https://doi.org/10.1016/j.trb.2016.08.003

A polynomial-time algorithm for sailing speed optimization with containership resource sharing

Shuaian Wang[a], Xinchang Wang[b,*]

[a] *Department of Logistics & Maritime Studies, The Hong Kong Polytechnic University, Kowloon, Hong Kong, wangshuaian@gmail.com*

[b] *Department of Marketing, Quantitative Analysis, and Business Law, Mississippi State University, Mississippi 39762, U.S.A., xwang@business.msstate.edu*

Abstract

The sailing speed optimization problem aims to determine the optimal cruising speeds of ships by balancing the number of ships required on services, the fuel consumption, and the level of service provided for customers. The level of service can be incorporated into a sailing speed optimization model from the perspective of supply chain management or from the perspective of shipping lines. We design a polynomial-time algorithm workable to solve the two models based on bi-section search methods. The novelties of the algorithm include constructing a new parameter on which the bi-section search will be executed and deriving a near-optimal solution by taking advantage of the problem structure. We also provide theoretical results that guarantee the validity of the polynomial-time algorithm.

Keywords: Bi-section search, Containership, Sailing speed, Bunker fuel, Transit time, Polynomial-time algorithm

1. Introduction

In this paper, we study the well-known *containership sailing speed optimization problem*, in which a container shipping line needs to determine the number of ships to be deployed on each service (or equivalently, ship route) as well as the sailing speed for each leg on each

*Corresponding author.

service to minimize the total cost.

To solve the problem, several trade-offs need to be balanced. First, the shipping line has a limited fleet of containerships, which causes that, if one service uses too many ships, the shipping line may fall in short of containerships for other services. Second, if one or more ships are removed from an existing service, the remaining ships have to sail faster in order to maintain a 7-day service frequency. As a result, the faster sailing speed will incur a higher fuel consumption rate, since it has been well known that the daily bunker consumption is approximately proportional to the sailing speed to the power three. On the other hand, the shipping line could reduce fuel consumption by utilizing as many ships as it can. However, doing so might create a need of running more ships and produce a potential cost for chartering extra ships. Third, a lower sailing speed saves fuel consumption but often leads to a longer port-to-port transit time, which increases the pipeline inventory of customers. Hence, a low sailing speed might not be favored from customers' point of view.

There are two different voices on how to address the impact of transit time (equivalently, sailing speed) in the problem. The first one advocates minimizing the *supply chain cost* that includes the container shipping line's cost (ship chartering cost and fuel cost) and the customers' cost (pipeline inventory cost) (Álvarez, 2012; Kim, 2014). In this approach, longer transit time is penalized by higher inventory costs. The rationale behind this approach is that by taking into account customers' inventory costs, the shipping line actually provides higher customer service levels. It will thereby be rewarded as customers are willing to pay higher freight rates and/or let it transport more of their cargoes. The other is solely from the perspective of a shipping line and suggests minimizing the sum of the chartering cost of ships and fuel cost while providing a certain level of service to customers by imposing a maximum port-to-port transit time constraint (Karsten et al., 2015). The idea is that it is very difficult for the shipping line to obtain accurate information on customers' inventories as there are too many customers, and even if it can, the shipping line will not be immediately rewarded. Therefore, the shipping line could simply impose a maximum port-to-port transit time constraint and exclude the cargo inventory costs from its objective function. Liner service planners from Orient Overseas Container Line (OOCL) told us that they determine the port-to-port transit time for key legs (e.g., the leg from the last port of call in Asia to the first port of call in North America on a trans-Pacific service) based on the prevailing transit time of the shipping market. Similar to Karsten et al. (2015, 2016), our study also defines the transit time constraints on each service individually.

Given a fleet of containerships to be deployed in a liner shipping network, Wang (2016) proposed a pseudo-polynomial-time algorithm to solve the sailing speed optimization model that is built from the perspective of supply chain management so as to minimize the supply chain cost. This paper formulates the problem as a mathematical programming model from the perspective of shipping lines, with the objective to minimize the sum of the chartering cost of ships and fuel cost subject to the maximum allowed transit times between ports on individual services. This paper extends the work of Wang (2016) and makes the following contributions to the literature on shipping service design:

(1) We show that the model from shipping lines' perspective can also be solved in pseudo-polynomial time in the size of the problem.

(2) We propose a polynomial-time algorithm workable for solving the speed optimization model formulated from the perspective of supply chain management or from the perspective of shipping lines based on bi-section search methods. The novelties of the algorithm include constructing a new parameter on which the bi-section search will be executed and deriving a near-optimal solution by taking advantage of the problem structure. The polynomial-time algorithm improves over the pseudo-polynomial-time algorithm in Wang (2016).

(3) We also provide theoretical results that guarantee the validity of the polynomial-time algorithm.

2. Literature Review

Unlike road transport in which the speeds of vehicles are determined by traffic conditions, in maritime transport the speeds of ships are mainly determined by economical considerations. In particular, the daily fuel consumption of a ship increases dramatically with the speed, often proportional to the speed cubed (Notteboom and Vernimmen, 2009) or even proportional to the speed to the power of four or higher (Du et al., 2011; Song and Dong, 2013; Meng et al., 2016). As a result, slow steaming saves fuel costs. On the other hand, slow steaming means more ships are required on a liner service in order to provide a weekly frequency. Hence, a natural choice of speed is to balance the trade-off between ship chartering costs and fuel costs in an optimal manner. To this end, Ronen (2011) optimized the speed of containerships for a liner service by enumerating all of the possible number of ships to be deployed; Wang and Meng (2012b) optimized the speed of containerships for a network consisting of many liner services by solving a mixed-integer nonlinear programming model. Du et al. (2015) proposed a practical fuel budget problem that aims to determine a group of bunker fuel budget values for a liner container ship over a round-trip voyage under uncertainties caused by severe weather conditions and addressed the problem with robust optimization techniques. Psaraftis and Kontovas (2013, 2014) have presented comprehensive reviews on ship speed optimization taxonomy, models, and algorithms.

There are also many models that integrate ship speed optimization with other planning decisions. As the ship speed affects the bunker consumption and thereby pollutant emission, a number of models for determining the sailing speeds while incorporating pollutant emission have been developed (Cariou, 2011; Kontovas and Psaraftis, 2011; Kim et al., 2012, 2013, 2014; Mansouri et al., 2015; Song et al., 2015; Wong et al., 2015). When different bunker fuel prices at different ports are taken into account, the sailing speed decisions must be made in combination with the choice of bunkering ports (Yao et al., 2012; Kim, 2014; Ghosh et al., 2015). The sailing speed is also closely related to the schedule design for liner services because once the planned arrival and departure time at each port of call is determined, the planning sailing speed from one port to the next is also determined. Schedule design may also be examined accounting fortransit time limits (Wang and Meng, 2012a) and port time uncertainty (Qi and Song, 2012; Wang and Meng, 2012a). In reality, ships are often not able to follow the planned schedule and in case of delay, ships often speed up. As a result, ship speed optimization is also used for analyzing schedule reliability (Song et al., 2015) and schedule recovery at the operational level (Li et al., 2015, 2016). Ship speed optimization is also modeled in the context of line network design (Karsten et al., 2016), tramp shipping (Hvattum et al., 2013), and transit-time-sensitive demand (Wang et al., 2013).

Slow steaming means a long port-to-port transit time, which increases the pipeline inventory of the customers. Hence, the sailing speed should not be too low from the customers' point of view. Álvarez (2012) argued that the level of service experienced by the shippers under different fleet configurations should be properly addressed, for which the inventory holding costs are used as a practical alternative to represent the shippers' level of service in a liner network. Kim (2014) presented an interesting Lagrangian heuristic to optimize the sailing speeds for a liner service while taking into account the time cost (inventory cost) of the containers in the objective function. Wang (2016) proposed a pseudo-polynomial-time algorithm to determine the optimal speed for each leg of each service in a liner network with the objective of minimizing the sum of chartering costs of ships, fuel costs, and inventory costs. Both Kim (2014) and Wang (2016) adopted the supply-chain approach for speed optimization.

