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Indefinite Mean-Field Type Linear-Quadratic Stochastic Optimal Control Problems ⋆

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Abstract

This paper focuses on indefinite stochastic mean-field linear-quadratic (MF-LQ, for short) optimal control problems, which allow the weighting matrices for state and control in the cost functional to be indefinite. The solvability of stochastic Hamiltonian system and Riccati equations is presented under indefinite case. The optimal controls in open-loop form and closed-loop form are derived, respectively. In particular, dynamic mean-variance portfolio selection problem can be formulated as an indefinite MF-LQ problem to tackle directly. Another example also sheds light on the theoretical results established.

Key words: Stochastic linear-quadratic problem, Mean-field, Hamiltonian system, Stochastic differential equation, Forward-backward stochastic differential equation, Riccati equation

1 Introduction

Historically, researchers have made many contributions to McKean-Vlasov type stochastic differential equation (SDE, for short) ([1,2,8,11,13,16,20]), which can be regarded as a kind of mean-field SDE (MF-SDE, for short). In recent years, stochastic mean-field optimal control problems, mean-field differential games and their applications have attracted researchers' attention. Andersson and Djehiche [4], and Buckdahn et al. [10] studied the maximum principle for SDEs of mean-field type, respectively. Buckdahn *et al.* [9] considered the mean-field backward SDE (MF-BSDE), Bensoussan *et al.* [7] obtained the unique solvability of mean-field type forward-backward SDE (MF-FBSDE). Recently, Duncan and Tembine [14] applied a direct method to discuss an MF-LQ game. Barreiro-Gomez et al. [5] investigated an MF-LQ game of jump-diffusion process with regime switching. This paper focuses on MF-LQ stochastic optimal control problems for the indefinite weighting case, which generalize the work of mean-field type optimal control problems with positive definite weighting case.

For the positive definite case, MF-LQ problems have been studied widely over the past decade. Yong [30] considered an MF-LQ problem with deterministic coefficients over a finite time horizon, and presented the optimal feedback using a system of Riccati equations. Recently, there are some related works following up Yong [30] (see [17,23,31,29,28]). Different from deterministic LQ problem, in the cost functional, the cost weighting matrices for the state and control are allowed to be indefinite. We notice that in the stochastic LQ setting, the cost functional with indefinite cost weighting matrices may still be convex in control variable. It is precisely this feature that determines whether an optimal control exists. Indefinite stochastic LQ theory has been extensively developed and has lots of interesting and important applications. Chen et al. [12] studied a kind of indefinite LQ problem based on Riccati equation. Ait Rami et al. [3] showed that the solvability of the generalized Riccati equation is sufficient and necessary condition for the well-posedness of the indefinite LQ problem. Subsequent research includes various cases, and refer to [19,26,27].

One of the motivations for indefinite MF-LQ problems comes from the mean-variance portfolio selection problem. Markowitz initially proposed and solved the mean-variance problem in the single-period setting in his Novel-Prize winning work [24,25], which is an important foundation of the development of modern finance. After Markowitz's pioneering work, the mean-variance

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model was extended to multi-period/continuous-time portfolio selection. When one attempts to solve the mean-variance portfolio selection, she has to face to two objectives: One is to minimize the difference between the terminal wealth and its expected value; the other one is to maximize her expected terminal wealth. Since there are two criteria in one cost functional, this stochastic control problem is significantly different from the classic LQ problem. The main reason is due to the variance term

$$\mathrm{Var}(X(T)) = \mathbb{E}\big[X(T) - \mathbb{E}[X(T)]\big]^2$$

essentially, which involves the nonlinear term of $(\mathbb{E}[X(T)])^2$. In general, for nonlinear utility function $U(\cdot)$, there exists an essential difference between $\mathbb{E}[U(X(T))]$ and $U(\mathbb{E}[X(T)])$, which leads to the fundamental difficulty to deal with the latter one by dynamic programming. Li and Zhou [21] embedded this problem into an auxiliary stochastic LQ problem, which actually is one of indefinite LQ problems. In this paper, we revisit the continuous-time mean-variance problem using the theoretical results of indefinite MF-LQ problems in a direct way (see the example in Section 4.1).

Besides the dynamic mean-variance portfolio selection problem, there are many phenomena in finance and engineering fields which involve indefinite weighting parameters in the integral term as well as the terminal term. Another motivation is inspired by multi-objective optimization problems involving mean value. These problems can be converted into a single-objective problem by putting weights on the different objectives, which essentially are the indefinite mean-field optimization problems. For example, in a moving high-speed train, the controller wants to improve the speed as high as possible. Except for speeding up the train, the controller also wants to improve the resistance to the stochastic disturbance, which means that the state $X(\cdot)$ of train cannot deviate too much from the mean value $\mathbb{E}[X(\cdot)]$. Therefore, there is a tradeoff between two objectives: One is to maximize the total speed $\mathbb{E}\int_0^T |u(t)|^2 dt$, the other one is to minimize the variance over interval $[0,T]$ measured by $\mathbb{E}\int_0^T |X(t)-\mathbb{E}[X(t)]|^2 dt$. We convert this multi-objective optimization problem into a single-objective problem as

$$J(u(\cdot)) = \mathbb{E}\int_0^T \Big\{\alpha\big|X(t) - \mathbb{E}[X(t)]\big|^2 - \beta|u(t)|^2\Big\}dt$$

with $\alpha, \beta > 0$. When the system is linear, this problem is a special case of indefinite MF-LQ problem.

In the literature on indefinite LQ problems, the standard matrix inverse is involved in the Riccati equation, requiring the related term to be nonsingular. However, sometimes, the theory of Riccati equation is abstract yet difficult. For example, the global solvability of Riccati equation (in the indefinite case or/and in the stochastic case) is often not simple. For this reason, we want to find another element with flexible restrictions instead of Riccati equation. Based on Yong [30] and inspired by Yu [33] and Huang and Yu [18], we generalize the results of positive definite MF-LQ problem to the indefinite case by introducing a *relaxed compensator*, which can be regarded as a generalization of the solution of Riccati equation. The presence of the relaxed compensator guarantees the well-posedness of MF-LQ problem. The open-loop and closed-loop optimal controls are also obtained under indefinite case. There are three main contributions of this paper:

- (i) Comparing with the solvability of Riccati equations, the relaxed compensator is defined under more flexible conditions (Condition (RC) in Section 3), which is more general.
- (ii) Based on the linear transformation involving relaxed compensator, we analyze the unique solvability of a kind of MF-FBSDEs, which does not satisfy the monotonicity condition in [7].
- (iii) We present the existence codition of relaxed compensator, which is a sufficient and necessary condition for the solvability of Riccati equations.

Recently, Sun [27] studied the MF-LQ problem under a uniform convexity condition, and showed that the convergence of a family of uniformly convex cost functionals is equivalent to the open-loop solvability of the MF-LQ problem. Different from the method in [27], this paper focuses on how to find a relaxed compensator to extend the condition of cost functional from positive case to the indefinite case.

The rest of this paper is organized as follows. We present some preliminaries and formulate an MF-LQ problem in Section 2. Section 3 studies on the indefinite MF-LQ problem, and derives the open-loop optimal control and the optimal feedback control. Section 4 illustrates some applications including the dynamic mean-variance problem.

2 Problem formulation and preliminaries

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $\mathbb{R}^{n\times m}$ be the set of all $(n \times m)$ matrices. Let $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ be the collection of all symmetric matrices. As usual, if a matrix $A \in \mathbb{S}^n$ is positive semidefinite (resp. positive definite; negative semidefinite; negative definite), we denote $A \geq 0$ (resp. > 0; ≤ 0; < 0). All the positive semidefinite (resp. negative semidefinite) matrices are collected by $\mathbb{S}^n_+$ (resp. $\mathbb{S}^n_-$). Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by all $\mathbb{P}$-null sets. For simplicity, we restrict ourselves to the case of one-dimensional standard Brownian motion. Some extensions to the case with multi-dimensional standard Brownian motion will be similarly derived for examples in Section 4. Let $T > 0$ be a finite time horizon. Let $\mathbb{H} = \mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{S}^n$, $\mathbb{S}^n_+$, etc. We introduce the following notation used in this paper:

- $L^\infty(0,T;\mathbb{H})$ is the space of $\mathbb{H}$-valued continuous functions $\varphi(\cdot)$ such that $\operatorname*{esssup}_{t\in[0,T]} |\varphi(t)| < \infty$;
- $C^1([0,T];\mathbb{H})$ is the space of $\mathbb{H}$-valued functions $\varphi(\cdot)$ such that $\dot\varphi(\cdot)$ is continuous;
- $L^2_{\mathcal{F}_T}(\Omega;\mathbb{H})$ is the space of $\mathbb{H}$-valued $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}[|\xi|^2] < \infty$;
- $L^2_{\mathbb{F}}(0,T;\mathbb{H})$ is the space of $\mathbb{H}$-valued $\mathbb{F}$-progressively measurable processes $\varphi(\cdot)$ such that $\mathbb{E}\int_0^T |\varphi(t)|^2 dt < \infty$;
- $L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{H}))$ is the space of $\mathbb{H}$-valued $\mathbb{F}$-progressively measurable processes $\varphi(\cdot)$ such that for almost all $\omega\in\Omega$, $r\mapsto\varphi(r,\omega)$ is continuous and $\mathbb{E}\Big[\sup_{t\in[0,T]} |\varphi(t)|^2\Big] < \infty$.

