On the Uniqueness of User Equilibrium Flow with Speed Limit

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Abstract

This technical note focuses on the link flow uniqueness of user equilibrium (UE) with speed limits. Under a mild assumption on the monotonicity of link travel time function, the UE link flow solutions are well recognized to be unique. However, the incorporation of speed limits in the network has undermined the strict monotonicity of link travel time functions, thus the UE flows on the links with speed limits may not be unique. This note addresses the uniqueness problem with two major contributions. First, a polyhedron defined on links is provided, and it is proven that the UE link flow is unique if and only if the polyhedron only contains one value. Second, two concise methods are proposed to mathematically check whether the polyhedron is a singleton, which can be easily solved and convenient for practical use.

Keywords: speed limits; user equilibrium; link flow; solution uniqueness; network-based analysis; polyhedron.

1 Introduction

Network-based speed limit evaluation and setting is addressed only very recently. Most of the existing studies on speed limit focus on the local impacts of speed limit on traffic safety and emission mitigation, see, e.g., Grabowski and Morrisey (2007); Bel et al. (2015); Islam and

El-Basyouny (2015), among many others. However, the impacts of speed limits on the traffic flows are system wide, which would affect the route choice of each traveler in the whole network (Ramaekers et al., 2013; Chu et al., 2015). To provide a methodology on the analysis of speed limits on the network flow reallocation, Yang et al. (2012) developed an optimization model for the user equilibrium (UE) flows under speed limits.

Wang et al. (2015) extended the work of Yang et al. (2012) by integrating the speed limit with road pricing scheme, with the objective of achieving a Pareto-efficient flow pattern. Yan et al. (2015) proposed a speed limit and reliability-based user equilibrium in a degradable transport network. Focusing on the dynamic and stochastic case, Zhu and Ukkusuri (2014) provided a link-based network loading model allowing the change of speed limits. Yang et al. (2015) examined the local and global impacts of speed limits with consideration of the commuters' non-obedient behavior. Apart from the flow equilibrium, the impacts of speed limits on some other network-wide issues are also investigated. Wang (2013) discussed the efficiency and equity of a transportation network with speed limits. Based on the network equilibrium models, some studies further analyzed the optimal design of speed limits in the transportation network. Considering multiple objectives, Yang et al. (2013) proposed a bi-level model for the optimal setting of speed limits scheme. Yu et al. (2014) then extended this work by proposing a robust model for the speed limits scheme design.

Recently, a number of studies further examined the impacts of variable speed limits on the traffic congestions. Focusing at the freeway bottlenecks, Chen et al. (2014) and Chen and Ahn (2015) investigated the speed limits as a control tool to mitigate queue around the bottleneck. In a similar context, Jin and Jin (2015) studied the control of a lane-drop bottleneck using variable speed limits. Liu et al. (2015) examined the variable speed limits from an equilibrium perspective with consideration of the commuters' long term responses to the variables speed limits. Grumert et al. (2015) analyzed the individualized speed limits using microscopic traffic simulation, where the communications between the infrastructure and the vehicles (V2I) are addressed. Khondaker and Kattan (2015) on the other hand addressed the variable speed limit control in a connected vehicle environment (V2V).

The macroscopic traffic assignment models provide quantitative and systematic measures of the speed limits, which are important evidence to the transport authorities when planning or changing the speed limit settings. It is of considerable importance to know whether the UE solution (equilibrium link flows) to the model is unique (Bahat and Bekhor, 2017; Liu and Huang, 2017). This is because if the solution is not unique, it implies that one set of speed limits may give rise to different network conditions, and some of these possible/potential conditions may not satisfy the targets of transport authorities. Therefore, a methodology is needed to check the uniqueness of the traffic assignment problems with speed limits. The uniqueness of UE link flows and some other traffic assignment models (e.g., stochastic user equilibrium) are well resolved, based on the convexity of the models; see, e.g., Sheffi (1985); Liu et al., (2014); Iryo (2015); Hwang and Cho (2017).

