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Fundamental Properties and Pseudo-Polynomial-Time Algorithm for Network Containership Sailing Speed Optimization

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Abstract

In container liner shipping, bunker cost is an important component of the total operating cost, and bunker consumption increases dramatically when the sailing speed of containerships increases. A higher speed implies higher bunker consumption (higher bunker cost), shorter transit time (lower inventory cost), and larger shipping capacity per ship per year (lower ship cost). Therefore, a container shipping company aims to determine the optimal sailing speed of containerships in a shipping network to minimize the total cost. We derive analytical solutions for sailing speed optimization on a single ship route with a continuous number of ships. The advantage of analytical solutions lies in that it unveils the underlying structure and properties of the problem, from which a number of valuable managerial insights can be obtained. Based on the analytical solution and the properties of the problem, the optimal integer number of ships to deploy on a ship route can be obtained by solving two equations, each in one unknown, using a simple bi-section search method. The properties further enable us to identify an optimality condition for network containership sailing speed optimization. Based on this optimality condition, we propose a pseudo-polynomial-time solution algorithm that can efficiently obtain an epsilon-optimal solution for sailing speed of containerships in a liner shipping network.

Key Words: Transportation; Liner shipping; Containership; Sailing speed; Bunker fuel

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1 Introduction

Liner shipping companies deploy containerships on regularly scheduled services to transport containers. Unlike tramp shipping, containerships in liner shipping have to sail according to the planned schedule no matter whether they are fully loaded or not (Christiansen et al., 2004, 2013). Once designed, the liner services are operated for a period of three to six months. Therefore, it is important for liner shipping companies to design efficient services as a large proportion of the total operating cost is fixed once the services are designed (Brouer et al., 2014; Mulder and Dekker, 2014; Ng, 2014; Plum et al., 2014; Zheng et al., 2014).

Bunker cost is a significant component in the total operating cost of a liner shipping company. Ronen (2011) estimated that when bunker fuel price is around 500 $/ton, the bunker cost constitutes about three quarters of the operating cost of a large containership. In 2011, the bunker price in Singapore reached 647 $/ton (UNCTAD, 2012). On 10 March 2015, the bunker price at Rotterdam dropped to 296 $/ton (Bunkerworld, 2012), which considerably cut down the costs for liner shipping companies.

The bunker consumption is largely affected by the sailing speed of containerships. When the speed increases, the bunker consumption increases more than linearly. Studies usually assume that daily bunker consumption is approximately proportional to the sailing speed cubed (or bunker consumption per unit of distance is proportional to the sailing speed squared). Wang and Meng (2012) calibrated the exponent to be between 2.7 and 3.3 using historical operating data of containerships, which supports the power of three approximation. Suggested by a ship engine manufacturing company, Du et al. (2011) adopted the exponent of 3.5 for feeder containerships, 4 for medium-sized containerships, and 4.5 for jumbo containerships. Kontovas and Psaraftis (2011) suggested using an exponent of 4 or greater when the speed of containerships is greater than 20 knots. As a result of the high bunker price and the sensitivity of bunker consumption on sailing speed, slow-steaming is a common technique to curb bunker consumption. After 2007, many liner shipping companies adopted the slow-steaming strategy to reduce bunker expenditure (UNCTAD, 2011). However, shippers are unhappy about slow steaming because it increases the transit time of cargoes from origin to destination.

In fact, on one hand, slow-steaming reduces bunker consumption and thereby bunker cost; on the other hand, it also decreases the effective shipping capacity and increases the transit time. Liner shipping companies usually provide a weekly service frequency, which means that each port of call is visited on the same day every week (Bell et al., 2011; Brouer et al., 2011, 2013). For example, Fig. 1 shows the North & Central China East Coast Express (NCE) operated by Orient Overseas Container Line (OOCL, 2013) which has a weekly frequency, meaning that the round-trip journey time (weeks) is equal to the number of ships deployed. For example, the round-trip journey time of NCE is 63 days, and hence 9 ships should be deployed to maintain a weekly service. If, for example, slow-steaming increases the round-trip journey time to 70 days, then 10 ships must be deployed, which leads to higher ship operating costs (manning, maintenance, insurance, consumables, etc.). Moreover, slow-steaming results in a longer transit time of containers. Consequently, the inventory cost for customers will be higher. For instance, Notteboom (2006) estimated that one day delay of a 4000-TEU (20-foot equivalent unit) ship implies a total cost of 57,000 Euros associated with the cargos in the containers. Therefore, liner shipping companies must design the speed to balance the trade-off between ship cost, bunker cost, and inventory cost.

It should be noted that a more straightforward approach for shipping lines is to set a maximum transit time for each port pair. The maximum transit time approach and the inventory costs approach have similarities and differences. On one hand, we could consider the maximum transit time as a “soft” constraint in some cases: a 30 days’ maximum transit time does not mean there are no cargoes when the real transit time is 30.1 days, but means there are fewer cargoes due to customers’ loss. Similarly, a 30 days’ maximum transit time does not mean there is no difference for the customers whether the real transit time is 29 days or 1 day. In the maximum transit time approach, if we penalize transit time longer than the maximum one and reward shorter transit time, the model will be the same as the inventory cost approach where the inventory cost rate is equal to the penalty rate and the reward rate. In fact, both Alvarez (2012) and Kim (2014) have used the inventory costs as a surrogate for level of service provided by shipping lines. On the other hand, if we treat the maximum transit time as a “hard” constraint to account for perishable products, then the shipping line will not provide a very short transit time even when the bunker price is low because nothing is gained by fast steaming. By contrast, when the bunker price is low, in the inventory costs approach the speed will be higher than that in the maximum transit time approach because the inventory cost implicitly assumes that the liner shipping company gains more revenue when the transit time is shorter by charging a higher freight rate or receiving more demand from customers.

Port Rotation
Eastbound: Pusan - Qingdao - Ningbo - Shanghai - New York - Norfolk - Savannah
Westbound: New York - Norfolk – Savannah - Pusan - Qingdao - Ningbo - Shanghai

Transit Time (Days) Asia to North America (Eastbound)			
From \ To	New York (Thu)	Norfolk (Sat)	Savannah (Mon)
Shanghai (Sat)	26	28	30
Ningbo (Thu)	28	30	32
Qingdao (Wed)	29	31	33
Pusan (Mon)	31	33	35

Transit Time (Days) North America to Asia (Westbound)				
From \ To	Pusan (Sun)	Qingdao (Tue)	Ningbo (Thu)	Shanghai (Fri)
Savannah (Tue)	26	28	30	31
Norfolk (Sat)	29	31	33	34
New York (Fri)	30	32	34	35

Fig. 1 NCE service provided by OOCL (2013)

There are a number of studies that are devoted to the optimization of sailing speed in different contexts of maritime transportation: shipping network design (Alvarez, 2009), ship fleet deployment (Meng and Wang, 2010; Gelareh and Meng, 2010), ship schedule construction (Bell and Bichou, 2008; Qi and Song, 2012), selection of bunkering port and volume (Yao et al., 2012; Kim, 2014), emission control (Kontovas and Psaraftis, 2011; Psaraftis and Kontovas, 2010, 2013; Kim et al., 2012, 2013), berth allocation (Du et al., 2011;

Zhen et al., 2011a, b), and minimizing bunker cost (Fagerholt et al., 2010; Norstad et al., 2011; Ronen, 2011; Wang and Meng, 2012; Hvattum et al., 2013; Kim, 2014; Kim et al., 2014). These studies have developed various mathematical models and optimization algorithms. Fagerholt et al. (2010) and Norstad et al. (2011) discretized the possible sailing speed and used dynamic programming to find the optimal speed to adopt in a tramp shipping environment. Du et al. (2011) investigated a joint berth allocation and speed optimization problem. They transformed the power relation between sailing speed and bunker consumption rate to second-order cone programming (SOCP) constraints and took advantage of state-of-the-art solvers to solve the SOCP problem. Wang and Meng (2012) generated piecewise linear functions and developed a mixed-integer linear programming approximation model to optimize the speed of containerships in a liner shipping network with container routing. Yao et al. (2012) studied not only the speed of containerships, but also the selection of bunkering ports and the amount of bunker to take at the refill port. They discretized the sailing speed into many small intervals and introduced binary decision variables to indicate the adopted sailing speed interval. Kim (2014) addressed a similar problem to Yao et al. (2012) by presenting an interesting Lagrangian heuristic.