Another possible approach for speed optimization is solely from the perspective of the shipping line: minimizing the sum of chartering costs of ships and fuel costs while imposing a maximum port-to-port transit time constraint. A relevant study is Karsten et al. (2015), which decides how to transport containers considering a maximum port-to-port transit time constraint without optimizing the speeds for containerships. Karsten et al. (2016) extended their previous research by designing a liner shipping network considering a maximum port-to-port transit time constraint on individual services.

Features that distinguish our research from most of the existing studies include: (1) for the deployment of vessels on the services the vessels are taken from a pool shared by all the services and therefore this optimization of the individual services is interdependent; (2) we examine two models that incorporate the level of service from the perspective of supply chain management and from the perspective of shipping lines, respectively; in the latter the transit time constraints are only defined on each service individually; (3) most importantly, we propose a polynomial-time algorithm for obtaining the optimal speeds for both the models.

3. Problem Description

We list the notation used in the paper below:

Sets

R	Set of services in a liner shipping network; $r \in R$ refers to a service
V	Set of ship types; $v \in V$ refers to a ship type
R_v	Set of services that use ships of type $v \in V$
I_r	Set of legs on service $r \in R$; $i \in I_r$ denotes the leg from the i-th port of call to the $(i+1)$-th port of call
$\mathbb{Z}_+$	Set of nonnegative integers

Parameters

α	Bunker fuel price
c_v	Chartering cost of a ship of type $v \in V$ per week
v_{ri}	Sailing speed of ships on leg $i \in I_r$, where $r \in R$
L_{ri}	Travel distance along leg $i \in I_r$, where $r \in R$

$g_{ri}(v_{ri}) := a_{ri}(v_{ri})^{b_{ri}}$. Fuel consumption per unit distance on leg i of service r as a function of sailing speed v_{ri}. Note that a_{ri} and b_{ri} are both parameters. Thus, the total fuel consumption on the leg is computed as $L_{ri} \cdot g_{ri}(v_{ri}) = L_{ri} \cdot a_{ri}(L_{ri}/t_{ri})^{b_{ri}} = a_{ri}(L_{ri})^{1+b_{ri}}(t_{ri})^{-b_{ri}}$, where $b_{ri} > 1$.

h_{ri} Inventory cost of containers on leg $i \in I_r$ of service $r \in R$ per unit travel time

M_v Maximum number of ships of type $v \in V$ in the fleet that can be chartered

$\hat{t}_{ri}$ Time spent at the i-th port of call on service $r \in R$

$t_{ri}^{\min}$ Minimum possible sailing time of leg $i \in I_r$ on service $r \in R$, which is equal to L_{ri} divided by the maximum ship speed obtainable

$t_{rij}^{\max}$ Maximum transit time allowed from the i-th port of call to the j-th port of call on service r, where $i, j \in I_r$ with $i \neq j$, which is the elapsed time from the departure of a ship at the i-th port of call to the arrival of the ship at the j-th port of call. If there is no transit time requirement for the two ports of call, then we can simply set $t_{rij}^{\max}$ to be a large number.

v_r Type of ships deployed on service r with $v_r \in V$, where $r \in R$

Decision variables

m_r Number of ships to be deployed on service $r \in R$ to maintain a weekly service frequency

t_{ri} Sailing time on leg i of service $r \in R$, which determines the sailing speed on the leg

Quantities to be calculated

$C^*(v)$ Optimal objective function value (8) of model [P1-v]

$C_r(m_r)$ Optimal objective function value (9), which is the minimum sum of ship chartering costs and fuel costs of service r given m_r ships are deployed on r

m_r^* Minimizer of function $C_r(m_r)$, i.e., $m_r^* \in \arg\min_{m_r \in \{1,2,\ldots,M_v\}} C_r(m_r)$, which can be understood as the number of ships to be deployed on service $r \in R_v$ to minimize $C_r(m_r)$ without considering other services

$m_r^{\min}$ Minimum number of ships to be deployed on service $r \in R_v$ such that $C_r(m_r)$ is finite, i.e., $m_r^{\min} := \min\{m_r \in \{1, 2, \ldots, M_v\} | C_r(m_r) < +\infty\}$

$m_r(\theta)$ Number of ships to be deployed on service $r \in R_v$ for a given θ as defined in Lemma 7

$\hat{m}_r^*$ Optimal number of ships assigned on service $r \in R$ by solving model [P1-v]

$\bar{m}_r^*$ Number of ships deployed on service $r \in R$ in an ϵ-approximation solution to [P1-v]

The speed optimization problem for a liner shipping network solely from the perspective of shipping lines can be formulated as a mixed-integer nonlinear optimization model with decision variables m_r and t_{ri}:

$$[\text{P1-shipping line}] \quad \min_{m_r, t_{ri}} \sum_{v \in V} \sum_{r \in R_v} c_v m_r + \alpha \sum_{r \in R} \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri})^{-b_{ri}} \tag{1}$$

subject to:

$$\sum_{i \in I_r} t_{ri} + \sum_{i \in I_r} \hat{t}_{ri} = 168 m_r, \forall r \in R \tag{2}$$

$$\sum_{k=i}^{j-1} t_{rk} + \sum_{k=i+1}^{j-1} \hat{t}_{rk} \leq t_{rij}^{\max}, \forall r \in R, \forall i \in I_r, \forall j \in I_r, j > i \tag{3}$$

$$\sum_{k=i}^{|I_r|} t_{rk} + \sum_{k=1}^{j-1} t_{rk} + \sum_{k=i+1}^{|I_r|} \hat{t}_{rk} + \sum_{k=1}^{j-1} \hat{t}_{rk} \leq t_{rij}^{\max}, \forall r \in R, \forall i \in I_r, \forall j \in I_r, j < i \tag{4}$$

$$\sum_{r \in R_v} m_r \leq M_v, \forall v \in V \tag{5}$$

$$t_{ri} \geq t_{ri}^{\min}, \forall r \in R, \forall i \in I_r \tag{6}$$

$$m_r \in \mathbb{Z}_+, \forall r \in R. \tag{7}$$

The objective function (1) minimizes the sum of chartering costs of ships and fuel costs. Constraints (2) ensure the number of ships deployed could ensure a weekly frequency, in which "168" is the number of hours in a week and we use "hours" as time units. Constraints (3) and (4) guarantee a certain level of service to customers in terms of maximum port-to-port transit times. Constraints (5) are the resource sharing constraints enforcing that the total number of ships of each type deployed cannot exceed the number of available ships in the fleet. Constraints (6) define the minimum sailing time on each leg and Constraints (7) require the number of ships deployed on each service is a nonnegative integer.

The speed optimization problem from the supply chain perspective in Wang (2016) is similar to [P1-shipping line] except that the level-of-service constraints (3) and (4) are replaced by a term in the objective function to represent the inventory cost:

$$\text{[P1'-supply chain]} \quad \min_{m_r, t_{ri}} \sum_{v \in V} \sum_{r \in R_v} c_v m_r + \alpha \sum_{r \in R} \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri})^{-b_{ri}} + \sum_{r \in R} \sum_{i \in I_r} h_{ri} t_{ri}$$

subject to Constraints (2), (5), (6) and (7). All of the algorithms we propose for [P1-shipping line] are also applicable to [P1'-supply chain] with minimum revision. Hence, we will only analyze [P1-shipping line] in the sequel.

