
\nabla\cdot\mathbf{f}&=-\partial_t\phi+\Delta\phi+\nabla\cdot\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=0.
\end{aligned}$$

Furthermore, in view of (2.14), we have

$$(\mathbf{f},\nabla\times\varphi_k)=\sum_{j=1}^m\alpha_j'(t)(\nabla\times\varphi_j,\nabla\times\varphi_k)+\left(\mathrm{Re}\left[\overline{\psi}\left(\tfrac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right],\nabla\times\varphi_k\right)=0.$$

By the Hodge decomposition, the last three equalities imply $\mathbf{f}=0$, and this proves that the solution of (2.9)-(2.14) also satisfies (1.7). Since $\phi=-\nabla\cdot\mathbf{A}\in L^2(0,T;H^1)$, it follows that (2.9) is equivalent to (1.6). The regularity (2.22) follows from Theorem 2.1 (i) and (4.74).

Under the regularity (2.22), the uniqueness of weak solutions of (1.6)-(1.7) can be proved in the same way as [28, section 3.3]. This proves Theorem 2.1 (iii). The uniqueness of weak solutions of (1.6)-(1.7) also imply Theorem 2.1 (ii).

### 4.7 Proof of Theorem 3.1 (ii)

Overall, we have proved Theorem 2.1 and the following statement: any sequence

$$(\psi_{h_m,\tau_m},B_{h_m,\tau_m},\phi_{h_m,\tau_m},u_{h_m,\tau_m},v_{h_m,\tau_m},\alpha_{h_m,\tau_m},\mathbf{A}_{h_m,\tau_m})$$

with $h_m,\tau_m\to0$ contains a subsequence converging to the unique solution $(\psi,B,\phi,u,v,\alpha,\mathbf{A})$ of the PDE problem (2.9)-(2.14) in the sense of (4.36)-(4.50). This implies that

$$(\psi_{h,\tau},B_{h,\tau},\phi_{h,\tau},u_{h,\tau},v_{h,\tau},\alpha_{h,\tau},\mathbf{A}_{h,\tau}) \text{ converges to } (\psi,B,\phi,u,v,\alpha,\mathbf{A}) \text{ as } h,\tau\to0$$

in the sense of (3.29). This proves Theorem 3.1 (ii).

## 5 Numerical experiments

In this section, we present several numerical examples by comparing the following different numerical methods in the numerical simulation of the vortex dynamics of the TDGL in convex polygons, nonconvex polygons and multi-connected nonsmooth domains, respectively.

Method I: solving the TDGL (1.6)-(1.8) directly by the Galerkin FEM. The magnetic field can be computed by $B_h^n = \nabla \times \mathbf{A}_h^n$.

Method II: solving the reformulated system (1.9)-(1.13) by the Galerkin FEM. The magnetic field can be computed by $B_h^n = H - p_h^n - (u_h^n - u_h^{n-1})/\tau$. This is the method introduced and analyzed in [27,28].

Method III: solving the new system (2.9)-(2.16) by the proposed method (3.18)-(3.25).

From a mathematical point of view, Method I can produce correct solutions for $\psi$ and $\mathbf{A}$ in smooth domains or convex nonsmooth domains. While in a domain with reentrant corners, the solution $\mathbf{A}$ of (1.6)-(1.8) is no longer in $H^1(\Omega) \times H^1(\Omega)$ and, as a consequence, Method I may yield incorrect solutions. As an improved method, Method II can produce correct solutions in simply connected nonsmooth and nonconvex domains. However, in a multi-connected nonsmooth domain, the Hodge decomposition used in [27] no longer holds, and the new system in [27] is no longer equivalent to the old system (1.6)-(1.8). In this case, Method III should give correct solutions, because it solves an equivalent system of equations whose solutions are all in $H^1(\Omega)$. Moreover, Method III has the magnetic field $B$ as an unknown solution. Hence, it can approximate the magnetic field $B$ better than Method II (which does not directly produce $B$ as a solution) even in simply connected domains.