Nevertheless, in the context of speed limits, only Yang et al. (2012) has initiated a discussion on the link flow uniqueness of traffic assignment problem with speed limits. Based on the mathematical conditions of UE, Yang et al. (2012) concluded that the UE link flows on the not binding links are unique, while on the binding links the flow uniqueness problem is equivalent to checking the number of feasible values in a polyhedron. However, this is not an easy task, as the polyhedron is defined on path flows and requires path enumeration.

To further solve the uniqueness problem of UE with speed limits, this technical note has two contributions: first, as an extension of Yang et al. (2012), we build a link-based polyhedron, which avoids path enumeration and storage; second, two methods are proposed to check whether the polyhedron is a singleton, and answers the link flow uniqueness problem accordingly. It is also proven that these two methods can be done in polynomial time with regard to the size of the network, thus they are convenient for practical use.

2 User Equilibrium under Speed Limits

For the ease of presentation, we provide the following list of notation, which in general follows that of Yang et al. (2012). Furthermore, the assumptions in Yang et al. (2012) also hold in this paper.

A	Set of directed links.
C_a	Capacity of link $a \in A$.
d_w	Travel demand between OD pair $w \in W$.
$f_{r,w}$	Traffic flow on path $r \in R_w$ between OD pair $w \in W$. $f = \left(f_{r,w}, r \in R_w, w \in W \right)^{\mathrm{T}}$ is the vector of all the path flows.
N	Set of nodes.
R_w	Set of all the paths between OD pair $w \in W$.
R_w^+	Set of equilibrated paths between O-D pair $w \in W$, i.e. these paths have the shortest travel time in the UE condition.
$\bar{s}_a$	Speed limit on link $a \in A$. The speed limits on all the links are grouped in vector $\bar{s} = \left(\bar{s}_a, a \in A \right)^{\mathrm{T}}$.
$t_a\left(v_a\right)$	Travel time function on link $a \in A$ considering the speed limit (if there is).
$\tilde{t}_a\left(v_a\right)$	Original travel time function of link $a \in A$ without speed limit.
$\bar{t}_a$	Travel time on link $a \in A$ at speed $\bar{s}_a$.
v_a	Traffic flow on link $a \in A$, which are grouped into vector $v = \left(v_a, a \in A \right)^{\mathrm{T}}$.
$\bar{v}_a$	Traffic flow on link $a \in A$ at speed $\bar{s}_a$, $\tilde{t}_a\left(\bar{v}_a\right) = \bar{t}_a$, $\bar{v}_a \le C_a$.
W	Set of the origin-destination (OD) pairs.
$\delta_{a,r}$	Link-path incidence relationship $\delta_{a,r} = 1$ if path r traverses link a, $\delta_{a,r} = 0$ otherwise.

Consider a strongly connected transport network, denoted by $G = \left(N, A \right)$. The OD demand d_w, path flow $f_{r,w}$ and link flow v_a should fulfill the following flow conservation equations.

$$\sum_{r \in R_w} f_{r,w} = d_w, \quad w \in W \tag{1}$$

$$f_{r,w} \geq 0, \; r \in R_w, w \in W \tag{2}$$

$$\sum_{w \in W} \sum_{r \in R_w} f_{r,w} \delta_{a,r} = v_a, \quad a \in A \tag{3}$$

In addition, the flow on each link is constrained by its physical capacity C_a. The flow conservation equations and link capacity constraints define the feasible set of flows. Let Ω_f be the set of all feasible path flows, which is

$$\Omega_f = \left\{ f \middle| f_{r,w} \geq 0, \sum_{r \in R_w} f_{r,w} = d_w, \; r \in R_w, w \in W; \sum_{w \in W} \sum_{r \in R_w} f_{r,w} \delta_{a,r} \leq C_a, a \in A \right\} \tag{4}$$

Consequently, the set of all feasible link flows Ω_v is given by

$$\Omega_v = \left\{ v \middle| v_a = \sum_{w \in W} \sum_{r \in R_w} f_{r,w} \delta_{a,r}, v_a \leq C_a, a \in A, f \in \Omega_f \right\} \tag{5}$$

where $\delta_{a,r}$ equals 1 if link a is on path r, and 0 otherwise.