It is not difficult to see that [P1-shipping line] can be decomposed for each ship type $v \in V$:

$$\text{[P1-}v\text{]} \quad \min_{m_r, t_{ri}} \sum_{r \in R_v} c_v m_r + \alpha \sum_{r \in R_v} \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri})^{-b_{ri}} \tag{8}$$

subjec to:

$$\begin{aligned} \sum_{i \in I_r} t_{ri} &= 168 m_r - \sum_{i \in I_r} \hat{t}_{ri}, \forall r \in R_v \\ \sum_{k=i}^{j-1} t_{rk} &\leq t_{rij}^{\max} - \sum_{k=i+1}^{j-1} \hat{t}_{rk}, \forall r \in R_v, \forall i, j \in I_r, j > i \end{aligned}$$

$$
\begin{aligned}
\sum_{k=i}^{|I_r|} t_{rk} + \sum_{k=1}^{j-1} t_{rk} &\leq t_{rij}^{\max} - \left(\sum_{k=i+1}^{|I_r|} \hat{t}_{rk} + \sum_{k=1}^{j-1} \hat{t}_{rk} \right), \forall r \in R_v, \forall i, j \in I_r, j < i \\
\sum_{r \in R_v} m_r &\leq M_v \\
t_{ri} &\geq t_{ri}^{\min}, \forall r \in R_v, \forall i \in I_r \\
m_r &\in \mathbb{Z}_+, \forall r \in R_v.
\end{aligned}
$$

As [P1-shipping line] involves solving $|V|$ models of [P1-v], if [P1-v] can be solved in polynomial time, [P1-shipping line] can also be solved in polynomial time. We thus focus our attention on how to solve [P1-v] throughout the rest of the paper.

4. A Pseudo-polynomial-time Algorithm

4.1. Properties of the optimal cost of a service with a given number of ships

We first investigate the optimal sailing time t_{ri} on each leg $i \in I_r$ of service $r \in R$ with a given number of ships m_r. We have the following nonlinear programming model [P2(r, m_r)].

$$
[\text{P2}(r,\, m_r)] \quad C_r(m_r) := c_{v_r} m_r + \min_{t_{ri}} \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri})^{-b_{ri}} \tag{9}
$$

subject to:

$$
\begin{aligned}
\sum_{i \in I_r} t_{ri} &= 168 m_r - \sum_{i \in I_r} \hat{t}_{ri} && (10) \\
\sum_{k=i}^{j-1} t_{rk} &\leq t_{rij}^{\max} - \sum_{k=i+1}^{j-1} \hat{t}_{rk}, \forall i \in I_r, \forall j \in I_r, j > i && (11) \\
\sum_{k=i}^{|I_r|} t_{rk} + \sum_{k=1}^{j-1} t_{rk} &\leq t_{rij}^{\max} - \left(\sum_{k=i+1}^{|I_r|} \hat{t}_{rk} + \sum_{k=1}^{j-1} \hat{t}_{rk} \right), \forall i \in I_r, \forall j \in I_r, j < i && (12) \\
t_{ri} &\geq t_{ri}^{\min}, \forall r \in R_v, \forall i \in I_r. && (13)
\end{aligned}
$$

We assume that [P2(r, m_r)] is feasible for at least one $m_r \in \{1, 2, \ldots, M_v\}$ for all $r \in R$ and define $C_r(m_r) := +\infty$ for all $m_r \in [0, M_v]$ such that [P2(r, m_r)] is infeasible.

As the following Lemma 1 shows, the parametric optimal objective function value of [P2(r, m_r)] turns out to be strictly convex in $m_r \in [0, M_v]$.

Lemma 1. *We temporarily assume that the parameter m_r in model [P2(r, m_r)] can take fractional quantities. For a given service r, $C_r(m_r) : [0, M_v] \mapsto \mathbb{R}$ is a strictly convex function of m_r.*

Proof. Given $m_r^{(1)}$, $m_r^{(3)}$, $0 < \lambda < 1$, and $m_r^{(2)} := \lambda m_r^{(1)} + (1-\lambda) m_r^{(3)}$, denote by $(t_{ri} = t_{ri}^{(1)}, i \in I_r)$ and $(t_{ri} = t_{ri}^{(3)}, i \in I_r)$ the optimal sailing times in models [P2(r, $m_r^{(1)}$)] and [P2(r, $m_r^{(3)}$)], respectively. Then, $(t_{ri} = t_{ri}^{(2)} := \lambda t_{ri}^{(1)} + (1-\lambda) t_{ri}^{(3)}, i \in I_r)$ is a feasible solution to [P2(r, $m_r^{(2)}$)] because all of the constraints in [P2(r, m_r)] are linear. We thus have

$$\begin{aligned}
C_r(m_r^{(2)}) &= c_{v_r} m_r^{(2)} + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri}^{(2)})^{-b_{ri}} \\
&= c_{v_r} [\lambda m_r^{(1)} + (1-\lambda) m_r^{(3)}] + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} \left(\lambda t_{ri}^{(1)} + (1-\lambda) t_{ri}^{(3)} \right)^{-b_{ri}} \\
&< c_{v_r} [\lambda m_r^{(1)} + (1-\lambda) m_r^{(3)}] + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} \left[\lambda (t_{ri}^{(1)})^{-b_{ri}} + (1-\lambda) (t_{ri}^{(3)})^{-b_{ri}} \right] \\
&= \lambda \left[c_{v_r} m_r^{(1)} + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri}^{(1)})^{-b_{ri}} \right] + \\
&\quad (1-\lambda) \left[c_{v_r} m_r^{(3)} + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri}^{(3)})^{-b_{ri}} \right] \\
&= \lambda C_r(m_r^{(1)}) + (1-\lambda) C_r(m_r^{(3)}),
\end{aligned}$$

where the inequality holds because function $x^{-b_{ri}}$ is strictly convex as $b_{ri} > 1$ and $x > 0$. □

Lemma 1 implies

Corollary 1. *Consider integer values of* m_r*. For a given service* r*,* $C_r(m_r) : \{0, 1, ..., M_v\} \mapsto \mathbb{R}$ *satisfies* $C_r(m_r + 2) - C_r(m_r + 1) > C_r(m_r + 1) - C_r(m_r)$*,* $m_r = 0, 1, ..., M_v - 2$*.*

Note that model [P2(r, m_r)] minimizes a separable convex function subject to linear constraints. Thus, we have Lemma 2 that follows from the time complexity analysis of the proposed scaling algorithm in Theorem 12 of Chubanov (2016) and its subsequent discussion "the scaling algorithm is polynomial, provided that we use a polynomial algorithm for LP".

Lemma 2. *Model [P2(*r*,* m_r*)] can be solved in polynomial time with regard to the size of the input using interior point methods.*

4.2. Definitions and domain of the number of ships to deploy on a ship route

Definition 1. Define m_r^* as the best number of ships deployed on service $r \in R_v$ without considering other services. In case of tie, choose the smallest m_r^*. That is

$$m_r^* = \min \{m_r \in \{1, 2, \dots, M_v\} | C_r(m_r) \leq C_r(m_r'), \forall m_r' \in \{1, 2, \dots, M_v\}\}.$$

Definition 2. Define $(m_r = \hat{m}_r^*, r \in R_v)$ as the optimal solution to [P1-v].

It is very easy to see that if $\sum_{r \in R_v} m_r^* \leq M_v$, then $(\hat{m}_r^* = m_r^*, r \in R_v)$ is an optimal solution for [P1-v]. Therefore, unless otherwise specified, in the following we assume $\sum_{r \in R_v} m_r^* > M_v$.

Definition 3. Define $m_r^{\min}$ as the smallest number of ships to be deployed on service $r \in R_v$ such that [P2(r, m_r)] is feasible. That is

$$m_r^{\min} := \min\{m_r \in \{1, 2, \dots, M_v\} | C_r(m_r) < +\infty\}.$$

Definition 3 implies that if $\sum_{r \in R_v} m_r^{\min} > M_v$, then [P1-$v$] is infeasible; if $\sum_{r \in R_v} m_r^{\min} = M_v$, then the only feasible solution to [P1-v] is $(m_r = m_r^{\min}, r \in R_v)$, which is of course optimal. Hence, in the sequel we always assume that $\sum_{r\in R_v} m_r^{\min} \leq M_v - 1$. Moreover, if for a service $r \in R_v$ we have $m_r^{\min} = m_r^*$, then in at least one optimal solution to [P1-v] the number of ships deployed on the service is $m_r^{\min}$ and hence this service can be excluded from the model. Therefore, we also assume that $m_r^{\min} \leq m_r^* - 1$ for all $r \in R_v$. Naturally, The optimal solution to [P1-v], $(m_r = \hat{m}_r^*, r \in R_v)$, satisfies $m_r^{\min} \leq \hat{m}_r^* \leq m_r^*, r \in R_v$.

