On the Properties of Yield Distributions in Random Yield Problems: Conditions, Class of Distributions and Relevant Applications

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Abstract

In this paper, we propose two technical assumptions to ensure the unimodality of the objective functions in two classes of price and quantity decision problems with one procurement opportunity under supply random yield and deterministic demand in a price-setting environment. The first class of problems involves a decentralized supply chain/assembly system under different configurations, and the second class focuses on a single firm's price and quantity decisions under different contracts, payment schemes and supplier portfolios. We provide appealing economic interpretations and easy-to-verify sufficient conditions for our proposed technical assumptions. We show that both assumptions are preserved under truncation and positive scale, and satisfied by most commonly used continuous yield distributions. Moreover, similar to the role that the increasing generalized failure rate (IGFR) property plays in analyzing operations problems with demand uncertainty, our Assumption 1 plays a fundamental role in regulating the behaviors of the objective functions for both classes of random yield problems. Assumption 2 is more general than both Assumption 1 and the IGFR property and is used to analyze the second class of problems. Finally, we discuss the difference between random yield and random demand problems, and explain the rationale for the need of different technical assumptions.

Key words: Random yield, price and quantity decisions, price elasticity functions

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1 Introduction

Supply risk is prevalent in today's global economy. Among various supply risks, production random yield is a significant one for many industries, such as agribusiness, semiconductor industry, vaccine manufacturing, etc. For firms in these industries, how to effectively mitigate the potential yield shortage is an important issue. In the operations management literature, supply uncertainty, especially the production yield uncertainty, has been extensively studied (see, e.g., Yano and Lee, 1995; Grosfeld-Nir and Gerchak, 2004, for a comprehensive review). The primary focus in this line of research is how to design operational strategies to effectively mitigate the supply yield risk (see, e.g., Tang, 2006, for a general discussion). For example, when facing production yield uncertainty, firms could inflate production and/or hold extra inventory to hedge against yield risk (see Henig and Gerchak, 1990), diversify their supply base to exploit the risk pooling effect and reduce supply output variability (see Anupindi and Akella, 1993; Dada et al., 2007; Federgruen and Yang, 2008, 2009a,b, 2011, 2014; Tang and Kouvelis, 2011). They can also exert effort to improve their suppliers' production reliabilities (see Wang et al., 2009, 2014; Tang et al., 2014), use backup production to mitigate the impact of yield losses (Xu and Lu, 2013), or postpone the pricing decision to better match demand with supply (Dong et al., 2015).

Despite the extensive literature on supply yield uncertainty, less research effort has been spent on firms' optimal price and quantity decisions in a single period price-setting environment under supply random yield. In general, there are two classes of basic price and quantity decision problems with one procurement opportunity under supply random yield that have not been well studied in the existing literature. The first class of problems involves a decentralized supply chain/assembly system with random yield of the upstream firm(s), and with the firms engaging in wholesale price and order/production quantity decisions as part of a leader-follower strategy game. The second class of problems is regarding a single firm's price and quantity decisions under supply random yield.

The key technical challenge in solving the above two classes of problems is to characterize conditions on the random yield distributions to ensure the uniqueness of the optimal decisions. In this paper, we contribute to the random yield literature by identifying two technical assumptions on the yield distribution to ensure the unimodality of the objective functions in the two classes of problems. More specifically, Assumption 1 ensures that the price elasticity of the production inflation rate (or the order quantity) with respect to the wholesale price is increasing, and that with respect to the sales price is decreasing. Assumption 2 ensures that the price elasticity of the overproduction probability with respect to the sales price is decreasing, and that with respect to the production cost is increasing. Moreover, Assumption 1 is sufficient for Assumption 2, and plays a fundamental role to ensure the unimodality of the objective functions for both classes of random yield problems. We further show that Assumption 1 directly assists the analysis of the first class of problems whereas Assumption 2 facilitates the analysis of the second class of problems.

In addition, both assumptions are on the yield distribution only, and are independent from model parameters. We show that they are preserved under truncation and positive scale. Moreover, we provide several easy-to-verify sufficient conditions for our proposed technical assumptions. By applying these sufficient conditions, we show that most commonly used continuous yield distributions (e.g., Uniform, Beta, truncated Normal, Gamma, Weibull, Lognormal, etc.) satisfy both of our technical assumptions. We also show that: Our Assumption 1 is independent from the increasing generalized failure rate (IGFR) property; and our Assumption 2 is more general than the IGFR property and more robust under distribution operations, such as shifting, convolution, and mixing, etc. By comparing the structural differences between random yield and random demand problems, we provide rationale for the need of different technical assumptions. We also argue that our Assumption 1 performs a similar role to random yield problems as that of the IGFR property to random demand problems.

Moreover, we provide several examples to show the wide applicability of our proposed technical assumptions. Specifically, our Assumption 1 directly assists the analysis of a class of random yield problems involving a decentralized supply chain/assembly system under different configurations. Our Assumption 2 is widely applicable to a class of random yield problems involving a single firm's price and quantity decisions under different contracts, payment schemes and supplier portfolios. Our results indicate that our technical assumptions are general enough to tackle the challenge brought by random yield in supply chain research. We expect them to be productively used in such future research.

Finally, we conclude this section by positioning our contributions in the related literature. The first class of problems analyzed in our paper is related to the selling to the newsvendor problem, which has been well studied under demand uncertainty and serves as a fundamental building block for the supply chain contracting literature (see Lariviere and Porteus, 2001; Cachon, 2004; Dong and Zhu, 2007, etc.). Supply chain management with supply uncertainty has also been studied in the existing literature. For example, Babich et al. (2007) analyze a monopoly firm's sourcing decision with a set of unreliable suppliers under correlated disruption risks and endogenous wholesale prices. Gurnani and Gerchak (2007) and Tang and Kouvelis (2014) investigate the coordination issues within an assembly system and a bilateral supply chain under random yield, respectively. However, in the existing literature, the selling to the newsvendor problem under yield uncertainty has not been studied. Our paper studies a bilateral supply chain firms' quantity and wholesale pricing decisions under different wholesale price contract arrangements, and provides a unified condition to ensure the unimodality of the Stackelberg leader's profit function. Our proposed condition is quite general and applies to a large class of random yield supply chain settings under different configurations. Recently, Pan and So (2016) study a decentralized assembly system with one assembler and two component suppliers under vendor managed inventory (VMI) arrangement and supply random yield. Their analysis depends on a specific condition on the yield distribution. They show that the condition is satisfied by uniform and exponential distributions, but may not be easily verified for general yield distributions. As an example, we show in Appendix B.2 that all of their results continue to hold under our proposed assumption, which is satisfied by most commonly used continuous yield distributions.

The second class of problems analyzed in our paper is relevant to the literature of the joint pricing and inventory management, which has been extensively studied under demand uncertainty, see Yano and Gilbert (2003), Chan et al. (2004), and Chen and Simchi-Levi (2012), for a comprehensive review. For multi-period setting, Federgruen and Heching (1999) show that a base-stock/list-price policy is optimal. This line of literature has grown rapidly since Federgruen and Heching (1999). For example, Chen and Simchi-Levi (2004a,b, 2006) study inventory control and pricing strategies with fixed setup costs and show the optimality of (s, S, p) policy for the finite horizon, the infinite horizon and the continuous review models. Feng et al. (2014) investigate the dynamic pricing and inventory management problem under a general demand function, which includes both additive and multiplicative random demand as special cases. When incorporating supply uncertainty, Li and Zheng (2006) and Feng (2010) revisit the dynamic pricing and inventory management problem with stochastic proportional yield and stochastic capacity, respectively, and show that a reorder-point/list-price policy is optimal. Feng and Shi (2012) and Tan et al. (2016) investigate the impact of supply diversification on a firm's dynamic pricing and inventory policy under random capacity. For single period setting, Whitin (1955) and Mills (1959) are among the first to study the inventory planning problem under price dependent demand. Petruzzi and Dada (1999) review and extend the classic price-setting newsvendor problem under demand uncertainty. Kocabykolu and Popescu (2011) offer a unifying perspective by introducing a measure of elasticity of stochastic demand, and characterize the structural results under general assumptions of such measure.

On the other hand, less research has been conducted to investigate the single period price-setting newsvendor problem when the uncertainty arises from production yield. Pan and So (2010) study an assembler's pricing and inventory decisions with supply random yield, and provide a sufficient condition on the yield distribution to characterize the optimal price and inventory decisions. Kazaz and Webster (2015) use the elasticity measurement approach to study the price-setting newsvendor problem under both supply and demand uncertainty and provide a condition on both the demand function and yield distribution to ensure the joint concavity of the objective function. Different from above papers, our analysis requires more general technical conditions, because our results are obtained for quasiconcave objective functions. Moreover, our condition has the following desirable features: (1) it has an appealing economic interpretation; (2) it is independent from model parameters; (3) it is easy to verify; and (4) it applies to a large class of random yield problems involving a single firm's price and quantity decisions. Finally, we remark that Xu and Lu (2013) analyze a price-setting newsvendor's pricing and production decisions under both supply random yield and multiplicative demand uncertainty with emergent backup production. They show that demand function with increasing price elasticity (IPE) property is sufficient to ensure the unimodality of the objective function. We differ from Xu and Lu (2013) as our second class of problems assumes no backup production and, thus, needs additional assumption on the yield distribution to ensure unimodality.

The rest of this paper is organized as follows. We propose two technical assumptions and reveal their economic interpretations in Section 2. In Section 3, we provide sufficient conditions for our proposed technical assumptions and verify that they are satisfied by most commonly used continuous yield distributions. In Sections 4 and 5, we provide several examples to show the wide applicability of our proposed technical assumptions. In Section 6, we compare random yield problems and random demand problems, and explain the rationale for the need of different technical conditions. Section 7 summarizes our findings. All proofs and supplemental results are relegated to Appendices A-F.

2 Two Assumptions on Yield Distributions

In this section, we propose our two technical assumptions on the yield distributions and discuss their underlying economic interpretations. They are presented within a simple one procurement quantity decision problem with random yield. More specifically, consider a centralized firm with deterministic demand $D > 0$. The firm's production suffers from yield uncertainty, i.e., the output quantity is only a random fraction ξ of the input production quantity. We assume the yield factor, ξ, is a continuous random variable with support on the interval $[l, u]$, where $0 \leq l < u \leq 1$. Let $g(\xi)$, $G(\xi)$, $\bar{G}(\xi)$, and μ denote the probability density function (p.d.f.), the cumulative distribution function (c.d.f.), the complementary cumulative distribution function (c.c.d.f.), and the mean of the yield distribution, respectively. The per unit sales price and production cost are p and c, respectively, with $p > c/\mu$. For expositional convenience, we assume there is no salvage value or lost sales penalty, which can be easily added to the model without changing any structural results.

The firm needs to decide the production quantity Q to maximize its ex ante profit:

$$\Pi(Q) = pE \min\{D, Q\xi\} - cQ.$$

Straightforward calculation yields the firm's optimal production quantity as:

$$Q^* = \frac{D}{\delta}, \quad \text{where } \int_l^\delta \xi g(\xi) d\xi = \frac{c}{p}.$$

The above calculation clearly indicates that the firm's optimal production quantity is a linear inflation of the demand D. The inflation rate $1/\delta$ only depends on the yield distribution and the cost-and-price ratio, but independent from the demand. The firm's optimal profit is $\Pi^* = \Pi(Q^*) = pD\bar{G}(\delta)$, where $\bar{G}(\delta)$ is the overproduction probability measuring the chance of getting the riskless revenue pD through production inflation.

The above linear inflation plays a fundamental role in analyzing complex operations/supply chain management problems involving production random yield. As discussed in Section 1, there are two classes of problems that have not been well studied in the literature. The first class of problems involves a decentralized supply chain/assembly system under random yield with endogenous wholesale price and order/production quantity decisions. The second class focuses on a single firm's price and quantity decisions. In the rest of this section, we identify a technical assumption for each of the two classes of problems and provide their economic interpretations.

2.1 Assumption 1

For the first class of problems, in which firms engage in a decentralized supply chain setting, the Stackelberg leader's pricing decision affects the follower's order/production quantity decision and, thus, affects its own profit. From the leader's perspective, it matters how the follower's order/production quantity responds to its pricing decision. To quantify such impact, we calculate the price elasticity of the inflation rate $1/\delta$ with respect to both the sales price and production cost as follows:

$$e_1(p) := \frac{\frac{\partial}{\partial p}\frac{1}{\delta}}{\frac{1}{\delta p}} = \frac{\int_l^\delta \xi g(\xi)d\xi}{\delta^2 g(\delta)};$$

$$e_1(c) := \frac{\frac{\partial}{\partial c}\frac{1}{\delta}}{\frac{1}{\delta c}} = -\frac{\int_l^\delta \xi g(\xi)d\xi}{\delta^2 g(\delta)}.$$

The property of the above two elasticity functions critically depends on the attributes of the function $\frac{x^2 g(x)}{\int_l^x \xi g(\xi)d\xi}$. To facilitate the analysis of the first class of problem, we introduce the following technical assumption on the yield distribution.

Assumption 1. $\frac{x^2 g(x)}{\int_l^x \xi g(\xi)d\xi}$ weakly decreases in $x \in [l, u]$.

Assumption 1 ensures the monotonicity of the above two price elasticity functions. The appealing economic interpretation of Assumption 1 is that as the sales price [production cost] becomes larger, the same percentage increase in the sales price [production cost] results in a smaller [larger] percentage increase [decrease] in the inflation rate. In other words, when the base sales price [production cost] is higher, the inflation rate is less [more] sensitive to the sales price [production cost] change.