The exact solution of the TDGL is unknown, but Theorem 3.1 indicates that Method III yields correct solutions (i.e. the numerical solutions converge to the PDEs' solution). Therefore, by comparing Methods I and II with Method III, we will see the performance of Methods I and II in the following numerical examples. In Examples 5.1–5.3 we show the difference of numerical solutions among the three different numerical methods in convex, nonconvex and multi-connected domains, respectively. In Example 5.4 we show the difference between the old and new Hodge decomposition methods in approximating the magnetic field $B$.

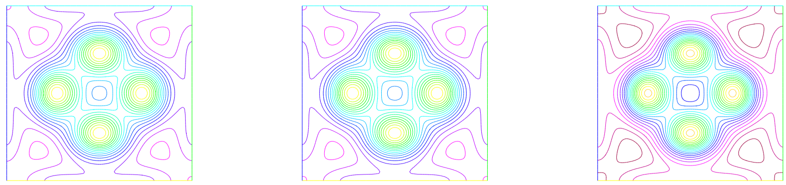
**Example 5.1** We solve the TDGL using Methods I–III with quadratic finite elements (with common mesh and time-step size) in a convex domain $\Omega = (0,1) \times (0,1)$ with the physical parameters $\eta = 1.0$, $\kappa = 10.0$ and $H = 5.0$, and the initial condition

$$\psi_0 = 1.0, \qquad \mathbf{A}_0 = (0,0).$$

Numerical simultation of the superconductivity density $|\psi|^2$ in such rectangular domains have been tested in [10].

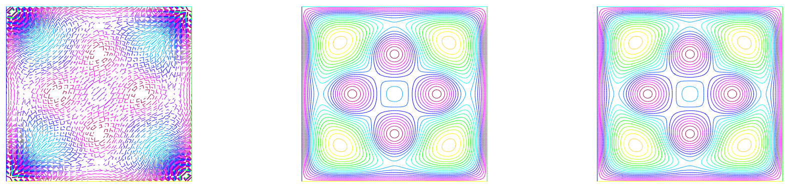
We present the contour plots of different numerical solutions of $|\psi|^2$ and $B$ in Figures 5.1 and 5.2, with $\tau = h = 1/32$. Theoretically, the numerical solutions corresponding to three methods are all convergent. Numerically, we see that the solution of $|\psi|^2$ given by the three methods agree well, while the solution of $B$ given by Method I is less accurate than that given by Methods II–III. This example shows that, in convex polygonal domains, Methods II and III yield comparably accurate superconductivity density $|\psi|^2$ as Method I, but are superior than Method I in computing the magnetic field $B$. This phenomenon can be explained as follows.

In a convex polygon, the solutions $\psi$ and $\mathbf{A}$ of the PDE problem (1.6)-(1.8) are both in $H^1(\Omega)$. Therefore, all three numerical methods can approximate $\psi$ and $\mathbf{A}$ correctly. However, Method I approximates the magnetic field $B$ by differentiating the numerical solution of $\mathbf{A}$, i.e. $B_h^n = \nabla \times \mathbf{A}_h^n$, which loses accuracy in the spatial direction. By using Methods II one can approximate the magnetic field directly with $B_h^n = H - p_h^n - (u_h^n - u_h^{n-1})/\tau$, without losing accuracy in the spatial direction (but it loses accuracy in the time direction). Using Method III one can approximate the magnetic field directly with $B_h^n$, which does not lose accuracy (as there is no differentiation in either space or time).



(a) Method I (b) Method II (c) Method III

Figure 5.1: Contour of $|\psi|^2$ at $t=50$.



(a) Method I (b) Method II (c) Method III

Figure 5.2: Contour of $B$ at $t=50$.