Lemma 1 implies

Corollary 2. *$C_r(m_r) < +\infty$ for all $m_r = m_r^{\min}, m_r^{\min}+1, ..., m_r^*$ because m_r is a convex combination of $m_r^{\min}$ and m_r^*.*

Lemma 3. *The value of $m_r^{\min}$, if exists (i.e., [P2(r, m_r)] is feasible for at least one $m_r \in \{1, 2, \ldots, M_v\}$), can be determined in the following manner:*

$$m_r^{\min} = \min\left\{ m_r \in \{1, 2, \ldots, M_v\} | 168 m_r \geq \sum_{i\in I_r} t_{ri}^{\min} + \sum_{i\in I_r} \hat{t}_{ri} \right\}. \tag{14}$$

Equivalently,

$$m_r^{\min} = \left\lceil \frac{\sum_{i\in I_r} t_{ri}^{\min} + \sum_{i\in I_r} \hat{t}_{ri}}{168} \right\rceil, \tag{15}$$

where $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Proof. Evidently, no m_r smaller than $m_r^{\min}$ defined in Eq. (15) is feasible. Hence, we just need to prove that $C_r(m_r^{\min}) < +\infty$. Suppose that $C_r(m_r^{\min}) = +\infty$ and there exists an $m_r' > m_r^{\min}$ such that $C_r(m_r') < +\infty$. Let $(t_{ri} = t_{ri}', i \in I_r)$ be the optimal solution to [P2(r, m_r')]. Then we can construct a feasible solution to [P2(r, $m_r^{\min}$)] in the following manner:

$$t_{ri} = t_{ri}^{\min} + (t_{ri}' - t_{ri}^{\min}) \frac{168 m_r^{\min} - \sum_{i\in I_r} t_{ri}^{\min} - \sum_{i\in I_r} \hat{t}_{ri}}{168 m_r' - \sum_{i\in I_r} t_{ri}^{\min} - \sum_{i\in I_r} \hat{t}_{ri}}, i \in I_r,$$

meaning that [P2(r, $m_r^{\min}$)] is feasible and thereby $C_r(m_r^{\min}) < +\infty$. □

Lemma 4. *Checking whether $C_r(m_r^{\min}) < +\infty$, in which $m_r^{\min}$ is defined in Eq. (15), can be done by solving the following linear programming model and hence can be completed in polynomial time.*

$$\min 0 \tag{16}$$

subject to Constraints (11), (12), (13) and

$$\sum_{i \in I_r} t_{ri} = 168 m_r^{\min} - \sum_{i \in I_r} \hat{t}_{ri}.$$

4.3. Solving [P1-v] in pseudo-polynomial time

Based on Corollary 1, Wang (2016) proved the following Theorem 1.

Theorem 1. *Solution $(m_r = \hat{m}_r^*, r \in R_v)$ is optimal to [P1-v] if and only if: $\sum_{r \in R_v} \hat{m}_r^* = M_v$ and for any two services $r_1 \in R_v$ and $r_2 \in R_v$, we have $C_{r_1}(\hat{m}_{r_1}^* - 1) - C_{r_1}(\hat{m}_{r_1}^*) \geq C_{r_2}(\hat{m}_{r_2}^*) - C_{r_2}(\hat{m}_{r_2}^* + 1)$. In words, shifting one ship from service r_1 to service r_2 cannot reduce the total cost.*

Based on Theorem 1 and similar to Wang (2016), we can develop the following pseudo-polynomial-time Algorithm 1 for [P1-v].

Remark 1 asserts that Algorithm 1 is of pseudo-polynomial computational time.

Remark 1. In Step 0 of Algorithm 1, [P2(r, m_r)] is solved $|R_v|M_v$ times. Lemma 2 implies that [P2(r, m_r)] can be solved in polynomial time of the input. Therefore, the time complexity of Step 0 is $|R_v|M_v$ times the complexity of the scaling algorithm of Chubanov (2016). Step 2 of Algorithm 1 is repeated at most M_v times, each of which has a complexity of $|R_v|$. Therefore, Algorithm 1 can find an optimal solution to [P1-v] in pseudo-polynomial time as the time complexity depends on the value of M_v.

Algorithm 1: A PSEUDO-POLYNOMIAL-TIME ALGORITHM FOR [P1-v]

Step 0. For each service $r \in R_v$, calculate $m_r^{\min}$ by Eq. (15). Check whether [P2(r, $m_r^{\min}$)] is feasible (Lemma 4). If there is an $r' \in R_v$ such that [P2(r', $m_{r'}^{\min}$)] is infeasible, [P1-v] is infeasible and stop.

Step 1. For each service $r \in R_v$, obtain $C_r(m_r)$ for each $m_r \in \{m_r^{\min}, m_r^{\min}+1, \ldots, M_v\}$ by solving [P2(r, m_r)]. Find $m_r^* \in \arg\min_{m_r \in \{1,2,\ldots,M_v\}} C_r(m_r)$. Define $\hat{m}_r := m_r^*$.

Step 2. If $\sum_{r \in R_v} \hat{m}_r \leq M_v$, then $(m_r = \hat{m}_r, r \in R_v)$ is the optimal solution to [P1-v] and stop.

Step 3. Set $C_r(m_r^{\min} - 1) \leftarrow +\infty$. Find a service r^* satisfying

$$r^* \in \arg\min_{r \in R_v} [C_r(\hat{m}_r - 1) - C_r(\hat{m}_r)].$$

That is, reducing one ship on service r^* leads to the smallest increase in the total cost.

Set $\hat{m}_{r^*} \leftarrow \hat{m}_{r^*} - 1$. Go to **Step 2**.

5. A polynomial-time algorithm to solve [P1-v]

We strengthen the results in the above section as well as the results in Wang (2016) by proposing a polynomial-time algorithm based on a bi-section search scheme. To this end, we need to construct a parameter that is amenable to the bi-section search and closely related to the optimal solution to model [P1-v]. Prior to this step, we examine more properties of model [P2(r, m_r)].

5.1. A parameter θ that is amenable to bi-section search

Theorem 1 can be restated as:

Lemma 5. *Solution $(m_r = \hat{m}_r^*, r \in R_v)$ is optimal to [P1-v] if and only if $\sum_{r \in R_v} \hat{m}_r^* = M_v$ and there exists a value θ^* such that $C_r(\hat{m}_r^* - 1) - C_r(\hat{m}_r^*) \geq \theta^* \geq C_r(\hat{m}_r^*) - C_r(\hat{m}_r^* + 1)$ for all services $r \in R_v$. (We define $C_r(M_v + 1) := +\infty$.)*

Proof. The "if" part is proved first. For any two services $r_1 \in R_v$ and $r_2 \in R_v \setminus \{r_1\}$, the definition of θ^* implies

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) \geq \theta^* \geq C_{r_2}(\hat{m}^*_{r_2})-C_{r_2}(\hat{m}^*_{r_2}+1),$$

which yields

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) \geq C_{r_2}(\hat{m}^*_{r_2})-C_{r_2}(\hat{m}^*_{r_2}+1).$$

Then, it follows from Theorem 1 that $(m_r = \hat{m}^*_r, r \in R_v)$ is an optimal solution to model [P1-v].