2.2 Assumption 2

For the second class of problems, in which a single firm makes multiple levels of operational decisions, the decision on price affects not only the production quantity but also the probability of obtaining the riskless revenue. From the firm's perspective, the overproduction probability $\bar{G}(\delta)$ is the critical factor to consider when making its pricing decision. Calculating the price elasticity of the overproduction probability with respect to both the sales price and production cost, we have:

$$e_2(p) := \frac{p\frac{\partial}{\partial p}\bar{G}(\delta)}{\bar{G}(\delta)} = \frac{\int_l^\delta \xi g(\xi)d\xi}{\delta \bar{G}(\delta)};$$

$$e_2(c) := \frac{c\frac{\partial}{\partial c}\bar{G}(\delta)}{\bar{G}(\delta)} = -\frac{\int_l^\delta \xi g(\xi)d\xi}{\delta \bar{G}(\delta)}.$$

The property of the above two elasticity functions critically depends on the attributes of the function $\frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi}$. To facilitate the analysis of the second class of problems, we introduce the following technical assumption on the yield distribution.

Assumption 2. $\frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi}$ weakly decreases in $x \in [l, u]$.

Assumption 2 is designed to ensure the monotonicity of the above two price elasticity functions. The appealing economic interpretation is that as the sales price [production cost] increases, the overproduction probability becomes less [more] sensitive to the sales price [production cost] change, i.e., the same percentage increase [decrease] in the sales price [production cost] leads to a smaller [larger] percentage increase in the overproduction probability.

2.3 The relationship between Assumptions 1 and 2

Recall that Assumption 1 is imposed to regulate the behavior of the inflation rate in response to the changes of sales prices and/or procurement costs whereas Assumption 2 is imposed to regulate that of the corresponding overproduction probability. As the change of the inflation rate naturally affects the resulting overproduction probability, we proceed to discuss the relationship between Assumptions 1 and 2 in the following proposition.

Proposition 1. *Let X be a continuous random variable with support on the interval $[l, u] \subseteq [0, 1]$. If X satisfies Assumption 1, then it also satisfies Assumption 2.*

Proposition 1 shows that Assumption 1 is a sufficient condition of Assumption 2, implying that the monotonicity of the elasticity of the inflation rate automatically induces the same monotonicity of that of the overproduction probability. In the following sections, we characterize sufficient conditions for both assumptions to hold and show that they are quite general and are satisfied by many commonly used continuous distributions. We also argue that similar to the role that the IGFR property plays in analyzing operations problems with random demand, Assumption 1 plays a fundamental role to ensure the unimodality of the objective functions for both classes of random yield problems discussed above. Moreover, Assumption 1 directly assists the analysis of the first class of problems involving a decentralized supply chain/assembly system under different configurations, whereas Assumption 2 further facilitates the analysis of the second class of problems involving a single firm's price and quantity decisions under different contracts, payment schemes and supplier portfolios.

3 Discussion on Assumptions 1 and 2

In this section, we provide several sufficient conditions for Assumptions 1 and 2, and verify that both assumptions are satisfied by most of the commonly used continuous yield distributions. Moreover, we also compare our Assumptions 1 and 2 with the IGFR property in Appendix A.

3.1 Sufficient conditions

Both Assumptions 1 and 2 are required to hold on the support of the yield factor ξ, i.e., $[l, u] \subseteq [0, 1]$. In the following analysis, we first focus on a general random variable X with support on the interval $[L, U]$, where $0 \leq L < U \leq \infty$. Since the yield factor cannot be negative, we require the lower bound of the distribution be nonnegative. When the random variable X satisfies Assumption $i = 1, 2$, Assumption i holds on its *entire* support $x \in [L, U]$. The p.d.f. and c.d.f. of the random variable X are $g(x)$ and $G(x)$, respectively. Moreover, we assume that $g(x)$ is differentiable. We begin our analysis by showing some preservation properties for both assumptions in the following proposition.

Proposition 2. *Let X be a continuous random variable with support on the interval $[L, U]$, where $0 \leq L < U \leq \infty$. If X satisfies Assumption $i = 1, 2$, i.e., $\frac{x^2 g(x)}{\int_L^x \xi g(\xi) d\xi}$ or $\frac{x\bar{G}(x)}{\int_L^x \xi g(\xi) d\xi}$ decreases in $x \in [L, U]$, the following statements hold:*

(i) The truncation of X on the interval $[l, u] \subseteq [L, U]$ satisfies Assumption i.

(ii) For any $k > 0$, kX on the interval $[kL, kU]$ satisfies Assumption i.

Proposition 2 shows that both assumptions are preserved under truncation and positive scale. As a consequence, we can truncate any nonnegative (and possibly unbounded) continuous distribution with support on $[L, U] \subseteq [0, \infty)$ to the interval $[l, u] \subseteq [0, 1]$ as the yield distribution. We can also scale a bounded distribution on $[L, U]$ to the interval $[l, u] \subseteq [0, 1]$ as the yield distribution. As long as the original distribution satisfies Assumption $i = 1, 2$, the resulting yield distribution from truncation and/or positive scale also satisfies Assumption i. Therefore, for the rest of this section, we discuss sufficient conditions for each assumption on the interval $[L, U] \subseteq [0, \infty)$. Next, we provide sufficient conditions for each assumption to hold in the following proposition.

Proposition 3. *Let X be a continuous random variable with support on the interval $[L, U]$, where $0 \leq L < U \leq \infty$. The following statements hold:*

(i) Assumption 1 holds, if $xg'(x)/g(x)$ decreases in $x \in [L, U]$.

(ii) Assumption 2 holds, if one of the following conditions is satisfied for all $x \in [L, U]$: (a) $xg(x) \geq G(x)$; (b) $xg(x) \geq \bar{G}(x)$; (c) X is IGFR; (d) $2g^2(x) + g'(x)\bar{G}(x) \geq 0$; (e) $xg'(x)/g(x)$ decreases in x.

Proposition 3(i) shows that if $xg'(x)/g(x)$ decreases in x, Assumption 1 holds. On the other hand, Proposition 3(ii) provides several sufficient conditions for Assumption 2. Condition (a) ensures that $\frac{x}{\int_l^x \xi g(\xi) d\xi}$ decreases in x, so Assumption 2 holds. We remark that this condition has been adopted by Pan and So (2010). Condition (b) requires the generalized failure rate be larger than 1 for all $x \in [L, U]$, which ensures that $x\bar{G}(x)$ decreases in x. Condition (c) further shows that the IGFR property is sufficient for Assumption 2 to hold. Condition (d) is more general than assuming increasing failure rate (IFR) distributions, and different from assuming IGFR distributions. Condition (e) follows from part (i) and Proposition 1.

To conclude this subsection, we provide sufficient conditions under which both assumptions are preserved under shifting in Proposition 4 below. When the density function is log-concave, Assumption 2 holds and is preserved under any shifting. When the density function is log-concave and $xg'(x)/g(x)$ decreases in $x \in [L, U]$, Assumption 1 holds and is preserved under any positive shifting (to the right). Note that the condition that $xg'(x)/g(x)$ decreases in x does not necessarily imply that the density function is log-concave.

Proposition 4. *Let X be a continuous random variable with support on the interval $[L, U]$, where $0 \leq L < U \leq \infty$. For any $a \in \mathbb{R}$, let random variable $Y = X + a$ be a shifting of X. If the density function of X, $g(x)$, is log-concave for $x \in [L, U]$, the following statements hold:*

(i) Both random variables X and Y satisfy Assumption 2.

(ii) If $a > 0$ and $xg'(x)/g(x)$ decreases in $x \in [L, U]$, both random variables X and Y satisfy Assumption 1.

3.2 Remarks on specific continuous distributions

We now apply the sufficient conditions in Proposition 3 to commonly used continuous distributions, and check whether they satisfy Assumptions 1 and 2. The results are summarized in Table 1. The distributions with support on the entire real axis have been truncated to the interval $[0, \infty)$ to reflect the fact that the yield factor is non-negative, and the density functions have been adjusted accordingly. In Table 1, we denote the truncation constant $C = \frac{1}{1-G(0)}$, and multiply it to the orginal p.d.f. as the truncated p.d.f.. Note that all the distributions listed in Table 1 satisfy the IGFR property, see Banciu and Mirchandani (2013). Thus, Assumption 2 holds by part (ii)(c) of Proposition 3. In Appendix E, we check Assumption 1 for the distributions listed in Table 1. We remark that by Proposition 2, the results in Table 1 are preserved when the above distributions are truncated and/or positively scaled to the interval $[l, u] \subseteq [0, 1]$.

In Appendix A, we further discuss the relationship between IGFR and the two assumptions. Recall from Propositions 3(ii)(c) and 2 that Assumption 2 is more general than IGFR and is preserved under truncation and positive scale. On the other hand, the IGFR property is not preserved under many distribution operations, such as shifting, convolution, and mixing, see, e.g., Paul (2005); Al-Zahrani and Stoyanov (2008); Banciu and Mirchandani (2013). In Appendix A.1, we give three examples to show that Assumption 2 is more robust under many distribution operations. In Appendix A.2, we provide examples of continuous distributions that satisfy either Assumption 1 or IGFR but not both. Thus, Assumption 1 and IGFR are two independent properties. Section 6 further comments that they are tailored to tackle the difficulties brought by different types of uncertainties.

4 Applications of Assumption 1

In this section, we show the wide applicability of Assumption 1 with two classic examples in supply chain management under random yield: (1) bilateral supply chain under wholesale price contract and random yield; and (2) bilateral supply chain under vendor managed inventory (VMI) contract and random yield. In addition, we provide other applications on an assembly system with one reliable and one unreliable component supplier in Appendix B. We remark that these applications either haven't been studied, or have been studied under specific yield distributions. However, their counterparts under demand uncertainty have been well developed in the supply chain management literature.

4.1 Bilateral supply chain under wholesale price contract and random yield

Consider a bilateral supply chain where a downstream retailer procures from an upstream supplier to meet deterministic demand D. The supplier offers the wholesale price w first, and the retailer responds by deciding the order quantity q. The supplier's production suffers from random yield with yield factor ξ, which is a continuous random variable with support on the interval $[l, u] \subseteq [0, 1]$. Let $g(\xi)$, $G(\xi)$, $\bar{G}(\xi)$, and μ denote the probability density function (p.d.f.), the cumulative distribution function (c.d.f.), the complementary cumulative distribution function (c.c.d.f.), and the mean of the yield distribution, respectively. The sales price is p and the production cost is c, with $c/\mu \leq p$ to

Distribution (Parameter)	Support	p.d.f	Assumption 1 holds?	Assumption 2 holds?
Beta ($\alpha > 0, \beta > 0$)	$[0, 1]$	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$	When $\beta \geq 1$	Yes
Chi ($k \in \mathbb{N}$)	$[0, \infty)$	$\frac{x^{k-1}e^{-\frac{x^2}{2}}}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}$	Yes	Yes
Chi-squared ($k \in \mathbb{N}$)	$[0, \infty)$	$\frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}$	Yes	Yes
Erlang ($\alpha > 0, k \in \mathbb{N}$)	$[0, \infty)$	$\frac{1}{\alpha^k(k-1)!}x^{k-1}e^{-\frac{x}{\alpha}}$	Yes	Yes
Exponential ($\lambda > 0$)	$[0, \infty)$	$\lambda e^{-\lambda x}$	Yes	Yes
Gamma ($\alpha > 0, \beta > 0$)	$[0, \infty)$	$\frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$	Yes	Yes
Laplace ($b > 0, \mu \in \mathbb{R}$)	$[0, \infty)$	$\frac{C}{2b}e^{-\frac{\lvert x-\mu\rvert}{b}}$	When $x > \mu$	Yes
Log-Logistic ($\alpha > 0, \beta > 0$)	$[0, \infty)$	$\frac{(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}}{(1+(\frac{x}{\alpha})^\beta)^2}$	Yes	Yes
Log-Normal ($\mu \in \mathbb{R}, \sigma > 0$)	$[0, \infty)$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}$	Yes	Yes
Normal ($\mu \in \mathbb{R}, \sigma > 0$)	$[0, \infty)$	$\frac{C}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	When $x > \frac{\mu}{2}$	Yes
Pareto ($b > 0,\ k > 0$)	$[b, \infty)$	$\frac{kb^k}{x^{k+1}}$	Yes	Yes
Power ($k > 0$)	$[0, 1]$	kx^{k-1}	Yes	Yes
Rayleigh ($\sigma > 0$)	$[0, \infty)$	$\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$	Yes	Yes
Uniform ($0 \leq a < b < \infty$)	$[a, b]$	$\frac{1}{b-a}$	Yes	Yes
Weibull ($k > 0, \lambda > 0$)	$[0, \infty)$	$\frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}e^{-(\frac{x}{\lambda})^k}$	Yes	Yes

Table 1: Remarks on Specific Continuous Distribution: $C = \frac{1}{1-G(0)}$ is the truncation constant

avoid the trivial case of no order/production. The retailer pays for its order quantity q at the per unit wholesale price w, which is suitable for the situations when the yield loss is driven by uncontrollable factors, such as weather conditions (Li and Zheng, 2006; Xu and Lu, 2013).[1] In this bilateral supply

[1]We acknowledge that the pay-by-order-quantity payment scheme, though heavily adopted in the literature, may have some practical limitations as the buyer completely bears the yield risk and may pay more than final delivery.

chain, the downstream retailer bears the yield risk alone, and we show that the Stackelberg game has unique equilibrium wholesale price and order quantity under Assumption 1.

First, given any wholesale price w, the retailer's optimal order quantity is a linear inflation of the demand, i.e., $q^*(w) = D/\delta$, where $\int_l^\delta \xi g(\xi)d\xi = w/p$. Next, the supplier's profit as a function of the wholesale price is $\Pi_s(w) = (w - c)q^*(w)$. Direct analysis of $\Pi_s(w)$ is involved. Thus, we adopt the approach from Lariviere and Porteus (2001) to change the decision variable from w to q, because there is a one-to-one monotone mapping between them, i.e., $w(q) = p\int_l^{\frac{D}{q}} \xi g(\xi)d\xi$. The supplier's profit function can be rewritten as $\Pi_s(q) = (w(q) - c)q$.