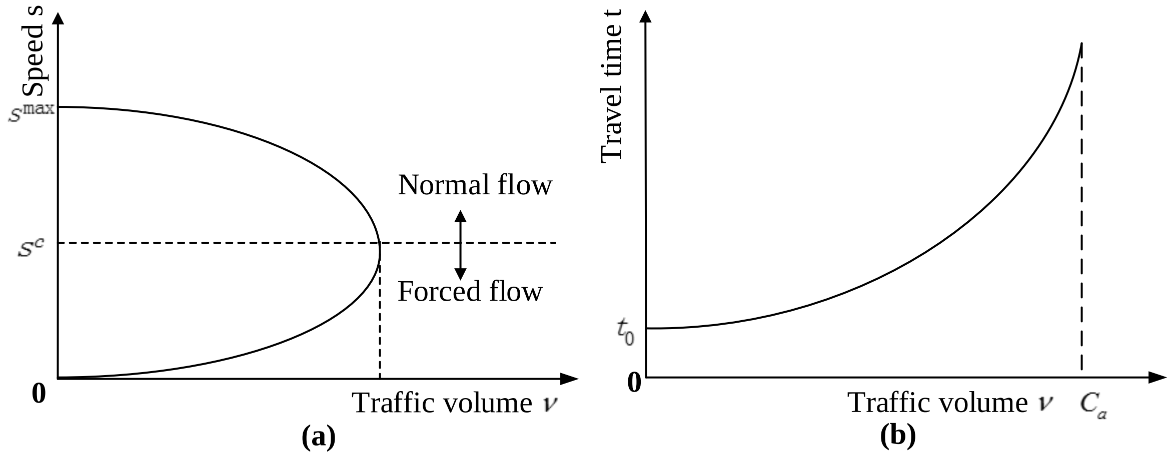

Fig. 1. Speed-flow and travel time-flow relationship without speed limit

When there is no speed limit, Figure 1 gives the speed-flow relationship (1a) and the link travel time function (1b), which are widely adopted in the studies of transport network modelling (Sheffi, 1985; Liu et al., 2013). Let $\tilde{t}_a(v_a)$ denote this *original* link travel time function in Figure 1b. For instance, the following function form, known as Bureau of Public Roads (BPR) functions are commonly used:

$$\tilde{t}_a(v_a) = t_a^0\left(1.0 + 0.15\left(\frac{v_a}{C_a}\right)^4\right), a \in A \tag{6}$$

where t_a^0 denotes the free flow travel time and C_a is the capacity of each link. Note that our definitions and models do not relate to this BPR function.

As discussed in Yang et al. (2012), for the network-wide analysis of speed limits, we focus on the normal flow regime of the speed-flow relationship, which is the upper part of Figure 1a. Hence, when there is a speed limit $\bar{s}_a$ on link a, the speed-flow relationship becomes the one in Figure 2(a). Not that the upper bound of travel speed $S^{\max}$ (see Figure 1a) corresponds to the free flow travel time t_0 (see Figure 1b), where the traffic volume is zero on the road and it is a free flow travel condition. Hence, the speed limit then imposes a lower bound to the link travel time function. Correspondingly, the link travel time function turns to be the one shown in Figure 2(b). In such case, the link travel time function $t_a(v_a)$ takes the following form:

$$t_a(v_a) = \begin{cases} \bar{t}_a, & 0 \le v_a \le \bar{v}_a \\ \tilde{t}_a(v_a), & \bar{v}_a < v_a \le C_a \end{cases} \tag{7}$$

where $\bar{t}_a$ is the travel time on link $a \in A$ at speed $\bar{s}_a$, and $\bar{v}_a$ is the traffic flow such that $\tilde{t}_a(\bar{v}_a) = \bar{t}_a$. It is assumed that there exists at least one link $a \in A$ whose speed limit $\bar{s}_a$ is within the range $\left[s_a^{\mathrm{c}}, s_a^{\max}\right)$, where $s_a^{\max}$ and s_a^{c} are defined in Figure 1(a) and Figure 2(a) as the upper-bound and lower-bound of the speed limit, respectively. It should be pointed out that, as shown in Figure 1a, there is also a lower bound for the travel speed S^C, which is reflected by the volume capacity. Since the flow on each link cannot exceed its capacity, such a lower bound is always fulfilled.

