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Polynomial-Time Solvability of Dynamic Lot Size Problems

Chung-Lun Li
Department of Logistics and Maritime Studies
The Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
Email: chung-lun.li@polyu.edu.hk

Qingying Li
Glorious Sun School of Business and Management
Donghua University
1882 West Yan'an Road, Shanghai, China, 200051
Email: liqingying@dhu.edu.cn

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Abstract

There has been a lot of research on dynamic lot sizing problems with different nonlinear cost structures due to capacitated production, minimum order quantity requirements, availability of quantity discounts, etc. Developing optimal solutions efficiently for dynamic lot sizing models with nonlinear cost functions is a challenging topic. In this paper we present a set of sufficient conditions such that if a single-item dynamic lot sizing problem satisfies these conditions, then the existence of a polynomial-time solution method for the problem is guaranteed. Several examples are presented to demonstrate the use of these sufficient conditions.

Keywords: Inventory; dynamic lot sizing; dynamic programming; polynomial-time algorithms

1 Introduction

Dynamic lot sizing (DLS) models are important inventory management tools for handling time-varying demands. They have been used extensively in production planning systems. However, in many real-life applications of DLS, the replenishment cost function has a nonlinear structure, which can make the analysis of the DLS model highly challenging. For example, product prices with quantity discounts are quite common in practice. These include all-units discounts, incremental discounts, truckload discounts, and other discount schemes. These price discounts make the replenishment cost function of the DLS model quite complicated, which in turn increases the complexity of the model. The cost function becomes even more complicated when there are additional requirements such as minimum order quantity (MOQ) constraints. Another example is when the production cost is concave (due to economy of scale of production) but the production capacity is limited. Combining a concave production cost with a finite capacity results in a complicated cost structure. In some DLS applications, the inventory holding cost function is also nonlinear. Hence, DLS models with a nonlinear cost structure cover a broad range of applications.

Since Wagner and Whitin (1958) introduced the DLS model in their seminal paper, many optimal and heuristic algorithms have been developed for different variants of the model. These variants cover single versus multiple items, single versus multiple stages of production, stationary versus time-varying demands, backlogging versus no shortages, uncapacitated versus capacitated production, as well as other additional features such as lost sales, resales, bounded inventory, demand time windows, etc. See Brahimi *et al.* (2006), Jans and Degraeve (2007, 2008), Buschkühl *et al.* (2010), and Suwondo and Yuliando (2012) for recent reviews on various DLS problems and solution approaches. Among these DLS problems, a substantial amount of research has been devoted to single-stage, single-item problems of the following form:

$$
\begin{aligned}
\mathbf{P}: \quad & \text{minimize} \quad && \sum_{i=1}^{n} \big[P_i(X_i) + H_i(I_i)\big] \\
& \text{subject to} \quad && I_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, n) \\
& && I_0 = \bar{I}_0 \\
& && I_n \le M \\
& && X_i \ge 0 \quad (i = 1, 2, \ldots, n)
\end{aligned}
$$

where

n = the number of time periods in the planning horizon;

d_i = the given demand in period i ($d_i \geq 0$);

X_i = the replenishment quantity in period i (X_i is a decision variable);

I_i = the ending inventory of period i (I_i is a decision variable);

$P_i(X_i)$ = replenishment cost in period i ($P_i(X_i) \geq 0$);

$H_i(I_i)$ = cost of holding I_i units of inventory from period i to period $i+1$ ($H_i(I_i) \geq 0$);

$\bar{I}_0$ = a given initial inventory level at the beginning of period 1; and

M = a given upper bound on the inventory level at the end of period n. Specifically, a lot of research has been conducted for problem **P** with different replenishment cost functions $P_i(\cdot)$ and holding cost functions $H_i(\cdot)$. Depending on these cost functions, some problems are computationally intractable (i.e., NP-hard), while some of them are solvable in polynomial time.

Note that in problem **P**, functions $P_i(\cdot)$ and $H_i(\cdot)$ are not required to be monotone, concave, or convex, and variable I_i is not required to be nonnegative. Note also that in problem **P**, we may disallow a certain replenishment quantity X_i by setting $P_i(X_i) = +\infty$. Hence, production capacity constraints may be represented by setting $P_i(X_i) = +\infty$ whenever X_i exceeds the production capacity. Similarly, we may disallow a certain inventory level I_i by setting $H_i(I_i) = +\infty$. Problems with backlogging are represented by setting $H_i(I_i)$ equal to the backordering cost whenever $I_i < 0$. Problems with no backlogging allowed are represented by setting $H_i(I_i) = +\infty$ whenever $I_i < 0$.

Traditionally, it is quite common that a custom-made solution method is developed for each particular DLS problem. This is particularly true for those problems that are solvable optimally in polynomial time with a high order of complexity, where custom-made algorithms comprising sophisticated techniques (e.g., dynamic programming with sophisticated data structures and/or complicated computational procedures) are often employed to lower the computational burden. It is also true when the researcher wishes to show that a DLS problem is polynomial-time solvable. Unlike the mainstream DLS literature, in this paper we present a set of sufficient conditions, such that if problem **P** satisfies these conditions, the existence of a polynomial-time solution method for the problem is guaranteed. These sufficient conditions enable us to identify the polynomial-solvability of DLS problems easily.

Problem **P** covers many DLS models studied by other researchers. For example, the special case where the holding cost function is linear and the replenishment cost function covers a fixed setup cost and a linear production cost is the classical DLS model. In this classical model, $H_i(I_i) = h_i I_i$

if $I_i \geq 0$, $P_i(X_i) = s_i + p_i X_i$ if $X_i > 0$, and $P_i(X_i) = 0$ if $X_i = 0$, for $i = 1, 2, \ldots, n$, where h_i is the unit holding cost, p_i is the unit production cost, and s_i is the fixed setup cost in period i. Highly efficient polynomial-time algorithms have been developed for this special case (and its variant with backlogging allowed); see Federgruen and Tzur (1991), Wagelmans *et al.* (1992), and Aggarwal and Park (1993). Extensions of this classical DLS model with concave costs have also been studied by Wagner (1960) and Zangwill (1966, 1969).