Under Assumption 1, the supplier's profit is unimodal in q, and the optimal order quantity and the optimal wholesale price can be uniquely determined, as shown in the following proposition. This result serves as a fundamental building block for many important supply chain problems under supply random yield, and Assumption 1 plays a crucial role in analyzing these problems.

Proposition 5. *Assume the yield distribution satisfies Assumption 1. In the bilateral supply chain under random yield, the supplier's profit is unimodal in q. The optimal q^* and, thus, w^* can be uniquely determined.*

To conclude, we remark that in the random yield literature, there is another commonly used payment scheme, in which the retailer only pays for the delivered quantity at per unit wholesale price $\hat{w}$. Under this scheme, there are two treatments regarding the supplier's production decision. In the first treatment, which is heavily adopted in the random yield literature (Federgruen and Yang, 2009a), the supplier is assumed to produce only the order quantity Q (or buyer controls the production quantity). The analysis of the Stackelberg game under this treatment is equivalent to the above model subject to a linear transformation on the wholesale price with $\hat{w} = w/\mu$ (Li and Zheng, 2006; Xu and Lu, 2013). In the second treatment, the supplier could choose to produce above the order quantity via inflation. In this treatment, even without Assumption 1, the three-stage Stackelberg game (supplier offers wholesale price, retailer decides order quantity, and supplier decides production quantity) has unique equilibrium in which the supplier charges $w^* = p$ and the retailer orders $q^* = D$. In this case, the supply chain is coordinated with the supplier obtaining all the channel profit.

4.2 Bilateral supply chain under VMI contract and random yield

In this section, we consider the same supply chain as that in Section 4.1, but with a VMI arrangement. The downstream retailer offers the wholesale price w, and then the upstream supplier decides the production quantity Q. When the yield is realized, the retailer pulls from available supply to satisfy demand D and pays for the used quantity at the per unit wholesale price w (a similar setting for an assembly system has been discussed in Pan and So, 2016, to model the situation that the production leadtime is long and the supplier has only one production opportunity with the possibility to not

Consequently, it is usually adopted in the situation when the supply yield risk is not controllable with strong buyers to absorb the risk. One practical example of this payment scheme used in agribusiness is called "acreage contract" in which the producer pays the farmer before harvest based on the acreage planted.

fully meet demand due to random yield.). In this VMI system, we assume that there is no shortage penalty or leftover salvage value for the supplier.[2] Next, we will show that the Stackelberg game has unique equilibrium wholesale price and production quantity under Assumption 1.

First, given any wholesale price w, the supplier's optimal production quantity is a linear inflation of the demand, i.e., $Q^*(w) = D/\delta$, where $\int_l^\delta \xi g(\xi)d\xi = c/w$. Given the supplier's production quantity, the retailer's ex ante profit as a function of w is $\Pi_r(w) = (p - w)E\min\{D, Q^*(w)\xi\}$, which is challenging to analyze directly. Since there is a one-to-one monotone mapping between w and Q, i.e., $w = c/\int_l^{\frac{D}{Q}} \xi g(\xi)d\xi$, we can change the decision variable from w to Q and rewrite the retailer's profit function as $\Pi_r(Q) = (p - w(Q))E\min\{D, Q\xi\}$.

Under Assumption 1, the retailer's profit is concave in Q, and the optimal production quantity and the optimal wholesale price can be uniquely determined, as shown in the following proposition. This result also serves as a fundamental building block for complex supply chain problems involving VMI systems under supply random yield.

Proposition 6. *Assume the yield distribution satisfies Assumption 1. In a bilateral VMI system under random yield, the retailer's profit is concave in Q. The optimal Q^* and, thus, w^* can be uniquely determined.*

5 Applications of Assumption 2

In this section, we show the wide applicability of Assumption 2 with a classic example: price-setting retailer's problem with deterministic price sensitive demand and an unreliable supplier under random yield. We consider both pay-by-order-quantity and pay-by-delivery-quantity payment schemes. In Appendix C, we provide an additional example on Assumption 2, i.e., selling to the price-setting retailer under random yield and buyback contract. Note that the above problems either haven't been studied, or have been studied under some specific yield distributions. However, their counterparts under demand uncertainty have all been well studied in the literature. Assumption 2 directly assists the analyses of the above problems. Moreover, Assumption 1 is also sufficient to ensure the unimodality of all the examples discussed in this section, because it implies Assumption 2.

Consider a monopoly retailer, which sources from an unreliable supplier and sells directly to the market with deterministic price sensitive demand $d(p)$. The supplier's production suffers from yield uncertainty with yield factor ξ, which is a continuous random variable with support on the interval $[l, u] \subseteq [0, 1]$. Let q be the downstream retailer's order quantity and Q be the upstream supplier's production quantity. Let $p_0 \le \infty$ be the maximal allowable price that induces 0 demand, i.e., $d(p_0) = 0$. Let c be the supplier's unit production cost. We consider two payment schemes: (1) the pay-by-order-quantity scheme, in which the retailer pays for the entire order quantity; and

[2] We acknowledge that in practice, many VMI programs impose some service level agreement between the supplier and the buyer to ensure that the service level is not too low. Incorporating per unit shortage penalty cost to boost supplier's production is one practical mechanism. Let δ be the exogenously given per unit shortage cost, then the supplier's and retailer's expected profits are: $\Pi_s(Q) = (w + \delta)E\min\{D, Q\xi\} - cQ - D\delta$ and $\Pi_r(w) = (p - (w + \delta))E\min\{D, Q\xi\} + D\delta$, respectively. We can change the decision variable from w to $w + \delta$, and the results in the subsequent analysis remain the same.

(2) the pay-by-delivery-quantity scheme, in which the retailer pays for the delivered quantity. Both payment schemes are commonly adopted in the random yield literature. Let w denote the per unit wholesale price under both payment schemes. To avoid trivial cases, we assume $p_0\mu \geq w \geq c$ for the pay-by-order-quantity scheme, and $p_0 \geq w \geq c/\mu$ for the pay-by-delivery-quantity scheme.

5.1 Monopoly pricing and ordering under pay-by-order-quantity payment scheme

We start our analysis with the pay-by-order-quantity scheme. In this payment scheme, the upstream supplier produces the order quantity q from the downstream retailer. Therefore, the downstream retailer bears all the production yield risk, and its ex ante profit function is:

$$\Pi_r(p,q) = pE\min\{d(p), q\xi\} - wq.$$

In general, $\Pi_r(p,q)$ is not jointly concave in (p,q). Thus, we adopt the sequential optimization approach. Given any price-induced demand $d(p)$, the firm's order quantity is $q^*(p) = d(p)/\delta$, where $\int_l^\delta \xi g(\xi)d\xi = w/p$. Plugging in $q^*(p)$, we have $\Pi_r(p,q^*(p)) = pd(p)(1-G(\delta))$.

Under Assumption 2, the unimodality of $\Pi_r(p,q^*(p))$ can be established if the demand function has the increasing price elasticity (IPE) property, where the price elasticity of demand is:

$$\eta(p) \doteq -\frac{pd'(p)}{d(p)}. \tag{1}$$

As shown in Yao et al. (2006) and Xu and Lu (2013), most of the demand functions in the literature (e.g., linear, iso-elastic, concave, and log-concave demand functions) satisfy the IPE property.

Proposition 7. *Assume that the demand function $d(p)$ satisfies the IPE property, and that the yield distribution satisfies Assumption 2. For the price setting newsvendor problem under the pay-by-order-quantity scheme, $\Pi_r(p,q^*(p))$ is quasi-concave, and the unique optimal price p^* is defined by the first order condition, which is equivalent to*

$$1 + \frac{\int_l^\delta \xi g(\xi)d\xi}{\delta(1-G(\delta))} + \frac{pd'(p)}{d(p)} = 0.$$

To conclude, we remark that Pan and So (2010) study an assembler's optimal pricing and sourcing decisions when facing one reliable and one unreliable component supplier under the pay-by-order-quantity payment scheme. Their problem can be formulated as the above problem, since the order quantity to the reliable component supplier always equals to the price-induced demand. To facilitate the analysis, Pan and So (2010, page 1794) assume the yield distribution satisfies the property that $\frac{x}{\int_l^x \xi dG(\xi)}$ decreases in x, and comment that "It is difficult to verify Assumption 1 (in their paper) directly for general supply reliability distributions". In addition, they provide conditions on yield distributions under which their assumption holds. Note that the Pan and So's assumption is a special case of our Assumption 2, because $\overline{G}(x)$ decreases in x. As shown in Section 3, our Assumption 2 is quite general and is satisfied by most commonly used yield distributions, including all IGFR distributions as well as their truncations and positive scales.

5.2 Monopoly pricing and ordering under pay-by-delivery-quantity payment scheme

Now, we consider the retailer's optimal pricing and ordering decisions under the pay-by-delivery-quantity payment scheme. When the payment is contingent upon the delivery performance, the supplier has incentive to produce more than needed to compensate for potential yield loss (i.e., we adopt the second treatment discussed after Proposition 5). We first solve the supplier's production decision by maximizing its ex ante profit $\Pi_s(Q|q) = wE\min\{q, Q\xi\} - cQ$. Clearly, $Q^*(q) = q/\delta$, where $\int_l^\delta \xi g(\xi)d\xi = c/w$. Taking into consideration of the supplier's inflation behavior, the retailer's ex ante profit is:

$$\Pi_r(p, q|Q^*(q)) = pE\min\left\{d(p), q, \frac{q\xi}{\delta}\right\} - wqE\min\left\{1, \frac{\xi}{\delta}\right\}.$$

Similarly, $\Pi_r(p, q|Q^*(q))$ is not jointly concave in (p, q), and we adopt the sequential approach. Define $\hat{p} \doteq w^2 E\min\{\delta, \xi\}/c$. It can be shown that $\hat{p} > w$. Given any price-induced demand $d(p)$,

$$q^*(p) = \begin{cases} d(p) & \text{if } \ p \in [w, \hat{p}] \\ d(p)\delta/\hat{\delta} & \text{if } \ p \in (\hat{p}, \max\{\hat{p}, p_0\}], \end{cases}$$

where $\hat{\delta}$ satisfies $\int_l^{\hat{\delta}} \xi g(\xi)d\xi = wE\min\{\delta, \xi\}/p$. Plugging in $q^*(p)$, the retailer's ex ante profit is:

$$\Pi_r(p, q^*(p)) = \begin{cases} (p-w)d(p)E\min\{1, \frac{\xi}{\delta}\} & \text{if } \ p \in [w, \hat{p}] \\ pd(p)(1-G(\hat{\delta})) & \text{if } \ p \in (\hat{p}, \max\{\hat{p}, p_0\}]. \end{cases}$$

Different from its counterpart under the pay-by-order-quantity scheme, the retailer's ex ante profit as a function of sales price under the pay-by-delivery-quantity scheme is a piecewise function. However, Proposition 8 shows that, under IPE demand function and Assumption 2, $\Pi_r(p, q^*(p))$ is well behaved.

Proposition 8. *Assume that the demand function $d(p)$ satisfies the IPE property, and that the yield distribution satisfies Assumption 2. For the price-setting newsvendor problem with the pay-by-delivery-quantity scheme, $\Pi_r(p, q^*(p))$ is continuously differentiable and quasi-concave. The unique optimal price p^* is defined by the first order condition, which is equivalent to:*

$$\begin{cases} (p-w)d'(p) + d(p) = 0 & \textit{if } \ p^* \le \hat{p} \\ 1 + \frac{\int_l^{\hat{\delta}} \xi g(\xi)d\xi}{\delta(1-G(\hat{\delta}))} + \frac{pd'(p)}{d(p)} = 0 & \textit{if } \ p^* > \hat{p}. \end{cases}$$

In addition, we consider a slightly different setting with a price-setting assembler procuring from one reliable and one unreliable component supplier to assemble the final product. With pay-by-delivery-quantity scheme, the assembler's problem can be modeled as the above model, because the order quantity to the reliable component supplier always equals to the price-induced demand.

To conclude, we remark that both examples in Section 5 assume exogenously given wholesale price from the supplier. One interesting yet challenging extension would be investigating the optimal wholesale price decision from the supplier taking into consideration of the retailer's optimal decisions. The counterpart of the above Stackelberg game under multiplicative demand uncertainty has been investigated by Granot and Yin (2008), which comments that "for a general distribution of ϵ (the multiplicative noise), the manufacturer's expected profit function in Stage 1, taking into account of retailer's reaction function, may not be well behaved". A similar comment has also been made by Shi et al. (2013). A common treatment for this type of problem in the literature is to consider a buyback contract, in which the retailer decides order quantity q and sales price p, and the supplier decides wholesale price w and buyback price b. After changing the decision variables from (w, b) to (p, q), the supplier's problem is well behaved under multiplicative demand uncertainty. In Appendix C.1, we show that such a Stackelberg game under yield uncertainty is also well behaved under either of our assumptions.

6 Assumptions for Random Yield versus Random Demand Models

In this section, we discuss the differences between random yield problems and random demand problems, and explain the need of distinct technical assumptions. Consider the standard newsvendor setting under random demand D with pdf $f(D)$, cdf $F(D)$, mean d, and variance σ^2, respectively. The retail price is p and the procurement cost is c with $p \geq c$. The firm needs to decide the order quantity Q by maximizing its expected profit:

$$\hat{\Pi}(Q) = pE_D \min\{D, Q\} - cQ = pE_{\hat{D}} \min\{d, Q + \hat{D}\} - cQ,$$

where $\hat{D} = d - D$ is a transformed distribution with $E\hat{D} = 0$ and $Var\hat{D} = \sigma^2$. After the transformation, the random demand newsvendor problem can be viewed as an equivalent random supply newsvendor problem with delivered quantity subject to a random noise $\hat{D}$. Consequently, the difference between the transformed random demand newsvendor problem and the random yield newsvendor problem is exactly the difference between additive and multiplicative uncertainty, which has been documented to have significant impacts on the firm's operational decisions (Petruzzi and Dada, 1999). Specifically, comparing the transformed random demand newsvendor problem and the random yield newsvendor problem, the differences lie in both the variance and the coefficient of variation of the delivered quantity. On one hand, the variance of the delivered quantity is independent from Q for the former and is proportional to Q^2 for the latter. On the other hand, the coefficient of variation of the delivered quantity decreases in Q for the former but is independent from Q for the latter.