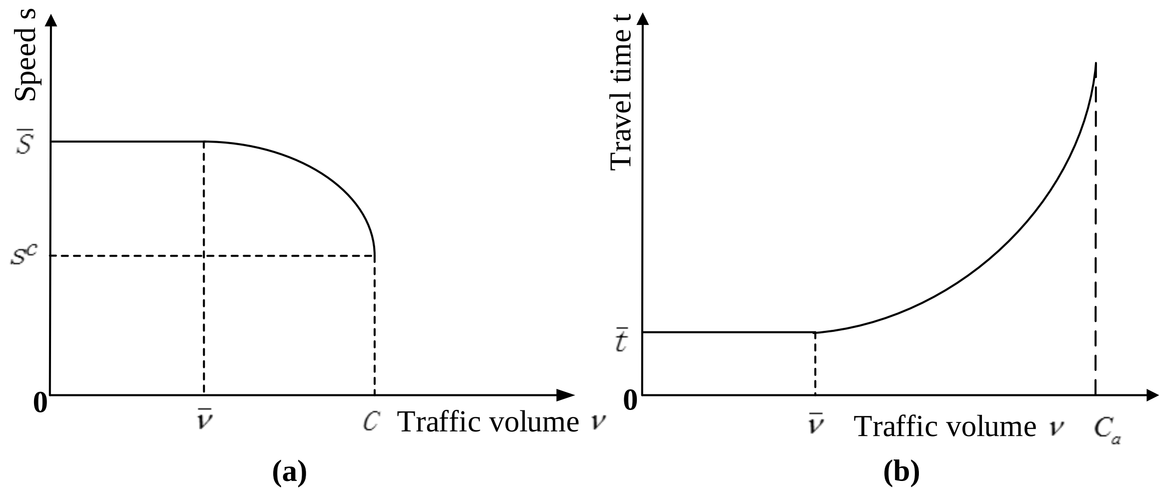

Fig.2. Speed-flow and travel time-flow relationship with a speed limit

Based on the travel time function $t_a\left(v_a\right)$, the following user equilibrium (UE) conditions are then given:

$$\begin{aligned} &\sum_{a\in A} t_a\left(v_a^*\right)\delta_{a,r} = \mu_w^*, \quad \text{if } f_{r,w}^* > 0, \ r\in R_w, w\in W \\ &\sum_{a\in A} t_a\left(v_a^*\right)\delta_{a,r} \geq \mu_w^*, \quad \text{if } f_{r,w}^* = 0, \ r\in R_w, w\in W \end{aligned} \tag{8}$$

where μ_w^* is the minimal travel time between O-D pair w.

Assumption 1: We adopt the fundamental assumption in the transportation discipline that there is at least one link flow vector v^* that fulfills the UE conditions (8) and satisfies $v_a^* < C_a$ for all $a \in A$.

Any link flow vector that fulfills the UE conditions (8) and satisfies $v_a^* \leq C_a$ for all $a \in A$ is equilibrium flows. In the sequel, when we mention v^*, it is referred to the one that fulfills the UE conditions (8) and satisfies $v_a^* < C_a$ for all $a \in A$. *As a result of Assumption 1*, the following optimization model proposed by Beckmann is well-known to be equivalent to the UE conditions (Sheffi, 1985):

$$\min_{v\in\Omega_v} \sum_{a\in A}\int_0^{v_a} t_a\left(\omega\right)\mathrm{d}\omega \tag{9}$$

Furthermore, the optimization model (9) is equivalent to the following variational inequality problem (VIP):

$$\sum_{a\in A} t_a(v_a^*)(\tilde{v}_a - v_a^*) \geq 0, \forall \tilde{v} \in \Omega_v \tag{10}$$

3 Yang et al. (2012)'s Path-based Condition for the Uniqueness of UE Link Flows with Speed Limit

As discussed in the Introduction, it is necessary to examine the existence and uniqueness of the UE flows. As the feasible set Ω_v is compact and nonempty, the objective function (9) is continuous, the existence of UE solutions can be guaranteed (Yang et al., 2012). However, with the existence of speed limit, the link travel time function (see Figure 2b) is not strictly increasing, thus there may be more than one UE solution for a particular network. For the ease of presentation, we have the following Definition 1.