Special cases with fixed setup costs, linear/concave holding and production costs, and production capacity have been studied. Polynomial-time algorithms have been proposed by Florian and Klein (1971), Van Hoesel and Wagelmans (1996), Ou (2012), and Piñeyro *et al.* (2013) for different variants of the problem when the production capacity is stationary. Special cases with fixed setup costs, linear/concave holding and production costs, and bounded inventory have also been studied. Polynomial-time algorithms have been proposed by Love (1973), Gutiérrez *et al.* (2002, 2007), Atamtürk and Küçükyavuz (2008), Liu (2008), Van den Heuvel and Wagelmans (2008), and Hwang and Van den Heuvel (2012).

MOQ requirements can be viewed as a special case of the general replenishment cost function $P_i(\cdot)$, where $P_i(X_i) = +\infty$ when replenishment quantity X_i is below the MOQ. Thus, many DLS models with MOQ requirements can be viewed as special cases of problem $\mathbf{P}$. Capacitated and uncapacitated DLS problems with MOQ and various cost functions have been studied by Lee (2004), Hwang (2010), Okhrin and Richter (2011a, 2011b), and Hellion *et al.* (2012, 2013).

Special cases with linear holding costs, fixed setup costs, and production costs with various quantity discount patterns have been studied extensively. These include models with all-units discount (Chung *et al.* 1987; Sohn and Hwang 1987; Federgruen and Lee 1990; Xu and Lu 1998; Chan *et al.* 2002; Li *et al.* 2012; Mirmohammadi and Eshghi 2012; Archetti *et al.* 2014), models with incremental discount (Federgruen and Lee 1990; Archetti *et al.* 2014), models with truckload discount (Li *et al.* 2004), and models with batch ordering cost (or stepwise production cost) structure (Vander Eecken 1968; Lippman 1969; Elmaghraby and Bawle 1972; Pochet and Wolsey 1993; Lee 1989; Li *et al.* 2004; Van Vyve 2007; Akbalik and Pochet 2009; Akbalik and Rapine 2012, 2013). A model with general piecewise concave replenishment and holding costs has been studied by Koca *et al.* (2014). A model with capacity reservation has been studied by Lee and Li (2013).

Problem $\mathbf{P}$ is known to be NP-hard for many special cases; see, for example, Florian *et al.* (1980) and Bitran and Yanasse (1982). Pseudo-polynomial time algorithms have been developed for some cases; see, for example, Swoveland (1975) and Shaw and Wagelmans (1998). Algorithms without polynomial time bounds have also been developed for some cases; see, for example, Chen *et al.* (1994). Fully polynomial time approximation schemes have been developed for the case with general monotone replenishment and holding cost functions; see, for example, Van Hoesel and Wagelmans (2001), Chubanov *et al.* (2006), Ng *et al.* (2010), Chubanov and Pesch (2012), and Halman *et al.* (2012).

Some of the abovementioned models satisfy our sufficient conditions, while some of them do not. Table 1 summarizes various known DLS models with nonlinear costs that satisfy our conditions (note: this table does not include those works on the classical DLS model with "fixed plus linear" replenishment costs and linear holding/backlogging costs).

In Section 2, we present the set of sufficient conditions mentioned above. We provide a framework for constructing a polynomial-time algorithm for those problems which satisfy these conditions. In Section 3, we present several examples to demonstrate the use of these sufficient conditions. Some concluding remarks are provided in Section 4.

2 Optimality Conditions

For simplicity, we assume that in problem **P** the initial inventory at the beginning of period 1 is zero and that the ending inventory at the end of period n is required to be zero. This assumption is made without loss of generality, as we can transform problem **P** into the following problem:

$$\begin{array}{lll} \mathbf{P}': & \text{minimize} & \displaystyle\sum_{i=1}^{n} \big[P_i(X_i) + H_i(I_i)\big] \\ & \text{subject to} & I_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, n) \\ & & I_0 = I_n = 0 \\ & & X_i \geq 0 \quad (i = 1, 2, \ldots, n) \end{array}$$

The transformation can be accomplished as follows: Consider any given instance of problem **P** with $\tilde{n}$ periods. If $\bar{I}_0 \geq 0$, then we add a dummy period 0 before period 1 with $d_0 = 0$,

$$P_0(X) = \begin{cases} 0, & \text{if } X = \bar{I}_0; \\ +\infty, & \text{if } X \neq \bar{I}_0; \end{cases}$$

Table 1: DLS models with nonlinear costs which fit into our framework.

Reference(s)	**Replenishment cost function**	**Holding & backordering cost functions**	**Running time of solution method(s)**
Wagner (1960)	Concave	Concave; no backlogging	$O(n^2)$
Zangwill (1966)	Concave	Concave; backlogging allowed	$O(n^3)$
Zangwill (1969)	Fixed plus linear cost	Concave; backlogging allowed	$O(n^2)$
Florian and Klein (1971)	Concave cost with constant capacities	Concave; backlogging allowed	$O(n^4)$
Bitran and Yanasse (1982)	Fixed plus linear cost; nonincreasing setup and unit production costs; constant capacities	Linear; no backlogging	$O(n^3)$
Pochet and Wolsey (1993)	Fixed plus linear cost with constant capacities C	Linear; no backlogging	$O(n^2 \min\{n, C\})$
Van Hoesel and Wagelmans (1996)	Concave cost with constant capacities	Linear; no backlogging	$O(n^3)$
Ou (2012)	Fixed plus linear cost with constant capacities	Concave; backlogging allowed	$O(n^3)$
Piñeyro *et al.* (2013)	Concave cost with constant capacities	Concave cost with non-speculative motives; no backlogging	$O(n^3)$
Hwang (2010)	Concave cost with MOQ requirement	Concave; backlogging allowed	$O(n^5)$
Okhrin and Richter (2011a)	Fixed cost; constant unit production cost; MOQ requirement; constant capacities	Linear; no backlogging	$O(n^3)$
Okhrin and Richter (2011b)	Fixed cost; constant unit production cost; MOQ requirement	Linear; no backlogging	$O(n^2)$
Hellion *et al.* (2012, 2013)	Concave cost with MOQ requirement and constant capacities	Concave; no backlogging	$O(n^6)$
Federgruen and Lee (1990); Xu and Lu (1998)	Fixed plus linear cost with all-units discount (single price breakpoint); nonincreasing product costs	Linear; no backlogging	$O(n^3)$
Sohn and Hwang (1987)	Fixed plus linear cost with all-units discount and resales (single price breakpoint q)	Linear; no backlogging	$O(n^3(\sum d_i/q)^2)$
Mirmohammadi and Eshghi (2012)	Fixed plus linear cost with all-units discount (single price breakpoint)	Linear; no backlogging	$O(n^4)$
Li *et al.* (2012)	Fixed plus linear cost with all-units discount and resales (single price breakpoint)	Linear; no backlogging	$O(n^2)$
Li *et al.* (2012)	Fixed plus linear cost with all-units discount and resales (m price breakpoints)	Linear; no backlogging	$O(n^{m+3})$
Federgruen and Lee (1990)	Fixed plus linear cost with incremental discount; nonincreasing setup and unit production costs	Linear; no backlogging	$O(n^2)$
Archetti *et al.* (2014)	Fixed plus linear cost with incremental discount	Linear; no backlogging	$O(n^2)$
Koca *et al.* (2014)	Piecewise concave with m breakpoints	Concave; backlogging allowed	$O(n^{2m+3})$
Lee and Li (2013)	Fixed plus linear cost with capacity reservation and constant capacities	Linear; no backlogging	$O(n^4)$