Our study aims to bridge the gap in the literature in three aspects. First, not many prior studies have considered the level of service provided to customers. The level of service can be formulated as maximum transit time (Karsten et al., 2015), or time window (Fagerholt et al., 2010; Norstad et al., 2011; Hvattum et al., 2013), or cargo inventory costs (Alvarez, 2012; Kim, 2014). We will use the cargo inventory costs as the surrogate for level of service. Second, many of the existing studies focus on just one route, and we will examine a shipping network consisting of a number of routes. Third, unlike the literature that uses numerical or approximate solution approaches, we develop analytical solutions and a pseudo-polynomial-time epsilon-optimal solution algorithm.

The objective of this paper is to develop mathematical models and analyze fundamental properties of network containership sailing speed optimization that minimizes the sum of ship cost, bunker cost, and inventory cost. We derive analytical solutions for sailing speed optimization on a single ship route with a continuous number of ships, i.e., an explicit relation between the input and the ouput of the (continuous) number of ships. The advantage of analytical solutions lies in that it unveils the underlying structure and properties of the problem, from which a number of valuable managerial insights can be obtained. Based on the analytical solution and the properties of the problem, the optimal integer number of ships to deploy on a ship route can be obtained by solving two equations, each in one unknown, using a simple bi-section search method. The properties further enable us to identify an optimality condition for network containership sailing speed optimization. Based on this optimality condition, we propose a pseudo-polynomial-time solution algorithm that can efficiently obtain an epsilon-optimal solution for sailing speed of containerships in a liner shipping network.

The remainder of this paper is organized as follows. Section 2 examines a single ship route assuming that the number of ships can be a fractional quantity. Section 3 investigates a single ship route with an integer number of ships. Section 4 derives the optimality condition and a pseudo-polynomial-time algorithm for network containership sailing speed optimization with a limited number of available ships. Section 5 reports a case study to demonstrate the applicability of the proposed algorithm. Conclusions are presented in Section 6.

2 A single ship route with a fractional number of ships

We first examine a single ship route such as the NCE service shown in Fig. 1. To begin with, we assume that the number of ships deployed on it, denoted by m, can be a fractional number.

The itinerary of the liner ship route forms a directed loop and a string of homogeneous containerships are deployed to maintain a weekly frequency. The ship route can be represented by its port calling sequence - $1 \rightarrow 2, \rightarrow \cdots \rightarrow N \rightarrow 1$ - where the number 1 denotes its first port of call and N is the number of ports of call, and these ports of call are grouped into a set $I := \{1, 2, \cdots, N\}$. The voyage between two consecutive ports of call on the liner ship route is referred to as a *leg*. The i^{th} leg is defined as the voyage from the i^{th} port of call to the $(i+1)^{\text{th}}$ port of call when $i = 1, 2, \cdots, N-1$ and the N^{th} leg is from the N^{th} port of call to the 1st port of call. The distance of the i^{th} leg is denoted by L_i (n mile).

Represent by α ($/ton) the price of bunker for the main engine and let c ($/week) be the fixed operating cost of a ship (excluding bunker cost for the main engine). Represent further by $\hat{T}$ (h) the total time spent at ports during a round-trip journey. The sum of $\hat{T}$ and sailing time at sea is the round-trip journey time. We denote by h_i ($/h) the inventory cost rate of containers on leg i. We note that h_i depends on (i) the volume of containers on the leg, (ii) the value of the cargo per container, (iii) the characteristics of the cargo (e.g. the value of cargos with a short life-cycle depreciates fast), and (iv) the priority of customers' inventory cost on the liner shipping company's planning decision. The bunker consumption function on the i^{th} leg, denoted by $g_i(v_i)$ (tons/n mile), is dependent on the sailing speed v_i and has the following form when the speed is within the range of acceptable speeds:

$$g_i(v_i) = a_i(v_i)^{b_i}, \forall i \in I \tag{1}$$

where $a_i > 0$, $b_i > 1$ are two parameters calibrated from historical operating data and depend on the cargo load and sea conditions (Ronen, 2011; Wang and Meng, 2012).

2.1 A nonlinear optimization model

The speed optimization problem aims to determine the number of ships m to deploy and the sailing speed v_i on each leg to minimize the total operating cost including ship cost, bunker cost, and inventory cost, while maintaining a weekly frequency. For the ease of presentation, we use the sailing time $t_i := L_i / v_i$ on leg i as the decision variable. The bunker consumption per nautical mile on leg i then becomes $a_i(L_i / t_i)^{b_i}$. The speed optimization problem can be formulated as a nonlinear optimization model:

[P1]

$$\min_{t_i, m} cm + \alpha \sum_{i \in I} L_i a_i (L_i / t_i)^{b_i} + \sum_{i \in I} h_i t_i \tag{2}$$

subject to:

$$\sum_{i \in I} t_i - 168m = -\hat{T} \tag{3}$$

$$t_i \geq 0, \forall i \in I \tag{4}$$

$$m \geq 0 \tag{5}$$

The objective function (2) minimizes the total cost. Constraint (3) imposes the weekly frequency where the coefficient 168 means that each week has 168 hours. Constraints (4) and (5) enforce nonnegativity of variables.

2.2 Analytical solutions based on the Karush-Kuhn-Tucker (KKT) conditions

To analyze the properties of the optimal solution to the nonlinear model [P1], we derive its KKT conditions. Let λ, π_i and θ be the Lagrangian multiplier associated with Eqs. (3), (4), (5), respectively. Since all of the constraints (3)–(5) are linear, the linearity constraint qualification holds. As a result, the optimal solution to [P1] must satisfy the KKT conditions below (Boyd and Vandenberghe, 2004):

[KKT]

$$c - 168\lambda - \theta = 0 \tag{6}$$

$$\alpha(L_i)^{b_i+1} a_i(-b_i)(t_i)^{-b_i-1} + h_i + \lambda - \pi_i = 0, \forall i \in I \tag{7}$$

$$\pi_i t_i = 0, \forall i \in I \tag{8}$$

$$\theta m = 0 \tag{9}$$

$$\pi_i \geq 0, \forall i \in I \tag{10}$$

$$\theta \geq 0 \tag{11}$$

$$\sum_{i \in I} t_i - 168m = -\hat{T} \tag{12}$$

$$t_i \geq 0, \forall i \in I \tag{13}$$

$$m \geq 0 \tag{14}$$

where Eqs. (6) and (7) are the KKT equations, Eqs. (8) and (9) are complementary slackness conditions, Eqs. (10) and (11) are nonnegativity constraints on Lagrangian multipliers, and Eqs. (12)-(14) impose feasibility of the solution.

Evidently, at the optimal solution we must have $t_i > 0, i \in I$ and $m > 0$. Therefore, the complementary slackness conditions (8) and (9) imply that $\pi_i = 0, i \in I$ and $\theta = 0$, respectively. According to Eq. (6), we obtain

$$\lambda = \frac{c}{168} \tag{15}$$

Eq. (7) becomes:

$$t_i = L_i \left(\frac{h_i + c/168}{\alpha a_i b_i} \right)^{-\frac{1}{b_i+1}} \tag{16}$$

Equivalently,

$$v_i = \left(\frac{h_i + c/168}{\alpha a_i b_i} \right)^{\frac{1}{b_i+1}} \tag{17}$$

The optimal number of ships m can be derived from Eqs. (12) and (16):

$$m = \frac{\hat{T} + \sum_{i \in I} t_i}{168} = \frac{\hat{T} + \sum_{i \in I} L_i \left(\frac{h_i + c/168}{\alpha a_i b_i} \right)^{-\frac{1}{b_i+1}}}{168} \tag{18}$$

Note that by assumption, m can be a fractional number.

2.3 Managerial insights

The close-form expressions (16)-(18) of the optimal solution enable us to analyze useful managerial insights. First, Eq. (17) demonstrates that even if a_i and b_i do not change with voyage leg, the optimal speeds on different legs are still different due to the inventory cost. If the inventory cost is not considered, but a maximum transit time or port time windows are considered, the sailing speeds on different legs will also be different.

Second, we notice that the optimal speed on a leg v_i increases with the inventory cost rate h_i and the ship operating cost c, and decreases with the bunker price α and the bunker consumption coefficient a_i. This means that, for example, a lower bunker price and more valuable cargos should lead to a higher speed. However, the relation is not linear and depends on the coefficient b_i. When the daily bunker consumption is proportional to the sailing speed cubed, we will have $b_i = 2$. In this case, if the bunker price is increased by 50%, the resulting optimal speed should be $(1/1.5)^{1/3} = 0.87$ times the original speed. In other words, the sailing time at sea is 1.14 times as long as before. Take the NCE service in Fig. 1 as an example. Its round-trip journey time is 63 days and 9 ships are deployed. Suppose that in a round-trip journey, 2 weeks are port time and 7 weeks are sailing time. When the bunker price is increased by 50%, the sailing time should be increased to 7×1.14=7.98 weeks. This suggests that one more ship should be inserted to the existing string (service) of 9 ships.