We then prove the "only if" part. Consider a particular service $r_1 \in R_v$. Theorem 1 implies that

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) \geq C_{r_2}(\hat{m}^*_{r_2})-C_{r_2}(\hat{m}^*_{r_2}+1), \forall r_2 \in R_v \setminus \{r_1\}.$$

Corollary 1 implies

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) > C_{r_1}(\hat{m}^*_{r_1})-C_{r_1}(\hat{m}^*_{r_1}+1), \forall r_2 \in R_v.$$

Combining the above two equations,

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) \geq C_{r}(\hat{m}^*_{r})-C_{r_2}(\hat{m}^*_{r}+1), \forall r \in R_v.$$

That is,

$$C_{r_1}(\hat{m}^*_{r_1}-1)-C_{r_1}(\hat{m}^*_{r_1}) \geq \max_{r\in R_v}[C_r(\hat{m}^*_r)-C_r(\hat{m}^*_r+1)].$$

As r_1 can be any service, we have

$$\min_{r\in R_v}[C_r(\hat{m}^*_r-1)-C_r(\hat{m}^*_r)] \geq \max_{r\in R_v}[C_r(\hat{m}^*_r)-C_r(\hat{m}^*_r+1)].$$

Hence, $\theta^* := \max_{r\in R_v}[C_r(\hat{m}^*_r)-C_r(\hat{m}^*_r+1)]$ satisfies the lemma. □

As it will soon be clearer in the subsequent Eq. (20) and Algorithm 2, the main process of the polynomial-time algorithm uses a bi-section search scheme over the domain of parameter θ, which measures the (negative) marginal cost of a service with respect to the number of ships deployed on the service, i.e., $C_r(m_r) - C_r(m_r + 1)$. Hence, we need to find a finite domain of θ.

5.2. Upper bound on the domain of θ

When one more ship is deployed on a service $r \in R_v$, the ship chartering cost is increased by c_v and the fuel cost is reduced. When the ships sail at the highest speed, the fuel consumption is the highest and can be computed by $\alpha \sum_{i \in I_r} a_{ri}(L_{ri})^{1+b_{ri}}(t_{ri}^{\min})^{-b_{ri}}$. If $\alpha \sum_{i \in I_r} a_{ri}(L_{ri})^{1+b_{ri}}(t_{ri}^{\min})^{-b_{ri}} < c_v$, then the marginal ship chartering cost of deploying one more ship is always larger than the marginal benefit of fuel savings. As a result, the smallest number of ships $m_r^{\min}$ should be deployed on r. Otherwise, in Lemma 5, we must have

$$C_r(\hat{m}_r^*) - C_r(\hat{m}_r^* + 1) \leq \left[\alpha \sum_{i \in I_r} a_{ri}(L_{ri})^{1+b_{ri}}(t_{ri}^{\min})^{-b_{ri}}\right] - c_v, \forall r \in R_v.$$

Hence, the value of θ^* in Lemma 5 has an upper bound $\theta^{\max}$ defined as

$$\theta^{\max} := \max_{r \in R_v} \alpha \sum_{i \in I_r} a_{ri}(L_{ri})^{1+b_{ri}}(t_{ri}^{\min})^{-b_{ri}} - c_v. \tag{17}$$

5.3. Procedures and properties used for designing a polynomial-time algorithm

We now state a few procedures and properties that will be used for designing a polynomial-time algorithm.

Lemma 6. *Finding the best number of ships to be deployed on service $r \in R_v$ without considering other services, i.e., finding $m_r^* \in \arg\min_{m_r \in \{1,2,\ldots,M_v\}} C_r(m_r)$, can be completed in polynomial time.*

Proof. As $C_r(m_r)$ is convex, we can solve model [P2(r, m_r)] with different m_r's in a golden section search manner over $m_r = 1, 2, \ldots, M_v$. Because m_r is an integer, model [P2(r, m_r)] needs to be solved at most $O(\log M_v)$ times. Since model [P2(r, m_r)] can be solved in polynomial time, m_r^* can be found in polynomial time. □

Using $(m_r^*, r \in R_v)$, the smallest value of θ is defined as

$$\theta^{\min} := \max_{r \in R_v} (C_r(m_r^*) - C_r(m_r^* + 1)). \tag{18}$$

Since later we will use bi-section on θ over the domain $[\theta^{\min}, \theta^{\max}]$, we need a finite $\theta^{\min}$. If $C_r(m_r^* + 1) = +\infty$ for all $r \in R_v$, then without loss of generality, we can assume the $C_r(m_r^* + 1)$ are not infinity but a very large number. In particular, we define

$$C_{r'}(m_{r'}^* + 1) = \sum_{r \in R_v} \left[m_r^{\min} c_v + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri}^{\min})^{-b_{ri}} \right], r' \in R_v$$

and $\theta^{\min}$ is redefined as

$$\theta^{\min} := \max_{r \in R_v} C_r(m_r^*) - \sum_{r \in R_v} \left[m_r^{\min} c_v + \alpha \sum_{i \in I_r} a_{ri} (L_{ri})^{1+b_{ri}} (t_{ri}^{\min})^{-b_{ri}} \right]. \tag{19}$$

The following Lemma 7 can then be obtained.

Lemma 7. *Consider any $\theta \in [\theta^{\min}, \theta^{\max}]$. Then, for any $r \in R_v$, there exists $m_r(\theta) \in \{m_r^{\min}, m_r^{\min} + 1, \ldots, m_r^*\}$ (i.e., the number of ships to be deployed on service $r \in R_v$) such that*

$$C_r(m_r(\theta)) - C_r(m_r(\theta) + 1) \le \theta \le C_r(m_r(\theta) - 1) - C_r(m_r(\theta)). \tag{20}$$

Moreover, finding $m_r(\theta)$ can be completed in polynomial time with bi-section search.

Proof. The result holds because $C_r(m_r^*) - C_r(m_r^* + 1) \le \theta^{\min}$ and $C_r(m_r^{\min} - 1) - C_r(m_r^{\min}) = \infty$ (we define $C_r(0) = +\infty$). □

Recall that we assume $\sum_{r\in R_v} m_r^* > M_v$. Then, the following Lemma 8 holds.

Lemma 8. *It holds that* $\sum_{r\in R_v} m_r(\theta^{\min}) > M_v$.

Proof. By definition of m_r^*, $C_r(m_r^*-1) - C_r(m_r^*) > 0$ for all $r \in R_v$. Eq. (18) implies that $\theta^{\min} \le 0$. Then $C_r(m_r^*-1) - C_r(m_r^*) \ge \theta^{\min}$ for all $r \in R_v$. Together with the definition of $\theta^{\min}$ in Eq. (18) we have $m_r(\theta^{\min}) = m_r^*$ for all $r \in R_v$ and thereby $\sum_{r\in R_v} m_r(\theta^{\min}) > M_v$. □

Corollary 1 implies that

Lemma 9. *For a service* $r \in R_v$, $m_r(\theta)$ *decreases strictly monotonically with* θ.

The following Lemma 10 follows from Lemma 5 with Lemma 9.

Lemma 10. *Recall that* $(m_r = \hat{m}_r^*, r \in R_v)$ *is the (to be determined) optimal solution to [P1-v] with* $\sum_{r\in R_v} \hat{m}_r^* = M_v$. *Consider any* $\theta \in [\theta^{\min}, \theta^{\max}]$. *The following results hold:*

(1) If $\sum_{r\in R_v} m_r(\theta) < M_v$, *then* $m_r(\theta) \le \hat{m}_r^*$ *for all* $r \in R_v$.

(2) If $\sum_{r\in R_v} m_r(\theta) > M_v$, *then* $m_r(\theta) \ge \hat{m}_r^*$ *for all* $r \in R_v$.

Proof. We only need to prove (1). Result (2) follows from a similar argument. Lemma 5 implies that there exists θ^* such that $\hat{m}_r^* = m_r(\theta^*)$ and $\sum_{r\in R_v} m_r(\theta^*) = M_v$ for all $r \in R_v$. Lemma 9 implies that $\sum_{r\in R_v} m_r(\theta)$ decreases monotonically with θ. Hence, if $\sum_{r\in R_v} m_r(\theta) < M_v = \sum_{r\in R_v} m_r(\theta^*)$, we have $\theta > \theta^*$. Using Lemma 9 again, we have $m_r(\theta) \le m_r(\theta^*) = \hat{m}_r^*$ for all $r \in R_v$. □

The following Lemmas 11 and 12 are the most important for establishing the polynomial-time algorithm.