Let $\mathscr{U}[0,T] \equiv L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ denote the set of admissible controls. For any initial state $x\in\mathbb{R}^n$ and any admissible control $u(\cdot)\in\mathscr{U}[0,T]$, we consider the following controlled MF-SDE:

$$\begin{cases} dX(t) = \Big\{A(t)X(t) + \widetilde{A}(t)\mathbb{E}[X(t)] + B(t)u(t) \\ \qquad\qquad + \widetilde{B}(t)\mathbb{E}[u(t)]\Big\}dt + \Big\{C(t)X(t) + \widetilde{C}(t)\mathbb{E}[X(t)] \\ \qquad\qquad + D(t)u(t) + \widetilde{D}(t)\mathbb{E}[u(t)]\Big\}dW(t), \quad t\in[0,T], \\ X(0) = x, \end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ is the initial state, $A(\cdot)$, $\widetilde{A}(\cdot)$, $C(\cdot)$, $\widetilde{C}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times n})$ and $B(\cdot)$, $\widetilde{B}(\cdot)$, $D(\cdot)$, $\widetilde{D}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m})$. By Proposition 2.6 in Yong [30] (see also Proposition 2.1 in [31] and Proposition 2.2 in [29] for wider versions), the MF-SDE (1) admits a unique solution $X(\cdot)\equiv X(\cdot;x,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$. $X(\cdot)$ is called an admissible trajectory corresponding to $u(\cdot)$, and $(X(\cdot),u(\cdot))$ is called an admissible pair. Now, we present a cost functional as follows:

$$\begin{aligned} &J\big(x;u(\cdot)\big) \\ =&\mathbb{E}\bigg\{\int_0^T \Big[\big\langle Q(t)X(t),\ X(t)\big\rangle + \big\langle\widetilde{Q}(t)\mathbb{E}[X(t)],\ \mathbb{E}[X(t)]\big\rangle \\ &+ 2\big\langle S(t)u(t),\ X(t)\big\rangle + 2\big\langle\widetilde{S}(t)\mathbb{E}[u(t)],\ \mathbb{E}[X(t)]\big\rangle \\ &+ \big\langle R(t)u(t),\ u(t)\big\rangle + \big\langle\widetilde{R}(t)\mathbb{E}[u(t)],\ \mathbb{E}[u(t)]\big\rangle\Big]dt \\ &+ \big\langle GX(T),\ X(T)\big\rangle + \big\langle\widetilde{G}\mathbb{E}[X(T)],\ \mathbb{E}[X(T)]\big\rangle\bigg\}, \end{aligned} \tag{2}$$

where $Q(\cdot)$, $\widetilde{Q}(\cdot)\in L^\infty(0,T;\mathbb{S}^n)$, $S(\cdot)$, $\widetilde{S}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m})$, $R(\cdot)$, $\widetilde{R}(\cdot)\in L^\infty(0,T;\mathbb{S}^m)$, and G, $\widetilde{G}\in\mathbb{S}^n$. It is clear that, for given $x\in\mathbb{R}^n$ and any $u(\cdot)\in\mathscr{U}[0,T]$, $J(x;u(\cdot))$ is well-defined.

Problem (MF-LQ). We introduce a family of MF-LQ stochastic optimal control problems: find an admissible control $u^*(\cdot)\in\mathscr{U}[0,T]$ such that

$$J\big(x;u^*(\cdot)\big) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J\big(x;u(\cdot)\big).$$

Problem (MF-LQ) is called *well-posed* if the infimum of $J\big(x;u(\cdot)\big)$ over the set of admissible controls is finite. If Problem (MF-LQ) is well-posed and the infimum of the cost functional is achieved by an admissible control $u^*(\cdot)$, then Problem (MF-LQ) is said to be *solvable* and $u^*(\cdot)$ is called an *optimal control.* $X^*(\cdot)\equiv X\big(\cdot;x,u^*(\cdot)\big)$ is called the *optimal trajectory* corresponding to $u^*(\cdot)$, and $(X^*(\cdot),u^*(\cdot))$ is called an *optimal pair.*

For simplicity, we use the following notation in this paper: $\widehat{\Pi} = \Pi + \widetilde{\Pi}$ with $\Pi = A(\cdot),\ B(\cdot),\ C(\cdot),\ D(\cdot),\ Q(\cdot),\ S(\cdot),\ R(\cdot),\ G$, and

$$\begin{cases} \mathbf{Q}(t) = \begin{pmatrix} Q(t) & O \\ O & \widehat{Q}(t)\end{pmatrix}, \quad \mathbf{S}(t) = \begin{pmatrix} S(t) & O \\ O & \widehat{S}(t)\end{pmatrix}, \\ \mathbf{R}(t) = \begin{pmatrix} R(t) & O \\ O & \widehat{R}(t)\end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} G & O \\ O & \widehat{G}\end{pmatrix}, \end{cases}$$

where O denotes zero matrices with appropriate dimensions.

Similar to Yong [30], we present another version of (1) and (2). In detail, taking expectation $\mathbb{E}[\cdot]$ on both sides of (1), we have

$$\begin{cases} d\mathbb{E}[X(t)] = \Big\{\widehat{A}(t)\mathbb{E}[X(t)] + \widehat{B}(t)\mathbb{E}[u(t)]\Big\}dt,\ t\in[0,T], \\ \mathbb{E}[X(0)] = x. \end{cases} \tag{3}$$

Then, the difference between $X(\cdot)$ and $\mathbb{E}[X(\cdot)]$ satisfies

$$\begin{cases} d\big(X(t) - \mathbb{E}[X(t)]\big) \\ = \Big\{A(t)\big(X(t) - \mathbb{E}[X(t)]\big) + B(t)\big(u(t) - \mathbb{E}[u(t)]\big)\Big\}dt \\ + \Big\{C(t)\big(X(t) - \mathbb{E}[X(t)]\big) + \widehat{C}(t)\mathbb{E}[X(t)] \\ + D\big(u(t) - \mathbb{E}[u(t)]\big) + \widehat{D}(t)\mathbb{E}[u(t)]\Big\}dW(t),\ t\in[0,T], \\ X(0) - \mathbb{E}[X(0)] = 0. \end{cases} \tag{4}$$

It is clear that the system consisting of (4) and (3) is equivalent to the equation (1). We denote

$$\mathbf{X}(t) = \begin{pmatrix} X(t) - \mathbb{E}[X(t)] \\ \mathbb{E}[X(t)]\end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} u(t) - \mathbb{E}[u(t)] \\ \mathbb{E}[u(t)]\end{pmatrix}.$$

Also, cost functional (2) can be rewritten into the following form

$$J\big(x;u(\cdot)\big)=\mathbb{E}\bigg\{\int_0^T\Big[\big\langle \mathbf{QX},\ \mathbf{X}\big\rangle+2\big\langle \mathbf{Su},\ \mathbf{X}\big\rangle + \big\langle \mathbf{Ru},\ \mathbf{u}\big\rangle\Big]dt+\big\langle \mathbf{GX}(T),\ \mathbf{X}(T)\big\rangle\bigg\}. \tag{5}$$

For an $\mathbb{S}^n$-valued process $f(\cdot)$, if $f(t)\geq 0$ (resp. >0; ≤ 0; <0) for almost everywhere $t\in[0,T]$, then we denote $f(\cdot)\geq 0$ (resp. >0; ≤ 0; <0). Moreover, if there exists a constant $\delta>0$ such that $f(\cdot)-\delta I_n\geq 0$ (resp. $f(\cdot)+\delta I_n\leq 0$), then we denote $f(\cdot)\gg 0$ (resp. $f(\cdot)\ll 0$), where I_n denotes the $(n\times n)$ identity matrix. Now, for a given quadruple of $(\mathbf{Q}(\cdot),\mathbf{S}(\cdot),\mathbf{R}(\cdot),\mathbf{G})$, we introduce a positive definite (PD, for short) condition:

Condition (PD). $\begin{pmatrix}\mathbf{Q}(\cdot) & \mathbf{S}(\cdot)\\ \mathbf{S}(\cdot)^\top & \mathbf{R}(\cdot)\end{pmatrix}\geq 0,\quad \mathbf{R}(\cdot)\gg 0,\quad \mathbf{G}\geq 0,\qquad t\in[0,T].$

Here and hereafter, we use the superscript $\top$ to denote the transpose of a matrix (or a vector). It is clear that, if $(\mathbf{Q}(\cdot),\mathbf{S}(\cdot),\mathbf{R}(\cdot),\mathbf{G})$ satisfies Condition (PD), then we have $J(x;u(\cdot))\geq 0$ for any $x\in\mathbb{R}^n$ and any $u(\cdot)\in\mathscr{U}[0,T]$. Hence, Problem (MF-LQ) is well-posed.