The optimal speed v_i also decreases with the coefficient b_i, which can be shown below. The term $(h_i + c/168)/(\alpha a_i b_i)$ in Eq. (17) is greater than 1 in practice (otherwise the optimal speed is less than 1 knot). Hence, for two different b_i's satisfying $1 < b_{i1} < b_{i2}$, the following relation holds:

$$v_{i1} = \left(\frac{h_i + c/168}{\alpha a_i b_{i1}} \right)^{\frac{1}{b_{i1}+1}} > \left(\frac{h_i + c/168}{\alpha a_i b_{i2}} \right)^{\frac{1}{b_{i1}+1}} > \left(\frac{h_i + c/168}{\alpha a_i b_{i2}} \right)^{\frac{1}{b_{i2}+1}} = v_{i2} \tag{19}$$

The first ">" is correct because $y = x^{1/(b_{i1}+1)}, x > 0$ is an increasing function of x. The second ">" is correct because $y = p^x, p > 1, x > 0$ is an increasing function of x. Therefore, the optimal speed v_i decreases with both a_i and b_i.

Third, the optimal speed v_i has a marginal cost explanation. Suppose that the optimal sailing time on leg i is increased by a small value $\Delta t > 0$. Then the ship cost and inventory cost on the leg will increase by

$$\frac{c}{168}\Delta t + h_i \Delta t \tag{20}$$

At the same time, the speed will be reduced and the bunker cost on the leg will be reduced by

$$\left| \frac{d\left[\alpha L_i a_i (L_i / t_i)^{b_i} \right]}{dt_i} \right| \Delta t = \alpha a_i b_i (L_i)^{b_i+1} (t_i)^{-b_i-1} \Delta t \tag{21}$$

Replacing t_i in Eq. (21) by Eq. (16), the bunker cost will be reduced by

$$\left(\frac{c}{168} + h_i \right) \Delta t \tag{22}$$

which is identical to Eq. (20). This implies the marginal cost of adding/reducing sailing time on a leg is 0 in the optimal solution.

Fourth, we notice that the optimal sailing speed of a leg in Eq. (17) is independent of parameters on other legs. We stress that this results from the assumption that *m* can be a fractional number. It has the following implication. In [P1] we do not impose an upper limit on the sailing speed although in reality each ship has a maximum speed. This assumption is reasonable when containerships do not sail at the highest speed during periods of high bunker prices and oversupply of ship capacity. When the bunker price is low, the inventory costs are high, and the maximum sailing speed of ships is low, it is possible that the calculated speed of a leg exceeds the maximum sailing speed. In this case, we simply need to set the speed for that leg to the highest sailing speed and reoptimize the speed for other legs. The fact that the optimal sailing speed of a leg is independent of parameters on other legs means that such a simple revision leads to an optimal solution.

3 A single ship route with an integer number of ships

In reality, the number of ships m must be an integer. A natural question is, what is the integer number of ships that should be deployed if the optimal number of ships computed in [P1], denoted by m^*, in not an integer? What if $m^* = 5.3$ is obtained? What if $m^* = 5.5$ and what if $m^* = 5.9$? One might conjecture that 5 ships should be deployed if $m^* = 5.3$, 6 ships should be deployed if $m^* = 5.9$, and it makes no difference whether 5 or 6 ships are deployed if $m^* = 5.5$. Unfortunately, this conjecture is incorrect.

3.1 Optimal speed with a given number of ships

We first develop a method for computing the optimal sailing time t_i with a given number of ships m. As $m = \left(\hat{T} + \sum_{i \in I} t_i\right)/168$, we rewrite Eq. (2) as the summation of cost (ship cost, bunker cost and inventory cost) associated with each leg and consider t_i as the decision variable:

$$\min_{t_i > 0} \frac{\hat{T}c}{168} + \sum_{i \in I} \left[\frac{t_i}{168} c + \alpha (L_i)^{1+b_i} a_i (t_i)^{-b_i} + h_i t_i \right] \tag{23}$$

Note that the first term in Eq. (23) is constant. We define

$$c_i(t_i) := \frac{t_i}{168} c + \alpha (L_i)^{1+b_i} a_i (t_i)^{-b_i} + h_i t_i, \forall i \in I \tag{24}$$

Given the number of ships m, to minimize the total cost, we can formulate the following model by removing the constant term in Eq. (23):

[P2(m)]: $$C(m) := \min_{t_i > 0} \sum_{i \in I} c_i(t_i) \tag{25}$$

subject to:

$$\sum_{i\in I} t_i = 168m - \hat{T} \tag{26}$$

In the sequel, whenever we mention the total cost, we refer to the objective function Eq. (25) that has already excluded the constant term. Let λ be the Lagrangian multiplier associated with constraint (26). The KKT condition of model [P2(m)] is:

$$\frac{dc_i(t_i)}{dt_i} + \lambda = 0, \forall i \in I \tag{27}$$

$$\sum_{i\in I} t_i = 168m - \hat{T} \tag{28}$$

Eqs. (24) and (27) imply that

$$\frac{c}{168} + h_i - \alpha (L_i)^{1+b_i} a_i b_i (t_i)^{-b_i-1} = -\lambda, \forall i \in I \tag{29}$$

The left-hand side is $|I|$ expressions that are all equal to $-\lambda$. Therefore, any of the two expressions are equal and equal to the first expression $c/168 + h_1 - \alpha(L_1)^{1+b_1} a_1 b_1 (t_1)^{-b_1-1}$. Hence, all the t_i can be expressed as functions of the sailing time on the first leg t_1:

$$\frac{c}{168} + h_i - \alpha (L_i)^{1+b_i} a_i b_i (t_i)^{-b_i-1} = \frac{c}{168} + h_1 - \alpha (L_1)^{1+b_1} a_1 b_1 (t_1)^{-b_1-1}, \forall i \in I \tag{30}$$

$$t_i = \left[\frac{h_i - h_1 + \alpha (L_1)^{1+b_1} a_1 b_1 (t_1)^{-b_1-1}}{\alpha (L_i)^{1+b_i} a_i b_i} \right]^{-\frac{1}{b_i+1}}, \forall i \in I \tag{31}$$

Substituting Eq. (31) into Eq. (28), we have

$$\sum_{i\in I} \left[\frac{h_i - h_1 + \alpha (L_1)^{1+b_1} a_1 b_1 (t_1)^{-b_1-1}}{\alpha (L_i)^{1+b_i} a_i b_i} \right]^{-\frac{1}{b_i+1}} = 168m - \hat{T} \tag{32}$$

The left-hand side of Eq. (32) is a function that is strictly increasing with t_1. Therefore, bi-section search method could efficiently find the solution to Eq. (32), which is also the optimal solution to [P2(m)].

In sum, given the number of ships, the optimal sailing time on each leg and the minimum total operating cost can easily be obtained by a bi-section search process (Fox and Landi, 1970).

3.2 Minimum total cost function on a leg

We then analyze the property of the function $c_i(t_i)$ in Eq. (24). Its first, second, and third derivatives are respectively:

$$c_i'(t_i) = \frac{c}{168} + h_i - \alpha (L_i)^{1+b_i} a_i b_i (t_i)^{-b_i-1}, \forall i \in I \tag{33}$$

$$c_i''(t_i) = (b_i + 1)\alpha (L_i)^{1+b_i} a_i b_i (t_i)^{-b_i-2}, \forall i \in I \tag{34}$$

$$c_i'''(t_i) = -(b_i + 2)(b_i + 1)\alpha (L_i)^{1+b_i} a_i b_i (t_i)^{-b_i-3}, \forall i \in I \tag{35}$$

Let t_i^* be the optimal solution shown in Eq. (16) to model [P1]. It is easy to see that t_i^* is also a stationary point of $c_i(t_i)$, i.e.,

$$c_i'(t_i^*) = 0, \forall i \in I \tag{36}$$

Moreover, as $t_i > 0$, we have

$$c_i''(t_i) > 0, \forall i \in I \tag{37}$$

$$c_i'''(t_i) < 0, \forall i \in I \tag{38}$$

Eq. (37) implies that $c_i(t_i)$ is a strictly convex function. Eqs. (37) and (38) imply that $c_i'(t_i)$ is a strictly increasing and strictly concave function, as shown in Fig. 2. Therefore, we have

$$c_i'(t_i^* + \Delta t) > 0, \forall \Delta t > 0 \tag{39}$$

$$c_i'(t_i^* - \Delta t) < 0, \forall 0 < \Delta t < t_i^* \tag{40}$$

$$\left| c_i'(t_i^* - \Delta t) \right| > c_i'(t_i^* + \Delta t), \forall 0 < \Delta t < t_i^* \tag{41}$$