**Example 5.2** We solve the TDGL using Methods I–III with quadratic finite elements and mesh sizes $\tau = h = 1/32$ in an L-shape domain. We set physical parameters $\eta = 1.0$, $\kappa = 10.0$, $H = 5.0$ and the initial data

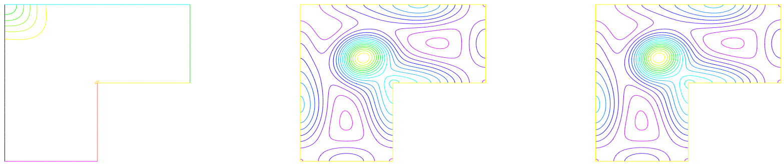
$$\psi_0 = 1.0, \qquad \mathbf{A}_0 = (0,0).$$

The contours of the numerical solutions of the superconductivity density $|\psi|^2$ and the magnetic field $B$ are presented in Figures 5.3–5.4. Theoretically, the numerical solutions corresponding to Methods II and III can be proved to be convergent (cf. [28] and Theorem 3.1), while the numerical solution of Method I may not converge due to the low regularity of the solution $\mathbf{A}$ in a nonconvex nonsmooth domain. Numerically, we see that the numerical solutions given by Methods II and III agree well, while the numerical solution given by Method I differ very much from the other two methods. This indicates that, indeed, Method I may yield spurious solutions in a nonconvex and nonsmooth domain. A more detailed explanation is given below.

When the computational domain contains reentrant corners, the solution $\mathbf{A}$ of the PDE problem is not in $H^1(\Omega) \times H^1(\Omega)$ in general, while the Galerkin finite element solution of $\mathbf{A}$ given by Method I still converges to some function in $H^1(\Omega) \times H^1(\Omega)$. This incorrect solution of $\mathbf{A}$ will also pollute the numerical solution of $\psi$ through the coupling of equations. Hence, both $|\psi|^2$ and $B$ given by Method I are polluted numerically. On the other hand, the numerical solutions of Methods II and III are expected to yield correct solutions in the presence of reentrant corners, because they solve equivalent systems of equations whose solutions are in $H^1(\Omega)$.

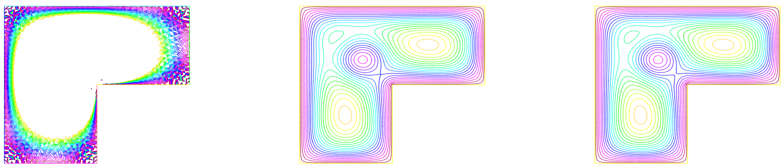
**Example 5.3** We compare Methods I–III in a multi-connected nonsmooth and nonconvex domain with common mesh sizes $\tau = h = 1/64$, as shown in Figure 5.5. Again, we set physical parameters $\eta = 1.0$, $\kappa = 10.0$, $H = 5.0$ and the initial data

$$\psi_0 = 1.0, \qquad \mathbf{A}_0 = (0,0).$$



(a) Method I (b) Method II (c) Method III

Figure 5.3: Contour of $|\psi|^2$ at $t=50$.



(a) Method I (b) Method II (c) Method III

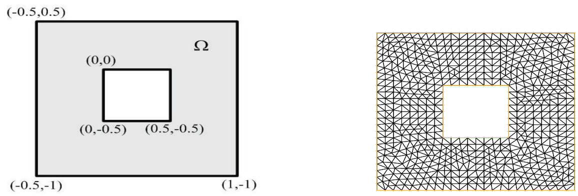
Figure 5.4: Contour of $B$ at $t=50$.

The contours of the numerical solutions given by Methods I–III are presented in Figures 5.6–5.7. Theoretically, only Method III has been proved to be convergent (see Theorem 3.1). Numerically, we see that the numerical solutions of Methods I-II differ very much from that of Method III. This indicates that both Method I and Method II may yield spurious solutions in a multi-connected nonconvex and nonsmooth domain.

**Example 5.4** Tables 5.1 and 5.2 contain errors of the numerical solutions of $|\psi|^2$ and $B$ given by Method I and II computed at $t=1$ for different mesh sizes with quadratic finite elements and $\tau=h$. The errors are computed by comparing the numerical solutions with that given by Method III under the same mesh size. The numerical results are shown for the three different domains considered in Examples 5.1–5.3, referred to as Domain I (convex and simply connected), Domain II (nonconvex and simply connected) and Domain III (nonconvex and mult-connected), respectively. Theoretically, Method III has been proved to be convergent in all the three types of domains, while the other two methods have been proved to be convergent only in some special cases. Numerical results in Tables 5.1 and 5.2 indicate that Method I yields reasonable convergence rates only in Domain I. Method II yields reasonable convergence rates for $|\psi|^2$ only in Domains I and II, and yields reasonable convergence rates for $B$ only in Domains I. The loss of accuracy for computing $B$ in Domain II may be due to the low regularity of the PDE's solution and the numerical differentiation in time when using the formula $B_h^n = H - p_h^n - (u_h^n - u_h^{n-1})/\tau$. This shows the superiority of Method III (compared with Method II) for approximating the magnetic field $B$ even in simply connected domains.