Next, we proceed to explain the need of different technical assumptions. We compare the optimal order quantities and their corresponding elasticities between the two problems in Table 2. Table 2 clearly shows the difference in the optimal order quantities between the random yield and the random demand newsvendor problems. The optimal order quantity Q^* balances the overage and underage risks in the random yield newsvendor problem by setting the expected yield in the underage case equal to the cost-price-ratio. On the other hand, the optimal order quantity Q^* balances the overage and underage risks in the random demand newsvendor problem by setting the stockout probability equal to the cost-price-ratio. The difference is driven by the different types of uncertainties discussed above. Consequently, Assumption 1 [the IGFR property] is proposed to regulate the monotonicity of the elasticity functions with respect to both the procurement cost and the retail price in the random yield [demand] newsvendor problem. As mentioned in Appendix A.2, Assumption 1 and the IGFR property are independent from each other, i.e., neither one strictly dominates the other, and they are tailored to tackle the difficulties brought by the different types of uncertainties.

Model	Q^*	$\frac{\partial Q^*}{\partial c} / \frac{Q^*}{c}$	$\frac{\partial Q^*}{\partial p} / \frac{Q^*}{p}$
Random Yield	$\int_l^{\frac{D}{Q}} \xi g(\xi) d\xi = \frac{c}{p}$	$-\frac{\int_l^{\frac{D}{Q}} \xi g(\xi) d\xi}{(\frac{D}{Q})^2 g(\frac{D}{Q})}$	$\frac{\int_l^{\frac{D}{Q}} \xi g(\xi) d\xi}{(\frac{D}{Q})^2 g(\frac{D}{Q})}$
Random Demand	$\bar{F}(Q) = \frac{c}{p}$	$-\frac{\bar{F}(Q)}{Qf(Q)}$	$\frac{\bar{F}(Q)}{Qf(Q)}$

Table 2: Comparisons of Optimal Order Quantities and Corresponding Elasticity Functions under Random Yield and Random Demand Problems

For the price-setting newsvendor under demand uncertainty, depending on the type of randomness, the required condition is different: Additive demand uncertainty requires the IFR property, whereas multiplicative demand uncertainty requires the IGFR property (see, e.g., Kocabykolu and Popescu, 2011). On the other hand, under yield uncertainty, our Assumption 1 continues to be effective in regulating the behavior of the objective function. Moreover, we also propose Assumption 2, which is more general than both Assumption 1 and the IGFR property and is more robust under distribution operations such as shifting, convolution, and mixing (see Appendix A.1).[3]

To sum up, Assumption 1 and IGFR property are two fundamental properties to tackle supply risk and demand risk, respectively. They are independent from each other and lead to the same order quantity elasticity implications. Moreover, Assumption 2 is more general than both Assumption 1 and IGFR property, and is more robust under distribution operations. It directly assists the analysis of price-setting newsvendor problem under either yield or multiplicative demand uncertainty.

7 Conclusion

In the random yield literature, there are two classes of price and quantity decision problems with one procurement opportunity under deterministic demand that are not well studied. The first class involves a decentralized supply chain/assembly system with the firms engaging in wholesale price and order/production quantity decisions as part of a leader-follower strategy game. The second class focuses on a single firm's price and quantity decisions under different contracts, payment schemes and supplier portfolios. In this paper, we propose two technical assumptions on the yield distribution to ensure the unimodality of the objective functions in two classes of problems. More specifically, Assumption 1 plays a fundamental role in regulating the behaviors of the objective functions for both classes of problems. Assumption 2 is more general and facilitates the analysis of the second class of

[3] We remark that for the price-setting newsvendor problem under multiplicative demand uncertainty, when the required IGFR property is relaxed to Assumption 2, the unimodality of the objective function and the uniqueness of the optimal price and quantity decisions continue to hold, see Appendix F.

problems. We provide appealing economic interpretations and easy-to-verify sufficient conditions for our proposed technical assumptions. We show that most commonly used continuous yield distributions satisfy both of our technical assumptions. Moreover, we provide several examples to show the wide applicability of our assumptions. Finally, we discuss the difference between random yield and random demand problems, and explain the need of different technical assumptions. We expect both assumptions, especially the first one, to play important roles in developing and proving results on random yield models and relevant applications.

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A Discussions on Assumptions 1, 2, and IGFR

A.1 On Assumption 2 and IGFR

In the operations management literature, the IGFR property has been widely used to regulate the behaviors of objective functions involving random demand. It is satisfied by most of the commonly used distributions. However, in the literature, researchers have found that the IGFR property is not preserved under many distribution operations, such as shifting, convolution, and mixing (see Paul, 2005; Al-Zahrani and Stoyanov, 2008; Banciu and Mirchandani, 2013, etc), which shows its potential limitations in supply chain models (Paul, 2005). On the other hand, Proposition 3(ii)(c) shows that being IGFR is a sufficient condition for Assumption 2 to hold, which indicates that Assumption 2 is more general than the IGFR property. In this subsection, we borrow the examples from the literature under which IGFR doesn't hold, and illustrate that they satisfy our Assumption 2.

Example 1 (Shifting). Paul (2005) shows that positive shifting of a Pareto distribution leads to a decreasing generalized failure rate (DGFR) distribution.[4] To be specific, let X follow Pareto(θ, k) with pdf $f(x) = \frac{\theta k^\theta}{x^{\theta+1}}$ and cdf $F(x) = 1 - (\frac{k}{x})^\theta$, $\forall x \geq k > 0$. For any $a > 0$, let $Y = X + a$ be a right shifting of X with pdf $\hat{f}(y) = \frac{\theta k^\theta}{(y-a)^{\theta+1}}$ and cdf $\hat{F}(y) = 1 - (\frac{k}{y-a})^\theta$. Simple calculation yields $\frac{\partial}{\partial y}\frac{y\hat{f}(y)}{1-\hat{F}(y)} = -\frac{a\theta}{(y-a)^2} < 0$. Thus, $Y = X + a$ is DGFR rather than IGFR.

We now show that for any Pareto distribution with finite mean (i.e., $\theta > 1$), $Y = X + a$ satisfies Assumption 2. We have $\frac{x(1-\hat{F}(x))}{\int_l^x \xi \hat{f}(\xi)d\xi} = \frac{y(\frac{k}{y-a})^\theta (y-a)^\theta(\theta-1)}{(y-a)^\theta(k\theta+a(\theta-1))+k^\theta(a-y\theta)}$. Taking derivative, $\frac{\partial}{\partial x}\frac{x(1-\hat{F}(x))}{\int_l^x \xi\hat{f}(\xi)d\xi} = M(ak^\theta(a-x)+(x-a)^\theta(a+(\theta-1)x)((\theta-1)a+k\theta))$, where $M = \frac{(\frac{k}{x-a})^\theta(x-a)^{\theta-1}(1-\theta)}{(a(k^\theta+(x-a)^\theta(a+(\theta-1)x)((\theta-1)a+k\theta)))^2} \leq 0$, since $\theta \geq 1$. Thus, to show $\frac{\partial}{\partial x}\frac{x(1-\hat{F}(x))}{\int_l^x \xi\hat{f}(\xi)d\xi} \leq 0$, it is equivalent to check $K := ak^\theta(a-x) + (x-a)^\theta(a + (\theta-1)x)((\theta-1)a + k\theta) \geq 0$, for all $\theta > 1$, $x \geq k + a$, and $k > 0, a \geq 0$. Easy to see $K \geq (x-a)k^{(\theta-1)}[(a + (\theta-1)x)((\theta-1)a + k\theta) - ka] \geq 0$, where the first inequality comes from the fact that $x - a \geq k$ and the second inequality is obvious since $\theta > 1$. Thus, $Y = X + a$ satisfies Assumption 2.

Example 2 (Convolution). Banciu and Mirchandani (2013) argue that even though Log-Logistic distribution is IGFR, the convolution of two Log-Logistic distributions may not be IGFR. In Figure A.1, we provide a numerical example illustrating that the convolution of two independent Log-Logistic(2,1) distributions is not IGFR, but satisfies our Assumption 2. Moreover, in our extensive numerical test on convolution of Log-Logistic distributions, we observe that Assumption 2 is satisfied by most instances which do not satisfy the IGFR property.

Example 3 (Mixing). Al-Zahrani and Stoyanov (2008) argue that the IGFR property is not preserved under mixing. To be specific, let F_i be the cdf of X_i, $i = 1, 2$. Let $H = pF_1 + (1-p)F_2$ be the mixed distribution of F_i with probability p. Then, H may not necessarily satisfy the IGFR property. They provide an example with X_1 following Exp(4), X_2 following Gamma$(1, 1)$, and probability $p = 0.95$. The mixture H is not IGFR. Figure A.2 numerically illustrates that the mixture H satisfies our Assumption 2. Moreover, in our extensive numerical test on the mixture of

[4]We remark that Pareto distribution is both IGFR and DGFR.

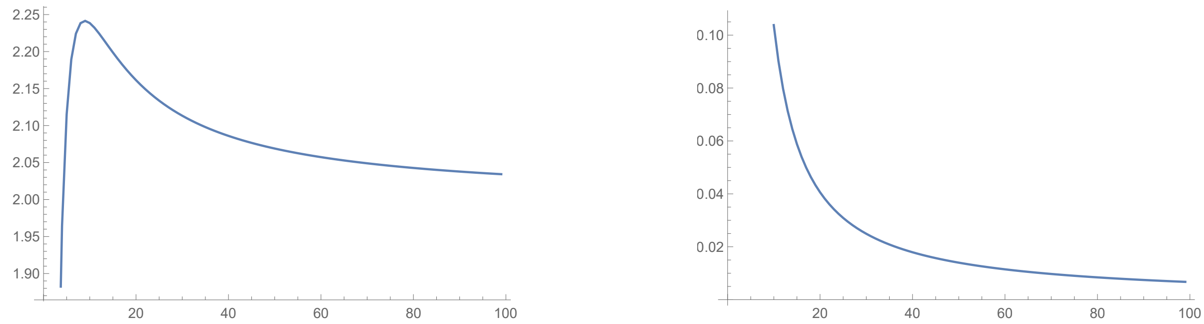

Figure A.1: $\frac{xf(x)}{\bar{F}(x)}$ (Left) and $\frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi}$ (Right) of Random Variable X: Convolution of Two Log-Logistic (2,1) Distributions.

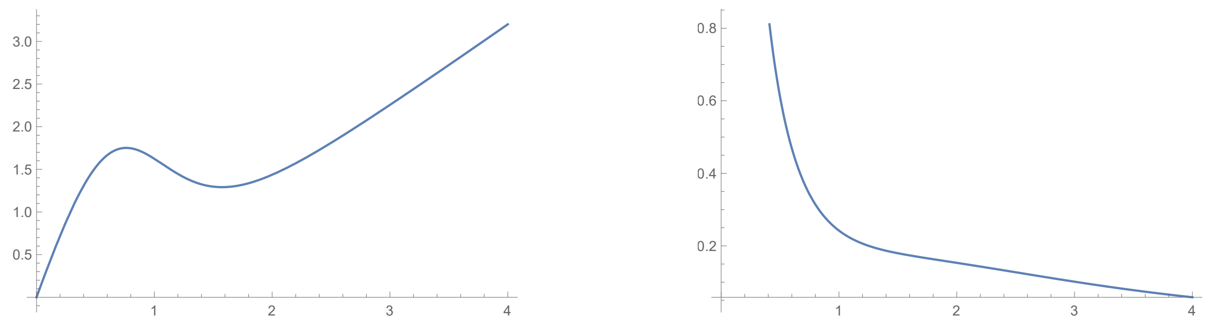

Figure A.2: $\frac{xf(x)}{\bar{F}(x)}$ (Left) and $\frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi}$ (Right) of Random Variable X: Mixture of Exp(4) and Gamma(1,1) with Probability $p = 0.95$.

an Exponential distribution and a Gamma distribution, we observe that Assumption 2 is satisfied by most instances which do not satisfy the IGFR property.

To sum up, our proposed Assumption 2 is more general than the IGFR property and more robust under many distribution operations. Consequently, it is an appealing condition to apply for our second class of random yield problems.

A.2 On Assumption 1 and IGFR

In this subsection, we provide examples to illustrate that our Assumption 1 and IGFR are two separate conditions and neither one strictly dominates the other. On one hand, for the distributions in Example 1 of Section A.1, i.e., positive shifts of a Pareto (θ, k) distribution with $\theta > 1$ and $k > 0$, they are DGFR and satisfy Assumption 2. Thus, they automatically satisfy Assumption 1 since $\frac{x^2 g(x)}{\int_l^x \xi g(\xi)d\xi} = \frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi}\frac{xg(x)}{\bar{G}(x)}$. On the other hand, Beta, Normal, and Laplace distributions are all IGFR distributions. However, from the proof of Table 1, they don't satisfy the sufficient condition that $xg'(x)/g(x)$ decreases in x with certain parameters. Consequently, they may violate Assumption 1. In Figure A.3, we provide two examples of Beta distributions with $\beta < 1$ that don't satisfy Assumption 1. In sum, the above examples clearly show that Assumption 1 and IGFR are two independent conditions without any dominating relationship.