Under the positive definite condition, Yong [30] studied the MF-LQ problem without cross-terms in the cost functional. For the case with cross-terms, there are some works of LQ problem in the existing literature, the corresponding results of positive definite MF-LQ problem can be obtained by the direct method introduced in Duncan and Pasik-Duncan [15], see also [22] for a different method. Recently, Barreiro-Gomez et al. [6] further developed a kind of nonlinear nonquadratic mean-field type game with cross-terms by the direct method. Inspired by the works in the positive definite case, we are interested in studying Problem (MF-LQ) under indefinite condition. Moreover, we solve the MF-FBSDEs and Riccati equations under more general conditions.

3 Relaxed compensators and Problem (MF-LQ) in the indefinite case

In this section, we are concerned about Problem (MF-LQ) under indefinite conditon. Relaxed compensator plays a key role to extend the positive definite case to the indefinite case. In detail, we introduce a space:

$$\Lambda[0,T]=\bigg\{F(\cdot)\ \bigg|\ F(t)=F(0)+\int_0^t f(s)ds,\ t\in[0,T],\ \text{where } f(\cdot)\in L^\infty(0,T;\mathbb{S}^n)\bigg\}.$$

For a given pair of functions $(H(\cdot),K(\cdot))\in\Lambda[0,T]\times\Lambda[0,T]$, we define (for simplicity of notation, the argument t is suppressed)

$$\begin{cases}\mathbf{Q}^{H,K}=\begin{pmatrix}Q^{H,K} & O\\ O & \widehat{Q}^{H,K}\end{pmatrix},\ \mathbf{S}^{H,K}=\begin{pmatrix}S^{H,K} & O\\ O & \widehat{S}^{H,K}\end{pmatrix},\\ \mathbf{R}^{H,K}=\begin{pmatrix}R^{H,K} & O\\ O & \widehat{R}^{H,K}\end{pmatrix},\ \mathbf{G}^{H,K}=\begin{pmatrix}G^{H,K} & O\\ O & \widehat{G}^{H,K}\end{pmatrix},\end{cases}$$

where

$$\begin{cases}Q^{H,K}=\dot{H}+HA+A^\top H+C^\top HC+Q,\\ \widehat{Q}^{H,K}=\dot{K}+K\widehat{A}+\widehat{A}^\top K+\widehat{C}^\top H\widehat{C}+\widehat{Q},\\ S^{H,K}=HB+C^\top HD+S,\\ \widehat{S}^{H,K}=K\widehat{B}+\widehat{C}^\top H\widehat{D}+\widehat{S},\\ R^{H,K}=D^\top HD+R,\quad \widehat{R}^{H,K}=\widehat{D}^\top H\widehat{D}+\widehat{R},\\ G^{H,K}=G-H(T),\quad \widehat{G}^{H,K}=\widehat{G}-K(T).\end{cases} \tag{6}$$

According to the notation given by (6), we introduce

$$J^{H,K}\big(x;u(\cdot)\big)=\mathbb{E}\bigg\{\int_0^T\Big[\big\langle \mathbf{Q}^{H,K}\mathbf{X},\ \mathbf{X}\big\rangle+2\big\langle \mathbf{S}^{H,K}\mathbf{u},\ \mathbf{X}\big\rangle + \big\langle \mathbf{R}^{H,K}\mathbf{u},\ \mathbf{u}\big\rangle\Big]dt+\big\langle \mathbf{G}^{H,K}\mathbf{X}(T),\ \mathbf{X}(T)\big\rangle\bigg\}.$$

Then, similar to Problem (MF-LQ), we propose an auxiliary MF-LQ with H,K as follows:

Problem $(\textbf{MF-LQ})^{H,K}$. For given $x\in\mathbb{R}^n$, the problem is to find an admissible control $u^{H,K}(\cdot)\in\mathscr{U}[0,T]$ such that

$$J^{H,K}\big(x;u^{H,K}(\cdot)\big)=\inf_{u(\cdot)\in\mathscr{U}[0,T]}J^{H,K}\big(x;u(\cdot)\big).$$

The following lemma shows the equivalence between $J(x;u(\cdot))$ and $J^{H,K}(x;u(\cdot))$, which plays a key role in our analysis.

Lemma 3.1 *Let $(H(\cdot),K(\cdot))\in\Lambda[0,T]\times\Lambda[0,T]$. For any $x\in\mathbb{R}^n$ and any $u(\cdot)\in\mathscr{U}[0,T]$,*

$$J(x;u(\cdot))=J^{H,K}(x;u(\cdot))+\langle K(0)x,\ x\rangle. \tag{7}$$

Proof Using Itô's formula to $\langle H(\cdot)(X(\cdot)-\mathbb{E}[X(\cdot)]),\ X(\cdot)-\mathbb{E}[X(\cdot)]\rangle$ and $\langle K(\cdot)\mathbb{E}[X(\cdot)],\ \mathbb{E}[X(\cdot)]\rangle$ on the interval $[0,T]$, respectively, we have

$$-\big\langle K(0)x,\ x\big\rangle=\mathbb{E}\bigg\{\int_0^T\Big[\big\langle(\mathbf{Q}^{H,K}-\mathbf{Q})\mathbf{X},\ \mathbf{X}\big\rangle + 2\big\langle(\mathbf{S}^{H,K}-\mathbf{S})\mathbf{u},\ \mathbf{X}\big\rangle+\big\langle(\mathbf{R}^{H,K}-\mathbf{R})\mathbf{u},\ \mathbf{u}\big\rangle\Big]dt + \big\langle(\mathbf{G}^{H,K}-\mathbf{G})\mathbf{X}(T),\ \mathbf{X}(T)\big\rangle\bigg\}, \tag{8}$$

adding (8) on both sides of (5) yields (7). The proof is completed. □

Definition 3.1 *If there exists a pair of functions* $(H(\cdot),K(\cdot)) \in \Lambda[0,T]\times\Lambda[0,T]$ *such that the quadruple of functions* $(\mathbf{Q}^{H,K}(\cdot),\mathbf{S}^{H,K}(\cdot),\mathbf{R}^{H,K}(\cdot),\mathbf{G}^{H,K})$ *satisfies Condition (PD), then we call* $(H(\cdot),K(\cdot))$ *a relaxed compensator for Problem (MF-LQ).*

By the definition of relaxed compensator and Lemma 3.1, we have

Corollary 3.1 *If there exists a relaxed compensator for Problem (MF-LQ), then Problem (MF-LQ) is well-posed.*

We introduce the condition of relaxed compensator, which is abbreviated as Condition (RC).

Condition (RC). Let $(H(\cdot),K(\cdot)) \in \Lambda[0,T]\times\Lambda[0,T]$. The following two groups of inequalities hold (the argument t is suppressed):

$$\text{(i).}\quad \begin{cases} \dot H + HA + A^\top H + C^\top HC + Q \\ \quad - \big[HB + C^\top HD + S\big]\big[D^\top HD + R\big]^{-1} \\ \quad \times \big[HB + C^\top HD + S\big]^\top \geq 0, \quad t\in[0,T], \\ H(T) \leq G, \\ D^\top HD + R \gg 0, \qquad t\in[0,T], \end{cases} \tag{9}$$

and

$$\text{(ii).}\quad \begin{cases} \dot K + K\widehat A + \widehat A^\top K + \widehat C^\top H\widehat C + \widehat Q \\ \quad - \big[K\widehat B + \widehat C^\top H\widehat D + \widehat S\big]\big[\widehat D^\top H\widehat D + \widehat R\big]^{-1} \\ \quad \times \big[K\widehat B + \widehat C^\top H\widehat D + \widehat S\big]^\top \geq 0, \quad t\in[0,T], \\ K(T) \leq \widehat G, \\ \widehat D^\top H\widehat D + \widehat R \gg 0, \qquad t\in[0,T]. \end{cases} \tag{10}$$

The following proposition means that Condition (RC) is the necessary and sufficient condition for a relaxed compensator, which can be proved by Schur's Lemma. We omit details.