and $H_0(X) = 0$ for any X. If $\bar{I}_0 < 0$, then we add a dummy period 0 before period 1 with $d_0 = -\bar{I}_0$,

$$P_0(X) = \begin{cases} 0, & \text{if } X = 0; \\ +\infty, & \text{if } X \neq 0; \end{cases}$$

and $H_0(X) = 0$ for any X. We also add a dummy period $\tilde{n}+1$ after period $\tilde{n}$ with $P_{\tilde{n}+1}(X) = H_{\tilde{n}+1}(X) = 0$ for any X and $d_{\tilde{n}+1} = M$. It is easy to see that in an optimal solution of the transformed problem, the beginning inventory level of period 1 must be $\bar{I}_0$, and the ending inventory level of period $\tilde{n}$ is allowed to be any value no greater than M. Thus, any optimal solution to the transformed problem is also optimal to problem $\mathbf{P}$ if we remove the replenishments in the two dummy periods. Note that this transformation does not affect the polynomial-time solvability of the problem. Hence, in the following we focus on analyzing problem $\mathbf{P}'$.

We first introduce a few definitions. A period i is said to be a *regeneration period* if $I_i = 0$. Hence, periods 0 and n of problem $\mathbf{P}'$ are regeneration periods. A period i is said to be a *replenishment period* if $X_i > 0$. Let $\{Y_1, Y_2, \ldots, Y_\ell\}$ be a set of *special replenishment quantities*, which are time-independent. Denote $Y_0 = 0$. A period is said to be a *regular period* if the replenishment quantity in that period is equal to Y_k for some $k = 0, 1, \ldots, \ell$. Otherwise, it is said to be an *irregular period*. A regular period with a replenishment quantity Y_k is called a *type-k regular period*. Thus, type-0 regular periods are non-replenishment periods.

Next, we introduce some conditions:

Condition 1: Functions $P_i(\cdot)$ and $H_i(\cdot)$, $i = 1, 2, \ldots, n$, can be evaluated in constant time.

Condition 2: There exists an optimal solution to $\mathbf{P}'$ in which for any two consecutive regeneration periods u and v $(u < v)$, at most one of the periods $u+1, u+2, \ldots, v$ is an irregular period.

Condition 3: The number of special replenishment quantities, ℓ, is fixed (i.e., a constant).

Condition 2 is a property which is often seen in DLS research, where irregular periods, depending on the context, are sometimes called "fractional production periods" (Van Hoesel and Wagelmans 1996) or "LTL replenishment periods" (Li *et al.* 2004). In the following, we show that a polynomial-time algorithm exists for problem $\mathbf{P}'$ when Conditions 1–3 are satisfied. To do so, we define Ψ_u as the optimal total cost to satisfy the demand in periods $u+1, u+2, \ldots, n$, given that period u is a regeneration period. For $0 \leq u < v \leq n$, we define $\psi_{u+1,v}$ as the optimal total cost to satisfy the demand in periods $u+1, u+2, \ldots, v$, given that periods u and v are (unnecessarily consecutive) regeneration periods and that the number of irregular periods in $\{u+1, u+2, \ldots, v\}$ is at most one. Clearly, the following dynamic program solves problem $\mathbf{P}'$ optimally:

(I) Recurrence relation: $\Psi_u = \min_{u<v\leq n}\{\psi_{u+1,v} + \Psi_v\}$ for $u = 0, 1, \ldots, n$.

(II) Boundary condition: $\Psi_n = 0$.

(III) Objective: Ψ_0.

The running time of this dynamic program is $O(n^2)$ if all the $\psi_{u+1,v}$ values have been determined. Therefore, it suffices to show that $\psi_{u+1,v}$ can be obtained in polynomial time. We will use the following notation: Let Z^+ denote the set of all nonnegative integers. Let $D_{i,j} = \sum_{k=i}^{j} d_k$ denote the cumulative demand in periods $i, i+1, \ldots, j$.

In the computation of the $\psi_{u+1,v}$ value, we consider the replenishment quantities in periods $u+1, u+2, \ldots, v$. According to Condition 2, among these periods, besides one period which we refer to as a *special period*, all the other $v-u-1$ periods are regular periods. The special period can be a regular period or an irregular period. In case there is no irregular period among periods $u+1, u+2, \ldots, v$, then the special period must be a regular period, and in such a case, we are free to select any one of these $v-u$ periods and refer to it as the special period.