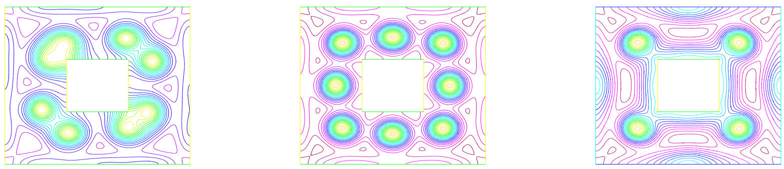
## 6 Conclusion

In this paper, we have proposed a new fully discrete Lagrange finite element method for solving the time-dependent Ginzburg–Landau equations in a bounded domain, possibly multi-connect, nonconvex, and nonsmooth. We have proved the convergence of numerical solutions based on the regularity of initial data $\psi_0$ and $\mathbf{A}_0$, and external magnetic field $H$, without requiring compatibility conditions between $\mathbf{A}_0$ and $H$. We have presented several numerical examples to compare the numerical results given by different numerical methods, to see the limitations of the some existing numerical methods. The numerical results show that the standard Lagrange finite element method only converges in convex domains;



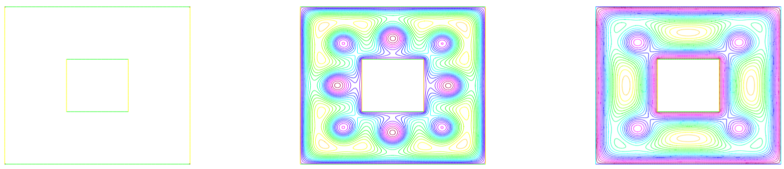

(a) The domain $\Omega$ (b) Triangulation

Figure 5.5: Illustration of the computational domain and the triangulation.



(a) Method I (b) Method II (c) Method III

Figure 5.6: Contour of $|\psi|^2$ at $t=50$.



(a) Method I (b) Method II (c) Method III

Figure 5.7: Contour of $B$ at $t=50$.

Table 5.1: Errors and convergence rates of numerical solutions of $|\psi|^2$

| | Method I | | | Method II | | |
|---|---|---|---|---|---|---|
| $h$ | Domain I | Domain II | Domain III | Domain I | Domain II | Domain III |
| 1/8 | 8.30E-02 | 5.93E-01 | 5.40E-01 | 2.33E-04 | 5.13E-04 | 9.34E-03 |
| 1/16 | 1.20E-02 | 5.93E-01 | 5.61E-01 | 4.49E-05 | 1.17E-04 | 1.10E-02 |
| 1/32 | 9.59E-04 | 5.82E-01 | 5.72E-01 | 6.95E-06 | 1.04E-04 | 1.19E-02 |
| 1/64 | 6.67E-05 | 5.49E-01 | 5.77E-01 | 9.73E-07 | 8.23E-05 | 1.24E-02 |
| 1/128 | 4.46E-06 | 5.09E-01 | 5.79E-01 | 1.29E-07 | 2.00E-05 | 1.27E-02 |
| rate | $O(h^{3.90})$ | $O(h^{0.10})$ | $O(h^{-0.01})$ | $O(h^{2.91})$ | $O(h^{2.04})$ | $O(h^{-0.03})$ |

the Hodge decomposition finite element method proposed in [28] converges in simply connected nonconvex domains for $|\psi|^2$, but only converges in convex domains for $B$. This may be due to the numerical differentiation $B_h^n = H - p_h^n - (u_h^n - u_h^{n-1})/\tau$ used in this second method, which makes the numerical solution not accurate in nonconvex domains.