Definition 1. Let Ω_v^* denote the set of all the UE link flow solutions under speed limit $\bar{s}$, $\Omega_v^* \subset \Omega_v$. Ω_v^* is the set of link flows that fulfill the UE conditions (8), or equivalently are the optimal solutions to the Beckmann's model (9), or equivalently satisfy the variational inequality (10). □

Under Assumption 1, the set Ω_v^* is non-empty and it may or may not be a singleton. When Ω_v^* is not a singleton, Yang et al. (2012) proved the results in Proposition 1 that characterize the relations between the UE link flow vectors.

Proposition 1. (i) All the UE link flow vectors in Ω_v^* have the same set of binding links $\bar{A}$ (links with binding speed limits) and not binding links $\bar{A}^c$. $\bar{A}^c$ is the complement set of $\bar{A}$, i.e., $A = \bar{A} \bigcup \bar{A}^c$. (ii) For any two vectors v^1 and v^2 in Ω_v^*, we have:

$$t_a\left(v_a^1\right) = t_a\left(v_a^2\right), a \in A \tag{11}$$

meaning that the travel time of each link is the same for different UE flow vectors. Moreover, the flow on each not binding link $a \in \bar{A}^c$ is also the same:

$$v_a^1 = v_a^2, a \in \bar{A}^c \tag{12}$$

(iii) On a binding link $a \in \bar{A}$, the UE link flow may not be unique, i.e. $v_a^1 \neq v_a^2$. This means it is possible that Ω_v^* contains more than one UE link flow vector, i.e., Ω_v^* is not a singleton. □

To analyze whether Ω_v^* contains more than one UE link flow, define R_w^+ as the set of equilibrated paths (i.e., shortest paths) between O-D pair $w \in W$ and R_w^{+c} the set of non-equilibrated paths within O-D pair $w \in W$, that is, the complement of R_w^+. Eqs. (11) imply that sets R_w^+ and R_w^{+c} are invariant for different UE link flow vectors. Suppose that $(v_a^*, a \in A)$ is an obtained UE link flow vector, i.e., $(v_a^*, a \in A) \in \Omega_v^*$. We define

Definition 2. $\bar{\Phi}$ is the polyhedral set of link flows $(v_a, a \in \bar{A})$ defined by the following conditions:

$$\sum_{w \in W} \sum_{r \in R_w^+} f_{r,w} \delta_{a,r} = v_a^*, \quad a \in \bar{A}^c \tag{13}$$

$$\sum_{w \in W} \sum_{r \in R_w^+} f_{r,w} \delta_{a,r} = v_a, \quad a \in \bar{A} \tag{14}$$

$$\sum_{r \in R_w^+} f_{r,w} = d_w, \quad w \in W \tag{15}$$

$$f_{r,w} \geq 0, \; r \in R_w^+, w \in W \tag{16}$$

$$f_{r,w} = 0, \; r \in R_w^{+c}, w \in W \tag{17}$$

$$v_a \leq \bar{v}_a, \quad a \in \bar{A} \tag{18}$$

Note that the fact of $\bar{v}_a \leq C_a$ and Eqs. (18) imply that $v_a \leq C_a, \;\; a \in A$. Note further that $\bar{\Phi}$ is a set of $|\bar{A}|$-dimensional vectors and Ω_v^* is a set of $|A|$-dimensional vectors. Yang et al. (2012) proved that

Proposition 2. Let v_a^* be a given UE flow vector $(v_a^*, a \in A)$. (i) If vector $(\hat{v}_a, a \in A) \in \Omega_v^*$, then $(v_a = \hat{v}_a, a \in \bar{A}) \in \bar{\Phi}$. (ii) If vector $(\breve{v}_a, a \in \bar{A}) \in \bar{\Phi}$, then $(v_a = \breve{v}_a, a \in \bar{A}; v_a = v_a^*, a \in \bar{A}^c) \in \Omega_v^*$. (iii) Ω_v^* is a singleton if and only if $\bar{\Phi}$ is a singleton. □

In sum, Yang et al. (2012) proved that to check whether the UE link flow is unique is equivalent to checking whether the polyhedron $\bar{\Phi}$ is a singleton.