For $0 \leq u < v \leq n$, and for $N_0, N_1, \ldots, N_\ell \in Z^+$ such that $\sum_{k=0}^{\ell} N_k = v-u-1$ and $\sum_{k=0}^{\ell} N_k Y_k \leq D_{u+1,v}$, define $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)$ as the optimal total cost to satisfy the demand in periods $u+1, u+2, \ldots, v$, given that: (i) periods u and v are (unnecessarily consecutive) regeneration periods; (ii) the set $\{u+1, u+2, \ldots, v\}$ includes one special period, say, period t; and (iii) among the periods in $\{u+1, u+2, \ldots, v\} \setminus \{t\}$, N_k of them are type-k regular periods ($k = 0, 1, \ldots, \ell$). Clearly,

$$\psi_{u+1,v} = \min_{N_0,N_1,\ldots,N_\ell \in Z^+ \text{ s.t. } \sum_{k=0}^{\ell} N_k = v-u-1 \text{ and } \sum_{k=0}^{\ell} N_k Y_k \leq D_{u+1,v}} \Big\{\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)\Big\}. \tag{1}$$

Thus, we focus on determining the value of $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)$ for any given $u, v, N_0, N_1, \ldots, N_\ell$ such that $\sum_{k=0}^{\ell} N_k = v-u-1$ and $\sum_{k=0}^{\ell} N_k Y_k \leq D_{u+1,v}$. Here, the technique of computing the $\psi_{u+1,v}$ value via evaluating all possible combinations of $(N_0, N_1, \ldots, N_\ell)$ is adopted from Li *et al.* (2012).

Consider any given $i = u+1, u+2, \ldots, v$. There are either 0 or 1 special periods among periods $i, i+1, \ldots, v$, and the rest are all regular periods. For $k = 0, 1, \ldots, \ell$, we let n_k denote the number of type-k regular period among the non-special periods in $\{i, i+1, \ldots, v\}$. Then,

$$v - i \leq \sum_{k=0}^{\ell} n_k \leq v - i + 1.$$

Note that $v-i+1-\sum_{k=0}^{\ell} n_k$, which is the number of special periods among periods $i, i+1, \ldots, v$, is equal to either 0 or 1. By using the state $(n_0, n_1, \ldots, n_\ell)$ to keep track of the current status, we develop the following dynamic program for calculating $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)$. Here, the state $(n_0, n_1, \ldots, n_\ell)$ represents the scenario where $N_k - n_k$ non-special type-k regular periods have been assigned to periods $u+1, u+2, \ldots, i-1$, so that there are n_k type-k regular periods left for those non-special periods in $\{i, i+1, \ldots, v\}$, for $k = 0, 1, \ldots, \ell$.

For $i = u+1, u+2, \ldots, v$ and $n_k = 0, 1, \ldots, N_k$ (for $k = 0, 1, \ldots, \ell$) such that $v-i \le \sum_{k=0}^{\ell} n_k \le v-i+1$, define $f_{u+1,v,i}(n_0, n_1, \ldots, n_\ell)$ as the optimal total cost to satisfy the demand in periods $i, i+1, \ldots, v$, given that: (i) periods u and v are (unnecessarily consecutive) regeneration periods; (ii) among those non-special periods within $\{i, i+1, \ldots, v\}$, n_k of them are type-k regular periods $(k = 0, 1, \ldots, \ell)$; and (iii) there are $v-i+1-\sum_{k=0}^{\ell} n_k$ special periods within $\{i, i+1, \ldots, v\}$.

(I) Recurrence relation: For $i = u+1, u+2, \ldots, v$ and $n_k = 0, 1, \ldots, N_k$ $(k = 0, 1, \ldots, \ell)$ such that $v-i \le \sum_{k=0}^{\ell} n_k \le v-i+1$,

$$
\begin{aligned}
&f_{u+1,v,i}(n_0, n_1, \ldots, n_\ell) \\
&= \begin{cases}
\min\Big\{ P_i(D_{u+1,v} - \sum_{k=0}^{\ell} N_k Y_k) + H_i(D_{i+1,v} - \sum_{k=0}^{\ell} n_k Y_k) + f_{u+1,v,i+1}(n_0, n_1, \ldots, n_\ell), \\
\quad \min\limits_{k=0,1,\ldots,\ell} \Big\{ P_i(Y_k) + H_i\big(\sum_{j=0}^{\ell}(N_j - n_j)Y_j + Y_k - D_{u+1,i}\big) + f_{u+1,v,i+1}(n_0, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_\ell) \Big\} \Big\}, \\
\hfill \text{if } \sum_{k=0}^{\ell} n_k = v-i; \\
\min\limits_{k=0,1,\ldots,\ell} \Big\{ P_i(Y_k) + H_i(D_{i+1,v} - \sum_{j=0}^{\ell} n_j Y_j + Y_k) + f_{u+1,v,i+1}(n_0, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_\ell) \Big\}, \\
\hfill \text{if } \sum_{k=0}^{\ell} n_k = v-i+1.
\end{cases}
\end{aligned}
$$

(II) Boundary conditions:

$$
f_{u+1,v,v+1}(n_0, n_1, \ldots, n_\ell) = \begin{cases} 0, & \text{if } (n_0, n_1, \ldots, n_\ell) = (0, 0, \ldots, 0); \\ +\infty, & \text{otherwise}; \end{cases}
$$

$f_{u+1,v,i}(n_0, n_1, \ldots, n_\ell) = +\infty$ if $n_k < 0$ for some $k = 0, 1, \ldots, \ell$ (for $i = u+1, u+2, \ldots, v$).

(III) Objective: $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell) = f_{u+1,v,u+1}(N_0, N_1, \ldots, N_\ell)$.

The recurrence relation is divided into two cases, depending on whether a special period is assigned to one of the periods $i, i+1, \ldots, v$ or not.

In the first case, $\sum_{k=0}^{\ell} n_k = v - i$. Then, $v - i + 1 - \sum_{k=0}^{\ell} n_k = 1$, and therefore a special period is to be assigned to one of the periods $i, i+1, \ldots, v$. In this case, we need to decide if we should assign the special period to period i.