Finally, we mention that the Lagrange finite element method proposed in this paper is limited to two-dimensional domains. There exist mixed finite element methods which have been proved to be convergent in general three-dimensional domains, either under regularity assumptions on the initial data and external magnetic fields [25], or under reasonable regularity assumptions on the solutions [22]. The construction of convergent Lagrange finite element methods for the Ginzburg–Landau equations in general three-dimensional domains (possibly nonconvex and nonsmooth) is still challenging.

Table 5.2: Errors and convergence rates of numerical solutions of $B$

| $h$ | Method I | | | Method II | | |
|---|---|---|---|---|---|---|
| | Domain I | Domain II | Domain III | Domain I | Domain II | Domain III |
| 1/8 | 2.56E-01 | 5.71E-01 | 3.32E-01 | 1.39E-04 | 6.25E-04 | 1.43E-03 |
| 1/16 | 1.28E-01 | 4.81E-01 | 1.71E-01 | 1.75E-05 | 3.87E-04 | 1.16E-03 |
| 1/32 | 6.40E-02 | 4.59E-01 | 9.42E-02 | 2.40E-06 | 4.10E-04 | 1.10E-03 |
| 1/64 | 3.20E-02 | 4.55E-01 | 6.04E-02 | 3.28E-07 | 3.68E-04 | 1.14E-03 |
| 1/128 | 1.60E-02 | 4.55E-01 | 4.85E-02 | 4.42E-08 | 3.71E-04 | 1.17E-03 |
| rate | $O(h^{1.00})$ | $O(h^{0.00})$ | $O(h^{0.31})$ | $O(h^{2.89})$ | $O(h^{-0.12})$ | $O(h^{-0.04})$ |

# A Proof of the Hodge decomposition (2.1)-(2.3)

Let $u$, $v$ and $\varphi_j$ be the solutions of (2.2)-(2.3). Then (2.2) implies

$$\frac{\partial(A_2+\partial_1 u)}{\partial x_1}+\frac{\partial(-A_1+\partial_2 u)}{\partial x_2}=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}+\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=\nabla\times\mathbf{A}+\Delta u=0, \tag{A.1}$$

which means that the vector field

$$\mathbf{w}:=\left(A_2+\frac{\partial u}{\partial x_1},-A_1+\frac{\partial u}{\partial x_2}\right) \tag{A.2}$$

is divergence-free.

In [7, equation (1.2) and section 2], Brenner et. al. have proved a different type of Hodge decomposition for divergence-free vector fields, which implies that $\mathbf{w}$ has the decomposition

$$\mathbf{w}=\nabla\times\phi-\textstyle\sum_{j=1}^{m}\alpha_j\nabla\varphi_j \tag{A.3}$$

for some constants $\alpha_j$, $j=1,\ldots,m$, where $\phi$ is the solution of the following problem (cf. [7, equation (1.4)]),

$$\begin{cases}\nabla\times(\nabla\times\phi)=\nabla\times\mathbf{w}=-\nabla\cdot\mathbf{A} & \text{in}\,\Omega,\\ \mathbf{n}\times(\nabla\times\phi)=0 & \text{on}\,\partial\Omega.\end{cases} \tag{A.4}$$

with the normalization condition $\int_\Omega\phi(x)\mathrm{d}x=0$. Since $\mathbf{n}\times(\nabla\times\phi)=-\nabla\phi\cdot\mathbf{n}$, the equation above can be equivalently written as

$$\begin{cases}\Delta\phi=\nabla\cdot\mathbf{A} & \text{in}\,\Omega,\\ \nabla\phi\cdot\mathbf{n}=0 & \text{on}\,\partial\Omega,\end{cases} \tag{A.5}$$

which implies $\phi=v$ upon comparing this equation with (2.2). Substituting (A.2) and $\phi=v$ into (A.3), we have

$$(A_2,-A_1)=-\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right)+\left(\frac{\partial v}{\partial x_2},-\frac{\partial v}{\partial x_1}\right)-\sum_{j=1}^{m}\alpha_j\left(\frac{\partial\varphi}{\partial x_1},\frac{\partial\varphi}{\partial x_2}\right), \tag{A.6}$$

which is equivalent to

$$(A_1,A_2)=\left(\frac{\partial u}{\partial x_2},-\frac{\partial u}{\partial x_1}\right)+\left(\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2}\right)+\sum_{j=1}^{m}\alpha_j\left(\frac{\partial\varphi}{\partial x_2},-\frac{\partial\varphi}{\partial x_1}\right). \tag{A.7}$$

This completes the proof of the Hodge decomposition (2.1).