Proposition 3.1 *A pair of functions* $(H(\cdot),K(\cdot)) \in \Lambda[0,T]\times\Lambda[0,T]$ *is a relaxed compensator for Problem (MF-LQ) if and only if Condition (RC) holds.*

Remark 3.1 *Significantly different from the classic LQ problem, because of the presence of the mean field term* $\mathbb{E}[X]$ *in system,* $K(\cdot)$ *plays a key role as one of the compensator. For saving space, see details in Remark 4.12 in [22].*

By definition of relaxed compensator, we present the open-loop optimal control of Problem (MF-LQ) under indefinite condition. Denote

$$\begin{aligned} M^2_{\mathbb F}(0,T) = {}& L^2_{\mathbb F}(\Omega;C([0,T];\mathbb R^n)) \times \mathscr U[0,T] \\ & \times L^2_{\mathbb F}(\Omega;C([0,T];\mathbb R^n)) \times L^2_{\mathbb F}(0,T;\mathbb R^n). \end{aligned}$$

Theorem 3.1 *If there exists a relaxed compensator* $(H(\cdot),K(\cdot)) \in \Lambda[0,T]\times\Lambda[0,T]$*, then for any initial state* x*, the following stochastic Hamiltonian system*

$$\begin{cases} 0 = \Psi\big(\Theta^*,\mathbb E[\Theta^*]\big), \quad t\in[0,T], \\ dX^* = \Big\{AX^* + \widetilde A\mathbb E[X^*] + Bu^* + \widetilde B\mathbb E[u^*]\Big\}dt \\ \qquad + \Big\{CX^* + \widetilde C\mathbb E[X^*] + Du^* + \widetilde D\mathbb E[u^*]\Big\}dW(t), \\ dY = -g\big(\Theta^*,\mathbb E[\Theta^*]\big)dt + ZdW(t), \quad t\in[0,T], \\ X^*(0) = x, \quad Y(T) = GX^*(T) + \widetilde G\mathbb E[X^*(T)] \end{cases} \tag{11}$$

with

$$\begin{cases} g(t,\theta,\tilde\theta) = Q(t)x + \widetilde Q(t)\tilde x + S(t)u + \widetilde S(t)\tilde u \\ \qquad + A(t)^\top y + \widetilde A(t)^\top\tilde y + C^\top(t)z + \widetilde C(t)^\top\tilde z, \\ \Psi(t,\theta,\tilde\theta) = S(t)^\top x + \widetilde S(t)^\top\tilde x + R(t)u + \widetilde R(t)\tilde u \\ \qquad + B(t)^\top y + \widetilde B(t)^\top\tilde y + D(t)^\top z + \widetilde D(t)^\top\tilde z \end{cases}$$

admits a unique solution $\Theta^*(\cdot) = (X^*(\cdot),u^*(\cdot),Y(\cdot),Z(\cdot)) \in M^2_{\mathbb F}(0,T)$*. Moreover,* $(X^*(\cdot),u^*(\cdot))$ *is the unique optimal pair of Problem (MF-LQ).*

Proof If $(H(\cdot),K(\cdot))$ is a relaxed compensator, by Definition 3.1, the quadruple $(\mathbf{Q}^{H,K}(\cdot),\mathbf{S}^{H,K}(\cdot),\mathbf{R}^{H,K}(\cdot),\mathbf{G}^{H,K})$ satisfies Condition (PD). By Theorem 3.2 in [22], the stochastic Hamiltonian system

$$\begin{cases} 0 = \Psi^{H,K}\big(\Theta^{H,K},\mathbb E[\Theta^{H,K}]\big), \quad t\in[0,T], \\ dX^{H,K} = \Big\{AX^{H,K} + \widetilde A\mathbb E[X^{H,K}] + Bu^{H,K} \\ \qquad + \widetilde B\mathbb E[u^{H,K}]\Big\}dt + \Big\{CX^{H,K} + \widetilde C\mathbb E[X^{H,K}] \\ \qquad + Du^{H,K} + \widetilde D\mathbb E[u^{H,K}]\Big\}dW, \quad t\in[0,T], \\ dY^{H,K} = -g^{H,K}\big(\Theta^{H,K},\mathbb E[\Theta^{H,K}]\big)dt + Z^{H,K}dW(t), \\ X^{H,K}(0) = x, \\ Y^{H,K}(T) = G^{H,K}X^{H,K}(T) \\ \qquad + \big(\widehat G^{H,K} - G^{H,K}\big)\mathbb E[X^{H,K}(T)] \end{cases} \tag{12}$$

with

$$\begin{cases} g^{H,K}(r,\theta,\bar\theta) = Q^{H,K}(t)x + \big(\widehat Q^{H,K}(t) - Q^{H,K}(t)\big)\bar x \\ \qquad + S^{H,K}(t)u + \big(\widehat S^{H,K}(t) - S^{H,K}(t)\big)\bar u \\ \qquad + A(t)^\top y + \widetilde A(t)^\top\bar y + C^\top(t)z + \widetilde C(t)^\top\bar z, \\ \Psi^{H,K}(r,\theta,\bar\theta) = \big(S^{H,K}(t)\big)^\top x + \big(\widehat S^{H,K}(t) - S^{H,K}(t)\big)^\top\bar x \\ \qquad + R^{H,K}(t)u + \big(\widehat R^{H,K}(t) - R^{H,K}(t)\big)\bar u \\ \qquad + B(t)^\top y + \widetilde B(t)^\top\bar y + D(t)^\top z + \widetilde D(t)^\top\bar z \end{cases}$$

related to Problem (MF-LQ)^{H,K} admits a unique solution $\Theta^{H,K}(\cdot) = (X^{H,K}(\cdot),u^{H,K}(\cdot),Y^{H,K}(\cdot),Z^{H,K}(\cdot))$. Moreover $(X^{H,K}(\cdot),u^{H,K}(\cdot))$ is the unique optimal pair of Problem (MF-LQ)^{H,K}.

For any given $x \in \mathbb{R}^n$, we prove the equivalent unique solvability between the Hamiltonian systems (11) and (12). If $\Theta^{H,K}(\cdot) = (X^{H,K}(\cdot), u^{H,K}(\cdot), Y^{H,K}(\cdot), Z^{H,K}(\cdot))$ is a solution to (12), then a straightforward calculation leads to

$$\begin{cases} X^* = X^{H,K}, \qquad u^* = u^{H,K}, \\ Y = Y^{H,K} + H\big(X^* - \mathbb{E}[X^*]\big) + K\mathbb{E}[X^*], \\ Z = Z^{H,K} + H\big(CX^* + \widetilde{C}\mathbb{E}[X^*] + Du^* + \widetilde{D}\mathbb{E}[u^*]\big) \end{cases} \tag{13}$$

is a solution to (11). On the other hand, if $\Theta^*(\cdot) = (X^*(\cdot), u^*(\cdot), Y(\cdot), Z(\cdot))$ is a solution to (11), then due to the invertibility, the transformation (13) also yields a solution to (12). Therefore, the existence and uniqueness between (11) and (12) are equivalent.