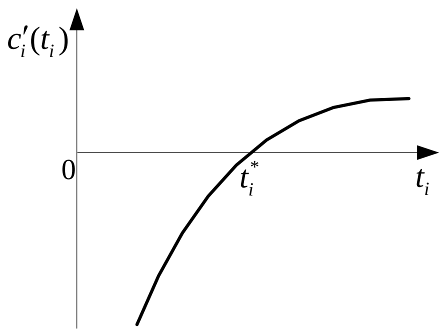

Fig. 2 Graph of function $c_i'(t_i)$

Eqs. (39)-(41) imply that $c_i(t_i)$ increases faster when t_i is decreased from t_i^* to $t_i^* - \Delta t$ than when t_i is increased from t_i^* to $t_i^* + \Delta t$, $0 < \Delta t < t_i^*$. In other words, the total cost is more sensitive when the sailing time is reduced than when it is increased. Mathematically, we write $c_i(t_i^* - \Delta t)$ and $c_i(t_i^* + \Delta t)$ using Taylor theorem:

$$c_i(t_i^* - \Delta t) = c_i(t_i^*) + \frac{c_i'(t_i^*)}{1!}(-\Delta t) + \frac{c_i''(t_i^*)}{2!}(-\Delta t)^2 + \frac{c_i'''(t_i^* - h_1 \Delta t)}{3!}(-\Delta t)^3, h_1 \in [0,1] \tag{42}$$

$$c_i(t_i^* + \Delta t) = c_i(t_i^*) + \frac{c_i'(t_i^*)}{1!}\Delta t + \frac{c_i''(t_i^*)}{2!}(\Delta t)^2 + \frac{c_i'''(t_i^* + h_2 \Delta t)}{3!}(\Delta t)^3, h_2 \in [0,1] \tag{43}$$

Note that $c_i'(t_i^*) = 0$. Hence,

$$c_i(t_i^* - \Delta t) - c_i(t_i^* + \Delta t) = -\frac{c_i'''(t_i^* - h_1 \Delta t)}{3!}(\Delta t)^3 - \frac{c_i'''(t_i^* + h_2 \Delta t)}{3!}(\Delta t)^3 \tag{44}$$

Eqs. (38) and (44) imply that

$$c_i(t_i^* - \Delta t) > c_i(t_i^* + \Delta t), \forall 0 < \Delta t < t_i^* \tag{45}$$

as shown in Fig. 3.

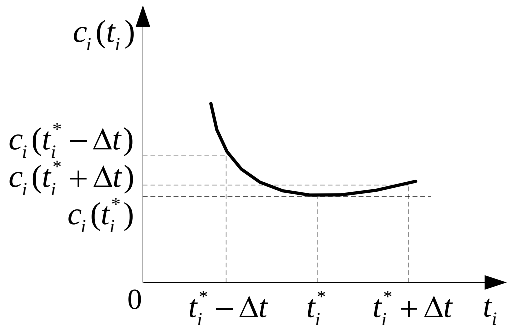

Fig. 3 Relation between the sailing time and the total cost of a leg

3.3 Minimum total cost with a given number of ships

Recall that $C(m)$ is the optimal objective value of [P2(m)] in Eq. (25), which is the minimum total cost of the ship route if m ships are deployed on it. Let m^* and t_i^* be the optimal solution to [P1] (m^* may be a fractional number), and $\overline{t_i}$ be the optimal solution to [P2(m)].

Theorem 1: [P2(m)] has the following properties:

(i) $\overline{t_i}$ strictly increases with m. In particular, if $m < m^*$, then $\overline{t_i} < t_i^*, \forall i \in I$; if $m > m^*$, then $\overline{t_i} > t_i^*, \forall i \in I$.

(ii) If $m < m^*$, $C(m)$ strictly decreases with m; if $m > m^*$, $C(m)$ strictly increases with m.

(iii) For $0 < \Delta m < m$, we have $C(m - \Delta m) + C(m + \Delta m) > 2C(m)$.

(iv) For $0 < \Delta m < m^*$, we have $C(m^* - \Delta m) > C(m^* + \Delta m)$.

The proof of Theorem 1 can be found in Appendix 1.

The form of $C(m)$ is similar to $c_i(t_i)$, and is shown in Fig. 4.

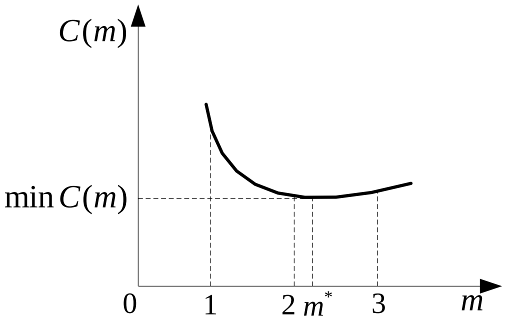

Fig. 4 Relation between the number of ships and the minimum total cost of a ship route

3.4 Optimal integer number of ships on a single ship route

Theorem 2: Let m^* be the optimal solution to [P1]. We assume that m^* is not an integer. We further assume that $m^* > 1$ because otherwise it is evident that one ship should be deployed. Define $\lfloor x \rfloor$ to be the largest integer not greater than x. Then, (i) if $m^* - \lfloor m^* \rfloor \geq 0.5$, deploying $\lfloor m^* \rfloor + 1$ ships is the unique optimal integer solution; (ii) if $0 < m^* - \lfloor m^* \rfloor < 0.5$, it is possible that deploying $\lfloor m^* \rfloor + 1$ ships is still preferable to $\lfloor m^* \rfloor$ ships. □

The proof of Theorem 2 can be found in Appendix 2.

When $0 < m^* - \lfloor m^* \rfloor < 0.5$, to find the optimal integer number of ships to deploy, we need to solve two problems [P2($\lfloor m^* \rfloor$)] and [P2($\lfloor m^* \rfloor + 1$)]. Note that to solve each problem, only a one-dimensional bi-section search is needed.

4 A liner shipping network with limited numbers of ships

This section extends a single ship route to a liner shipping network that consists of many ship routes such as the one shown in Fig. 5. Moreover, in contrast to the previous two sections that assume an unlimited number of ships available, here we consider a limited number of available ships in the ship fleet of the liner shipping company.

The set of ship routes is denoted by R and the legs on ship route $r \in R$ are denoted by set I_r. The distance of leg $i \in I_r$ of ship route r is L_{ri}, the sailing speed on the leg is denoted by v_{ri}, and the inventory cost rate is h_{ri}. The sailing time $t_{ri} = L_{ri} / v_{ri}$. The fixed port time on ship route r is $\hat{T}_r$ and a total of m_r ships are deployed to maintain a weekly frequency. Both m_r and t_{ri} are decision variables.

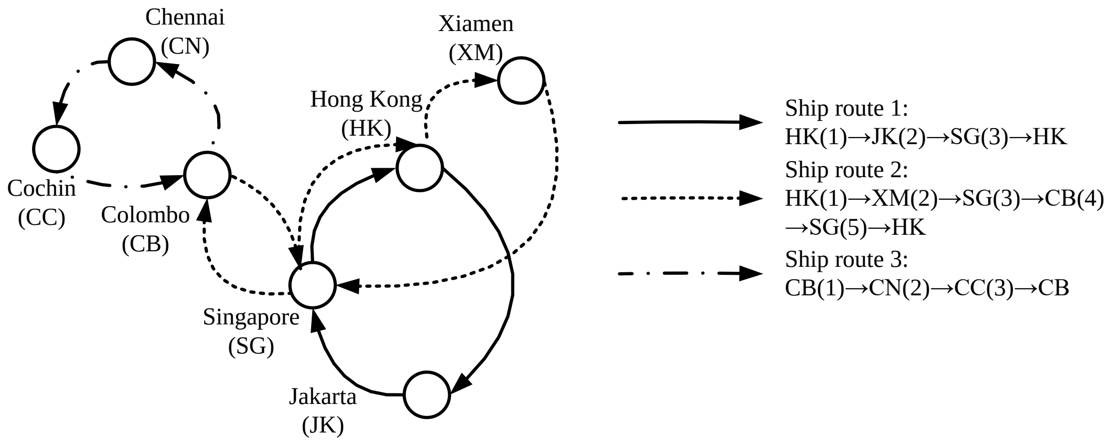

Fig. 5 An illustrative liner shipping network (Wang et al., 2015)

Different ship routes may have different types of ships. We let V be the set of ship types. We assume that a limited number of M_v ships in type $v \in V$ are available. The fixed operating cost of a ship in type v is denoted by c_v. Moreover, we represent by R_v the set of ship routes that use ships in type v. The bunker consumption function on leg $i \in I_r$ of ship route $r \in R$ is $g_{ri}(v_{ri}) = a_{ri}(v_{ri})^{b_{ri}}$.