## B Maximal $\ell^p$-regularity of discretized parabolic equations

Let $X_h = (\mathring{V}_h, \|\cdot\|_{H^{-1}})$. The domain $D(\Delta_h^D)$ of the operator $\Delta_h^D : X_h \to X_h$ is equipped with the graph norm, i.e.,

$$\|\phi_h\|_{D(\Delta_h^D)} = \|\Delta_h^D \phi_h\|_{H^{-1}} \sim \|\phi_h\|_{H_0^1}, \tag{B.1}$$

where "$\sim$" indicates norm equivalence.

Consider the discretized parabolic problem

$$\begin{cases} \dfrac{\phi_h^n - \phi_h^{n-1}}{\tau} - \Delta_h^D \phi^n = f_h^n, & \text{with } f_h^n \in \mathring{V}_h,\ n = 1,\dots,N, \\ \phi_h^0 = P_h g, & \text{with } g \in H^{-1}, \end{cases} \tag{B.2}$$

where $P_h$ denotes the $L^2$ projection onto $\mathring{V}_h$ (which has a bounded extension $P_h : H^{-1} \to \mathring{V}_h$). It is well known that the operator $\Delta_h^D$ generates a bounded analytic semigroup in the Hilbert space $X_h$, satisfying

$$\|e^{t\Delta_h^D} v_h\|_{X_h} + t\|\Delta_h^D e^{t\Delta_h^D} v_h\|_{X_h} \le C\|v\|_{X_h}, \quad \forall v_h \in X_h, \tag{B.3}$$

where the constant $C$ is independent of $h$. This implies the maximal $L^p$-regularity of the semi-discrete problem (see [15, 16])

$$\begin{cases} \partial_t \phi_h - \Delta_h^D \phi_h = f_h, & \text{with } f_h \in L^p(0,T;\mathring{V}_h),\ 1 < p < \infty, \\ \phi_h(0) = 0. \end{cases} \tag{B.4}$$

By [24, Theorem 3.1], the maximal $L^p$-regularity of the semi-discrete problem above implies the maximal $L^p$-regularity of the fully discrete problem. Namely, the solution of (B.2) with $g = 0$ satisfies

$$\|(\tfrac{\phi_h^n - \phi_h^{n-1}}{\tau})_{n=1}^N\|_{\ell^p(X_h)} + \|(\Delta_h^D \phi^n)_{n=1}^N\|_{\ell^p(X_h)} \le C\|(f_h^n)_{n=1}^N\|_{\ell^p(X_h)}, \tag{B.5}$$

which implies

$$\|(\tfrac{\phi_h^n - \phi_h^{n-1}}{\tau})_{n=1}^N\|_{\ell^p(H^{-1})} + \|(\phi^n)_{n=1}^N\|_{\ell^p(H_0^1)} \le C\|(f_h^n)_{n=1}^N\|_{\ell^p(H^{-1})}, \text{ if } g = 0. \tag{B.6}$$

In the case $f_h^n = 0$, the solution of (B.2) satisfies (as a consequence of [34, Lemma 7.3])

$$\begin{aligned} \|\Delta_h^D \phi_h^n\|_{X_h} &\le C\|\Delta_h^D P_h g\|_{X_h} \le C\|g\|_{H_0^1} \\ \|\Delta_h^D \phi_h^n\|_{X_h} &\le C t_n^{-1}\|P_h g\|_{X_h} \le C t_n^{-1}\|g\|_{H^{-1}}, \end{aligned} \tag{B.7}$$

We denote by $E_\tau : H^{-1} \to L^\infty(\mathbb{R}_+, X_h)$ the operator which maps $g$ to the piecewise constant (in time) function

$$E_\tau g = \phi_h^n \quad \forall t \in (t_{n-1}, t_n], \quad n = 1, 2, \dots$$

