https://doi.org/10.1016/j.trc.2019.03.019

Mixed-Integer Second-Order Cone Programming Model for Bus Route Clustering Problem

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Abstract: Bus route clustering problem (BRCP) concerns the assignment of bus routes to different boarding locations of a bus station with the objective of minimizing passenger waiting time. In this study, we formulate the BRCP as a mixed-integer second-order cone program (MISOCP). Simulations are conducted, in which the MISOCP model is applied to a major bus station in Hong Kong based on the network of actual bus routes. Experiments are tested for large size instances under different scenarios. Results show that the complexity of the BRCP is highly dependent on the overlapping degree of bus networks, while other factors, including the number of bus routes, destinations, and boarding locations have a joint effect; the influence is instance-specific based on different overlapping topologies of bus route networks.

Keywords: bus route cluster, mixed-integer second-order cone program, bus passenger.

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1 Introduction

Although options for travel modes have increased in many cities, bus transport remains the dominant transportation mode for people in cities such as Hong Kong, and especially in those cities with limited land or road spaces. Ninety percent of residents in Hong Kong commute by public transport in their daily lives, and over fifty percent commute by bus (Gov HK, 2017). A primary benefit of bus transport is that it alleviates road congestion; the number of vehicles traveling on roads correspondingly decreases when people are transported in the same vehicle. An efficient bus transport system increases the number of people who use public transport, but if a bus transport system is inappropriately designed, traffic congestion can be a significant issue. The direct impact of an inefficient bus transport system on passengers leads to a relatively long waiting time. If the number of people who need to travel is high, but the bus frequency is extremely low, passengers tend to utilize private vehicles, resulting in more vehicles traveling at peak hours and greater traffic congestion. An efficient bus transport system is determined by many factors, including bus fleet size, routes, timetables, and frequencies.

A bus route clustering problem (BRCP) seeks to establish an efficient bus transport system. BRCP assigns a set of bus routes to a set of bus station boarding locations in a way that minimizes the total waiting time of passengers. Related definitions are given first since the BRCP is a newly proposed problem. The BRCP is defined in the background that the capacity of one boarding location is insufficient, and the investigated bus station has more than one boarding location. A bus route cluster refers to a set of bus routes that are assigned to the same bus station boarding location. Figure 1(a) shows four boarding locations of the bus station “Beijing South Railway Station South Square,” near the largest railway station in Beijing, China. Figure 1(b) shows two boarding locations of the bus station “Fish Market,” near Norway’s most visited outdoor market in Bergen. Distinguishing boarding locations from bus stations is essential to understand the BRCP. Different boarding locations belonging to one bus station are very close to each other (e.g., 100 meters), and passengers waiting in one boarding location may even catch sight of another boarding location. However, different bus stations should be separated for a certain distance (e.g., 5 minutes walking duration) between two adjacent bus stations, since they need to cover passengers from different origins. With given passenger demands of a focused bus station, the BRCP resolves how passengers choose a boarding location; it focuses only on one bus station and no interaction exists between boarding locations of two different bus stations.

In the BRCP, passengers are considered from the same region with a similar walking distance from passengers' origins to each boarding location. Then, different waiting times become the most important issue for passengers choosing a boarding location. Waiting times for passengers to arrive at their destinations depend on how bus routes are clustered at various boarding locations. For instance, if all the bus routes reaching destination "A" are assigned to one boarding location, bus frequencies to "A" are centralized, then passengers to "A" will have the minimum waiting time. Sometimes, bus routes toward "A" cannot be centralized due to the capacity limitation of the boarding location. In such a case, these bus routes should be partitioned to different clusters. If bus routes reaching destination "A" are assigned to different boarding locations, bus frequencies to "A" are dispersive and most passengers to "A" will correspondingly have a longer waiting time. If bus routes are partitioned inappropriately, then passenger waiting times may largely increase. Thus, the BRCP aims to appropriately assign bus routes to different boarding locations of the focused bus station, in order to minimize the total waiting time of passengers toward their destinations.