The speed optimization problem for a liner shipping network can be formulated as a mixed-integer nonlinear optimization model with variables m_r and t_{ri}:

[P3] $$\min_{t_{ri},m_r} \sum_{v\in V}\sum_{r\in R_v} c_v m_r + \alpha \sum_{r\in R}\sum_{i\in I_r} L_{ri} a_{ri} (L_{ri}/t_{ri})^{b_{ri}} + \sum_{r\in R}\sum_{i\in I_r} h_{ri} t_{ri} \quad (46)$$

subject to:

$$\sum_{i\in I_r} t_{ri} - 168 m_r = -\hat{T}_r, \forall r \in R \quad (47)$$

$$\sum_{r\in R_v} m_r \le M_v, \forall v \in V \quad (48)$$

$$t_{ri} \ge 0, \forall r \in R, \forall i \in I_r \quad (49)$$

$$m_r \ge 1 \text{ and integer}, \forall r \in R \quad (50)$$

where constraint (48) requires that the number of ships in each type used cannot exceed the number of available ships in the fleet.

It is not difficult to see that [P3] can be decomposed for each ship type $v \in V$:

[P3- v] $$\min_{t_{ri},m_r} \sum_{r\in R_v} c_v m_r + \alpha \sum_{r\in R_v}\sum_{i\in I_r} L_{ri} a_{ri} (L_{ri}/t_{ri})^{b_{ri}} + \sum_{r\in R_v}\sum_{i\in I_r} h_{ri} t_{ri} \quad (51)$$

subject to:

$$\sum_{i\in I_r} t_{ri} - 168 m_r = -\hat{T}_r, \forall r \in R_v \quad (52)$$

$$\sum_{r\in R_v} m_r \le M_v \quad (53)$$

$$t_{ri} \ge 0, \forall r \in R_v, \forall i \in I_r \quad (54)$$

$$m_r \ge 1 \text{ and integer}, \forall r \in R_v \quad (55)$$

Let m_r^* be the optimal number of ships obtained by model [P1] for ship route $r \in R_v$ and m_r^* may not be an integer. Provided that there are sufficient ships, the optimal integer number of ships for ship route r , denoted by $\tilde{m}_r$, can easily be obtained according to Section 3. Note that if there are two optimal integer solutions, i.e., $\lfloor m_r^* \rfloor$ and $\lfloor m_r^* \rfloor + 1$, the method in Section 3 can find both solutions and we define $\tilde{m}_r$ as the smaller one, i.e., $\lfloor m_r^* \rfloor$. If $\sum_{r\in R_v} \tilde{m}_r \le M_v$, then $(\tilde{m}_r, r \in R_v)$ is the optimal solution to [P3- v]. Otherwise, in model [P3- v], constraint (53) must be binding at the optimal solution.

Now we investigate the optimal number of ships on each ship route when $\sum_{r\in R_v}\tilde{m}_r > M_v$. Similar to $C(m)$ in Eq. (25), we define function $C_r(m)$ as the minimum total cost of ship route $r \in R_v$ if m ship are deployed on it.

[P3-v] can thus be reformulated as:

[P4-v]
$$\min_{m_r} \sum_{r\in R_v} C_r(m) \tag{56}$$

subject to:

$$\sum_{r\in R_v} m_r \le M_v \tag{57}$$

$$m_r \ge 1 \text{ and integer}, \forall r \in R_v \tag{58}$$

4.1 Optimality condition

Theorem 3: Assuming that $\sum_{r\in R_v}\tilde{m}_r > M_v$ where $\tilde{m}_r$ is the optimal integer number of ships for ship route r provided that there are sufficient ships, a solution denoted by $(\hat{m}_r, r \in R_v)$ to [P4-v] when there are only M_v ships is optimal if and only if it satisfies the following condition: For any two ship routes $r_1 \in R_v, r_2 \in R_v$ such that $\hat{m}_{r_1} \ge 2$, we have $C_{r_1}(\hat{m}_{r_1} - 1) - C_{r_1}(\hat{m}_{r_1}) \ge C_{r_2}(\hat{m}_{r_2}) - C_{r_2}(\hat{m}_{r_2} + 1)$. In other words, shifting one ship from ship route r_1 to ship route r_2 cannot reduce the total cost.□

The proof of Theorem 3 can be found in Appendix 3.

It should be noted that even if the optimal integer solution to [P1] is unique, the optimal solution to [P4-v] may not be unique. For instance, suppose that a total of 7 ships are available and R_v consists of two identical ship routes, and the optimal integer number of ships on each ship route is 5. Hence, there are two optimal solutions: one ship route has 3 ships, the other has 4; or vice versa. We further have the property:

Theorem 4: The optimal solution $(\hat{m}_r, r \in R_v)$ to model [P4-v] satisfies $\hat{m}_r \le \tilde{m}_r$, $r \in R_v$. □

The proof of Theorem 4 can be found in Appendix 4.

4.2 A pseudo-polynomial-time algorithm

Based on the above optimality condition, we develop a pseudo-polynomial-time algorithm for solving the mixed-integer nonlinear programming model [P3-v] or [P4-v] for network containership sailing speed optimization.

A pseudo-polynomial-time algorithm for solving [P3-v]:

Step 1. For each ship route $r \in R_v$, compute the optimal number of ships m_r^* by model [P1]. If m_r^* is an integer, set $\tilde{m}_r := m_r^*$; else if $m_r^* < 1$, set $\tilde{m}_r := 1$; else if $m_r^* - \lfloor m_r^* \rfloor \geq 0.5$, set $\tilde{m}_r := \lfloor m_r^* \rfloor + 1$; else, solve the two models [P2($\lfloor m_r^* \rfloor$)] and [P2($\lfloor m_r^* \rfloor + 1$)] for the ship route using bi-section search and set $\tilde{m}_r := \lfloor m_r^* \rfloor$ if $C_r(\lfloor m_r^* \rfloor) \leq C_r(\lfloor m_r^* \rfloor + 1)$ and set $\tilde{m}_r := \lfloor m_r^* \rfloor + 1$ otherwise.

Step 2. If $\sum_{r \in R_v} \tilde{m}_r \leq M_v$, then $(\tilde{m}_r, r \in R_v)$ is the optimal solution to [P3-v] and stop. Otherwise compute $C_r(m)$ for each ship route $r \in R_v$ and each $m = \lceil \hat{T}_r / 168 \rceil, \lceil \hat{T}_r / 168 \rceil + 1, \cdots, \tilde{m}_r - 1$ where $\lceil \bullet \rceil$ is the smallest integer greater than or equal to $\bullet$, set $C_r(\tilde{m}_r + 1) = +\infty$, $r \in R_v$, and define $\hat{m}_r := \tilde{m}_r$, $r \in R_v$.

Step 3. Find one ship route r^* satisfying

$$r^* \in \arg\min_{r \in R_v, \tilde{m}_r \geq 2} \left[C_r(\hat{m}_r - 1) - C_r(\hat{m}_r) \right] \tag{59}$$

That is, reducing one ship on ship route r^* leads to the smallest increase in the total cost. Set $\hat{m}_{r^*} := \hat{m}_{r^*} - 1$

Step 4. If $\sum_{r \in R_v} \hat{m}_r \leq M_v$, $(\hat{m}_r, r \in R_v)$ is the optimal solution and stop. Otherwise go to Step 3. □

We note that to solve [P2(m)] using bi-section search, we can specify a predetermined tolerance for t_{r1} (the sailing time on leg 1 of ship route r) denoted by ε. Since the domain of t_{r1} is a subset of $(0, 168M)$, the bi-section search method ends in $\log_2(168M / \varepsilon)$ iterations. In each iteration we need to calculate the sailing time for $|I_r|$ legs (i.e., the number of legs that the route consists of). Therefore, the time required for solving [P2(m)] is bounded by $O(|I^{\max}| \log_2(168M / \varepsilon))$, where $|I^{\max}|$ is the maximum number of legs a route contains. The optimal number of ships in [P1] can straightforwardly be found using Eq. (18). Hence, both [P2(m)] and [P1] can be solved polynomially.

In the above algorithm, [P1] is solved at most $|R_v|$ times and [P2(m)] is solved at most $O(|R_v|)$ times in Step 1 and Step 2. Step 3 and Step 4 require at most $O(|R_v|^2)$ comparisons. However, Step 3 may be repeated at most M_v times. Moreover, model [P3-v] needs to be solved $|V|$ times. Therefore, the above algorithm can find an ε-optimal solution to network containership sailing speed optimization problem in $O(|R_v||I^{\max}||V|\log_2(168M/\varepsilon)+M|R_v|^2|V|)$ time, which is pseudo-polynomial with regard of the input length. The algorithm is pseudo-polynomial rather than polynomial because the expression is bounded by the number of ships M.