4 A Link-based Condition for the Uniqueness of UE Link Flows with Speed Limit

As per Proposition 2, the problem of link flow uniqueness is then converted to checking the number of vectors in the polyhedron $\bar{\Phi}$. However, it can be seen that the polyhedron $\bar{\Phi}$ is defined on path flows, because Eqs. (13)-(17) contain path flow variables $f_{r,w}$. Hence, it requires path enumeration and storage, which is a very tedious work on large networks. To further investigate the uniqueness of UE flows, the first focus/contribution of this technical note is to build a new polyhedron that is only defined on link flows. To this end, we first introduce a set of link-based flow conservation equations. Let O denote the set of all the origins, $W(o)$ the set of OD pairs whose origin is o, and v_a^o the traffic flow on link $a \in A$ originating from $o \in O$. The set of link-based flow conservation equations are (Akamatsu, 1997):

$$\sum_{a \in A_h(n)} v_a^o - \sum_{a \in A_t(n)} v_a^o = d_w \delta_{wn} - \sum_{w \in W(o)} d_w \bar{\delta}_{nw}, \forall o \in O, n \in N \quad (19)$$

$$v_a = \sum_{o \in O} v_a^o, \forall a \in A \quad (20)$$

$$v_a^o \geq 0, \forall o \in O, \forall a \in A \quad (21)$$

Eqs. (19) define the flow conservation equation on each node n, where $A_t(n)$ and $A_h(n)$ denote the set of links whose tail or head nodes are node n, respectively. δ_{wn} ($\bar{\delta}_{nw}$) reflects the topological incidence relationships between node n and the destination (origin) of OD pair w, i.e. $\delta_{wn} = 1$ if n is the destination of w and $\delta_{wn} = 0$ otherwise. Eqs. (20) define the total flow on a link and Eqs. (21) impose non-negativity of link flows.

Definition 3. Φ is the polyhedral set of link flows $(v_a, a \in \bar{A})$ defined by the following conditions:

$$\sum_{a \in \bar{A}} \bar{t}_a v_a = \sum_{a \in \bar{A}} \bar{t}_a v_a^* \quad (22)$$

$$v_a \leq \bar{v}_a, a \in \bar{A} \quad (23)$$

$$v_a = \sum_{o \in O} v_a^o, \forall a \in \bar{A} \quad (24)$$

$$v_a^* = \sum_{o \in O} v_a^o, \forall a \in \bar{A}^c \quad (25)$$

and Eqs. (19) and (21). □

Similar to Proposition 2, we have:

Proposition 3. Let v_a^* be an given UE flow vector $(v_a^*, a \in A)$, (i) If vector $(\hat{v}_a, a \in A) \in \Omega_v^*$, then $(v_a = \hat{v}_a, a \in \bar{A}) \in \Phi$. (ii) If vector $(\breve{v}_a, a \in \bar{A}) \in \Phi$, then $(v_a = \breve{v}_a, a \in \bar{A}; v_a = v_a^*, a \in \bar{A}^c) \in \Omega_v^*$. (iii) Ω_v^* is a singleton if and only if Φ is a singleton. □.

Proof. We first prove (i). If $(\hat{v}_a, a \in A) \in \Omega_v^*$, Eqs. (12) state

$$\hat{v}_a = v_a^*, a \in \bar{A}^c \tag{26}$$

Eqs. (11) mean the link travel times of different UE flows are the same. Eqs. (11) and (8) demonstrate that

$$\sum_{a \in \bar{A}} \bar{t}_a \hat{v}_a + \sum_{a \in \bar{A}^c} t_a(v_a^*)\hat{v}_a = \sum_{a \in \bar{A}} \bar{t}_a v_a^* + \sum_{a \in \bar{A}^c} t_a(v_a^*) v_a^* \tag{27}$$

Eqs. (26) and (27) imply Eq. (22).