By the above analysis, the stochastic Hamiltonian system (11) related to Problem (MF-LQ) admits a unique solution $\Theta^*(\cdot)$. Moreover, $(X^*(\cdot), u^*(\cdot)) = (X^{H,K}(\cdot), u^{H,K}(\cdot))$. By the equivalence between the cost functionals $J^{H,K}(x; u(\cdot))$ and $J(x; u(\cdot))$ (see Lemma 3.1), the unique optimal pair $(X^*(\cdot), u^*(\cdot)) = (X^{H,K}(\cdot), u^{H,K}(\cdot))$ of Problem (MF-LQ)^{H,K} is also the unique optimal pair of Problem (MF-LQ). The proof is completed. $\square$

Remark 3.2 *Theorem 3.1 presents the open-loop optimal control of the MF-LQ problem under indefinite condition. Moreover, it also gives a new condition about the solvability of MF-FBSDEs, in which* (13) *plays an important role. See an example about MF-FBSDEs not satisfying monotonicity condition in [7] in Section 4.2 for details.*

Next, we present the closed-loop optimal control in the indefinite case. For simplicity of notation, let us define $\Gamma : [0,T] \times \mathbb{S}^n \to \mathbb{R}^{m\times n}$ and $\widehat{\Gamma} : [0,T] \times \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}^{m\times n}$ by

$$\begin{cases} \Gamma\big(t, P\big) = -\big[D(t)^\top PD(t) + R(t)\big]^{-1}\big[PB(t) \\ \qquad\qquad + C(t)^\top PD(t) + S(t)\big]^\top, \\ \widehat{\Gamma}\big(t, P, \widehat{P}\big) = -\big[\widehat{D}(t)^\top P\widehat{D}(t) + \widehat{R}(t)\big]^{-1}\big[\widehat{P}\widehat{B}(t) \\ \qquad\qquad + \widehat{C}(t)^\top P\widehat{D}(t) + \widehat{S}(t)\big]^\top. \end{cases}$$

Theorem 3.2 *If there exists a relaxed compensator $(H(\cdot), K(\cdot)) \in \Lambda[0,T] \times \Lambda[0,T]$, then the following (decoupled) system of Riccati equations (with t suppressed)*

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad - \Gamma(P)^\top\big[D^\top PD + R\big]\Gamma(P) = 0, \quad t \in [0,T], \\ P(T) = G, \\ D^\top PD + R \gg 0, \qquad t \in [0,T], \end{cases} \tag{14}$$

and

$$\begin{cases} \dot{\widehat{P}} + \widehat{P}\widehat{A} + \widehat{A}^\top\widehat{P} + \widehat{C}^\top P\widehat{C} + \widehat{Q} \\ \quad - \widehat{\Gamma}(P,\widehat{P})^\top\big[\widehat{D}^\top P\widehat{D} + \widehat{R}\big]\widehat{\Gamma}(P,\widehat{P}) = 0, \quad t \in [0,T], \\ \widehat{P}(T) = \widehat{G}, \\ \widehat{D}^\top P\widehat{D} + \widehat{R} \gg 0, \qquad t \in [0,T] \end{cases} \tag{15}$$

admits a unique pair of solutions $(P(\cdot), \widehat{P}(\cdot))$ taking values in $\mathbb{S}^n \times \mathbb{S}^n$. Moreover, for a given $x \in \mathbb{R}^n$, the unique optimal control $u^*(\cdot)$ of Problem (MF-LQ) has the following feedback form:

$$u^* = \Gamma\big(P\big)\big(X^* - \mathbb{E}[X^*]\big) + \widehat{\Gamma}\big(P,\widehat{P}\big)\mathbb{E}[X^*], \quad t \in [0,T], \tag{16}$$

where $X^*(\cdot)$ is determined by

$$\begin{cases} dX^* = \Big\{\big(A + B\Gamma(P)\big)\big(X^* - \mathbb{E}[X^*]\big) \\ \quad + \big(\widehat{A} + \widehat{B}\widehat{\Gamma}(P,\widehat{P})\big)\mathbb{E}[X^*]\Big\}dt \\ \quad + \Big\{\big(C + D\Gamma(P)\big)\big(X^* - \mathbb{E}[X^*]\big) \\ \quad + \big(\widehat{C} + \widehat{D}\widehat{\Gamma}(P,\widehat{P})\big)\mathbb{E}[X^*]\Big\}dW(t), \\ X^*(0) = x. \end{cases} \tag{17}$$

Moreover,

$$\inf_{u(\cdot)\in\mathscr{U}[0,T]} J(x; u(\cdot)) = J\big(x; u^*(\cdot)\big) = \big\langle \widehat{P}(0)x, x\big\rangle.$$

Proof If there exists a relaxed compensator $(H(\cdot), K(\cdot))$, then the quadruple $(\mathbf{Q}^{H,K}(\cdot), \mathbf{S}^{H,K}(\cdot), \mathbf{R}^{H,K}(\cdot), \mathbf{G}^{H,K})$ satisfies Condition (PD). By Theorem 3.3 in [22], the following system of Riccati equations

$$\begin{cases} \dot{P}^{H,K} + P^{H,K}A + A^\top P^{H,K} + C^\top P^{H,K}C + Q^{H,K} \\ -\Gamma^{H,K}(P^{H,K})^\top\big[D^\top P^{H,K}D + R^{H,K}\big]\Gamma^{H,K}(P^{H,K}) = 0, \\ P^{H,K}(T) = G^{H,K}, \\ D^\top P^{H,K}D + R^{H,K} \gg 0, \qquad t \in [0,T], \end{cases} \tag{18}$$

and

$$\begin{cases} \dot{\widehat{P}}^{H,K} + \widehat{P}^{H,K}\widehat{A} + \widehat{A}^\top\widehat{P}^{H,K} + \widehat{C}^\top P^{H,K}\widehat{C} + \widehat{Q}^{H,K} \\ -\widehat{\Gamma}^{H,K}(P^{H,K},\widehat{P}^{H,K})^\top\big[\widehat{D}^\top P^{H,K}\widehat{D} \\ + \widehat{R}^{H,K}\big]\widehat{\Gamma}^{H,K}(P^{H,K},\widehat{P}^{H,K}) = 0, \quad t \in [0,T], \\ \widehat{P}^{H,K}(T) = \widehat{G}^{H,K}, \\ \widehat{D}^\top P^{H,K}\widehat{D} + \widehat{R}^{H,K} \gg 0, \qquad t \in [0,T] \end{cases} \tag{19}$$

with

$$\begin{cases} \Gamma^{H,K}\big(t,P\big) = -\big[D(t)^\top PD(t) + R^{H,K}(t)\big]^{-1}\big[PB(t) \\ \qquad\qquad + C(t)^\top PD(t) + S^{H,K}(t)\big]^\top, \\ \widehat{\Gamma}^{H,K}\big(t,P,\widehat{P}\big) = -\big[\widehat{D}(t)^\top P\widehat{D}(t) + \widehat{R}^{H,K}(t)\big]^{-1}\big[\widehat{P}\widehat{B}(t) \\ \qquad\qquad + \widehat{C}(t)^\top P\widehat{D}(t) + \widehat{S}^{H,K}(t)\big]^\top \end{cases}$$

admits a unique solution. If $(P^{H,K}(\cdot), \widehat{P}^{H,K}(\cdot))$ taking values in $\mathbb{S}^n_+ \times \mathbb{S}^n_+$ is a solution to (18)-(19), then by a straightforward calculation,

$$P(\cdot) = P^{H,K}(\cdot) + H(\cdot),\ \widehat{P}(\cdot) = \widehat{P}^{H,K}(\cdot) + K(\cdot) \quad (20)$$

are solutions to (14)-(15). On the other hand, if $(P(\cdot), \widehat{P}(\cdot)) \in \mathbb{S}^n \times \mathbb{S}^n$ is a solution to (14)-(15), then the inverse transformation of (20) provides a solution to (18)-(19). Therefore, the existence and uniqueness of solutions between (14)-(15) and (18)-(19) is equivalent.

Let

$$\begin{aligned} u^{H,K} &= \Gamma^{H,K}\big(P^{H,K}\big)\big(X^{H,K} - \mathbb{E}[X^{H,K}]\big) \\ &\quad + \widehat{\Gamma}^{H,K}\big(P^{H,K}, \widehat{P}^{H,K}\big)\mathbb{E}[X^{H,K}], \quad t \in [0,T], \end{aligned} \quad (21)$$

where $X^{H,K}(\cdot)$ satisfies

$$\begin{cases} dX^{H,K} = \Big\{\big(A + B\Gamma^{H,K}(P^{H,K})\big)\big(X^{H,K} - \mathbb{E}[X^{H,K}]\big) \\ \qquad + \big(\widehat{A} + \widehat{B}\widehat{\Gamma}^{H,K}(P^{H,K}, \widehat{P}^{H,K})\big)\Big\}dt \\ \qquad + \Big\{\big(C + D\Gamma^{H,K}(P^{H,K})\big)\big(X^{H,K} - \mathbb{E}[X^{H,K}]\big) \\ \qquad + \big(\widehat{C} + \widehat{D}\widehat{\Gamma}^{H,K}(P^{H,K}, \widehat{P}^{H,K})\big)\Big\}dW(t),\ t \in [0,T], \\ X^{H,K}(0) = x. \end{cases} \quad (22)$$

Theorem 3.3 in [22] implies that the admissible pair $(X^{H,K}(\cdot), u^{H,K}(\cdot))$ is optimal for Problem (MF-LQ)^{H,K}. It is easy to verify that

$$\Gamma\big(P\big) = \Gamma^{H,K}\big(P^{H,K}\big),\ \ \widehat{\Gamma}\big(P, \widehat{P}\big) = \widehat{\Gamma}^{H,K}\big(P^{H,K}, \widehat{P}^{H,K}\big).$$