The BRCP has significant practical importance since the bus station of the BRCP is commonly located in the predominant location with heavy traffic flow, in which an inefficient bus system will severely intensify traffic pressure and congestion. The above-mentioned "Beijing South Railway Station South Square" (i.e., the largest railway station in Beijing) and "Fish Market" (i.e., the most visited outdoor market in Bergen) in Figure 1 are both places with geographic significance. Correspondingly, a geographically important bus station is commonly one of the busiest stations in the city, passed through by a large number of bus routes with intensive bus frequencies; hence, the capacity of one boarding location is insufficient. Due to the limited capacity of one boarding location and heavy traffic flow, several boarding locations should be set for a bus station. The BRCP should efficiently partition the set of bus routes passing through the bus station to several bus route clusters, each of which is assigned to one boarding location. The BRCP is important to be discussed since the focused bus station is passed through by complicated and overlapping bus routes. Hundreds of destinations are reachable in the downstream of the bus station under study. Each bus route has its bus frequency, each boarding location has its capacity, and different destinations also have different travel demands. Thus, an efficient method of bus route clustering is necessary. The need is especially great when facing a highly utilized network with overlapping bus routes. Operators should make right decisions on allocating which bus route to which boarding location, concerning limited capacity of each boarding location.

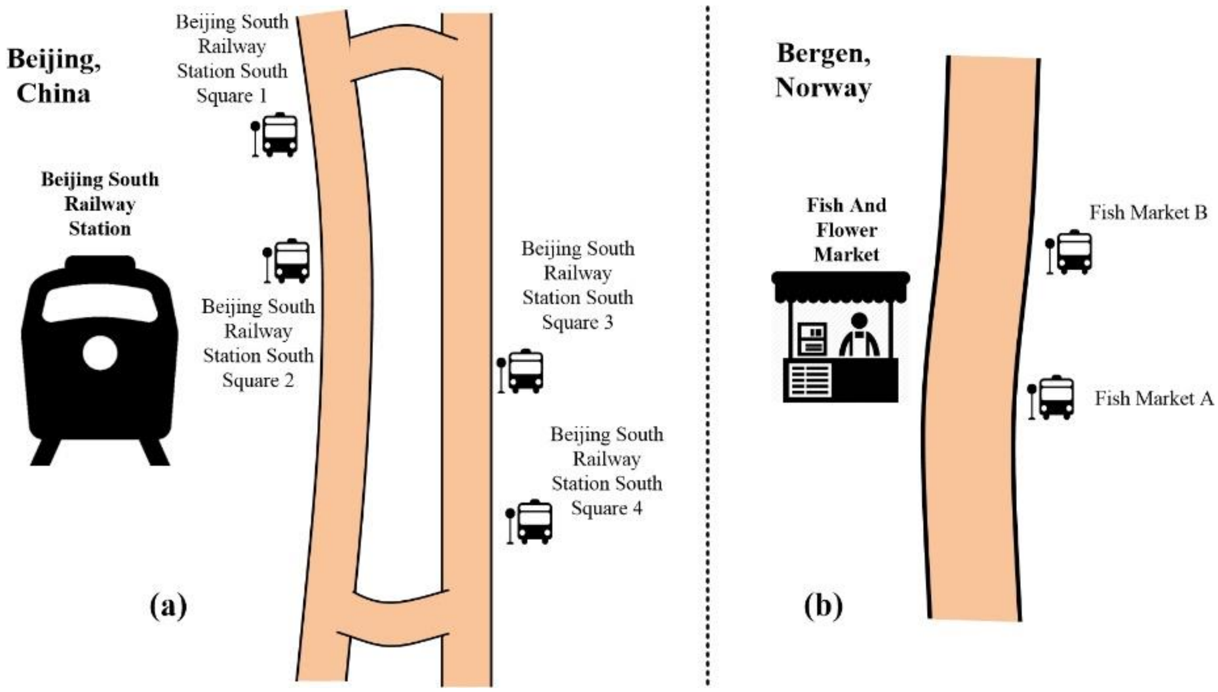

Figure 1. Bus route clusters of (a) Beijing South Railway Station South Square and (b) Fish Market.