Finally, we note that the pseudo-polynomial is fast enough for practical purposes as in reality the number of ships deployed on a route has an upper bound M, for example, $M=500$. This can be substantiated by the fact that the largest container shipping company in the world Maersk Line had 453 ships in total in 2012 (UNCTAD, 2013).

5 Case study

We use an Asia-Europe-Oceania shipping network shown in Fig. 6 provided by a global liner shipping company to demonstrate the applicability of the pseudo-polynomial-time algorithm. This network has 46 ports and 11 ship routes, as shown in Table 1. Because ship routes with different types of ships can be optimized separately, we assume that the same type of ships – 8000-TEU ships – is deployed on all the ship routes. The other parameters are as follows: $c_v = 200,000$, $\alpha = 500$, $a_{ri} = 0.001$ and $b_{ri} = 2$ for all legs of all of the ship routes; h_{ri} is equal to 0.5 times the volume of containers on the leg, and we assume that on each ship route the first leg has 4000 TEUs of containers, the last leg has 8000 TEUs, and the differences of the flows on leg $i \in I_r \setminus \{1\}$ and leg $i-1$ are the same; the port time is 24 hours at each port of call; and $M_v = 35$.

The pseudo-polynomial time algorithm for [P3-v] and the bi-section search algorithm for [P1] are coded in MATLAB on a 3.5 GHz Eight Core PC with 16 GB of RAM. In the bi-section search algorithm, $\varepsilon = 0.001$ h. The model [P3-ν], with a total of 11 integer variables and 87 continuous variables, can be solved in 1 second.

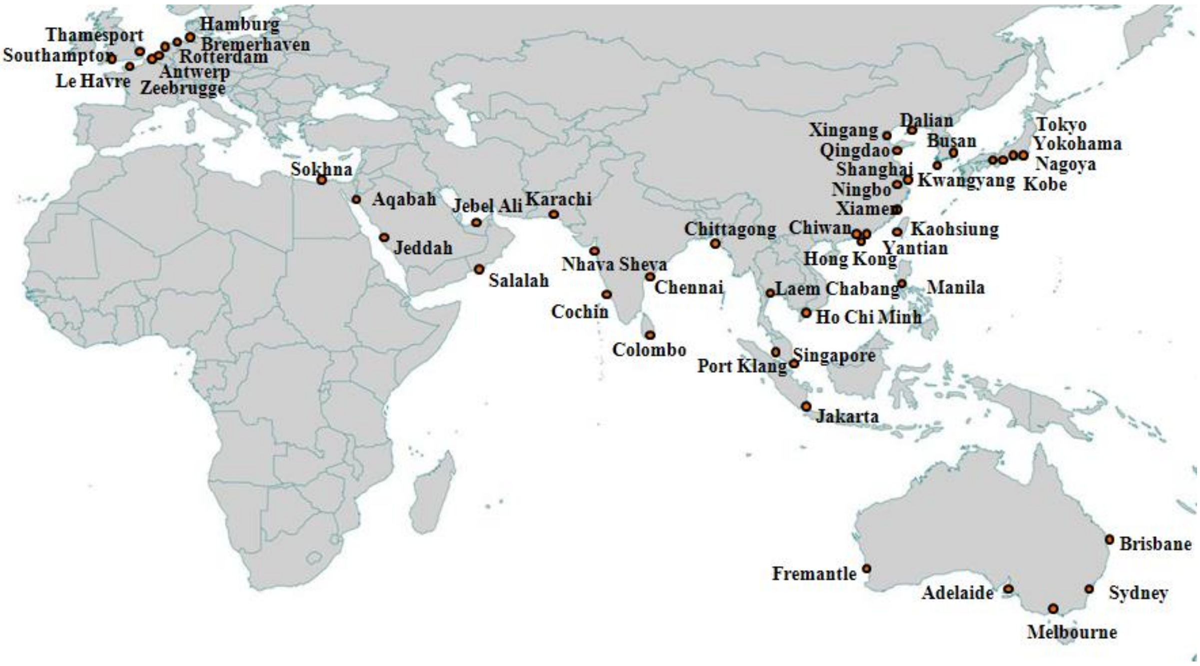

Fig. 6 An Asia-Europe-Oceania liner shipping network (Wang et al., 2015)

Table 1 Ship routes

No.	Ports of call
1	Singapore → Brisbane → Sydney → Melbourne → Adelaide → Fremantle
2	Xiamen → Chiwan → Hong Kong → Singapore → Port Klang → Salalah → Jeddah → Aqabah → Salalah → Singapore
3	Yokohama → Tokyo → Nagoya → Kobe → Shanghai
4	Ho Chi Minh → Laem Chabang → Singapore → Port Klang
5	Brisbane → Sydney → Melbourne → Adelaide → Fremantle → Jakarta → Singapore
6	Manila → Kaohsiung → Xiamen → Hong Kong → Yantian → Chiwan → Hong Kong
7	Dalian → Xingang → Qingdao → Shanghai → Ningbo → Shanghai → Kwangyang → Busan
8	Chittagong → Chennai → Colombo → Cochin → Nhava Sheva → Cochin → Colombo → Chennai
9	Sokhna → Aqabah → Jeddah → Salalah → Karachi → Jebel Ali → Salalah
10	Southampton → Thamesport → Hamburg → Bremerhaven → Rotterdam → Antwerp → Zeebrugge → Le Havre
11	Southampton → Sokhna → Salalah → Colombo → Singapore → Hong Kong → Xiamen → Shanghai → Busan → Dalian → Xingang → Qingdao → Shanghai → Hong Kong → Singapore → Colombo → Salalah

To optimize the total cost, we first compute m_r^*, and the result is shown in Table 2. The optimal integer numbers of ships on ship routes 6, 7, 8, 10, 11 can easily be computed as they satisfy $m_r^* - \lfloor m_r^* \rfloor \geq 0.5$. We need to calculate $C_r(\lfloor m_r^* \rfloor)$ and $C_r(\lfloor m_r^* \rfloor + 1)$ for the other ship routes. Their values (unit: 10^6 \$/week) and the optimal integer numbers of ships are shown in Table 2. It can be seen that on ship routes 3 and 4, although $m_r^* - \lfloor m_r^* \rfloor < 0.5$, deploying $\lfloor m_r^* \rfloor + 1$ ships are preferable. This again validates part (ii) of Theorem 2.

The total cost of all of the ship routes with the (infeasible) solution $(\tilde{m}_r, r \in R_v)$ is 32.02×10^6 \$/week. As $\sum_{r \in R_v} \tilde{m}_r = 41 > M_v$, we compute $C_r(m)$ for each ship route $r \in R_v$ and each $m = \lceil \hat{T}_r / 168 \rceil, \lceil \hat{T}_r / 168 \rceil + 1, \cdots, \tilde{m}_r - 1$. Table 3 reports the value of $C_r(m)$ (unit: 10^6 \$/week) for each ship route $r \in R_v$ (the rows) and each $m = \lceil \hat{T} / 168 \rceil, \lceil \hat{T} / 168 \rceil + 1, \cdots, \tilde{m}_r$ (the columns). Steps 3 and 4 of the algorithm then gives the optimal solution $(\hat{m}_r, r \in R_v) = (3,5,1,1,4,2,2,3,3,2,9)$, highlighted in grey shadings in Table 3. The minimum total cost is 33.42×10^6 \$/week.