First, suppose we assign the special period to period i (which is the first term of the minimization). Then, the replenishment quantity in period i must be $D_{u+1,v} - \sum_{k=0}^{\ell} N_k Y_k$. The ending inventory level of period i is equal to $D_{i+1,v} - \sum_{k=0}^{\ell} n_k Y_k$, because the total demand in periods $i+1, i+2, \ldots, v$ exceeds the total replenishment quantity in those periods by this quantity. Hence, the holding cost incurred in period i is $H_i(D_{i+1,v} - \sum_{k=0}^{\ell} n_k Y_k)$. Also, because the special period is assigned to period i, the minimum possible total cost incurred in periods $i+1, i+2, \ldots, v$ is $f_{u+1,v,i+1}(n_0, n_1, \ldots, n_\ell)$.

Next, suppose we do not assign the special period to period i (which is the second term of the minimization). Then, we need to decide if the replenishment quantity in period i should be $Y_0, Y_1, \ldots$, or Y_ℓ. If we select an replenishment quantity of Y_k, then the replenishment cost incurred in period i is $P_i(Y_k)$. Furthermore, the ending inventory level of period i is $\sum_{j=0}^{\ell}(N_j - n_j)Y_j + Y_k - D_{u+1,i}$, because the total replenishment quantity in periods $u+1, u+2, \ldots, i$ is $\sum_{j=0}^{\ell}(N_j - n_j)Y_j + Y_k$ and the total demand in these periods is $D_{u+1,i}$. Hence, in this case, the holding cost incurred in period i is $H_i\big(\sum_{j=0}^{\ell}(N_j - n_j)Y_j + Y_k - D_{u+1,i}\big)$, and the minimum possible total cost incurred in periods $i+1, i+2, \ldots, v$ is $f_{u+1,v,i+1}(n_0, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_\ell)$.

In the second case, $\sum_{k=0}^{\ell} n_k = v - i + 1$. Then, $v - i + 1 - \sum_{k=0}^{\ell} n_k = 0$, and therefore no special period is assigned to periods $i, i+1, \ldots, v$. In such a case, we just need to decide the replenishment quantity for period i, which is equal to either $Y_0, Y_1, \ldots$, or Y_ℓ. If we select a replenishment quantity of Y_k, then the replenishment cost incurred in period i is $P_i(Y_k)$, and the ending inventory level of period i is $D_{i+1,v} - \sum_{j=0}^{\ell} n_j Y_j + Y_k$, because the total demand in periods $i+1, u+2, \ldots, v$ is $D_{i+1,v}$, while the total replenishment quantity in these periods is $\sum_{j=0}^{\ell} n_j Y_j - Y_k$. Also, the minimum possible total cost incurred in periods $i+1, i+2, \ldots, v$ is $f_{u+1,v,i+1}(n_0, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_\ell)$.

Note that in the above dynamic programming formulation, a state $(n_0, n_1, \ldots, n_\ell)$ is used to keep track of the current status. This enables us to attain a polynomial running time when ℓ is fixed, that is, when Condition 3 is satisfied. To analyze the computational complexity of the above solution method, we first note that the values in $\{D_{i,j} \mid 1 \leq i, j \leq n\}$ can be pre-computed in $O(n^2)$ time.

Suppose we have pre-computed all of the $D_{i,j}$ values. Then, consider the above dynamic program for calculating $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)$. (i) There are $O(n)$ stages; (ii) in each stage there are $O(n)$ possible values of each n_k, for $k = 0, 1, \ldots, \ell - 1$; (iii) if $n_0, n_1, \ldots, n_{\ell-1}$ have been chosen, then there are only two possible values of n_ℓ; and (iv) it requires a constant time to evaluate each $f_{u+1,v,i}(n_0, n_1, \ldots, n_\ell)$ in the recurrence relation (by Conditions 1 and 3). Thus, this dynamic program can be executed in $O(n^{\ell+1})$ time. In other words, given $u, v, N_0, N_1, \ldots, N_\ell$, the value of $\psi'_{u+1,v}(N_0, N_1, \ldots, N_\ell)$ can be determined in $O(n^{\ell+1})$ time. Note also that, given u and v, the value of $\psi_{u+1,v}$ can be determined from equation (1), where there are $O(n^\ell)$ possible values of $(N_0, N_1, \ldots, N_\ell)$ such that $\sum_{k=0}^{\ell} N_k = v - u - 1$. Hence, the value of $\psi_{u+1,v}$ can be determined in $O(n^{\ell+1} \cdot n^\ell) = O(n^{2\ell+1})$ time. Therefore, predetermining all $\psi_{u+1,v}$ values can be done in $O(n^{2\ell+1} \cdot n^2) = O(n^{2\ell+3})$ time, which implies that the overall running time needed for solving problem $\mathbf{P}'$ via this method is $O(n^{2\ell+3})$.

Summarizing the above analysis, we have the following theorem, which implies that problem $\mathbf{P}$ is solvable in polynomial time.

Theorem 1 *If Conditions 1–3 are satisfied, then an $O(n^{2\ell+3})$ time algorithm can be constructed for problem $\mathbf{P}$.*

3 Examples of Problems Satisfying Conditions 1–3

In this section, we demonstrate the sufficient conditions developed in Section 2 with several examples.

3.1 Example 1: DLS with Production Capacity and Backlogging

The first example is a capacitated DLS problem with fixed setup costs, linear production costs, production capacities, backlogging allowed, and concave holding/backordering costs. Such a capacitated DLS problem can be viewed as a special case of problem $\mathbf{P}$ with replenishment cost

$$P_i(X_i) = \begin{cases} 0, & \text{if } X_i = 0; \\ s_i + p_i X_i, & \text{if } 0 < X_i \le C_i; \\ +\infty, & \text{if } X_i > C_i; \end{cases}$$

where s_i is a fixed setup cost of production, p_i is the unit production cost, and C_i is a time-varying production capacity (see Figure 1(a)). In this problem, $H_i(0) = 0$ and $H_i(\cdot)$ is concave over intervals

$(-\infty, 0]$ and $[0, \infty)$ (see Figure 1(b)). This problem is known to be NP-hard in general (Florian *et al.* 1980; Bitran and Yanasse 1982), and the special case with stationary production capacity can be solved in $O(n^3)$ time (Ou 2012). This problem has the property that there always exists an optimal solution such that there is at most one fractional production period between any two consecutive regeneration periods, where a fractional production period i is a period in which $0 < X_i < C_i$ (see Florian and Klein 1971). Thus, Condition 2 is satisfied.