Lemma 11. *For any* θ, *if* $\sum_{r\in R_v} m_r(\theta) < M_v$, *then it follows from (20) that increasing the number of ships deployed on* r *from* $m_r(\theta)$ *to* $m_r(\theta)+1$ *reduces the total cost by at most* θ. *The convexity of* $C_r(m_r)$ *implies that increasing the number of ships deployed on* r *from* $m_r(\theta)$ *to*

$\hat{m}_r^*$ reduces the total cost by at most $\theta(\hat{m}_r^* - m_r(\theta))$, i.e., $C_r(m_r(\theta)) - C_r(\hat{m}_r^*) \le \theta(\hat{m}_r^* - m_r(\theta))$. Hence,

$$\sum_{r\in R_v} C_r(m_r(\theta)) - \sum_{r\in R_v} C_r(\hat{m}_r^*) \le \sum_{r\in R_v} \theta(\hat{m}_r^* - m_r(\theta)) = \theta(M_v - \sum_{r\in R_v} m_r(\theta)).$$

Lemma 12. *Consider any $\theta_1 > \theta_2$ such that $\sum_{r\in R_v} m_r(\theta_1) < M_v$ and $\sum_{r\in R_v} m_r(\theta_2) > M_v$. Then, there exists an integer vector $(m_r := \bar{m}_r^*, r \in R_v)$ with $m_r(\theta_1) \le \bar{m}_r^* \le m_r(\theta_2)$ such that $\sum_{r\in R_v} \bar{m}_r^* = M_v$ and*

$$\sum_{r\in R_v} C_r(m_r(\theta_1)) - \sum_{r\in R_v} C_r(\bar{m}_r^*) \ge \sum_{r\in R_v} \theta_2(\bar{m}_r^* - m_r(\theta_1)) = \theta_2\left(M_v - \sum_{r\in R_v} m_r(\theta_1)\right). \tag{21}$$

Proof. Let $\hat{r}$ be such that

$$\hat{r} \quad := \quad \min\left\{ r' \in \{1,2,\ldots,|R_v|\} \,\middle|\, \sum_{r=1}^{r'} m_r(\theta_2) + \sum_{r=r'+1}^{|R_v|} m_r(\theta_1) \ge M_v \right\}. \tag{22}$$

Note that $\hat{r} \in R_v$ exists since $\sum_{r\in R_v} m_r(\theta_2) > M_v$. Define

$$\bar{m}_r^* \quad = \quad \begin{cases} m_r(\theta_2) & \text{for} \quad r = 1,2,\ldots,\hat{r}-1, \\ m_r(\theta_1) & \text{for} \quad r = \hat{r}+1, \hat{r}+2, \ldots, |R_v|, \\ M_v - \sum_{r\in R_v\setminus\{\hat{r}\}} \bar{m}_r^* & \text{for} \quad r = \hat{r}. \end{cases} \tag{23}$$

It follows from (22) that

$$\sum_{r=1}^{\hat{r}-1} m_r(\theta_2) + m_{\hat{r}}(\theta_2) + \sum_{r=\hat{r}+1}^{|R_v|} m_r(\theta_1) \quad \ge \quad M_v, \tag{24}$$

$$\sum_{r=1}^{\hat{r}-1} m_r(\theta_2) + m_{\hat{r}}(\theta_1) + \sum_{r=\hat{r}}^{|R_v|} m_r(\theta_1) \quad < \quad M_v. \tag{25}$$

It follows from (24) that $\bar{m}^*_{\hat{r}} = M_v - \sum_{r\in R_v\setminus\{\hat{r}\}} \bar{m}^*_r \le m_{\hat{r}}(\theta_2)$ and it follows from (25) that $\bar{m}^*_{\hat{r}} = M_v - \sum_{r\in R_v\setminus\{\hat{r}\}} \bar{m}^*_r > m_{\hat{r}}(\theta_1)$. Thus, $m_{\hat{r}}(\theta_1) \le \bar{m}^*_{\hat{r}} \le m_{\hat{r}}(\theta_2)$. Thus, $(m_r := \bar{m}^*_r, r \in R_v)$ is an integer vector with $m_r(\theta_1) \le \bar{m}^*_r \le m_r(\theta_2)$. Clearly, $\sum_{r\in R_v} \bar{m}^*_r = M_v$.

It follows from (20) that increasing the number of ships deployed on r by 1, as long as the number after increase does not exceed $m_r(\theta_2)$, leads to a cost reduction at least θ_2, which further gives (21). □

We present the polynomial-time algorithm in Algorithm 2.

Algorithm 2: A POLYNOMIAL-TIME ALGORITHM FOR SOLVING [P1-v]

Input: [P1-v] model, [P2(r, m_r)] model for all $r \in R_v$, $\epsilon > 0$. Set index $\kappa \leftarrow 1$.

Output: ϵ-optimal solution $(\hat{m}^*_r, r \in R_v)$ to the [P1-v] model

Step 0. Pre-processing. Execute Algorithm 3.

Step 1. Set $\theta \leftarrow (UB^\kappa + LB^\kappa)/2$. For each $r \in R_v$, use bi-section search to find the number of ships to be deployed, denoted by $m_r(\theta)$, such that (20) holds (Lemma 7).

Step 2. If $\sum_{r\in R_v} m_r(\theta) = M_v$, then $(\hat{m}^*_r = m_r(\theta), r \in R_v)$ is the optimal solution to [P1-v] and stop (Lemma 5).

Step 3. If $\sum_{r\in R_v} m_r(\theta) > M_v$, then $(m_r = m_r(\theta), r \in R_v)$ is infeasible to [P1-v]. We thus need to increase the value of θ (Lemma 9). Set $LB^{\kappa+1} \leftarrow \theta$, $UB^{\kappa+1} \leftarrow UB^\kappa$, $\kappa \leftarrow \kappa + 1$, and go to **Step 1**.

Step 4. If $\sum_{r\in R_v} m_r(\theta) < M_v$, then $(m_r = m_r(\theta), r \in R_v)$ is feasible but not optimal. We first check the optimality gap.

(4.1) If $(\theta - LB^\kappa)M_v \le \epsilon$, i.e., if $|\theta - LB^\kappa| \le \epsilon/M_v$, find an integer vector $(m_r := \bar{m}^*_r, r \in R_v)$ such that $m_r(\theta) \le \bar{m}^*_r \le m_r(LB^\kappa)$ and $\sum_{r\in R_v} \bar{m}^*_r = M_v$ according to (23). Then, $(\hat{m}^*_r := \bar{m}^*_r, r \in R_v)$ is an ϵ-approximation solution and stop.

(4.2) If $(\theta - LB^\kappa)M_v \ge \epsilon$, set $UB^{\kappa+1} \leftarrow \theta$, $LB^{\kappa+1} \leftarrow LB^\kappa$, $\kappa \leftarrow \kappa + 1$, and go to **Step 1**.

Recall that $C^*(v)$ is the (unknown) optimal objective function value of [P1-v]. Define tolerance error $\epsilon > 0$ and the algorithm will stop if we find a solution with objective value

Algorithm 3: THE PRE-PROCESSING IN ALGORITHM 2

Input: [P1-v] model, [P2(r, m_r)] model for all $r \in R_v$.

Output: UB^1, LB^1, $(m_r^{\min}, r \in R_v)$, $(m_r^*, r \in R_v)$

Step 0. For each service $r \in R_v$, calculate $m_r^{\min}$ by Eq. (15). Check whether [P2(r, $m_r^{\min}$)] is feasible (Lemma 4). If there is an $r' \in R_v$ such that [P2(r', $m_{r'}^{\min}$)] is infeasible, [P1-v] is infeasible and stop.