Table 2 Optimal number of ships assuming sufficient ships in the fleet

No.	m_r^*	$C_r(\lfloor m_r^* \rfloor)$	$C_r(\lfloor m_r^* \rfloor + 1)$	$\tilde{m}_r$
1	4.07	3.25	3.43	4
2	6.26	5.46	5.54	6
3	1.47	2.30	1.12	2
4	1.45	1.57	1.10	2
5	4.09	3.57	3.80	4
6	1.61			2
7	1.88			2
8	2.98			3
9	3.32	2.65	2.74	3
10	1.59			2
11	10.81			11

Table 3 The value of $C_r(m)$

No.	$m=1$	2	3	4	5	6	7	8	9	10	11
1	538.27	9.19	3.86	3.25							
2	N.A.	129.68	18.27	8.32	6.00	5.46					
3	2.30	1.12									
4	1.57	1.10									
5	1.2E10	12.01	4.36	3.57							
6	3.6E9	0.75									
7	N.A.	0.80									
8	N.A.	3.52	1.92								
9	3.6E9	5.39	2.65								
10	N.A.	0.60									
11	N.A.	N.A.	626.65	83.93	32.76	18.58	13.04	10.55	9.40	8.94	8.85

6 Conclusions

This paper has examined the fundamental properties of the network containership sailing speed optimization problem that minimizes the sum of ship cost, bunker cost, and inventory cost and thereby developed a pseudo-polynomial-time solution algorithm. We have derived analytical solutions for sailing speed optimization on a single ship route with a continuous number of ships. The analytical solutions demonstrate that the optimal sailing speeds on different legs are generally different due to the different inventory cost rates. The optimal speed on a leg increases with the inventory cost rate and the ship operating cost, and decreases with the bunker price and the bunker consumption coefficients. Nevertheless, the rates of change are not constant. If the optimal continuous number of ship to deploy is not an integer, and its part on the right of the decimal place is not smaller than 0.5, then the optimal integer number of ships can be obtained by simply rounding up the optimal continuous number to the nearest integer. However, there is no straightforward answer if the part on the right of the decimal place is smaller than 0.5. In this case, to find the optimal integer number of ships, we need to solve two equations, each in one unknown, using a simple bi-section search method. The properties of two functions, namely, the function of the total cost on a leg with regard to the sailing time on it and the function of the total cost of a ship route with regard to the number of ships deployed on it, enabled us to identify an optimality condition for network containership sailing speed optimization. Based on this optimality condition, we proposed a pseudo-polynomial-time solution algorithm that can efficiently obtain an epsilon-optimal solution for sailing speed of containerships in a liner shipping network. The proposed pseudo-polynomial-time solution algorithm was applied to an Asia-Europe-Oceania liner shipping network.

Appendix 1: Proof of Theorem 1

Property (i) is correct as a result of Eqs. (31) and (32). Property (ii) is correct because of Property (i) and because $c_i(t_i)$ strictly increases with t_i when $t_i > t_i^*$, and strictly decreases with t_i when $t_i < t_i^*$.

Property (iii) is the convexity of $C(m)$. To prove it, suppose that $(\bar{t}_i^{m-\Delta m}, \forall i \in I)$ is the optimal solution to [P2($m-\Delta m$)] and $(\bar{t}_i^{m+\Delta m}, \forall i \in I)$ is the optimal solution to [P2($m+\Delta m$)]. Hence, $(0.5\bar{t}_i^{m-\Delta m} + 0.5\bar{t}_i^{m+\Delta m}, \forall i \in I)$ is a solution to [P2(m)]. Because of the strict convexity of $c_i(t_i)$, $c_i(0.5\bar{t}_i^{m-\Delta m} + 0.5\bar{t}_i^{m+\Delta m}) < 0.5c_i(\bar{t}_i^{m-\Delta m}) + 0.5c_i(\bar{t}_i^{m+\Delta m})$. Therefore, the objective value of [P2(m)] with the solution $(0.5\bar{t}_i^{m-\Delta m} + 0.5\bar{t}_i^{m+\Delta m}, \forall i \in I)$ is less than $0.5C(m-\Delta m) + 0.5C(m+\Delta m)$. Hence, $C(m) < 0.5C(m-\Delta m) + 0.5C(m+\Delta m)$.

To prove Property (iv), suppose that the optimal solution to [P2($m^* - \Delta m$)] is $\bar{t}_i$, $\bar{t}_i < t_i^*, \forall i \in I$. Then the vector $(t_i = 2t_i^* - \bar{t}_i, i \in I)$ is a feasible solution to [P2($m^* + \Delta m$)], and Eq. (45) implies that the objective value of [P2($m^* + \Delta m$)] with the solution $(t_i = 2t_i^* - \bar{t}_i, i \in I)$ is smaller than the optimal value of [P2($m^* - \Delta m$)]. Therefore, $C(m^* + \Delta m) < C(m^* - \Delta m)$.□

Appendix 2: Proof of Theorem 2

(i) When $m^* - \lfloor m^* \rfloor = 0.5$, Property (iv) of Theorem 1 immediately implies that $C(\lfloor m^* \rfloor + 1) < C(\lfloor m^* \rfloor)$. When $m^* - \lfloor m^* \rfloor > 0.5$, Property (iv) of Theorem 1 implies that $C(\lfloor m^* \rfloor + 1) < C(2m^* - \lfloor m^* \rfloor - 1)$ (e.g., $C(6) < C(5.2)$ when $m^* = 5.6$). Property (ii) of

Theorem 1 indicates that $C(\lfloor m^* \rfloor) > C(2m^* - \lfloor m^* \rfloor - 1)$ (e.g., $C(5) > C(5.2)$ when $m^* = 5.6$). Consequently, $C(\lfloor m^* \rfloor + 1) < C(\lfloor m^* \rfloor)$.

(ii) Consider a ship route with two legs with identical parameters. $c = 168{,}000$, $L_i = 5{,}000$, $a_i = 0.0005$, $b_i = 2$, $\alpha = 500$, $h_i = 3000$, $\hat{T} = 84$. The optimal sailing time on both legs are $t_i^* = 250$ according to Eq. (16) and the optimal number of ships $m = 3.48$ according to Eq. (18). The total cost calculated by Eq. (2) is \$3,000,000/week. If three ships are deployed, the total cost is \$3,097,234; if four ships are deployed, the minimum total cost in Eq. (2) becomes \$3,075,078. This example demonstrates that it is possible that when $m^* - \lfloor m^* \rfloor < 0.5$, deploying $\lfloor m^* \rfloor + 1$ ships is still preferable to deploying $\lfloor m^* \rfloor$ ships. □

Appendix 3: Proof of Theorem 3

The necessity of the condition is evident because otherwise $(m_r = \hat{m}_r, r \in R_v \setminus \{r_1, r_2\}; m_{r_1} = \hat{m}_{r_1} - 1; m_{r_2} = \hat{m}_{r_2} + 1)$ is a better solution. We now prove its sufficiency. Suppose that $(\bar{m}_r, r \in R_v) \neq (\hat{m}_r, r \in R_v)$ is an optimal solution. Then there exists a ship route $r_1 \in R_v$ such that $1 \le \bar{m}_{r_1} < \hat{m}_{r_1}$ and another ship route $r_2 \in R_v$ such that $\bar{m}_{r_2} > \hat{m}_{r_1} \ge 1$. By definition,

$$C_{r_1}(\hat{m}_{r_1} - 1) - C_{r_1}(\hat{m}_{r_1}) \ge C_{r_2}(\hat{m}_{r_2}) - C_{r_2}(\hat{m}_{r_2} + 1) \tag{60}$$

As $(\bar{m}_r, r \in R_v)$ is the optimal solution and the above condition is a necessary condition for optimality, we have

$$C_{r_2}(\bar{m}_{r_2} - 1) - C_{r_2}(\bar{m}_{r_2}) \ge C_{r_1}(\bar{m}_{r_1}) - C_{r_1}(\bar{m}_{r_1} + 1) \tag{61}$$

Property (iii) of Theorem 1 implies that $C_r(m-1) - C_r(m)$ strictly decreases with m. Therefore,

$$C_{r_1}(\bar{m}_{r_1}) - C_{r_1}(\bar{m}_{r_1} + 1) \ge C_{r_1}(\hat{m}_{r_1} - 1) - C_{r_1}(\hat{m}_{r_1}) \tag{62}$$

$$C_{r_2}(\hat{m}_{r_2}) - C_{r_2}(\hat{m}_{r_2} + 1) \ge C_{r_2}(\bar{m}_{r_2} - 1) - C_{r_2}(\bar{m}_{r_2}) \tag{63}$$

Note that in Eqs. (62) and (63) we use "≥" rather than ">" as in Property (iii) of Theorem 1 because it is possible that $\bar{m}_{r_1} = \hat{m}_{r_1} - 1$ and $\hat{m}_{r_2} = \bar{m}_{r_2} - 1$. Adding up Eqs. (60)-(63) we obtain

$$0 \ge 0 \tag{64}$$

This indicates that all the "≥" in Eqs. (60)-(63) should be "=". Hence, Eq. (61) indicates that the solution $(m_r = \bar{m}_r, r \in R_v \setminus \{r_1, r_2\}; m_{r_1} = \bar{m}_{r_1} + 1; m_{r_2} = \bar{m}_{r_2} - 1)$ is also optimal. Eqs. (62) and (63) imply that $\bar{m}_{r_1} = \hat{m}_{r_1} - 1$ and $\hat{m}_{r_2} = \bar{m}_{r_2} - 1$. Consequently, the solution $(m_r = \bar{m}_r, r \in R_v \setminus \{r_1, r_2\}; m_{r_1} = \hat{m}_{r_1}; m_{r_2} = \hat{m}_{r_2})$ is optimal. Repeating the above process, we can see that the solution $(\hat{m}_r, r \in R_v)$ is optimal. □