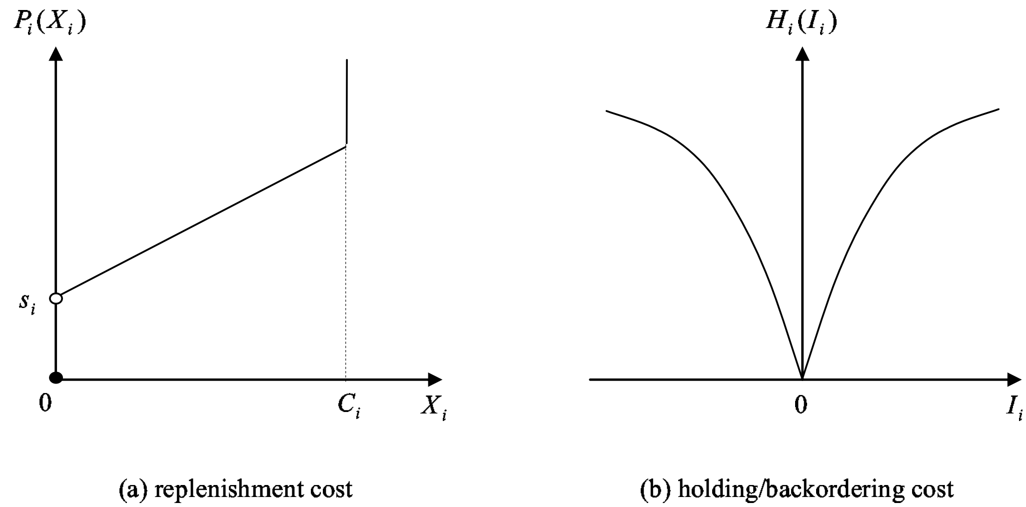


Figure 1: DLS with production capacity and backlogging.

Consider the case in which the number of distinct values of $C_1, C_2, \ldots, C_n$ is fixed. Let $C'_1, C'_2, \ldots, C'_m$ be the distinct values of $C_1, C_2, \ldots, C_n$, where m is a fixed value. Let $\{C'_1, C'_2, \ldots, C'_m\}$ be the set of special replenishment quantities. Then, Condition 3 is satisfied with $\ell = m$. If the functions $H_i(\cdot)$, $i = 1, 2, \ldots, n$, can be evaluated in constant time, then Condition 1 is also satisfied and, by Theorem 1, the problem is solvable in $O(n^{2\ell+3})$ time. This also implies that the special case with stationary production capacity is solvable in $O(n^5)$ time.

3.2 Example 2: Capacitated DLS with MOQ Requirement

The next example is a capacitated DLS problem with concave production and holding costs, stationary production capacity, no backlogging, and an MOQ requirement. The problem can be formulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} \big[p_i(X_i) + h_i(I_i)\big] \\
\text{subject to} \quad & I_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, n) \\
& LY_i \le X_i \le UY_i \quad (i = 1, 2, \ldots, n) \\
& I_0 = 0 \\
& I_i \ge 0 \quad (i = 1, 2, \ldots, n) \\
& Y_i \in \{0, 1\} \quad (i = 1, 2, \ldots, n)
\end{aligned}
$$

where $p_i(\cdot)$ and $h_i(\cdot)$ are concave functions, $L > 0$ is the MOQ, and $U > 0$ is the production capacity. In this formulation, the concave replenishment cost function $p_i(\cdot)$ may include a setup cost. In some applications, the setup cost is ignored, as the MOQ already prohibits the replenishment quantities to go below a certain level (see Okhrin and Richter 2011a, 2011b). Hellion *et al.* (2012, 2013) have developed an $O(n^6)$ algorithm for this problem.

This problem can be viewed as a special case of problem **P** with replenishment cost

$$
P_i(X_i) = \begin{cases} p_i(X_i), & \text{if } L \le X_i \le U; \\ +\infty, & \text{if } X_i < L \text{ or } X_i > U; \end{cases}
$$

(see Figure 2) and inventory holding cost

$$
H_i(I_i) = \begin{cases} h_i(I_i), & \text{if } I_i \ge 0; \\ +\infty, & \text{if } I_i < 0; \end{cases}
$$

The problem has a property that there always exists an optimal solution such that there is at most one fractional production period between any two consecutive regeneration periods, where a fractional production period i is a period in which $L < X_i < U$ (see Property 2 in Hellion *et al.* 2012).

Let $\{L, U\}$ be the set of special replenishment quantities. Then, Condition 2 is satisfied. The number of special replenishment quantities, ℓ, equals 2. Hence, Condition 3 is satisfied. If the functions $p_i(\cdot)$ and $h_i(\cdot)$, $i = 1, 2, \ldots, n$, can be evaluated in constant time, then Condition 1 is also satisfied and, by Theorem 1, the problem is solvable in $O(n^7)$ time.

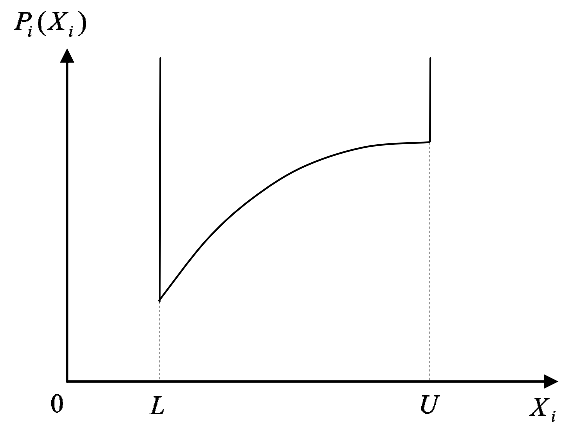


Figure 2: Capacited DLS with MOQ requirement.