Then (B.7) implies

$$\|E_\tau g\|_{L^\infty(\mathbb{R}_+, D(\Delta_h^D))} \le C\|g\|_{H_0^1} \quad \text{and} \quad \|E_\tau g\|_{L^{1,\infty}(\mathbb{R}_+, D(\Delta_h^D))} \le C\|g\|_{H^{-1}}. \tag{B.8}$$

The real interpolation of the last two estimates yields

$$\|E_\tau g\|_{(L^{1,\infty}(\mathbb{R}_+, D(\Delta_h^D)), L^\infty(\mathbb{R}_+, D(\Delta_h^D)))_{1-\frac{1}{p},p}} \le c\|g\|_{(H^{-1}, H_0^1)_{1-\frac{1}{p},p}}, \forall p \in (1, \infty).$$

Since $(L^{1,\infty}(\mathbb{R}_+,D(\Delta_h^D)),L^\infty(\mathbb{R}_+,D(\Delta_h^D)))_{1-\frac{1}{p},p}=L^p(\mathbb{R}_+,D(\Delta_h^D))$ (cf. [5, Theorem 5.2.1]), the inequality above further implies

$$\|(\phi_h^n)_{n=1}^N\|_{\ell^p(D(\Delta_h^D))}\leq C\|g\|_{(H^{-1},H_0^1)_{1-\frac{1}{p},p}}. \tag{B.9}$$

Substituting the estimate above into (B.2) and using the norm equivalence (B.1), we obtain

$$\Big\|\Big(\frac{\phi_h^n-\phi_h^{n-1}}{\tau}\Big)_{n=1}^N\Big\|_{\ell^p(H^{-1})}+\|(\phi_h^n)_{n=1}^N\|_{\ell^p(H_0^1)}\leq C\|g\|_{(H^{-1},H_0^1)_{1-\frac{1}{p},p}}, \quad \text{if } f_h^n=0. \tag{B.10}$$

By setting $\phi_h^n=B_h^n-H$, $g=B_h^0-H$ and combining (B.6) and (B.10), we obtain (4.16). The proof of (4.17) is similar and omitted.

# C Real interpolation between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$

From [31, Theorems 3.30 and 3.33] we know that

$$\widetilde{H}^{-1}(\Omega)=H^1(\Omega)', \quad H_0^1(\Omega)=\widetilde{H}^1(\Omega) \quad \text{and} \quad H^{-1}(\Omega)=H_0^1(\Omega)'.$$

Then [31, Theorems B.8 and B.9] imply

$$\begin{aligned} L^2(\Omega)=(\widetilde{H}^{-1}(\Omega),\widetilde{H}^1(\Omega))_{1/2,2}=(H^1(\Omega)',H_0^1(\Omega))_{1/2,2} &\hookrightarrow (H_0^1(\Omega)',H_0^1(\Omega))_{1/2,2} \\ &\hookrightarrow (H_0^1(\Omega)',H^1(\Omega))_{1/2,2} \\ &=(H^{-1}(\Omega),H^1(\Omega))_{1/2,2}=L^2(\Omega) \end{aligned}$$

Hence, we have $(H^{-1}(\Omega),H_0^1(\Omega))_{1/2,2}=(H_0^1(\Omega)',H_0^1(\Omega))_{1/2,2}=L^2(\Omega)$. By the reiteration theorem [4, Theorem 3.5.4], for $0<2\theta-1<1/2$ and $q\geq 2$ we have

$$\begin{aligned} (H^{-1}(\Omega),H_0^1(\Omega))_{\theta,q}=(L^2(\Omega),H_0^1(\Omega))_{2\theta-1,q} \hookleftarrow (L^2(\Omega),H_0^1(\Omega))_{2\theta-1,2}=\widetilde{H}^{2\theta-1}(\Omega) \\ =H^{2\theta-1}(\Omega). \end{aligned} \tag{C.1}$$

where the last equality is a consequence of [31, Theorems 3.33 and 3.40].

# Acknowledgments

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