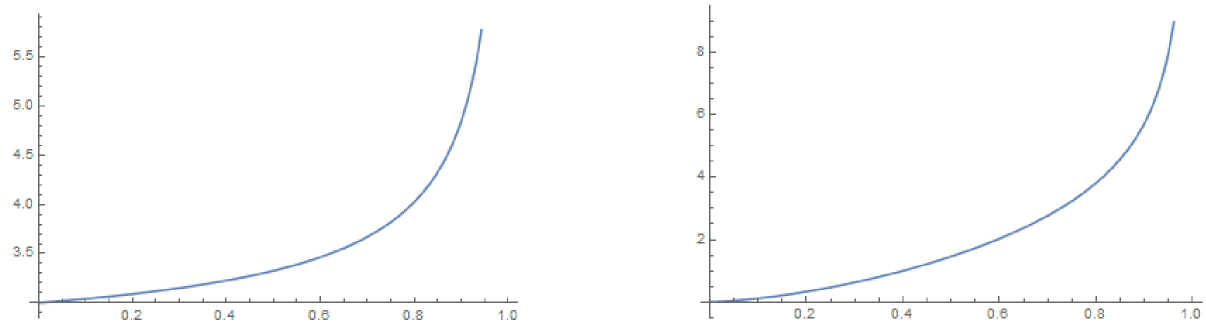

Figure A.3: $\frac{x^2 g(x)}{\int_l^x \xi g(\xi) d\xi}$ for Beta Distributed Random Variable: Beta$(2, 0.5)$ (Left) and Beta$(0.5, 0.5)$ (Right).

B Additional Applications of Assumption 1

In this section, we provide addition examples of Assumption 1 by studying an assembly system with deterministic demand D and two component suppliers. Supplier 1 suffers from production random yield with per unit production cost c_1, and supplier 2 is perfectly reliable with per unit production cost c_2. We study two decentralized settings. In the first setting, the component suppliers serve as the Stackelberg leader and set component wholesale prices. In the second setting, the assembler serves as the Stackelberg leader and sets component wholesale prices.

B.1 The first setting

In this case, both component suppliers first decide the wholesale price w_i, and then the assembler decides the order quantity q_i, $i = 1, 2$. For simplicity, we assume the assembler pays for its entire order quantity. Given any wholesale price pair (w_1, w_2), the assembler decides its order quantities to maximize its ex ante profit:

$$\Pi_a(q_1, q_2 | w_1, w_2) = pE \min\{D, q_1\xi, q_2\} - w_1 q_1 - w_2 q_2.$$

It can be shown that

$$(q_1^*, q_2^*) = \begin{cases} (D/\delta, D) & \text{if } \; w_1 \le p\mu \text{ and } w_2 \le p(1 - G(\delta)) \\ (0, 0) & \text{otherwise,} \end{cases}$$

where $\int_l^\delta \xi g(\xi) d\xi = w_1/p$.

Back to the wholesale price decisions, the reliable supplier produces its component only if $w_2^* \ge c_2$. In this case, the reliable supplier always charges $w_2^* = p(1 - G(\delta))$, which decreases in w_1. Let $\hat{w}$ satisfy $p(1 - G(\delta)) = c_2$. Clearly, the unreliable supplier should charge wholesale price $w_1 \le \hat{w}$, otherwise $q_1^* = q_2^* = 0$. There are two cases to consider. If $\hat{w} \le c_1$, no order/production occurs. If $\hat{w} > c_1$, $\Pi_1(w_1) = (w_1 - c_1) q_1^*(w_1)$. By Proposition 5, the optimal wholesale price can be uniquely determined under Assumption 1, i.e., $w_1^* = \min\{\hat{w}, w^*\}$, where w^* is the unique unconstrained optimum. Consequently, the assembly system has unique equilibrium wholesale prices and order quantities.

B.2 The second setting

In this case, the assembler first sets the wholesale prices w_i for both component suppliers, and the suppliers respond by deciding the production quantities Q_i, $i = 1, 2$. When the yield risk is resolved, the assembler assembles from the available components to satisfy the demand D. This important and challenging problem has been first studied by Pan and So (2016). Define $\hat{w}_2 := \frac{c_2}{1-\int_l^\delta G(\xi)d\xi/\delta}$. They show that given the component wholesale price pair (w_1, w_2), the optimal production quantities are

$$(Q_1^*, Q_2^*) = \begin{cases} (D/\delta, D) & \text{if } \; w_1 \geq c_1/\mu \text{ and } w_2 \geq \hat{w}_2 \\ (0, 0) & \text{otherwise,} \end{cases}$$

where δ satisfies $\int_l^\delta \xi g(\xi)d\xi = c_1/w_1$. Then $w_2^* = \hat{w}_2$, and the assembler chooses w_1 by maximizing its ex ante profit function:

$$\Pi_a(w_1) = (p - w_1 - \hat{w}_2)E\min\{D, Q_1^*\xi, Q_2^*\} = (p - w_1)\left(1 - \frac{\int_l^\delta G(\xi)d\xi}{\delta}\right)D - c_2D.$$

Let $\hat{p} := \min\{w_1 + \hat{w}_2 : w_1 \geq c_1/\mu, \Pi_a(w_1) \geq 0\}$ be the deterministic threshold price, below which it is not profitable for the assembler to assemble the product. To facilitate the analysis, Pan and So (2016) assume that $\frac{x^3 g(x)}{(\int_l^x \xi g(\xi)d\xi)^3}$ decreases in $x \in [l, u]$. Under this assumption, the expression of $\hat{p}$ can be determined depending on model parameters, and $\Pi_a(w_1)$ is concave in w_1 when $p > \hat{p}$ (see Propositions 2 and 3 in Pan and So 2016). Moreover, they show that this assumption is satisfied by generalized uniform distributions and truncated exponential distributions, but is hard to verify for other yield distributions (see Proposition 4 in Pan and So 2016).

Since there is a one-to-one monotone mapping between w_1 and Q_1, we change the decision variable from w_1 to Q_1 and obtain:

$$\Pi_a(Q_1) = (p - w_1(Q_1))E\min\{D, Q_1\xi\} - c_2D. \tag{B.1}$$

After the change of decision variable, we show, below, that Propositions 2 and 3 in Pan and So (2016) can be obtained under Assumption 1. Therefore, the results and managerial insights in Section 4 of Pan and So (2016) are further generalized and proved to hold for a variety of continuous yield distributions.

Proof of Proposition 2 in Pan and So (2016): To derive the expression of $\hat{p}$, it is equivalent to derive the minimum of the function $T(w_1) := w_1 + \hat{w}_2$, where $\hat{w}_2 = \frac{c_2}{1-\int_l^\delta G(\xi)d\xi/\delta}$ and $\delta = D/Q_1$ satisfies $\int_l^\delta \xi g(\xi)d\xi = c_1/w_1$. To facilitate the analysis, we define the following functions: $S(Q) := E\min\{D, Q\xi\}$, $J(Q) := \frac{S(Q)}{\int_l^{\frac{D}{Q}} \xi dG(\xi)}$, and $f(Q) := \frac{D^2 g(\frac{D}{Q})}{Q^2 \int_l^{\frac{D}{Q}} \xi dG(\xi)}$. Under Assumption 1, $f(Q)$ increases in $Q \in [\frac{D}{u}, \frac{D}{l}]$.

It is easy to calculate that $S(Q_1(w_1)) = E\min\{D, Q_1(w_1)\xi\} = D(1 - \int_l^\delta G(\xi)d\xi/\delta)$. Thus,

$T(w_1) = w_1 + c_2 D/S(Q_1(w_1))$. Since there is a one-to-one monotone mapping between w_1 and Q_1, we change the decision variable from w_1 to Q_1. After some necessary simplification, we have:

$$T(Q_1) = \frac{c_1}{\int_l^{\frac{D}{Q_1}} \xi dG(\xi)} + \frac{c_2 D}{S(Q_1)}$$
$$= \frac{c_1}{\int_l^{\frac{D}{Q_1}} \xi dG(\xi)} \left(1 + \frac{c_2 D}{c_1 J(Q_1)}\right).$$

Taking derivative with respect to Q_1 and after some necessary simplification, we have:

$$\frac{\partial}{\partial Q_1} T(Q_1) = \frac{c_1}{\int_l^{\frac{D}{Q_1}} \xi dG(\xi)} \left(-\frac{c_2 D J'(Q_1)}{c_1 J(Q_1)} + \left(1 + \frac{c_2 D}{c_1 J(Q_1)}\right) \frac{f(Q_1)}{Q_1}\right)$$
$$= \frac{c_1}{J^2(Q_1) \int_l^{\frac{D}{Q_1}} \xi dG(\xi)} \left(-\frac{c_2 D}{c_1} + \frac{J^2(Q_1) f(Q_1)}{Q_1}\right),$$

where the second equality comes from the fact that $J'(Q_1) = 1 + \frac{f(Q_1)J(Q_1)}{Q_1}$. Based on the proof of Proposition 6, we have $J(Q_1)(\geq 0)$ increases in Q_1 and $\frac{f(Q_1)J(Q_1)}{Q_1}(\geq 0)$ increases in Q_1. Thus, $\frac{J^2(Q_1)f(Q_1)}{Q_1}$ increases in Q_1.

Next, checking the boundary conditions, we have $\lim_{Q_1 \to D/l} T'(Q_1) \to +\infty$ and $\lim_{Q_1 \to D/u} T'(Q_1) = -c_2 D/c_1 + ug(u)D/\mu$. If $c_2/c_1 > ug(u)/\mu$, then $T(Q_1)$ is unimodal in Q_1 with the minimizer Q_1^* satisfying the first order condition, i.e., $-\frac{c_2 D}{c_1} + \frac{J^2(Q_1)f(Q_1)}{Q_1} = 0$. In this case, $\hat{p} = T(Q_1^*)$ (after some necessary simplifications, it has the same expression as that in Part (ii) of Proposition 2 in Pan and So (2016)). Otherwise, $T(Q_1)$ increases in Q_1, and the minimum is obtained at $Q_1 = D/u$. In this case, $\hat{p} = c_1/\mu + c_2 u/\mu$. Thus, Proposition 2 in Pan and So (2016) continues to hold under our Assumption 1. □

Proof of Proposition 3 in Pan and So (2016): When $p > \hat{p}$, we have:

$$\Pi_a(w_1) = (p - w_1)\left(1 - \frac{\int_l^\delta G(\xi)d\xi}{\delta}\right) D - c_2 D = (p - w_1) E \min\{D, Q_1^*(w_1)\} - c_2 D.$$

Changing decision variable from w_1 to Q_1, we have:

$$\Pi_a(Q_1) = (p - w_1(Q_1)) E \min\{D, Q_1 \xi\} - c_2 D.$$

By Proposition 9, $\Pi_a(Q_1)$ is concave in Q_1 under Assumption 1 and the optimal Q_1^* and, thus, w_1^* can be uniquely determined. Moreover, checking boundary condition, we have $\frac{\partial \Pi_a(Q_1)}{\partial Q_1}|_{Q1=D/l} < 0$ and:

$$\frac{\partial \Pi_a(Q_1)}{\partial Q_1}|_{Q1=D/u} = \left(\left(p \int_l^{\frac{D}{Q_1}} \xi g(\xi) d\xi - c\right) J'(Q_1) - p\frac{D^2}{Q_1^3} g(\frac{D}{Q_1}) J(Q_1)\right)|_{Q_1=D/u}$$
$$= p\mu - c_1 - c_1 u^2 g(u)/\mu,$$

If $p > c_1/\mu + c_1 u^2 g(u)/\mu$, then $\frac{\partial \Pi_a(Q_1)}{\partial Q_1}|_{Q1=D/u} > 0$. Otherwise, $\frac{\partial \Pi_a(Q_1)}{\partial Q_1}|_{Q1=D/u} \leq 0$. The corner solutions and the boundary conditions can be discussed using the same arguments as these in Pan and So (2016) and, thus, omitted. Please refer to Pan and So (2016) for a complete solution of w_1^*. □

C Additional Applications of Assumption 2

In this section, we provide an additional example of Assumption 2, i.e., selling to the price-setting retailer under random yield and buyback contract. We remark that our Assumption 2 is also helpful in solving the price-setting retailer's problem with one reliable and one unreliable supplier, see Dong et al. (2015). Note that the objective functions of above examples are also well behaved under Assumption 1. The proof of the proposition in this section is relegated to Appendix D.

C.1 Selling to the price-setting retailer under random yield and buyback contract

In this section, we consider the setting with an upstream supplier selling to a downstream retailer under buyback contract. The supplier suffers from random yield, and the retailer faces deterministic price sensitive demand $d(p)$. The retailer pays for its order quantity at per unit wholesale price w. The leftover inventory will be bought back by the supplier at per unit buyback price b. In this case, the over-ordering risk is shared between the two firms. The sequence of events is: (1) the supplier decides the wholesale price w and buyback price b; (2) the retailer decides the sales price p and order quantity q; and (3) the supplier decides the production quantity Q. The purpose of this section is to show that under Assumption 2 and some general conditions of the demand function, the three-stage Stackelberg game has a unique equilibrium. We remark that the counterpart of the above problem under demand uncertainty and deterministic yield has been studied by Song et al. (2008).

We solve the game using backward induction. First, given any (w, b) and (q, p), the supplier decides its production quantity Q to maximize its ex ante profit

$$\Pi_s(Q|w, b, p, q) = wq - cQ - bE(\min\{q, Q\xi\} - d(p))^+,$$

which decreases in Q since $b > 0$. Thus, $Q^* = q$.