3.3 Example 3: DLS with Piecewise Concave Production Costs

The following DLS model is a generalization of the problem presented in Section 3.2:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} \left[P_i(X_i) + H_i(I_i) \right] \\
\text{subject to} \quad & I_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, n) \\
& I_0 = 0 \\
& X_i, I_i \geq 0 \quad (i = 1, 2, \ldots, n)
\end{aligned}
$$

where $P_i(\cdot)$ is a piecewise concave function (see Figure 3) and $H_i(\cdot)$ is a concave function over the

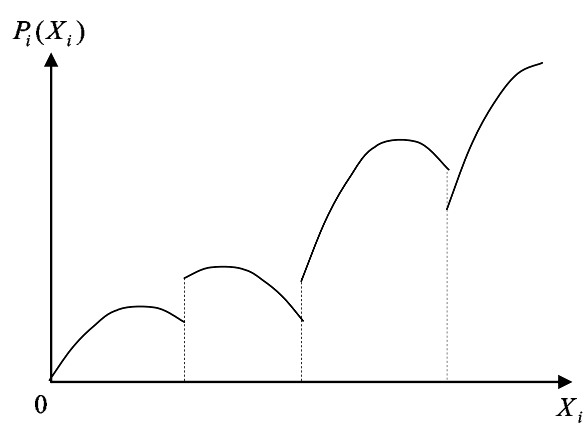


Figure 3: DLS with piecewise concave production costs.

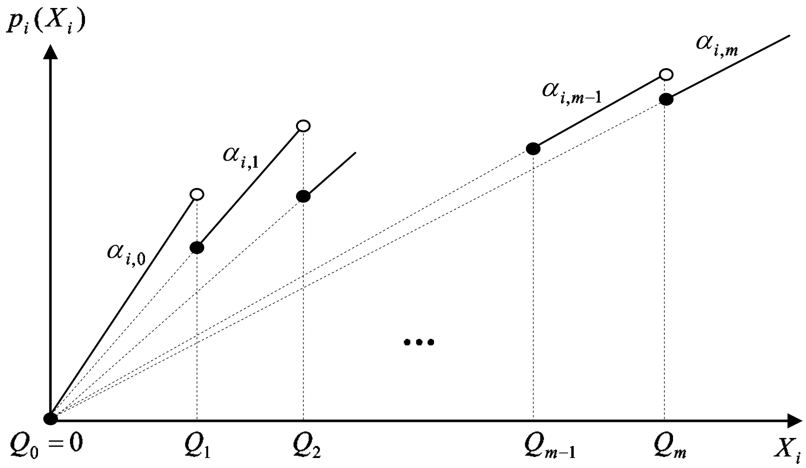
interval $[0, \infty)$. Koca *et al.* (2014) have developed an $O(n^{2m+3})$ algorithm for the case where the breakpoints of the production cost function are stationary and the number of breakpoints, m, is fixed.

This problem has a property that there always exists an optimal solution such that there is at most one fractional production period between any two consecutive regeneration periods, where a fractional production period i is a period in which X_i is not at a breakpoint of the curve $P_i(\cdot)$ (see Theorem 1 in Koca *et al.* 2014). Let the distinct breakpoints in the production cost functions $P_1(\cdot), P_2(\cdot), \ldots, P_n(\cdot)$ be the special replenishment quantities. Then, Condition 2 is satisfied. If the total number of distinct breakpoints in the production cost functions is fixed to m, then Condition 3 is satisfied with $\ell = m$. If, in addition, the functions $P_i(\cdot)$ and $H_i(\cdot)$, $i = 1, 2, \ldots, n$, can be evaluated in constant time, then Condition 1 is also satisfied and, by Theorem 1, the problem is solvable in $O(n^{2m+3})$ time.

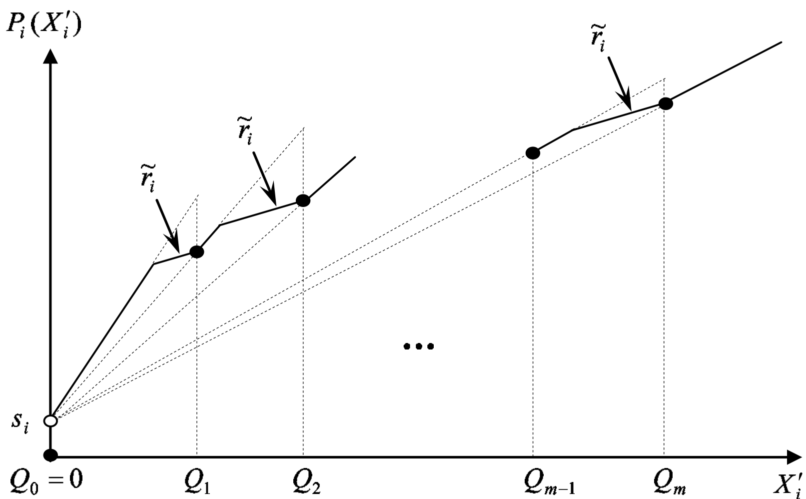
3.4 Example 4: DLS with All-Units Discount and Resales

Next, we consider a DLS problem with all-units discount and resales. Let s_i be the fixed ordering cost in period i. Let h_i be the unit holding cost in period i. Let $p_i(X_i)$ be the purchase price of the item in period i when the replenishment quantity is X_i. Function $p_i(\cdot)$ has an all-units discount price structure as shown in Figure 4(a), where the price breakpoints $Q_1, Q_2, \ldots, Q_m$ are stationary. Denote $Q_0 = 0$ and $Q_{m+1} = +\infty$. The unit price of the item in period i is $\alpha_{i,k}$ if the replenishment quantity is within $[Q_k, Q_{k+1})$, for $k = 0, 1, \ldots, m$. In period i, the decision maker may choose to resell part of the inventory and receive a revenue of r_i per unit. We assume that $\alpha_{i,m} + \sum_{q=i}^{j-1} h_q \geq r_j$ for $1 \leq i \leq j \leq n$. Without this assumption, the decision maker can purchase an unlimited amount of the item in period i, carry it, and resell it in period j to obtain an unlimited profit. Let S_i be the quantity of the item resold in period i. No backlogging is allowed. The problem can be formulated as follows:

$$
\begin{aligned}
\mathbf{P}_1: \quad & \text{minimize} \quad && \sum_{i=1}^{n} \left[s_i \delta(X_i) + p_i(X_i) + h_i I_i - r_i S_i \right] \\
& \text{subject to} \quad && I_i + S_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, n) \\
& && I_0 = I_n = 0 \\
& && X_i, I_i, S_i \geq 0 \quad (i = 1, 2, \ldots, n)
\end{aligned}
$$



(a) purchase price in period i



(b) replenishment cost function in problem $\mathbf{P}_1$

Figure 4: DLS with all-units discount and resales.

where $\delta(X_i) = 1$ if $X_i > 0$, and $\delta(X_i) = 0$ if $X_i = 0$. A replenishment period i is called a *fractional replenishment period* if $X_i \notin \{Q_1, Q_2, \ldots, Q_m\}$. A period i is called a *resale period* if $S_i > 0$.