We then prove (ii). If vector $(\breve{v}_a, a \in \bar{A}) \in \Phi$, then

$$\sum_{a \in \bar{A}} \bar{t}_a \breve{v}_a = \sum_{a \in \bar{A}} \bar{t}_a v_a^* \tag{28}$$

Since $\left(v_a^*, a \in A\right) \in \Omega_v^*$, Eq. (10) implies

$$\sum_{a \in \bar{A}} \bar{t}_a (\tilde{v}_a - v_a^*) + \sum_{a \in \bar{A}^c} t_a(v_a^*)(\tilde{v}_a - v_a^*) \geq 0, \forall \tilde{v} \in \Omega_v \tag{29}$$

That is,

$$\sum_{a \in \bar{A}} \bar{t}_a \tilde{v}_a - \sum_{a \in \bar{A}} \bar{t}_a v_a^* + \sum_{a \in \bar{A}^c} t_a(v_a^*)(\tilde{v}_a - v_a^*) \geq 0, \forall \tilde{v} \in \Omega_v \tag{30}$$

Combining Eqs. (28) and (30), we have

$$\sum_{a \in \bar{A}} \bar{t}_a \tilde{v}_a - \sum_{a \in \bar{A}} \bar{t}_a \breve{v}_a + \sum_{a \in \bar{A}^c} t_a(v_a^*)(\tilde{v}_a - v_a^*) \geq 0, \forall \tilde{v} \in \Omega_v \tag{31}$$

That is,

$$\sum_{a \in \bar{A}} \bar{t}_a (\tilde{v}_a - \breve{v}_a) + \sum_{a \in \bar{A}^c} t_a(v_a^*)(\tilde{v}_a - v_a^*) \geq 0, \forall \tilde{v} \in \Omega_v \tag{32}$$

Hence, $(v_a = \breve{v}_a, a \in \bar{A}; v_a = v_a^*, a \in \bar{A}^c) \in \Omega_v^*$. (iii) is implied by (i) and (ii). □

Proposition 2 and Proposition 3 imply:

Corollary 1. $\Phi = \bar{\Phi}$. □

Hence, checking whether the UE link flow is unique, i.e., whether Ω_v^* is a singleton, or equivalently, whether the polyhedron $\bar{\Phi}$ is a singleton, can be done by checking whether the polyhedron Φ is a singleton. The advantage of checking Φ is that it is not defined on paths, thus compared with $\bar{\Phi}$, path enumeration is not needed.

5 A New Method to Check the Uniqueness of UE Link Flows with Speed Limit

In practice, to check whether the UE link flow pattern is unique, one simply needs to check whether the polyhedron Φ is a singleton. If not, then the UE link flow pattern is not unique. For the macroscopic analysis of speed limits, this is still an open question. Hence, the next focus/contribution of this note is to propose a method to check whether the polyhedron Φ is a singleton. To this end, we proposed two methods, which are summarized in Theorem 1 and Theorem 2.

Theorem 1. To check whether the polyhedron Φ is singleton, we only need to solve at most $2|\bar{A}|$ linear programming models. For each $a' \in \bar{A}$, we solve two models:

[M1(a')] $$\min v_{a'} \qquad (33)$$

subject to

$$\left(v_a, a \in \bar{A}\right) \in \Phi \qquad (34)$$

and

[M2(a')] $$\max v_{a'} \qquad (35)$$

subject to

$$\left(v_a, a \in \bar{A}\right) \in \Phi \qquad (36)$$

If there exists an a' such that the optimal objective values to [M1(a')] and [M2(a')] are different, then the polyhedron is not a singleton, meaning that the link flow on arcs in the set $\bar{A}$ is not unique. Otherwise, the polyhedron is a singleton and the link flow on arcs in the set $\bar{A}$ is unique.

Proof. If there exists an a' such that the optimal objective values to [M1(a')] and [M2(a')] are different, then the optimal solutions to [M1(a')] and [M2(a')] are different. Since both optimal solutions are feasible solutions, they are in the polyhedron Φ. As a result, the polyhedron has an infinite number of points because any point on the line segment connecting the two solutions is in the polyhedron Φ.