Step 1. For each service $r \in R_v$, use bi-section search on $m_r \in \{m_r^{\min}, m_r^{\min}+1, \ldots, M_v\}$ to find the optimal solution to [P2(r, m_r)], denoted by m_r^* (Lemma 6). If $\sum_{r \in R_v} m_r^* \le M_v$, then we should deploy m_r^* ships on service r and stop.

Step 2. Check each service $r \in R_v$. If $m_r^* = m_r^{\min}$, then we should deploy m_r^* ships on it and hence we set $M_v \leftarrow M_v - m_r^*$ and $R_v \leftarrow R_v \setminus \{r\}$. If $R_v = \varnothing$, stop.

Step 3. Compute $\theta^{\max}$ by Eq. (17). Compute $\theta^{\min}$ by Eq. (18) or Eq. (19).

Step 4. Set upper bound $UB^1 := \theta^{\max}$, lower bound $LB^1 := \theta^{\min}$.

of at most $C^*(v) + \epsilon$. We define a upper bound on θ as $UB^1 := \theta^{\max}$ and a lower bound on θ as $LB^1 := \theta^{\min}$. Note that it follows from Lemma 8 that $\sum_{r \in R_v} m_r(LB^1) > M_v$.

The following Remark 2 guarantees the validity of the stopping criterion in Step (4.1).

Remark 2. If Algorithm 2 stops in Step (4.1), then $\sum_{r \in R_v} C_r(\bar{m}_r^*) - C^*(v) \le \epsilon$.

To see this, Lemma 11 implies that

$$\sum_{r \in R_v} C_r(m_r(\theta)) - C^*(v) \le \theta(M_v - \sum_{r \in R_v} m_r(\theta)). \tag{26}$$

Note that $(m_r = m_r(LB^\kappa), r \in R_v)$ is infeasible as $\sum_{r \in R_v} m_r(LB^\kappa) > M_v$. We can thus choose an integer vector $(m_r := \bar{m}_r^*, r \in R_v)$ such that $m_r(\theta) \le \bar{m}_r^* \le m_r(LB^\kappa)$ and $\sum_{r \in R_v} \bar{m}_r^* = M_v$ according to (23), then $(m_r := \bar{m}_r^*, r \in R_v)$ is feasible to [P1-v] and

Lemma 12 implies

$$\sum_{r\in R_v} C_r(m_r(\theta)) - \sum_{r\in R_v} C_r(\bar{m}_r^*) \geq LB^\kappa(M_v - \sum_{r\in R_v} m_r(\theta)). \tag{27}$$

Eqs. (26) and (27) lead to

$$\sum_{r\in R_v} C_r(\bar{m}_r^*) - C^*(v) \leq (\theta - LB^\kappa)(M_v - \sum_{r\in R_v} m_r(\theta)) \leq (\theta - LB^\kappa)M_v.$$

The following Remark 3 claims that Algorithm 2 is a polynomial-time algorithm with respect to precision $\epsilon > 0$.

Remark 3. **Step 1** of Algorithm 2 is implemented for $O\left(\log \frac{\theta^{\max}-\theta^{\min}}{\epsilon/M_v}\right)$ times, i.e.,

$$O\left(\log(M_v(\theta^{\max} - \theta^{\min})) + \log\frac{1}{\epsilon}\right)$$

times. Each iteration of **Step 1** needs to find $|R_v|$ values of $m_r(\theta)$, which can be completed in polynomial time according to Lemma 7. Therefore, Algorithm 2 can find an optimal solution to [P1-v] in polynomial time.

Remark 4. The sequence of solutions $(\hat{m}_r^* := \bar{m}_r^*, r \in R_v)$ obtained in Step (4.1) of Algorithm 2 in different iterations κ converges to the optimal solution with rate of convergence $O\left(1/2^\kappa\right)$, meaning that the sequence $(\hat{m}_r^* := \bar{m}_r^*, r \in R_v)$ in different iterations κ approximately linearly converges to the optimal solution.

6. Numerical experiments

In this section we report the results of computational experiments. The experiments are implemented on a PC equipped with 3.30GHz of Intel Core i5 CPU and 4GB of RAM. The algorithm is coded in Matlab 2011b.

6.1. Efficiency of Algorithm 2

The first group of test instances is on sailing speed optimization, i.e., Model [P1-v]. We solve the problems using Algorithm 2, in which Model [P2(r, m_r)] is solved by the interior point method of Matlab function "fmincon".

We consider ships with a capacity of 8,000 twenty-foot equivalent units with parameters $c_v = \$210,000$/week, $a_{ri} = 4.667 \times 10^{-4}$, $b = 2.118$, and $\hat{t}_{ri} = 24$ hours (Wang and Meng, 2012b). The bunker price $\alpha = \$200$/ton. We consider different combinations of the number of services $|R_v| \in \{5, 10, 15\}$ and maximum number of ports of call on a service $(\max_{r \in R_v} I_r) \in \{5, 10, 15\}$ (Ng, 2014). The number of ships of type v in the fleet is $M_v := \lceil 0.8 \times |R_v| \times \max_{r \in R_v} I_r \rceil$. The voyage distance of a leg L_{ri} is uniformly generated between 1 and 5000 nautical miles. The minimum sailing time $t_{ri}^{\min}$ is equal to L_{ri} divided by the maximum sailing speed, which is defined to be 25 knots. The maximum transit time $t_{rij}^{\max}$ is equal to twice the minimum possible transit time from the ith port of call to the jth on service r. For each combination of $|R_v|$ and $\max_{r \in R_v} I_r$, we randomly generate 20 instances, each of which has different numbers of ports of call on a service (uniformly generated between 2 and $\max_{r \in R_v} I_r$) and different voyage distances of a leg. The computation error ϵ in Algorithm 2 is set to be \$100/week.

The results of computation time are reported in Table 1. We can see that as expected, the computation time increases with the number of services and the maximum number of ports of call on a service. The number of services has a larger impact on the computation time than the maximum number of ports of call on a service. Overall, Algorithm 2 is very efficient: when there are 15 services with a maximum of 15 ports of call on a service and 180 ships in the fleet, the average computation time is less than half a minute. Finally, we note that most of the computation time is spent in solving Model [P2(r, m_r)] by the interior point method of Matlab function "fmincon".

Table 1: Average computation time per instance of the sailing speed optimization problem

$\|R_v\|$	$\max_{r\in R_v} I_r$	M_v	CPU time (s)	$\|R_v\|$	$\max_{r\in R_v} I_r$	M_v	CPU time (s)
5	5	20	1.7238	10	10	80	7.8328
5	10	40	1.5366	10	15	120	8.9498
5	15	60	2.1185	15	10	120	17.8301
10	5	40	4.5451	15	15	180	26.0646

6.2. Comparison between Algorithm 1 and Algorithm 2

The second group of test instances is conducted in order to show the superiority of Algorithm 2, the polynomial algorithm, over Algorithm 1, the pseudo-polynomial algorithm. We consider similar settings as the ones in Section 6.1. The random test instances have different numbers of services: $|R| \in \{5, 10, 20, 50, 100\}$. For each $|R|$, we randomly generate 20 instances, each of which has different numbers of ships. Moreover, different services have different voyage distances and time spent at port. The computation error ϵ is set to be 1.

We let all $t_{ri}^{\min}$ be 0 and all $t_{rij}^{\max}$ be infinity. As a result, given the number of ships to deploy on a service, we can easily solve [P2(r, m_r)]) as the optimal speeds on different legs are the same. Therefore, we compare the number of times $C_r(m_r)$ is computed (through solving [P2(r, m_r)]) when Algorithm 1 is used and that when Algorithm 2 is used. The results are reported in Table 2, where the column "#Pseudo-polynomial" means the average number of times $C_r(m_r)$ is computed per instance by Algorithm 1, the column "#Polynomial" means the average number of times $C_r(m_r)$ is computed per instance by Algorithm 2, and the column "Ratio" is the ratio of the computation times by the two algorithms. We stress again that we report the number of times $C_r(m_r)$ is computed because both algorithms are very efficient. From the results we can see that the polynomial algorithm significantly reduces the number of times $C_r(m_r)$ is computed. More importantly, when the problem size increases, the advantage of the polynomial algorithm over the pseudo-polynomial one is more evident. This provides strong evidence of the practical relevance of the polynomial algorithm.