Second, given (w, b), the retailer decides (p, q) to maximize its ex ante profit

$$\Pi_r(p, q|w, b) = pE\min\{d(p), q\xi\} + bE(q\xi - d(p))^+ - wq.$$

Applying the method in the proof of Proposition 7, it can be shown that (p^*, q^*) can be uniquely determined under Assumption 2 and IPE demand function (see the proof of Proposition C.1 for details). Moreover, they must satisfy the following set of first order conditions, where $\delta = d(p)/q$:

$$\frac{\partial}{\partial p}\Pi_r(p, q|w, b) = (p - b)d'(p)\bar{G}(\delta) + d(p)E\min\left\{1, \frac{\xi}{\delta}\right\} = 0;$$

$$\frac{\partial}{\partial q}\Pi_r(p,q|w,b) = (p-b)\int_l^\delta \xi g(\xi)d\xi - (w-b\mu) = 0.$$

Finally, it is quite involved to analyze the supplier's pricing decisions directly. Thus, we adopt the method from Song et al. (2008) and change the decision variables from (w,b) to (p,q) by solving (w,b) from the above set of first order conditions as:

$$b = \frac{d(p)}{d'(p)\bar{G}(\delta)} E\min\left\{1, \frac{\xi}{\delta}\right\} + p;$$

$$w = p\int_l^\delta \xi g(\xi)d\xi + \left(\frac{d(p)}{d'(p)\bar{G}(\delta)} E\min\left\{1, \frac{\xi}{\delta}\right\} + p\right)\left(\mu - \int_l^\delta \xi g(\xi)d\xi\right).$$

Plugging (w,b) into the supplier's profit function and after some necessary simplifications, we have:

$$\Pi_s(p,q) = pd(p)\left(1 + \frac{d(p)}{pd'(p)}\right)\left(\bar{G}(\delta) + \frac{1}{\delta}\int_l^\delta \xi dG(\xi)\right) - cq. \tag{C.1}$$

Under Assumption 2 and some regularity conditions on the demand function, the following proposition shows that $\Pi_s(p,q)$ is unimodal and thus, the three-stage Stackelberg game has the unique equilibrium (w^*, b^*, p^*, q^*).

Proposition C.1. *Assume Assumption 2 holds and the demand function $d(p)$ satisfies (1) IPE property; (2) $p(1-\frac{1}{\eta(p)})$ increases in p; and (3) $\frac{d'(p)(p+d(p)/d'(p))}{d(p)(p+d(p)/d'(p))'}$ decreases in p, where $\eta(p)$ is defined in Equation (1). For the selling to the price-setting retailer problem under random yield and buyback contract, the supplier's profit function $\Pi_s(p,q)$ in (C.1) is unimodal, and the equilibrium decisions (w^*, b^*, p^*, q^*) can be uniquely determined.*

To conclude, we remark that conditions (1)-(3) in Proposition C.1 are not restrictive. They are satisfied by many commonly used demand functions, such as (1) $d(p) = a - bp$, $a > 0, b > 0$; (2) $d(p) = (a-bp)^k$, $a > 0, b > 0, k > 0$; (3) $d(p) = ap^{-k}$, $a > 0, k > 1$, etc.

D Proofs of Statements

Proof of Proposition 1: Let $g(x)$ be the p.d.f. of random variable X. Taking derivative, we have

$$\begin{aligned}\frac{\partial}{\partial x}\frac{x\bar{G}(x)}{\int_l^x \xi g(\xi)d\xi} &= \frac{[\bar{G}(x) - xg(x)]\int_l^x \xi g(\xi)d\xi - x^2\bar{G}(x)g(x)}{(\int_l^x \xi g(\xi)d\xi)^2} \\ &= \frac{E\min\{x,\xi\}}{x\int_l^x \xi dG(\xi)} A(x),\end{aligned}$$

where $A(x) = \frac{x\bar{G}(x)}{E\min\{x,\xi\}} - f(x)$ and $f(x) = \frac{x^2 g(x)}{\int_l^x \xi dG(\xi)}$. To show Assumption 2 holds, it remains sufficient to show that $A(x) \le 0$, $\forall x \in [l,u]$. Clearly, $A(u) \le 0$. Taking derivative, we have:

$$A'(x) = \frac{\int_l^x \xi dG(\xi)}{xE\min\{x,\xi\}} A(x) - f'(x). \tag{D.1}$$

Now, define $K(x) = A(x)e^{\int_x^u \frac{\int_l^y \xi dG(\xi)}{yE\min\{y,\xi\}}dy}$. Taking derivative, we have:

$$K'(x) = \left(e^{\int_x^u \frac{\int_l^y \xi dG(\xi)}{yE\min\{y,\xi\}}dy}\right)\left(A'(x) - A(x)\frac{\int_l^x \xi dG(\xi)}{xE\min\{x,\xi\}}\right)$$
$$= \left(e^{\int_x^u \frac{\int_l^y \xi dG(\xi)}{yE\min\{y,\xi\}}dy}\right)(-f'(x)) \geq 0,$$

where the second equality comes from Equation (D.1) and the last inequality comes from the fact that X satisfies Assumption 1. Consequently, $K(x)$ increases in x, and $K(x) \leq K(u) = A(u) \leq 0$, $\forall x \in [l,u]$, which implies that $A(x) \leq 0$, $\forall x \in [l,u]$. □

Proof of Proposition 2: Part (i): Let $\bar{X}$ be the truncation of X within the interval $[l,u]$. Then, it has c.d.f. $F(x) = \frac{G(x)-G(l)}{G(u)-G(l)}$ and p.d.f. $f(x) = \frac{g(x)}{G(u)-G(l)}$, respectively.

For Assumption 1, $\frac{x^2 f(x)}{\int_l^x \xi f(\xi)d\xi} = \frac{x^2 g(x)}{\int_l^x \xi g(\xi)d\xi}$. Taking derivative, we have

$$\frac{\partial}{\partial x}\frac{x^2 f(x)}{\int_l^x \xi f(\xi)d\xi} = \frac{(2xg(x) + x^2 g'(x))\int_l^x \xi g(\xi)d\xi - x^3 g^2(x)}{(\int_l^x \xi g(\xi)d\xi)^2} \leq 0$$

iff $x^2 g^2(x) \geq (2g(x) + xg'(x))\int_l^x \xi g(\xi)d\xi$, $\forall x \in [l,u]$. On the other hand, Assumption 1 holds for random variable X, which implies that $x^2 g^2(x) \geq (2g(x) + xg'(x))\int_L^x \xi g(\xi)d\xi$ for all $x \in [L,U]$. For any $x \in [l,u]$, there are two cases to consider. *Case (1):* if $(2g(x) + xg'(x)) \leq 0$, $x^2 g^2(x) \geq (2g(x)+xg'(x))\int_l^x \xi g(\xi)d\xi$ automatically holds. *Case (2):* if $(2g(x)+xg'(x)) \geq 0$, $x^2 g^2(x) \geq (2g(x)+xg'(x))\int_L^x \xi g(\xi)d\xi \geq (2g(x) + xg'(x))\int_l^x \xi g(\xi)d\xi$, where the first inequality comes from the fact that Assumption 1 holds for random variable X, and the second one from $\int_L^x \xi g(\xi)d\xi \geq \int_l^x \xi g(\xi)d\xi$, for $l \geq L \geq 0$. Combining the two cases, $\frac{\partial}{\partial x}\frac{x^2 f(x)}{\int_l^x \xi f(\xi)d\xi} \leq 0$ holds, $\forall x \in [l,u]$. Therefore, $\bar{X}$ satisfies Assumption 1.

For Assumption 2, $\frac{x\bar{F}(x)}{\int_l^x \xi f(\xi)d\xi} = \frac{x(G(u)-G(x))}{\int_l^x \xi g(\xi)d\xi}$. Taking derivative, we have

$$\frac{\partial}{\partial x}\frac{x\bar{F}(x)}{\int_l^x \xi f(\xi)d\xi} = \frac{(G(u) - G(x) - xg(x))\int_l^x \xi g(\xi)d\xi - x^2 g(x)(G(u) - G(x))}{(\int_l^x \xi g(\xi)d\xi)^2} \leq 0$$

iff $x^2 g(x) \geq (1 - \frac{xg(x)}{G(u)-G(x)})\int_l^x \xi g(\xi)d\xi$, $\forall x \in [l,u]$. On the other hand, Assumption 2 holds for random variable X, which implies that $x^2 g(x) \geq (1-\frac{xg(x)}{1-G(x)})\int_L^x \xi g(\xi)d\xi$, $\forall x \in [L,U]$. Since $G(u) \leq 1$, $1-\frac{xg(x)}{G(u)-G(x)} \leq 1-\frac{xg(x)}{1-G(x)}$, $\forall x \in [l,u]$. For any $x \in [l,u]$, there are two cases to consider. *Case (1):* if $1-\frac{xg(x)}{1-G(x)} \leq 0$, $1-\frac{xg(x)}{G(u)-G(x)} \leq 0$, which immediately implies that $x^2 g(x) \geq (1-\frac{xg(x)}{G(u)-G(x)})\int_l^x \xi g(\xi)d\xi$. *Case (2):* if $1 - \frac{xg(x)}{1-G(x)} > 0$, $x^2 g(x) \geq (1 - \frac{xg(x)}{1-G(x)})\int_L^x \xi g(\xi)d\xi \geq (1 - \frac{xg(x)}{1-G(x)})\int_l^x \xi g(\xi)d\xi \geq (1 - \frac{xg(x)}{G(u)-G(x)})\int_l^x \xi g(\xi)d\xi$, where the first inequality comes from the fact that Assumption 2 holds for random variable X, the second comes from the fact that $\int_L^x \xi g(\xi)d\xi \geq \int_l^x \xi g(\xi)d\xi$, for $l \geq L \geq 0$, and the last comes from the fact that $1-\frac{xg(x)}{1-G(x)} \leq 1-\frac{xg(x)}{G(u)-G(x)}$. Combining the two cases, $\frac{\partial}{\partial x}\frac{x\bar{F}(x)}{\int_l^x \xi f(\xi)d\xi} \leq 0$ holds, $\forall x \in [l,u]$. Therefore, $\bar{X}$ satisfies Assumption 2.

Part (ii): Let $\bar{X} = kX$, then it has support on $[kL, kU]$. Simple calculation yields that $\bar{X}$

has p.d.f. $f(x) = \frac{1}{k} g(\frac{x}{k})$ and c.d.f. $F(x) = G(\frac{x}{k})$. For Assumption 1, $\frac{x^2 f(x)}{\int_{kL}^{x} \xi f(\xi) d\xi} = \frac{\frac{x^2}{k} g(\frac{x}{k})}{\int_{kL}^{x} \frac{\xi}{k} g(\frac{\xi}{k}) d\xi} = \frac{\frac{x^2}{k^2} g(\frac{x}{k})}{\int_{L}^{\frac{x}{k}} m g(m) dm}$, which decreases in $\frac{x}{k} \in [L, U]$. Thus, $\frac{x^2 f(x)}{\int_{kL}^{x} \xi f(\xi) d\xi}$ decreases in $x \in [kL, kU]$, i.e., Assumption 1 holds for $\bar{X}$. For Assumption 2, $\frac{x \bar{F}(x)}{\int_{kL}^{x} \xi f(\xi) d\xi} = \frac{x(1-G(\frac{x}{k}))}{\int_{kL}^{x} \frac{\xi}{k} g(\frac{\xi}{k}) d\xi}$. Let $m = \frac{\xi}{k} \in [L, \frac{x}{k}]$ with $dm = \frac{d\xi}{k}$, then $\frac{x \bar{F}(x)}{\int_{kL}^{x} \xi f(\xi) d\xi} = \frac{x(1-G(\frac{x}{k}))}{\int_{kL}^{x} \frac{\xi}{k} g(\frac{\xi}{k}) d\xi} = \frac{\frac{x}{k}(1-G(\frac{x}{k}))}{\int_{L}^{\frac{x}{k}} m g(m) dm}$, which decreases in $\frac{x}{k} \in [L, U]$. Thus, $\frac{x \bar{F}(x)}{\int_{kL}^{x} \xi f(\xi) d\xi}$ decreases in $x \in [kL, kU]$, i.e., Assumption 2 holds for $\bar{X}$. □

Proof of Proposition 3: Part (i): Let $f(x) = x g(x)$, then $\frac{\partial}{\partial x} \frac{x^2 g(x)}{\int_{L}^{x} \xi g(\xi) d\xi} = \frac{\partial}{\partial x} \frac{x f(x)}{\int_{L}^{x} f(\xi) d\xi} \leq 0$ iff $x f(x) \geq [1 + \frac{x f'(x)}{f(x)}] \int_{L}^{x} f(\xi) d\xi$. Using integration by parts, we have $x f(x) = L f(L) + \int_{L}^{x} f(\xi)[1 + \frac{\xi f'(\xi)}{f(\xi)}] d\xi$. Thus, $\frac{\partial}{\partial x} \frac{x^2 g(x)}{\int_{L}^{x} \xi g(\xi) d\xi} \leq 0$ iff $L f(L) + \int_{L}^{x} f(\xi)[1 + \frac{\xi f'(\xi)}{f(\xi)}] d\xi \geq [1 + \frac{x f'(x)}{f(x)}] \int_{L}^{x} f(\xi) d\xi$. Since $L f(L) \geq 0$, the inequality holds if $\frac{x f'(x)}{f(x)}$ decreases in x. Plugging $f(x) = x g(x)$, we have $\frac{x f'(x)}{f(x)} = 1 + \frac{x g'(x)}{g(x)}$, which decreases in x. Thus $\frac{x^2 g(x)}{\int_{L}^{x} \xi g(\xi) d\xi}$ decreases in x, i.e., Assumption 1 holds.