Therefore, the admissible pair $(X^*(\cdot), u^*(\cdot))$ defined by (16)-(17) is the same as $(X^{H,K}(\cdot), u^{H,K}(\cdot))$ defined by (21)-(22). By Lemma 3.1, the unique optimal pair $(X^*(\cdot), u^*(\cdot)) = (X^{H,K}(\cdot), u^{H,K}(\cdot))$ of Problem (MF-LQ)^{H,K} is also the unique optimal pair of Problem (MF-LQ). The proof is completed. □

If there exist nonhomogeneous terms in system (1) and linear terms in cost functional (2), these terms do not affect the well-posedness of Problem (MF-LQ). We can parallely derive the corresponding results similar to the theoretical ones established in this paper. For example, consider $J(x; u(\cdot))$ in the form of (2) plus a linear term $\langle g,\ \mathbb{E}[X(T)]\rangle$ with an n-dimensional constant vector g as

$$\widetilde{J}(x; u(\cdot)) = J(x; u(\cdot)) + 2\langle g,\ \mathbb{E}[X(T)]\rangle. \quad (23)$$

The similar results can be parallely obtained. We present the following corollary as one example in details. For convenience, we call MF-LQ problem with respect to system (1) and cost functional (23) as **Problem (MF-LQ)^L** and denote $\mathcal{D}(\varphi) = -\big[\widehat{D}^\top P \widehat{D} + \widehat{R}\big]^{-1}\widehat{B}^\top \varphi$.

Corollary 3.2 *If there exists a relaxed compensator $(H(\cdot), K(\cdot)) \in \Lambda[0,T] \times \Lambda[0,T]$, Problem (MF-LQ)^L admits a unique optimal feedback control*

$$u^* = \Gamma\big(P\big)\big(X^* - \mathbb{E}[X^*]\big) + \widehat{\Gamma}\big(P, \widehat{P}\big)\mathbb{E}[X^*] + \mathcal{D}(\varphi)$$

with φ satisfying

$$\begin{cases} \dot{\varphi} + \big[\widehat{A} + \widehat{B}\widehat{\Gamma}(P, \widehat{P})\big]^\top \varphi = 0, \quad t \in [0,T], \\ \varphi(T) = g, \end{cases}$$

and $(P, \widehat{P})$ being the unique solution of Riccati equations (14)-(15), where the optimal state $X^*(\cdot)$ satisfies

$$\begin{cases} dX^* = \Big\{\big(A + B\Gamma(P)\big)\big(X^* - \mathbb{E}[X^*]\big) \\ \quad + \big(\widehat{A} + \widehat{B}\widehat{\Gamma}(P, \widehat{P})\big)\mathbb{E}[X^*] \\ \quad - \widehat{B}\big[\widehat{D}^\top P \widehat{D} + \widehat{R}\big]^{-1}\widehat{B}^\top \varphi\Big\}dt \\ \quad + \Big\{\big(C + D\Gamma(P)\big)\big(X^* - \mathbb{E}[X^*]\big) \\ \quad + \big(\widehat{C} + \widehat{D}\widehat{\Gamma}(P, \widehat{P})\big)\mathbb{E}[X^*] \\ \quad - \widehat{D}\big[\widehat{D}^\top P \widehat{D} + \widehat{R}\big]^{-1}\widehat{B}^\top \varphi\Big\}dW(t), \quad t \in [0,T], \\ X^*(0) = x. \end{cases}$$

Based on the result of Theorem 5.2 in [27], this corollary can be proved, similar to the proof of Theorem 3.2. We omit the proof here.

Remark 3.3 *When there exists a relaxed compensator $(H(\cdot), K(\cdot)) \in \Lambda[0,T] \times \Lambda[0,T]$, from (20), we derive the following inequalities $H(\cdot) \le P(\cdot), K(\cdot) \le \widehat{P}(\cdot)$, where $(P(\cdot), \widehat{P}(\cdot))$ is the solution to the system of Riccati equations.*

Remark 3.4 *By comparing the system of Riccati equations (14)-(15) with the system of inequalities (9)-(10) in Condition (RC), we find the following two facts:*

(i). *If the system of Riccati equations (14)-(15) is solvable, then the solution $(P(\cdot), \widehat{P}(\cdot))$ is a relaxed compensator for Problem (MF-LQ). Consequently, in the indefinite case, the solvability of the system of Riccati equations (14)-(15) implies the solvability of Problem (MF-LQ).*

(ii). *The first two equations in (14) and two equations in (15) are relaxed into the corresponding inequalities in (9)-(10). The solvability of the system of inequalities (9)-(10) also implies the solvability of Problem (MF-LQ). This can be regarded as an explanation of the notion of relaxed compensators from the viewpoint of Riccati equations.*

Then, we present the relationship between relaxed compensator and solutions of Riccati equations by a corollary.

Corollary 3.3 *A relaxed compensator* $(H(\cdot), K(\cdot)) \in \Lambda[0,T] \times \Lambda[0,T]$ *exists if and only if the system of Riccati equations* (14) *and* (15) *admits a unique pair of solutions* $(P(\cdot), \widehat{P}(\cdot))$ *taking values in* $\mathbb{S}^n \times \mathbb{S}^n$*.*

At the end of this section, we present the relationship between Hamiltonian system (11) and Riccati equations (14)-(15).

Proposition 3.2 *Let*

$$\begin{cases} Y = P\big(X^* - \mathbb{E}[X^*]\big) + \widehat{P}\mathbb{E}[X^*], \\ Z = P\big(CX^* + \widetilde{C}\mathbb{E}[X^*] + Du^* + \widetilde{D}\mathbb{E}[u^*]\big), \end{cases} \tag{24}$$

where $(X^*(\cdot), u^*(\cdot))$ *defined by* (17)-(16) *and* $(P(\cdot), \widehat{P}(\cdot))$ *is the solution of Riccati equations* (14)-(15). *Then* $\Theta^*(\cdot) = (X^*(\cdot), u^*(\cdot), Y(\cdot), Z(\cdot))$ *defined by* (16), (17) *and* (24) *is a solution to the Hamiltonian system* (11).

For saving space, the proofs of Corollary 3.3 and Proposition 3.2 are given in [22].

4 Applications

4.1 Mean-variance Portfolio Selection Problem

In this subsection, a dynamic mean-variance portfolio problem is considered within the framework of indefinite MF-LQ. In the market, we suppose that there are $m+1$ assets traded continuously under self-financing assumption. One asset is risk-free (for example, a default-free bond without coupons), whose price process $S_0(t)$ is governed by the following ordinary differential equation (ODE):

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \quad t \in [0,T], \\ S_0(0) = s_0, \end{cases}$$

where $s_0 > 0$ is the initial price and $r(\cdot)$ is nonnegative bounded function and presents the interest rate of bond. Additionally, the other m assets are securities (for example, stocks), whose price processes $S_i(\cdot)$ $(i = 1, 2, \cdots, m)$ satisfy the following SDE:

$$\begin{cases} dS_i(t) = S_i(t)\Big\{\mu_i(t)dt + \displaystyle\sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)\Big\}, \ t \in [0,T], \\ S_i(0) = s_i, \end{cases}$$

where $s_i > 0$ is the initial price, $\mu(\cdot) := (\mu_1(\cdot), \mu_2(\cdot), \cdots, \mu_m(\cdot))^\top$ with $\mu_i(\cdot) > 0$ is the appreciation rate, and $\sigma_i(\cdot) := (\sigma_{i1}(\cdot), \sigma_{i2}(\cdot), \cdots, \sigma_{im}(\cdot))$ $(i = 1, 2, \cdots, m)$ is the volatility of stocks. Define the covariance matrix $\sigma(\cdot) := (\sigma_{ij}(\cdot))_{m\times m}$. Assume that $\mu(\cdot)$ and $\sigma(\cdot)$ are bounded functions. Furthermore, we assume that there exists a constant $\delta > 0$ such that

$$\sigma(t)\sigma(t)^\top \geq \delta I, \quad \text{for all } t \in [0,T],$$

where I denotes the identity $m \times m$ matrix.