Table 2: Average number of times $C_r(m_r)$ is computed per instance by the two algorithms

$\lvert R \rvert$	#Pseudo-polynomial	#Polynomial	Ratio
5	140.60	61.05	2.3
10	613.50	158.30	3.9
20	2372.70	376.50	6.3
50	16122.95	1150.55	14.0
100	63857.50	2581.75	24.7

7. Concluding comments

In this paper, we looked into the containership sailing speed optimization problem, in which a container shipping line needs to allocate its limited resources (i.e., containerships) over a network of services (i.e., ship routes). The problem can be formulated as a mixed-integer nonlinear programming model from the perspective of supply chain management and a model from the perspective of shipping lines. The main contribution of our research lies in that we show the sailing speed optimization problem with containership resource sharing is not NP-hard, but in P, by proposing a polynomial-time algorithm that can be used to solve both the models. The algorithm uses a bi-section search method over a finite domain of a parameter that measures the marginal cost of each service and finds an ϵ-approximation solution in polynomial time. We provided various theoretical results that justify the validity of the algorithm. While our algorithm was designed with the intention to solve the sailing speed optimization problem, it could potentially be applied to solve a general class of mathematical programming models that can be decomposed as a bunch of sub-models linked with a resource sharing constraint, such as how to allocate buses to different bus routes (Liu et al., 2016), and how to allocate trains to subway routes.

References

Álvarez, J. F., 2012. Mathematical expressions for the transit time of merchandise through a liner shipping network. Journal of the Operational Research Society 63 (6), 709–714.

Cariou, P., 2011. Is slow steaming a sustainable means of reducing CO_2 emissions from container shipping? Transportation Research Part D: Transport and Environment 16 (3), 260–264.

Chubanov, S., 2016. A polynomial-time descent method for separable convex optimization problems with linear constraints. SIAM Journal on Optimization 26 (1), 856–889.

Du, Y., Chen, Q., Quan, X., Long, L., Fung, R. Y., 2011. Berth allocation considering fuel consumption and vessel emissions. Transportation Research Part E: Logistics and Transportation Review 47 (6), 1021–1037.

Du, Y., Meng, Q., Wang, Y., 2015. Budgeting fuel consumption of container ship over round-trip voyage through robust optimization. Transportation Research Record: Journal of the Transportation Research Board (2477), 68–75.

Ghosh, S., Lee, L. H., Ng, S. H., 2015. Bunkering decisions for a shipping liner in an uncertain environment with service contract. European Journal of Operational Research 244 (3), 792–802.

Hvattum, L. M., Norstad, I., Fagerholt, K., Laporte, G., 2013. Analysis of an exact algorithm for the vessel speed optimization problem. Networks 62 (2), 132–135.

Karsten, C. V., Brouer, B. D., Desaulniers, G., Pisinger, D., 2016. Time constrained liner shipping network design. Transportation Research Part E: Logistics and Transportation Review in press.

Karsten, C. V., Pisinger, D., Ropke, S., Brouer, B. D., 2015. The time constrained multi-commodity network flow problem and its application to liner shipping network design. Transportation Research Part E: Logistics and Transportation Review 76, 122–138.

Kim, H., 2014. A Lagrangian heuristic for determining the speed and bunkering port of a ship. Journal of the Operational Research Society 65 (5), 747–754.

Kim, H.-J., Chang, Y.-T., Kim, K.-T., Kim, H.-J., 2012. An epsilon-optimal algorithm considering greenhouse gas emissions for the management of a ship's bunker fuel. Transportation Research Part D: Transport and Environment 17 (2), 97–103.

Kim, J.-G., Kim, H.-J., Lee, P. T.-W., 2013. Optimising containership speed and fleet size under a carbon tax and an emission trading scheme. International Journal of Shipping and Transport Logistics 5 (6), 571–590.

Kim, J.-G., Kim, H.-J., Lee, P. T.-W., 2014. Optimizing ship speed to minimize fuel consumption. Transportation Letters: the International Journal of Transportation Research 6 (3), 109–117.

Kontovas, C., Psaraftis, H. N., 2011. Reduction of emissions along the maritime intermodal container chain: operational models and policies. Maritime Policy & Management 38 (4), 451–469.

Li, C., Qi, X., Lee, C.-Y., 2015. Disruption recovery for a vessel in liner shipping. Transportation Science 49 (4), 900–921.

Li, C., Qi, X., Song, D., 2016. Real-time schedule recovery in liner shipping service with regular uncertainties and disruption events. Transportation Research Part B: Methodological in press.

Liu, Z., Wang, S., Chen, W., Zheng, Y., 2016. Willingness to board: A novel concept for modeling queuing up passengers. Transportation Research Part B: Methodological 90, 70–82.

Mansouri, S. A., Lee, H., Aluko, O., 2015. Multi-objective decision support to enhance environmental sustainability in maritime shipping: A review and future directions. Transportation Research Part E: Logistics and Transportation Review 78, 3–18.

Meng, Q., Du, Y., Wang, Y., 2016. Shipping log data based container ship fuel efficiency modeling. Transportation Research Part B: Methodological 83, 207–229.

Ng, M., 2014. Distribution-free vessel deployment for liner shipping. European Journal of Operational Research 238 (3), 858–862.

Notteboom, T. E., Vernimmen, B., 2009. The effect of high fuel costs on liner service configuration in container shipping. Journal of Transport Geography 17 (5), 325–337.

Psaraftis, H. N., Kontovas, C. A., 2013. Speed models for energy-efficient maritime transportation: A taxonomy and survey. Transportation Research Part C: Emerging Technologies 26, 331–351.

Psaraftis, H. N., Kontovas, C. A., 2014. Ship speed optimization: Concepts, models and combined speed-routing scenarios. Transportation Research Part C: Emerging Technologies 44, 52–69.

Qi, X., Song, D.-P., 2012. Minimizing fuel emissions by optimizing vessel schedules in liner shipping with uncertain port times. Transportation Research Part E: Logistics and Transportation Review 48 (4), 863–880.

Ronen, D., 2011. The effect of oil price on containership speed and fleet size. Journal of the Operational Research Society 62 (1), 211–216.

Song, D.-P., Dong, J.-X., 2013. Long-haul liner service route design with ship deployment and empty container repositioning. Transportation Research Part B: Methodological 55, 188–211.

Song, D.-P., Li, D., Drake, P., 2015. Multi-objective optimization for planning liner shipping service with uncertain port times. Transportation Research Part E: Logistics and Transportation Review 84, 1–22.

Wang, S., 2016. Fundamental properties and pseudo-polynomial-time algorithm for network containership sailing speed optimization. European Journal of Operational Research 250 (1), 46–55.

Wang, S., Meng, Q., 2012a. Liner ship route schedule design with sea contingency time and port time uncertainty. Transportation Research Part B: Methodological 46 (5), 615–633.

Wang, S., Meng, Q., 2012b. Sailing speed optimization for container ships in a liner shipping network. Transportation Research Part E: Logistics and Transportation Review 48 (3), 701–714.

Wang, S., Meng, Q., Liu, Z., 2013. Containership scheduling with transit-time-sensitive container shipment demand. Transportation Research Part B: Methodological 54, 68–83.

Wong, E. Y., Tai, A. H., Lau, H. Y., Raman, M., 2015. An utility-based decision support sustainability model in slow steaming maritime operations. Transportation Research Part E: Logistics and Transportation Review 78, 57–69.

Yao, Z., Ng, S. H., Lee, L. H., 2012. A study on bunker fuel management for the shipping liner services. Computers & Operations Research 39 (5), 1160–1172.