Part (ii): For condition **(a)**, to show Assumption 2 holds, it is sufficient to show that $\frac{x}{\int_{L}^{x} \xi g(\xi) d\xi}$ decreases in x. Taking derivative, we have $\frac{\partial}{\partial x} \frac{x}{\int_{L}^{x} \xi g(\xi) d\xi} = \frac{\int_{L}^{x} \xi g(\xi) d\xi - x^2 g(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2}$. Since $\int_{L}^{x} \xi g(\xi) d\xi = x G(x) - \int_{L}^{x} G(\xi) d\xi$, $\int_{L}^{x} \xi g(\xi) d\xi - x^2 g(x) \leq 0$ if $G(x) \leq x g(x)$. For condition **(b)**, it ensures that $x \bar{G}(x)$ decreases in x. Thus, Assumption 2 holds.

To prove condition **(c)**, taking derivative, we have:

$$
\begin{aligned}
\frac{\partial}{\partial x} \frac{x \bar{G}(x)}{\int_{L}^{x} \xi g(\xi) d\xi} &= \frac{[\bar{G}(x) - x g(x)] \int_{L}^{x} \xi g(\xi) d\xi - x^2 \bar{G}(x) g(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2} \\
&= \frac{\bar{G}(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2} \left[\left(1 - \frac{x g(x)}{\bar{G}(x)}\right) \left(x G(x) - \int_{L}^{x} G(\xi) d\xi\right) - x^2 g(x) \right] \\
&= \frac{\bar{G}(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2} A(x)
\end{aligned}
$$

where $A(x) := -x \bar{G}(x) + \left(1 - \frac{x g(x)}{\bar{G}(x)}\right) \left(L + \int_{L}^{x} \bar{G}(\xi) d\xi\right)$. Applying L' Hospital's rule, we have $A(L) = -L^2 g(L) \leq 0$, since $g(L) \geq 0$. Taking derivative, we have $A'(x) = -\left(\frac{x g(x)}{\bar{G}(x)}\right)' \left(L + \int_{L}^{x} \bar{G}(\xi) d\xi\right) \leq 0$, because X is IGFR and $L \geq 0$. Above argument implies that $\forall x \in [L, U]$, we have $A(x) \leq 0$. Thus, $\frac{\partial}{\partial x} \frac{x \bar{G}(x)}{\int_{0}^{x} \xi g(\xi) d\xi} \leq 0$ for all $x \in [L, U]$, and Assumption 2 holds.

To prove condition **(d)**, taking derivative and rearranging the terms, we have:

$$
\begin{aligned}
\frac{\partial}{\partial x} \frac{x \bar{G}(x)}{\int_{L}^{x} \xi g(\xi) d\xi} &= \frac{g(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2} \left[\left(\frac{\bar{G}(x)}{g(x)} - x\right) \left(x G(x) - \int_{L}^{x} G(\xi) d\xi\right) - x^2 \bar{G}(x) \right] \\
&= \frac{g(x)}{(\int_{L}^{x} \xi g(\xi) d\xi)^2} B(x)
\end{aligned}
$$

where $B(x) := \left(\frac{\bar{G}(x)}{g(x)} - x\right) \left(x G(x) - \int_{L}^{x} G(\xi) d\xi\right) - x^2 \bar{G}(x)$ Applying L' Hospital's rule, we have $B(L) = -L^2 \leq 0$. Taking derivative, we have $B'(x) = -x \bar{G}(x) - \frac{2 g^2(x) + g'(x) \bar{G}(x)}{g^2(x)} (x G(x) - \int_{L}^{x} G(\xi) d\xi) < 0$, if $2 g^2(x) + g'(x) \bar{G}(x) > 0$ for all $x \in [L, U]$. Thus, $B(x) \leq 0$ for all $x \in [L, U]$ and Assumption 2 holds. Part (e) is immediate from Proposition 1. $\square$

Proof of Proposition 4: Part (i): The fact that $g(x)$ is log-concave implies that X is IGFR (Lariviere, 2006). Hence, X satisfies Assumption 2 by part (i)(c) of Proposition 3. On the other hand, log-concavity is preserved under shifting, thus, $Y = X + a$ is also log-concave, and, thus, IGFR, which implies that Y satisfies Assumption 2.

Part (ii): By part (i) of Proposition 3, X satisfies Assumption 1. Let $Y = X + a \in [L+a, U+a]$. The p.d.f. and c.d.f. of Y are $g_Y(y) = g_X(y-a)$ and $G_Y(y) = G_X(y-a)$, respectively. Then, $\frac{yg'_Y(y)}{g_Y(y)} = \frac{yg'_X(y-a)}{g_X(y-a)} = \frac{(y-a)g'_X(y-a)}{g_X(y-a)} + \frac{ag'_X(y-a)}{g_X(y-a)}$. The first part decreases in $y - a \in [L, U]$ from the fact that $\frac{xg'_X(x)}{g_X(x)}$ decreases in $x \in [L, U]$, and the second part decreases in $y - a \in [L, U]$ from the fact that X is log-concave and $a > 0$. Thus, $\frac{yg'_Y(y)}{g_Y(y)}$ decreases in $y \in [L+a, U+a]$, and Y satisfies Assumption 1 by part (i) of Proposition 3. $\square$

Proof of Proposition 5: $\Pi_s(q) = (w(q) - c)q$ with $w(q) = p\int_l^{\frac{D}{q}} \xi dG(\xi)$. Taking derivative with respect to q, we have

$$\frac{\partial \Pi_s(q)}{\partial q} = p\int_l^{\frac{D}{q}} \xi dG(\xi) - p\frac{D^2}{q^2}g(\frac{D}{q}) - c = p\int_l^{\frac{D}{q}} \xi dG(\xi)\left(1 - \frac{\frac{D^2}{q^2}g(\frac{D}{q})}{\int_l^{\frac{D}{q}} \xi g(\xi)d\xi}\right) - c.$$

Under Assumption 1, $1 - \frac{\frac{D^2}{q^2}g(\frac{D}{q})}{\int_l^{\frac{D}{q}} \xi g(\xi)d\xi}$ decreases in q. Together with the fact that $p\int_l^{\frac{D}{q}} \xi dG(\xi)$ decreases in q, it is immediate that $\frac{\partial \Pi_s(q)}{\partial q}$ changes sign at most once from above. Thus, $\Pi_s(q)$ is quasi-concave in q with q^* (and w^*) be uniquely determined. $\square$

Proof of Proposition 6: To facilitate the analysis, we define the following functions: $S(Q) := E\min\{D, Q\xi\}$, $J(Q) := \frac{S(Q)}{\int_l^{\frac{D}{Q}} \xi dG(\xi)}$, and $f(Q) := \frac{D^2 g(\frac{D}{Q})}{Q^2\int_l^{\frac{D}{Q}} \xi dG(\xi)}$. Under Assumption 1, $f(Q)$ increases in $Q \in [\frac{D}{u}, \frac{D}{l}]$.

Let $w(Q) = c/\int_l^{\frac{D}{Q}} \xi dG(\xi)$, the retailer's profit function as a function of the production quantity Q is:

$$\Pi_r(Q) = (p - w(Q))S(Q) = \left(p\int_l^{\frac{D}{Q}} \xi g(\xi)d\xi - c\right)J(Q).$$

Taking derivative with respect to Q, we have:

$$\begin{aligned}\frac{\partial \Pi_r(Q)}{\partial Q} &= \left(p\int_l^{\frac{D}{Q}} \xi g(\xi)d\xi - c\right)J'(Q) - p\frac{D^2}{Q^3}g(\frac{D}{Q})J(Q)\\ &= p\int_l^{\frac{D}{Q}} \xi g(\xi)d\xi - c - c\frac{f(Q)J(Q)}{Q},\end{aligned}$$

where the last equality comes from the fact that $J'(Q) = 1 + \frac{f(Q)J(Q)}{Q}$. To show $\Pi_r(Q)$ is concave, it is sufficient to show that $\frac{f(Q)J(Q)}{Q}$ increases in Q. Since $f(Q)$ increases in Q by Assumption 1, it remains sufficient to show that $\frac{J(Q)}{Q}$ increases in Q. Simple calculation yields that:

$$\frac{J(Q)}{Q} = \frac{E\min\{Q\xi, D\}}{Q\int_l^{\frac{D}{Q}} \xi dG(\xi)} = \frac{E\min\{\xi, x\}}{\int_l^x \xi dG(\xi)} = 1 + \frac{x\bar{G}(x)}{\int_l^x \xi dG(\xi)},$$

which decreases in x by Proposition 1 and, thus, increases in Q, where $x := \frac{D}{Q} \in [l, u]$. As a consequence, $\Pi_r(Q)$ is concave in Q with unique Q^*. □

Proof of Proposition 7: Let $\Pi_r(p) = \Pi_r(p, q^*(p)) = pd(p)\bar{G}(\delta)$, where $\delta = \frac{d(p)}{q^*}$ satisfies $\int_l^\delta \xi g(\xi)d\xi = \frac{w}{p}$ with $p \geq \frac{w}{\mu}$. Taking derivative, we have $\frac{\partial\delta}{\partial p} = \frac{d'(p)q^* - d(p)\frac{\partial q^*}{\partial p}}{(q^*)^2} = -\frac{\int_l^\delta \xi g(\xi)d\xi}{p\delta g(\delta)}$ and $\frac{\partial}{\partial p}\Pi_r(p) = (d(p) + pd'(p))\bar{G}(\delta) + d(p)\frac{\int_l^\delta \xi g(\xi)d\xi}{\delta}$. Checking boundary conditions, we have $\frac{\partial}{\partial p}\Pi_r(p)|_{p=\frac{w}{\mu}} = d(\frac{w}{\mu})\frac{\mu}{u} > 0$ and $\frac{\partial}{\partial p}\Pi_r(p)|_{p=p_0} = p_0 d'(p_0)\bar{G}(\delta_0) < 0$. Thus, due to continuity, there must exist at least one $p^* \in (\frac{w}{\mu}, p_0)$ such that $\frac{\partial}{\partial p}\Pi_r(p)|_{p=p^*} = 0$.

Rearranging the terms, we have $\frac{\partial}{\partial p}\Pi_r(p) = d(p)\bar{G}(\delta)\left(1 + \frac{pd'(p)}{d(p)} + \frac{\int_l^\delta \xi g(\xi)d\xi}{\delta\bar{G}(\delta)}\right)$. By the IPE property and Assumption 2 with the fact that δ decreases in p, it is immediate that $\left(1 + \frac{pd'(p)}{d(p)} + \frac{\int_l^\delta \xi g(\xi)d\xi}{\delta\bar{G}(\delta)}\right)$ decreases in p, which, together with the boundary conditions, implies that $\frac{\partial}{\partial p}\Pi_r(p)$ crosses zero only once from above. Thus, $\Pi_r(p)$ is quasi-concave with p^* be uniquely determined via FOC. □

Proof of Proposition 8: Let $\hat{p} := \frac{w^2 E\min\{\delta,\xi\}}{c}$ and $\hat{\delta}$ satisfies $\int_l^{\hat{\delta}} \xi g(\xi)d\xi = \frac{wE\min\{\delta,\xi\}}{p}$, where $\int_l^\delta \xi g(\xi)d\xi = \frac{c}{w}$. It can be shown that $\hat{p} > w$. Define:

$$\Pi_r(p) := \Pi_r(p, q^*(p)) = \begin{cases} (p-w)d(p)E\min\{1, \frac{\xi}{\delta}\} & \text{if } \; p \in [w, \hat{p}] \\ pd(p)(1 - G(\hat{\delta})) & \text{if } \; p \in (\hat{p}, \max\{\hat{p}, p_0\}], \end{cases}$$

On one hand, as shown in Yao et al. (2006), $(p-w)d(p)$ is quasi-concave when $d(p)$ satisfies the IPE property. On the other hand, from the proof of Proposition 7, $pd(p)(1 - G(\hat{\delta}))$ is also quasi-concave under Assumption 2 and IPE demand. Also, it is easy to check that $\Pi_r(p)$ is continuous at point $p = \hat{p}$.

To show $\Pi_r(p)$ is continuously differentiable and unimodal, it is sufficient to show that $\frac{\partial}{\partial p}\Pi_r(p)|_{p=\hat{p}^-} = \frac{\partial}{\partial p}\Pi_r(p)|_{p=\hat{p}^+}$. Simple calculation yields that

$$\frac{\partial}{\partial p}\Pi_r(p)|_{p=\hat{p}^-} - \frac{\partial}{\partial p}\Pi_r(p)|_{p=\hat{p}^+} = d'(\hat{p})\left(\frac{\hat{p}c}{w\delta} - wE\min\{1, \frac{\xi}{\delta}\}\right) = 0$$

Consequently, $\Pi_r(p)$ is continuously differentiable and unimodal. Moreover, it is easy to check that $\frac{\partial}{\partial p}\Pi_r(p)|_{p=w} > 0$ and $\frac{\partial}{\partial p}\Pi_r(p)|_{p=p_0} < 0$. Thus, there exists a unique $p^* \in (w, p_0)$. □

Proof of Proposition C.1: We show Proposition C.1 in the following two steps. **Step 1:** given (w, b), the retailer's profit function is unimodal and it has unique optimal (p^*, q^*). **Step 2:** the

Stackelberg game has unique equilibrium (p^*, q^*, w^*, b^*).