We prove the second part of the theorem by contradiction. Suppose the optimal objective values to [M1(a')] and [M2(a')] are the same for all $a' \in \bar{A}$ but the polyhedron is not a singleton. Then there is a point in the polyhedron, denoted by $\left(v_a^{\#}, a \in \bar{A}\right)$, that is different from $\left(v_a^{*}, a \in \bar{A}\right)$. Hence, there is at least one coordinate in which the two vectors differ. Let $a'' \in \bar{A}$ be such a coordinate, i.e., $v_{a''}^{\#} \neq v_{a''}^{*}$. Then, the optimal objective values to [M1(a'')] and [M2(a'')] must be different, contradicting our assumption. This completes the proof. □

The method proposed in Theorem 1 is very easy to implement and easy to understand. Below we propose a more efficient yet slightly more complex method.

Theorem 2. To check whether Φ is a singleton, we only need to solve at most $|\bar{A}|+1$ linear programming models. For each $a' \in \bar{A}$, we solve the following model:

[M1(a')] $$\min v_{a'} \qquad (37)$$

subject to

$$\left(v_a, a \in \bar{A}\right) \in \Phi \tag{38}$$

and then we solve the following model:

[M3] $$\min \sum_{a \in \bar{A}} \left(-v_a\right) \tag{39}$$

subject to

$$\left(v_a, a \in \bar{A}\right) \in \Phi \tag{40}$$

If all of the above $|\bar{A}|+1$ models have the same optimal solution, then the polyhedron Φ is a singleton. Otherwise it is not a singleton.

Proof. Similar to the proof to Theorem 1, if the above $|\bar{A}|+1$ models have at least two different optimal solutions, then Φ has an infinite number of points because any point on the line segment connecting two solutions is in the polyhedron Φ.

We prove the first part of the theorem by contradiction. Suppose the above $|\bar{A}|+1$ models have the same optimal solution, denoted by $\left(v''_a, a \in \bar{A}\right)$, but the polyhedron is not a singleton. Then there is a point in the polyhedron, denoted by $\left(v^{\#}_a, a \in \bar{A}\right)$, that is different from $\left(v''_a, a \in \bar{A}\right)$. Since $\left(v''_a, a \in \bar{A}\right)$ is the optimal solution to the $|\bar{A}|+1$ models, we have

$$v''_a \le v^{\#}_a, a \in \bar{A} \tag{41}$$

$$\sum_{a \in \bar{A}} \left(-v''_a\right) \le \sum_{a \in \bar{A}} \left(-v^{\#}_a\right) \tag{42}$$

If all the inequalities in Eqs. (41) are equality, then we have $\left(v''_a, a \in \bar{A}\right) = \left(v^{\#}_a, a \in \bar{A}\right)$, contradicting our assumption. Otherwise, there exists a link $\hat{a} \in \bar{A}$ such that the inequality in Eqs. (41) is strict, then we have $\sum_{a \in \bar{A}} \left(-v''_a\right) > \sum_{a \in \bar{A}} \left(-v^{\#}_a\right)$, contradicting Eq. (42). □

Corollary 2. Checking whether Φ is a singleton can be done in polynomial time with regard to the size of the network.

Proof. The polyhedron Φ is described by a set of linear equations whose number is polynomial with a polynomial number of decision variables. Hence, each one of the models [M1(a')], [M2(a')], and [M3] can be solved in polynomial time. As we only need to solve at most $2|\bar{A}|$ such models using the method in Theorem 1 (or $|\bar{A}|+1$ such models using the method in Theorem 2), the whole process stops in polynomial time with regard to the size of the network. □

As per Corollary 2, it is not a difficult task to solve the uniqueness problem using the methods in Theorem 1 or Theorem 2. Hence, the uniqueness of UE link flow with speed limits is well addressed and solved.

6 Conclusion

This technical note focused on the link flow uniqueness of UE with speed limits. As an extension of Yang et al. (2012), we first provided a new polyhedron defined on links instead of paths. We rigorously proved that to check whether this polyhedron is a singleton is equivalent to the addressed uniqueness problem. Then, two methods were proposed to mathematically check whether the polyhedron is a singleton.

The work in Yang et al. (2012) also discussed about the integration of speed limits with congestion pricing scheme. It should be noted that the method provided in this note is also suitable for the integration of speed limit with other schemes including congestion pricing. Future efforts are needed to extend the discussions in this note to the cases of other traffic assignment models besides UE, including stochastic user equilibrium, dynamic traffic assignment, etc.

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