In financial investment, the investor's total wealth is denoted by $X(\cdot)$, and the amount of the wealth invested in the i-th stock is denoted by $\pi_i(\cdot)$ $(i = 1, 2, \cdots, m)$. Since the strategy $\pi(\cdot) := (\pi_1(\cdot), \pi_2(\cdot), \cdots, \pi_m(\cdot))^\top$ is used in a self-financing way, the wealth invested in the bond is $X(\cdot) - \sum_{i=1}^m \pi_i(\cdot)$. Then, the wealth process $X(\cdot)$ with the initial endowment x satisfies the following SDE

$$\begin{cases} dX(t) = \big[r(t)X(t) + b(t)^\top u(t)\big]dt + u(t)^\top dW(t), \\ X(0) = x, \end{cases}$$

where $x > 0$ is the initial wealth, $u(t) = \sigma(t)^\top \pi(t)$ and $b(t) = \sigma(t)^{-1}(\mu(t) - r(t)\mathbf{1})$ for all $t \in [0, T]$. Here, $\mathbf{1}$ denotes the vector of all entries with 1 and $W(\cdot) = (W^1(\cdot), W^2(\cdot), \cdots, W^m(\cdot))^\top$ is m-dimensional standard Brownian motion. All the theoretical results established in this paper hold true for m-dimensional standard Brownian motion case.

The mean-variance problem means that the investor's objective is to maximize the expected terminal wealth $\mathbb{E}[X(T)]$ as well as to minimize the variance of the terminal wealth $\mathrm{Var}(X(T))$. Let ν be a positive constant. Then, the cost functional is

$$J(x; u(\cdot)) = \frac{\nu}{2}\mathrm{Var}(X(T)) - \mathbb{E}[X(T)]. \tag{25}$$

Problem (MV). The mean-variance portfolio selection problem is to find an admissible control $u^*(\cdot) \in \mathscr{U}[0,T]$ satisfying

$$J(x; u^*(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J(x; u(\cdot)).$$

Such an admissible control $u^*(\cdot)$ is called an optimal control, and $X^*(\cdot) = X^*(\cdot; x, u^*(\cdot))$ is called the corresponding optimal trajectory.

We deal with Problem (MV) as a special case of Problem (MF-LQ)^L with indefinite matrices. In this example, (25) can be rewritten as

$$J(x; u(\cdot)) = \frac{\nu}{2}\mathbb{E}[X^2(T)] - \frac{\nu}{2}\big(\mathbb{E}[X(T)]\big)^2 - \mathbb{E}[X(T)],$$

then, $\mathbf{Q}(\cdot) = \mathbf{S}(\cdot) = \mathbf{R}(\cdot) = 0$, $G = \frac{\nu}{2}$, $\widetilde{G} = -\frac{\nu}{2}$ and $g = \frac{1}{2}$. From Corollary 3.2, we present the closed-loop form of optimal control by the following proposition.

Proposition 4.1 *Problem (MV) admits a unique optimal control in the following closed-loop form:*

$$\begin{aligned} u^*(t) = -b(t)\Big\{ & X^*(t) - \mathbb{E}[X^*(t)] \\ & - \frac{1}{\nu}\exp\Big[\int_t^T \Big(|b(s)|^2 - r(s)\Big)ds\Big]\Big\}, \quad t \in [0,T], \end{aligned}$$

where $X^*(\cdot)$ *satisfies*

$$\begin{cases} dX^* = \Big\{\big(r(t) - |b(t)|^2\big)X^* + |b(t)|^2\mathbb{E}[X^*] \\ \quad + \dfrac{|b(t)|^2}{\nu}\exp\Big[\displaystyle\int_t^T \Big(|b(s)|^2 - r(s)\Big)ds\Big]\Big\}dt \\ \quad - b(t)^\top\Big\{X^*(t) - \mathbb{E}[X^*(t)] \\ \quad - \dfrac{1}{\nu}\exp\Big[\displaystyle\int_t^T \Big(|b(s)|^2 - r(s)\Big)ds\Big]\Big\}dW(t), \\ \qquad\qquad t \in [0,T], \\ X(0) = x. \end{cases}$$

Proof The corresponding Riccati equations of Problem (MV) are

$$\begin{cases} \dot P(t) + 2r(t)P(t) - |b(t)|^2 P(t) = 0, \quad t \in [0,T], \\ P(T) = \dfrac{\nu}{2}, \end{cases}$$

$$\begin{cases} \dot{\widehat P}(t) + 2r(t)\widehat P(t) - \dfrac{|b(t)|^2\widehat P(t)^2}{P(t)} = 0, \quad t \in [0,T], \\ \widehat P(T) = 0, \end{cases}$$

which admit the solutions

$$P(t) = \frac{\nu}{2}\exp\Big(\int_t^T [2r(s) - |b(s)|^2]ds\Big), \quad t \in [0,T] \tag{26}$$

and

$$\widehat P(t) = 0, \quad t \in [0,T],$$

respectively. We choose $(P(\cdot), \widehat P(\cdot))$ as a relaxed compensator. By a direct calculation, we have $\Gamma(t,P) = -\frac{1}{P}\cdot b(t)P = -b(t)$, $\widehat\Gamma(t,P,\widehat P) = -\frac{1}{P}\cdot b(t)\widehat P = 0$ and $\mathcal{D}(t,\varphi) = -\frac{1}{P}\cdot b(t)\varphi$, by Corollary 3.2, Problem (MV) admits a unique optimal control:

$$u^*(t) = -b(t)\Big[X^*(t) - \mathbb{E}[X^*(t)] + \frac{\varphi(t)}{P(t)}\Big], \; t \in [0,T], \tag{27}$$

where $\varphi(\cdot)$ is the solution to

$$\begin{cases} \dot\varphi(t) + r(t)\varphi(t) = 0, \quad t \in [0,T], \\ \varphi(T) = -\dfrac{1}{2}. \end{cases}$$

Explicitly,

$$\varphi(t) = -\frac{1}{2}\exp\Big\{\int_t^T r(s)ds\Big\}, \quad t \in [0,T]. \tag{28}$$

Substituting (26) and (28) into (27) leads to the desired result. □

4.2 An Example about Problem (MF-LQ)

In this part, we consider an example about Problem (MF-LQ). In this example, we not only obtain the optimal control, but also obtain the unique solvability of a kind of MF-FBSDE not satisfying the monotonicity condition in [7]. Consider the following system

$$\begin{cases} dX(t) = \big\{a(t)X(t) + \tilde a(t)\mathbb{E}[X(t)] + b(t)u(t) \\ \qquad + \tilde b(t)\mathbb{E}[u(t)]\big\}dt + u(t)dW(t), \quad t \in [0,T], \\ X(0) = x, \end{cases}$$

and the cost functional

$$J(x;u(\cdot)) = \mathbb{E}\int_0^T \Big\{\alpha\big|X(t) - \mathbb{E}[X(t)]\big|^2 - \beta|u(t)|^2\Big\}dt + \gamma\mathbb{E}[X^2(T)],$$

where $x \in \mathbb{R}$, $a(t)$, $\tilde a(t)$, $b(t)$, $\tilde b(t)$ are 1-dimensional deterministic functions, and α, β, γ are constants. The coefficients satisfy $\alpha \ge 0$, $\gamma > \max\{\beta, 0\}$ and $a(t) \ge (b(t)^2\gamma)/(2(\gamma-\beta))$. Denote $\hat a(s) = a(s) + \tilde a(s)$ and $\hat b(s) = b(s) + \tilde b(s)$. The objective of this problem is to find an admissible control $u^*(\cdot) \in \mathscr{U}[0,T]$ such that

$$J\big(x;u^*(\cdot)\big) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J\big(x;u(\cdot)\big).$$

When $\beta < 0$, this MF-LQ problem is under positive definite case, no more tautology here. We mainly discuss the indefinite case. We verify that $H(t) = \gamma$ and

$$K(t) = \Big[\frac{1}{\gamma}\exp\Big\{-2\int_t^T \hat a(s)ds\Big\} + \int_t^T \frac{\hat b(s)^2}{\gamma - \beta}\exp\Big\{-2\int_t^s \hat a(\tau)d\tau\Big\}ds\Big]^{-1}$$

constitute a relaxed compensator. Therefore, this MF-LQ problem is well-posed.