Step 1: Given (w, b), the retailer's ex ante profit function is: $\Pi_r(p, q|w, b) = pE\min\{d(p), q\xi\} + bE(q\xi - d(p))^+ - wq$. Taking derivative with respect to q, we can show that $q^*(p) = d(p)/\delta_r$, where δ_r satisfies $\int_l^{\delta_r} \xi g(\xi)d\xi = (w - b\mu)/(p - b)$ with $p > w/\mu > b$. Plugging in $q^*(p)$, we have $\Pi_r(p) := \Pi_r(p, q^*(p)|w, b) = (p - b)d(p)\bar{G}(\delta_r)$. Taking derivative with respect to p and after some necessary simplifications, we have:

$$\frac{\partial \Pi_r(p)}{\partial p} = d(p)\bar{G}(\delta_r)\left(\frac{(p-b)d'(p)}{d(p)} + 1 + \frac{\int_l^{\delta_r} \xi g(\xi)d\xi}{\delta_r \bar{G}(\delta_r)}\right).$$

We first show that if $d(p)$ satisfies the IPE property, then $\forall\ p > b > 0$, $(p-b)d'(p)/d(p)$ decreases in $p > b$. Taking derivative, we have:

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{(p-b)d'(p)}{d(p)}\right) &= \frac{(d'(p) + (p-b)d''(p))d(p) - (p-b)(d'(p))^2}{d^2(p)} \\ &= \frac{(p-b)}{d^2(p)}A(b),\end{aligned}$$

where $A(b) := \left(\frac{d'(p)}{(p-b)} + d''(p)\right)d(p) - (d'(p))^2$. Easy to see that $A(b)$ decreases in $b < p$. Since the IPE property implies that $A(0) < 0$, we have $A(b) < A(0) < 0$. Thus, $(p - b)d'(p)/d(p)$ decreases in $p > b$.

Next, easy to check that $\frac{\partial \Pi_r(p)}{\partial p}|_{p=w/u} > 0$ and $\frac{\partial \Pi_r(p)}{\partial p}|_{p=p_0} < 0$. By a similar argument as used in the proof of Proposition 7, under Assumption 2, $\frac{\partial \Pi_r(p)}{\partial p}$ must cross zero only once from above with optimal p^* be determined via FOC. Thus, for any given (w, b), $\Pi_r(p, q|w, b)$ is unimodal. Define $\delta := d(p)/q$, the unique optimal (p^*, q^*) must also satisfy the set of first order conditions:

$$\frac{\partial}{\partial p}\Pi_r(p, q|w, b) = (p-b)d'(p)\bar{G}(\delta) + d(p)E\min\left\{1, \frac{\xi}{\delta}\right\} = 0;$$

$$\frac{\partial}{\partial q}\Pi_r(p, q|w, b) = (p-b)\int_l^{\delta} \xi g(\xi)d\xi - (w - b\mu) = 0.$$

Solving (w, b) from above set of first order conditions, we have the unique solution:

$$b = \frac{d(p)}{d'(p)\bar{G}(\delta)}E\min\left\{1, \frac{\xi}{\delta}\right\} + p;$$

$$w = p\int_l^{\delta} \xi g(\xi)d\xi + \left(\frac{d(p)}{d'(p)\bar{G}(\delta)}E\min\left\{1, \frac{\xi}{\delta}\right\} + p\right)\left(\mu - \int_l^{\delta} \xi g(\xi)d\xi\right).$$

Plugging in (w, b) and after some necessary simplifications, we have the supplier's ex ante profit function as a function of (p, q):

$$\Pi_s(p, q) = pd(p)\left(1 + \frac{d(p)}{pd'(p)}\right)\left(\bar{G}(\delta) + \frac{1}{\delta}\int_l^{\delta} \xi dG(\xi)\right) - cq.$$

Step 2: In this step, we show that $\Pi_s(p, q)$ is unimodal and has unique (p^*, q^*). Consequently, the Stackelberg game has unique equilibrium decisions (p^*, q^*, w^*, b^*). First, for any given p, it is immediate to show that $\Pi_s(p,q)$ is concave in q with optimal order quantity given as $q^* = d(p)/\delta_s$, where δ_s satisfies $\int_l^{\delta_s} \xi g(\xi) d\xi = c/(p(1-1/\eta(p)))$. To ensure that δ_s decreases in p, we assume that $p(1-1/\eta(p))$ increases in p. Also, we restrict $p \in [\bar{p}, p_0]$, where $\bar{p}$ satisfies $\bar{p}(1-1/\eta(\bar{p})) = c/\mu$. Plugging in $q^*(p)$, we have:

$$\Pi_s(p) := \Pi_s(p, q^*(p)) = pd(p)\left(1 + \frac{d(p)}{pd'(p)}\right)\bar{G}(\delta_s).$$

Taking derivative with respect to p and after some necessary simplifications, we have:

$$\frac{\partial \Pi_s(p)}{\partial p} = d(p)\left(1 + \left(\frac{d(p)}{d'(p)}\right)'\right)\bar{G}(\delta_s)\left(1 + \frac{d'(p)(p+\frac{d(p)}{d'(p)})}{d(p)(1+(\frac{d(p)}{d'(p)})')} + \frac{\int_l^{\delta_s}\xi g(\xi)d\xi}{\delta_s \bar{G}(\delta_s)}\right).$$

Checking the boundary derivatives, we have $\frac{\partial \Pi_s(p)}{\partial p}|_{p=\bar{p}} = d(\bar{p})\left(1 + \left(\frac{d(\bar{p})}{d'(\bar{p})}\right)'\right) = d(\bar{p})(\bar{p}(1-1/\eta(\bar{p})))' > 0$ and $\frac{\partial \Pi_s(p)}{\partial p}|_{p=p_0} = p_0 d'(p_0)\bar{G}(\delta_s(p_0)) < 0$. Assume $f(p) := \frac{d'(p)(p+\frac{d(p)}{d'(p)})}{d(p)(1+(\frac{d(p)}{d'(p)})')} = \frac{d'(p)(p+d(p)/d'(p))}{d(p)(p+d(p)/d'(p))'}$ decreases in p. By a similar argument as used in the proof of Proposition 7, under Assumption 2, $\frac{\partial \Pi_s(p)}{\partial p}$ must cross zero only once from above. Thus, $\Pi_s(p)$ is quasi-concave with unique p^* determined via FOC.

Putting together, assume Assumption 2 holds and the demand function $d(p)$ satisfies (1) IPE property; (2) $p(1-\frac{1}{\eta(p)})$ increases in p; and (3) $\frac{d'(p)(p+d(p)/d'(p))}{d(p)(p+d(p)/d'(p))'}$ decreases in p, then $\Pi_s(p,q)$ is unimodal and has unique (p^*, q^*). Consequently, the Stackelberg game has unique equilibrium decisions (p^*, q^*, w^*, b^*).

Finally, it is immediate to check that the conditions on the demand function are satisfied by many commonly used demand functions, including (1) $d(p) = a - bp$, $a > 0, b > 0$; (2) $d(p) = (a-bp)^k$, $a > 0, b > 0, k > 0$; (3) $d(p) = ap^{-k}$, $a > 0, k > 1$, etc. □

E Proofs of Statements in Table 1

Beta distribution [0, 1]: For Beta distribution, $g(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$. We have $xg'(x)/g(x) = \alpha - 1 + (1-\beta)x/(1-x)$, which decreases in x if and only if $\beta \geq 1$. Based on part (ii) of Proposition 3, Assumption 1 holds when $\beta \geq 1$.

Chi distribution [0, ∞): For Chi distribution, $g(x) = \frac{1}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}x^{k-1}e^{-\frac{x^2}{2}}$, $x \in [0,\infty)$. We have $xg'(x)/g(x) = k - 1 - x^2$, which decreases in x when $x > 0$. Based on part (ii) of Proposition 3, Assumption 1 holds.

Chi-squared distribution [0, ∞): For Chi-squared distribution, $g(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$, $x \in [0,\infty)$. We have $xg'(x)/g(x) = k/2 - 1 - x/2$, which decreases in x. Based on part (ii) of Proposition 3, Assumption 1 holds.

Erlang distribution [0, ∞): A special case of Gamma distribution. Thus, following the discussion for Gamma distribution below, Assumption 1 holds.

Exponential distribution [0, ∞): For Exponential distribution, $g(x) = \lambda e^{-\lambda x}$. We have $xg'(x)/g(x) = -\lambda x$, which decreases in $x \in [0, \infty)$. Based on part (ii) of Proposition 3, Assumption 1 holds.

Gamma distribution [0, ∞): For Gamma distribution, $g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$. We have $xg'(x)/g(x) = \alpha - 1 - \beta x$, which decreases in x since $\beta > 0$. Based on part (ii) of Proposition 3, Assumption 1 holds.

Laplace distribution [0, ∞): For truncated Laplace distribution on the interval $[0, \infty)$, $g(x) = \frac{C}{2b} e^{-\frac{|x-\mu|}{b}}$. When $x > \mu$, we have $xg'(x)/g(x) = -x/b$, which decreases in x. Based on part (ii) of Proposition 3, Assumption 1 holds when $x > \mu$.

Log-Logistic distribution [0, ∞): For Log-Logistic distribution, $g(x) = \frac{(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}}{(1+(\frac{x}{\alpha})^\beta)^2}$. We have $xg'(x)/g(x) = \beta - 1 - 2\frac{\beta(\frac{x}{\alpha})^\beta}{(1+(\frac{x}{\alpha})^\beta)}$, which decreases in x. Based on part (ii) of Proposition 3, Assumption 1 holds.

Log-Normal distribution [0, ∞): For Log-Normal distribution, $g(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$. We have $xg'(x)/g(x) = -1 - \frac{\ln x - \mu}{\sigma^2}$, which decreases in x. Based on part (ii) of Proposition 3, Assumption 1 holds.

Normal distribution [0, ∞): For truncated Normal distribution on the interval $[0, \infty)$, $g(x) = \frac{C}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. We have $xg'(x)/g(x) = -\frac{x(x-\mu)}{\sigma^2}$, which decreases in x when $x > \mu/2$. Based on part (ii) of Proposition 3, Assumption 1 holds when $x > \mu/2$.

Pareto distribution (b, ∞): For Pareto distribution, $g(x) = \frac{kb^k}{x^{k+1}}$, where $x > b > 0$. We have $xg'(x)/g(x) = -(k+1)$, which weakly decreases in x. Based on part (ii) of Proposition 3, Assumption 1 holds.

Power distribution [0, 1]: For power distribution, $g(x) = kx^{k-1}$, $k > 0$. Since $\frac{x^2 g(x)}{\int_0^x \xi g(\xi) d\xi} = k+1$, so Assumption 1 holds.

Rayleigh distribution [0, ∞): Rayleigh distribution is a special case of Weibull distribution. Thus, following the discussion for Weibull distribution below, Assumption 1 holds.

Uniform distribution [a, b]: Assumption 1 holds by simple calculations.

Weibull distribution [0, ∞): For Weibull distribution, $g(x|k, \lambda) = \frac{k}{\lambda}(\frac{x}{\lambda})^{k-1} \exp^{-(\frac{x}{\lambda})^k}$. We have $xg'(x)/g(x) = k - 1 - \frac{kx^k}{\lambda^{k-1}}$, which decreases in $x > 0$, since $k > 0$ and $\lambda > 0$. Based on part (ii) of Proposition 3, Assumption 1 holds.

F Price-Setting Newsvendor Problem with Multiplicative Demand Uncertainty

In this section, we show that for the price-setting newsvendor problem with multiplicative demand uncertainty, the unimodality of the objective function and the uniqueness of the optimal price and order quantity can be obtained when the mean demand function satisfies the IPE property and the random noise satisfies Assumption 2. To be more specific, the firm's objective function is:

$$\hat{\Pi}(p, Q) = pE_\xi \min\{d(p)\xi, Q\} - cQ,$$

where $\xi \in [0, \infty)$ is the random noise with pdf $f(\xi)$, cdf $F(\xi)$, and mean $\mu = 1$, respectively. We assume that ξ satisfies Assumption 2 and the expected demand $d(p)$ satisfies the IPE property.

For any fixed p, the optimal order quantity Q^* can be easily obtained via the FOC, which satisfies $\bar{F}(\frac{Q^*}{d(p)}) = \frac{c}{p}$. Plugging back, we have

$$\hat{\Pi}(p|Q^*(p)) = pd(p) \int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi).$$

Taking derivative, we have $\frac{\partial}{\partial p}\hat{\Pi}(p|Q^*(p)) = [pd'(p) + d(p)] \int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi) + pQ^* f(\frac{Q^*}{d(p)}) \frac{\partial}{\partial p}(\frac{Q^*}{d(p)})$. Since Q^* is defined as $\bar{F}(\frac{Q^*}{d(p)}) = \frac{c}{p}$, we have $\frac{\partial Q^*}{\partial p} = \frac{Q^* d'(p)}{d(p)} + \frac{cd(p)}{p^2 f(\frac{Q^*}{d(p)})}$ and $\frac{\partial}{\partial p}\frac{Q^*}{d(p)} = \frac{c}{p^2 f(\frac{Q^*}{d(p)})}$. Plugging back, we have

$$\begin{aligned}
\frac{\partial}{\partial p}\hat{\Pi}(p|Q^*(p)) &= [pd'(p) + d(p)] \int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi) + \frac{c}{p}Q^* \\
&= [pd'(p) + d(p)] \int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi) + Q^* \bar{F}(\frac{Q^*}{d(p)}) \\
&= d(p) \int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi) \left[\frac{pd'(p)}{d(p)} + 1 + \frac{\frac{Q^*}{d(p)} \bar{F}(\frac{Q^*}{d(p)})}{\int_0^{\frac{Q^*}{d(p)}} \xi dF(\xi)} \right]
\end{aligned}$$

Clearly, $\frac{\partial}{\partial p}\hat{\Pi}(p|Q^*(p))|_{p=0} > 0$ and $\frac{\partial}{\partial p}\hat{\Pi}(p|Q^*(p))|_{p=p_0} < 0$. By a similar argument as used in the proof of Proposition 7, under IPE demand and Assumption 2 together with the fact that $\frac{\partial}{\partial p}\frac{Q^*}{d(p)} = \frac{c}{p^2 f(\frac{Q^*}{d(p)})} > 0$, $\frac{\partial \hat{\Pi}(p)}{\partial p}$ must cross zero only once from above. Thus, $\hat{\Pi}(p)$ is quasi-concave with optimal p^* be uniquely determined via FOC. □