Appendix 4: Proof of Theorem 4

Suppose that there exists a ship route $r_1 \in R_v$ satisfying $\hat{m}_{r_1} > \tilde{m}_{r_1} \geq 1$. There must exists another ship route $r_2 \in R_v$ satisfying $\hat{m}_{r_2} < \tilde{m}_{r_2}$. Property (ii) of Theorem 1 indicates that

$$C_{r_1}(\hat{m}_{r_1} - 1) - C_{r_1}(\hat{m}_{r_1}) \leq 0 \tag{65}$$

$$C_{r_2}(\hat{m}_{r_2}) - C_{r_2}(\hat{m}_{r_2} + 1) > 0 \tag{66}$$

Note that in Eq. (65), "=" is true only if $C_{r_1}(\lfloor m^*_{r_1} \rfloor) = C_{r_1}(\lfloor m^*_{r_1} \rfloor + 1)$ and $\hat{m}_{r_1} = \lfloor m^*_{r_1} \rfloor + 1$. Eqs. (65) and (66) contradict the optimality condition in Theorem 3. □

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References

Alvarez, J.F., 2009. Joint routing and deployment of a fleet of container vessels. Maritime Economics & Logistics 11 (2), 186–208.

Alvarez, J.F., 2012. Mathematical expressions for the transit time of merchandise through a liner shipping network. Journal of the Operational Research Society 63 (6), 709–714.

Bell, M., Bichou, K., 2008. An analysis into speed and schedule stability for container liner services. Maritime Economics and Logistics 10 (1), 175–184.

Bell, M.G.H., Liu, X., Angeloudis, P., Fonzone, A., Hosseinloo, S.H., 2011. A frequency-based maritime container assignment model. Transportation Research Part B 45 (8), 1152–1161.

Boyd, S., Vandenberghe, L., 2004. Convex optimization. Cambridge University Press.

Brouer, B.D., Pisinger, D., Spoorendonk, S., 2011. Liner shipping cargo allocation with repositioning of empty containers. INFOR 49 (2), 109–124.

Brouer, B.D., Dirksen, J., Pisinger, D., Plum, C.E.M., Vaaben, B., 2013. The Vessel Schedule Recovery Problem (VSRP) - A MIP model for handling disruptions in liner shipping. European Journal of Operational Research 224 (2), 362–374.

Brouer, B., Alvarez, J. F., Plum, C., Pisinger, D., Sigurd, M., 2014. A base integer programming model and benchmark suite for linear shipping network design. Transportation Science 48 (2), 281–312.

Bunkerworld. Fuel prices at Rotterdam. http://www.bunkerworld.com/prices/port/nl/rtm/. Accessed 10 March 2015.

Christiansen, M., Fagerholt, K., Ronen, D., 2004. Ship routing and scheduling: Status and perspectives. Transportation Science 38 (1), 1–18.

Christiansen, M., Fagerholt, K., Nygreen, B., Ronen, D., 2013. Ship routing and scheduling in the new millennium. European Journal of Operational Research 228 (3), 467–478.

Du, Y., Chen, Q., Quan, X., Long, L., Fung, R.Y.K., 2011. Berth allocation considering fuel consumption and vessel emissions. Transportation Research Part E 47 (6), 1021–1037.

Fagerholt, K., Laporte, G., Norstad, I., 2010. Reducing fuel emissions by optimizing speed on shipping routes. Journal of the Operational Research Society 61 (3), 523–529.

Fox, B. L., Landi, D.M., 1970. Searching for the multiplier in one-constraint optimization problems. Operations Research 18(2), 253–262.

Gelareh, S., Meng, Q., 2010. A novel modeling approach for the fleet deployment problem within a short-term planning horizon. Transportation Research Part E 46 (1), 76–89.

Hvattum, L.M., Norstad, I., Fagerholt, K., Laporte, G., 2013. Analysis of an exact algorithm for the vessel speed optimization problem. Networks 62 (2), 132–135.

Karsten, C. V., Pisinger, D., Ropke, S., Brouer, B. D., 2015. The time constrained multi-commodity network flow problem and its application to liner shipping network design. Transportation Research Part E 76, 122–138.

Kim, H.J., 2014. A Lagrangian heuristic for determining the speed and bunkering port of a ship. Journal of the Operational Research Society 65(5), 747–754.

Kim, H.J., Chang, Y.T., Kim, K.T., Kim, H.J., 2012. An epsilon-optimal algorithm considering greenhouse gas emissions for the management of a ship's bunker fuel. Transportation Research Part D 17(2), 97–103.

Kim, J.G., Kim, H.J., Lee, P.T.W., 2013. Optimising containership speed and fleet size under a carbon tax and an emission trading scheme. International Journal of Shipping and Transport Logistics 5(6), 571–590.

Kim, J.G., Kim, H.J., Lee, P.T.W., 2014. Optimizing ship speed to minimize fuel consumption. Transportation Letters 6(3), 109–117.

Kontovas, C.A., Psaraftis, H.N., 2011. Reduction of emissions along the maritime intermodal container chain: operational models and policies. Maritime Policy and Management 38 (4), 451–469.

Meng, Q., Wang, T., 2010. A chance constrained programming model for short-term liner ship fleet planning problems. Maritime Policy and Management 37 (4), 329–346.

Mulder, J., Dekker, R., 2014. Methods for strategic liner shipping network design. European Journal of Operational Research, 235 (2), 367–377.

Ng, M.W., 2014. Distribution-free vessel deployment for liner shipping. European Journal of Operational Research 238 (3), 858–862.

Norstad, I., Fagerholt, K., Laporte, G., 2011. Tramp ship routing and scheduling with speed optimization. Transportation Research Part C 19 (5), 853–865.

Notteboom, T.E., 2006. The time factor in liner shipping services. Maritime Economics and Logistics 8 (1), 19–39.

OOCL. Service Routes. http://www.oocl.com/eng/ourservices/serviceroutes/tpt/Pages/default.aspx. Accessed 28 March 2013.

Plum, C.E.M., Pisinger, D., Sigurd, M.M., 2014. A service flow model for the liner shipping network design problem. European Journal of Operational Research 235 (2), 378–386.

Psaraftis, H.N., Kontovas, C.A., 2010. Balancing the economic and environmental performance of maritime transportation. Transportation Research Part D 15 (8), 458–462.

Psaraftis, H.N., Kontovas, C.A., 2013. Speed models for energy-efficient maritime transportation: a taxonomy and survey. Transportation Research Part C 26, 331–351.

Qi, X., Song, D.P., 2012. Minimizing fuel emissions by optimizing vessel schedules in liner shipping with uncertain port times. Transportation Research Part E 48 (4), 863–880.

Ronen, D., 2011. The effect of oil price on containership speed and fleet size. Journal of the Operational Research Society 62 (1), 211–216.

UNCTAD. Review of Maritime Transportation 2011. Paper presented at the United Nations Conference on Trade and Development. New York and Geneva. http://unctad.org/en/docs/rmt2011_en.pdf. Accessed March 25, 2013.

UNCTAD. Review of Maritime Transportation 2012. Paper presented at the United Nations Conference on Trade and Development. New York and Geneva. http://unctad.org/en/PublicationsLibrary/rmt2012_en.pdf

UNCTAD, 2013. Review of Maritime Transportation: Paper presented at the United Nations Conference on Trade and Development. New York and Geneva. http://unctad.org/en/publicationslibrary/rmt2013_en.pdf. Accessed on 4 June 2014.

Wang, S., Liu, Z., Bell, M.G.H., 2015. Profit-based maritime container assignment models for liner shipping networks. Transportation Research Part B (72), 59–76.

Wang, S., Meng, Q., 2012. Sailing speed optimization for container ships in a liner shipping network. Transportation Research Part E 48 (3), 701–714.

Yao, Z., Ng, S.H., Lee, L.H., 2012. A study on bunker fuel management for the shipping liner services. Computers & Operations Research 39 (5), 1160–1172.

Zhen, L., Chew, E.P., Lee, L.H., 2011a. An integrated model for berth template and yard template planning in transshipment hubs. Transportation Science 45 (4), 483–504.

Zhen, L., Lee, L.H., Chew, E.P., 2011b. A decision model for berth allocation under uncertainty. European Journal of Operational Research 212 (1), 54–68.

Zheng, J., Sun, Z., Meng, Q., 2014. Impact analysis of maritime cabotage legislations on liner hub-and-spoke shipping network design. European Journal of Operational Research 223 (3), 874–884.