Li *et al.* (2012) have shown that this problem is NP-hard in general and have developed an $O(n^{m+3})$ algorithm for the case where the number of price breakpoints, m, is fixed. To show that this problem satisfies Conditions 1–3, we first demonstrate that the decision variables $S_1, S_2, \ldots, S_n$

can be eliminated. Define

$$\tilde{r}_i = \max_{j=i,i+1,\ldots,n} \left\{ r_j - \sum_{q=i}^{j-1} h_q \right\},$$

which represents the resalable value of a unit of the item in period i if we select the best possible period among periods $i, i+1, \ldots, n$ to resell it. Li *et al.* (2012) have shown that solving the given problem is equivalent to solving the problem with the objective function of minimizing $\sum_{i=1}^{n} \left[s_i \delta(X_i) + p_i(X_i) + h_i I_i - \tilde{r}_i S_i \right]$. Using this new objective function, the problem has the following property (see Lemma 2 in Li *et al.* 2012): There exists an optimal solution in which for any two consecutive regeneration periods u and v $(u < v)$, (i) none of periods $u+2, u+3, \ldots, v$ is a resale period; and (ii) the number of fractional replenishment periods plus the number of resale periods included in $\{u+1, u+2, \ldots, v\}$ is at most one. This property implies that a resale must take place in a replenishment period. It also implies that among periods $u+1, u+2, \ldots, v$, except for at most one period, the *net replenishment quantity* (i.e., replenishment quantity less the resale quantity) in a period must be equal to one of $Q_0, Q_1, \ldots, Q_m$.

Let X'_i denote the net replenishment quantity in period i. Let k be the index such that $X'_i \in [Q_k, Q_{k+1})$. Then, the net replenishment cost, excluding the fixed setup cost, incurred in period i is

$$p'_i(X'_i) = \min \left\{ \alpha_{i,k} X'_i, \alpha_{i,k+1} Q_{k+1} - \tilde{r}_i (Q_{k+1} - X'_i), \alpha_{i,k+2} Q_{k+2} - \tilde{r}_i (Q_{k+2} - X'_i), \ldots, \alpha_{i,m} Q_m - \tilde{r}_i (Q_m - X'_i) \right\},$$

because the decision maker has a choice of purchasing X'_i units without reselling any of them, purchasing Q_{k+1} units and reselling $Q_{k+1} - X'_i$ units, purchasing Q_{k+2} units and reselling $Q_{k+2} - X'_i$ units, etc. Hence, the given problem can be transformed into the following problem:

$$\begin{aligned}
\mathbf{P}'_1: \quad \text{minimize} \quad & \sum_{i=1}^{n} \left[P_i(X'_i) + h_i I_i \right] \\
\text{subject to} \quad & I_i = I_{i-1} + X'_i - d_i \quad (i = 1, 2, \ldots, n) \\
& I_0 = I_n = 0 \\
& X'_i, I_i \geq 0 \quad (i = 1, 2, \ldots, n)
\end{aligned}$$

where $P_i(X'_i) = s_i + p'_i(X'_i)$ if $X'_i > 0$, and $P_i(X'_i) = 0$ if $X'_i = 0$ (see Figure 4(b)).

Let $\{Q_1, Q_2, \ldots, Q_m\}$ be the set of special replenishment quantities. The above property implies that there always exists an optimal solution to Problem $\mathbf{P}'_1$ such that between any two consecutive regeneration periods, there is at most one replenishment period not having a special replenishment quantity. Thus, Condition 2 is satisfied. If m is fixed, then function $P_i(\cdot)$ can be evaluated in constant time, and therefore Condition 1 is satisfied. Hence, problem $\mathbf{P}_1'$, and therefore problem $\mathbf{P}_1$, is solvable in $O(n^{2m+3})$ time when m is fixed.

4 Concluding Remarks

We have presented a simple framework for obtained polynomial time algorithms for DLS problems with nonlinear costs. This framework states that if Conditions 1–3 are satisfied, then a polynomial time algorithm exists.

Problem $\mathbf{P}$ has only two sets of decision variables, namely replenishment quantity variables and inventory variables. However, many DLS models involve more than these two sets of decision variables. When there is a third set of decision variables, depending on the problem structure, the model may be convertible into problem $\mathbf{P}$ or $\mathbf{P}'$ by consolidating two different sets of decision variables into one single set. For example, in the DLS problem with make-or-buy decisions presented by Lee and Zipkin (1989), the problem can be transformed into an equivalent DLS problem with only one set of replenishment variables. The transformed problem in Lee and Zipkin, in fact, satisfies Conditions 1–3 when the cost functions and production capacities are stationary. Another example is the DLS problem with all-units discount and resales presented in Section 3.4, where we have demonstrated how to combine the replenishment variable X_i with the resale variable S_i and convert the problem into problem $\mathbf{P}'$.

Note that Condition 2 is valid for only a subset of single-item, single-stage DLS problems. There are many problems which can be formulated as model $\mathbf{P}$ but do not satisfy Condition 2. For example, models with inventory bounds (see, e.g., Gutiérrez *et al.* 2007) are unlikely to satisfy this condition. Hence, an interesting future research direction is to extend the current framework so that it can be applied to more DLS problems. Another possible future research direction is to extend Conditions 1–3 to cover DLS models with more complicated structures such as DLS problems with multiple-items and DLS problems with multiple stages of production.

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