By Theorem 3.1, this MF-LQ problem admits a unique solution satisfying the stochastic Hamiltonian system

$$\begin{cases} 0 = -\beta u^*(t) + b(t)Y(t) + \tilde b(t)\mathbb{E}[Y(t)] + Z(t), \\ dX^*(t) = \big\{a(t)X^*(t) + \tilde a(t)\mathbb{E}[X^*(t)] + b(t)u(t) \\ \quad + \tilde b(t)\mathbb{E}[u(t)]\big\}dt + u(t)dW(t), \quad t \in [0,T], \\ dY(t) = -\big\{a(t)Y(t) + \tilde a(t)\mathbb{E}[Y(t)] + \alpha X^*(t)\big\} \\ \quad + Z(t)dW(t), \quad t \in [0,T], \\ X(0) = x, \quad Y(T) = \gamma X(T). \end{cases} \tag{29}$$

From the relationship (24) in Proposition 3.2, we decouple equation (29) as follows

$$\begin{cases} Y(t) = P(t)\big(X^*(t) - \mathbb{E}[X^*(t)]\big) + \widehat P(t)\mathbb{E}[X^*(t)], \\ \mathbb{E}[Y(t)] = \widehat P(t)\mathbb{E}[X^*(t)], \\ Z(t) = P(t)u^*(t), \quad \mathbb{E}[Z(t)] = P(t)\mathbb{E}[u^*(t)], \end{cases} \tag{30}$$

where $(P, \widehat P)$ is the unique solution to the following pair of Riccati equations

$$
\begin{cases}
\dot{P}(t)+2a(t)P(t)+\alpha-\dfrac{|b(t)|^2P(t)^2}{P(t)-\beta}=0,\\
P(T)=\gamma,\quad P(t)-\beta>0,
\end{cases}
\tag{31}
$$

and

$$
\begin{cases}
\dot{\widehat{P}}(t)+2\hat{a}(t)\widehat{P}(t)+\alpha-\dfrac{|\hat{b}(t)|^2\widehat{P}(t)^2}{P(t)-\beta}=0,\quad t\in[0,T],\\
\widehat{P}(T)=\gamma.
\end{cases}
\tag{32}
$$

Putting (30) into the first equation in (29) yields

$$
\begin{aligned}
-\beta u^*(t)+b(t)[P(t)\big(X^*(t)-\mathbb{E}[X^*(t)]\big)+\widehat{P}(t)\mathbb{E}[X^*(t)]]\\
+\tilde{b}(t)\widehat{P}(t)\mathbb{E}[X^*(t)]+Pu^*(t)=0,
\end{aligned}
$$

then the optimal control can be presented by

$$
\begin{aligned}
u^*(t)=-\frac{1}{P(t)-\beta}\big[b(t)P(t)\big(X^*(t)-\mathbb{E}[X^*(t)]\big)\\
+\big(b(t)+\tilde{b}(t)\big)\widehat{P}(t)\mathbb{E}[X^*(t)]\big)\big],
\end{aligned}
\tag{33}
$$

where $X^*(\cdot)$ satisfies the following equation

$$
\begin{cases}
dX^*(t)=\Big\{\big[a(t)-\dfrac{|b(t)|^2P(t)}{P(t)-\beta}\big]X^*(t)+\big[\tilde{a}(t)\\
\quad-\dfrac{1}{P(t)-\beta}\big(|b(t)+\tilde{b}(t)|^2\widehat{P}(t)\\
\quad-|b(t)|^2P(t)\big)\big]\mathbb{E}[X(t)]\Big\}dt\\
\quad-\dfrac{1}{P(t)-\beta}\big\{b(t)P(t)\big(X^*(t)-\mathbb{E}[X^*(t)]\big)\\
\quad+\big(b(t)+\tilde{b}(t)\big)\widehat{P}(t)\mathbb{E}[X^*(t)]\big)\big\}dW(t),\ t\in[0,T],\\
X^*(0)=x.
\end{cases}
\tag{34}
$$

We can see that the optimal control u^* is determined by the system states $X^*(\cdot)$, $\mathbb{E}[X^*(\cdot)]$ and the solutions $P(\cdot)$, $\widehat{P}(\cdot)$ of Riccati equations.

Moreover, combining (30) with (33), the unique solution $(Y,\mathbb{E}[Y],Z,\mathbb{E}[Z])$ of MF-FBSDE (36) can be represented as

$$
\begin{cases}
Y(t)=P(t)\big(X^*(t)-\mathbb{E}[X^*(t)]\big)+\widehat{P}(t)\mathbb{E}[X^*(t)],\\
\mathbb{E}[Y(t)]=\widehat{P}(t)\mathbb{E}[X^*(t)],\\
Z(t)=-\dfrac{P(t)}{P(t)-\beta}\big[b(t)P(t)\big(X^*(t)-\mathbb{E}[X^*(t)]\big)\\
\qquad+\big(b(t)+\tilde{b}(t)\big)\widehat{P}(t)\mathbb{E}[X^*(t)]\big)\big],\\
\mathbb{E}[Z(t)]=-\dfrac{P(t)}{P(t)-\beta}\big(b(t)+\tilde{b}(t)\big)\widehat{P}(t)\mathbb{E}[X^*(t)],
\end{cases}
\tag{35}
$$

which also can be expressed by $P(\cdot)$, $\widehat{P}(\cdot)$, $X^*(\cdot)$ and $\mathbb{E}[X^*(\cdot)]$. In fact, (34) and (35) provide an effective way for solving MF-FBSDE (29).

In addition, we would like to discuss more about Hamiltonian system (29) with cases $\beta>0$ and $\beta=0$.

Case I: When $\beta>0$, Hamiltonian system (29) is rewritten as

$$
\begin{cases}
dX^*(t)=\Big\{a(t)X^*(t)+\tilde{a}(t)\mathbb{E}[X^*(t)]+\dfrac{b(t)}{\beta}\big[b(t)Y(t)\\
\quad+\tilde{b}(t)\mathbb{E}[Y(t)]+Z(t)\big]+\dfrac{\tilde{b}(t)}{\beta}\big[(b(t)+\tilde{b}(t))\mathbb{E}[Y(t)]\\
\quad+\mathbb{E}[Z(t)]\big]\Big\}dt+\dfrac{1}{\beta}\big\{b(t)Y(t)+\tilde{b}(t)\mathbb{E}[Y(t)]\\
\quad+Z(t)\big\}dW(t),\quad t\in[0,T],\\
dY(t)=-\big\{a(t)Y(t)+\tilde{a}(t)\mathbb{E}[Y(t)]+\alpha X^*(t)\big\}dt\\
\quad+Z(t)dW(t),\quad t\in[0,T],\\
X(0)=x,\quad Y(T)=\gamma X(T).
\end{cases}
\tag{36}
$$

It is obvious that MF-FBSDE (36) does not satisfy the monotonicity condition in [7]. Based on the above discussion, it follows from Theorem 3.1 that equation (36) admits a unique solution. Moreover, the optimal control

$$
u^*(t)=\frac{1}{\beta}\big[b(t)Y(t)+\tilde{b}(t)\mathbb{E}[Y(t)]+Z(t)\big]\tag{37}
$$

can be expressed by $(Y(\cdot),\mathbb{E}[Y(\cdot)],Z(\cdot))$ in terms of (35). In fact, (37) is equivalent to (33).

Case II: When $\beta=0$, the Hamiltonian system (29) can be reduced to the following MF-FBSDE

$$
\begin{cases}
dX(t)=\{a(t)X(t)+\tilde{a}(t)\mathbb{E}[X(t)]+b(t)u(t)\\
\qquad+\tilde{b}(t)\mathbb{E}[u(t)]\}dt+u(t)dW(t),\quad t\in[0,T],\\
dY(t)=-\big\{a(t)Y(t)+\tilde{a}(t)\mathbb{E}[Y(t)]+\alpha X(t)\big\}dt\\
\qquad-\big\{b(t)Y(t)+\tilde{b}(t)\mathbb{E}[Y(t)]\big\}dW(t),\quad t\in[0,T],\\
X(0)=x,\quad Y(T)=\gamma X(T).
\end{cases}
\tag{38}
$$

In (38), there are three unknown processes $X(\cdot),Y(\cdot)$, $u(\cdot)$, and the diffusion of the backward equation depending on $Y(\cdot)$ and $\mathbb{E}[Y(\cdot)]$ while not $Z(\cdot)$. This implies that (38) is not a classic FBSDE. To the best of our knowledge, this kind of equations are largely underexplored. In this paper, because of the presence of relaxed compensator, from Theorem 3.1, MF-FBSDE (38) admits a unique solution. Moreover, the state solution $X^*(\cdot)$ is presented in (34), $Y(\cdot)$ is solved by (35), and the optimal feedback $u^*(\cdot)$ is in the form of (33) with $\beta=0$.

Let $\alpha\geq 0$, $\gamma>\max\{\beta,0\}$ and $a(t)\geq(b(t)^2\gamma)/(2(\gamma-\beta))$ as before. From another viewpoint, when $\gamma>\beta$, one can check the cost functional is uniformly convex in the control variable, which leads to the unique existence of optimal control. However, the uniform convexity is broken when $\gamma\leq\beta$. One can refer to Sun [27] for more details on this viewpoint.

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