This study aims at addressing a new and practical problem, the BRCP, by formulating optimization models and designing solution algorithms. The contributions of this study are threefold. First, we identify a novel decision problem that has not been addressed in the literature but is experienced by bus transport planners. The BRCP is practically important both for big cities like Beijing and Hong Kong and for small cities like Bergen. Moreover, the consequence of ignoring the problem is serious, since the BRCP constantly deals with the busiest station with geographic significance and faces a highly utilized network where bus routes are overlapping. An inefficient bus system will intensify traffic pressure and congestion. The BRCP efficiently assigns a set of bus routes passing through the bus station to its several boarding locations, to minimize passenger waiting times and to encourage green travel by bus. Second, we develop a mixed-integer second-order cone programming (MISOCP) formulation that can be solved by off-the-shelf solvers efficiently and effectively. Third, we apply the proposed models on a large bus station in Hong Kong with three clusters to determine several valuable and managerial insights. We identify that the optimal solution structure without capacity constraints for the BRCP is that the bus routes of different boarding locations are not overlapping. The MISOCP is then tested on larger-size instances with different overlapping scenarios to draw in-depth analyses. For instance, the computation time of the MISOCP is highly dependent on the overlapping degree of bus networks, while other factors (i.e., the number of bus routes, destinations, and boarding locations) jointly affect the complexity of the BRCP.

The remainder of this paper is organized as follows: Section 2 reviews the related literature. In Section 3, the BRCP is formulated as a mixed-integer second-order cone program (MISOCP). An extension model is described in Section 4. We then apply the MISOCP model to a major bus station in Hong Kong based on the network of actual bus routes in Section 5. Section 6 further tests the proposed models in large scale instances with different configurations of bus route networks. Section 7 concludes the study.

2 Literature review

BRCP has not been explicitly addressed in the literature yet, but we can draw some insights from the following two main aspects. (i) BRCP has analogical concerns or trade-offs with problems related to bus (route) system, i.e., the transit network design problem (TNDP), overlapping bus routes, and frequency-based transit assignment approach. (ii) BRCP is considered a variant of the generalized assignment problem (also called as clustering problem or partitioning problem in the literature). The above-mentioned two aspects of studies are reviewed in this subsection. Besides, an MISOCP formulation is proposed to solve the BRCP in this study, and reviews of SOCP are also given.

Related literature of bus (route) system is presented first, including the three categories of the transit network design problem (TNDP), overlapping bus routes, and frequency-based transit assignment approach. Methodologies in existing literature cannot be directly used to solve BRCP since each of them has different emphases on bus system problems with various considerations. A review of each category is followed by analogies or differences analyses compared with BRCP as follows.

TNDP is one of the three major activities in the transit network problem, along with frequency setting and timetabling problems of transit networks (Guihaire and Hao, 2008; Meng and Qu, 2013; Zhang et al., 2018; Varga et al., 2018; Gkiotsalitis and Cats, 2018; Kang et al., 2019). Numerous studies have been reported on the TNDP, typically to design bus route layout and destination locations (Ceder and Wilson, 1986; Saka, 2001; Meng and Yang, 2002; Ibeas et al., 2010; Liu et al., 2013). Studies have also proposed methods to manage route design and frequency setting problems simultaneously (Tom and Mohan, 2003; Szeto and Wu, 2011; Yan et al., 2013; Nikolić and Teodorović, 2014; Szeto and Jiang, 2014) or to handle route design and timetabling problems simultaneously (Yan et al., 2006; Yan and Tang, 2008; Zhao and Zeng, 2008; Chu, 2018; Lyu et al., 2019). Reviews and future research potential have been discussed by Farahani et al. (2013) and Ibarra-Rojas et al. (2015). If we classify the BRCP to a typical category of literature, the TNDP is the most related one, while differences between them should be declared. The TNDP attempts to output a set of bus lines and bus stops to construct an efficiency network, whereas in the BRCP, the network of bus routes and destination layouts are given. The BRCP solves a downstream problem after the network design and selects a boarding location of the focused bus station for each route passing through it, rather than to determine bus stops to be served along the route. We focus on only one bus station which has several boarding locations, as well as all the bus routes that visit the focused bus station.

Studies investigated overlapping bus routes in different aspects as follows. Overlapping bus routes share common bus stops and this means more than one bus route can serve the travel demand from the focused bus station to a destination (Yu et al., 2011). Fouilhoux et al. (2016) indicated that different bus routes converge at specific stops of the network, and buses are easily congested at these common bus stops of overlapping routes. They optimized the departure times of buses to avoid the arrival of two buses at a stop at the same time, thereby ensuring short waiting times and sufficient transfer times of passengers. Han and Wilson (1982) allocated a fleet of buses between routes in networks, where extensive bus routes overlap, and buses frequently operate close to capacity. The above literature discussed different effects of overlapping bus routes to be considered when designing an efficient bus system, while they fall into the category of bus frequency setting or timetabling problem, but not from the boarding location assignment perspective as pursued in this study. Still, similarities are included. For example, Han and Wilson (1982) allocated buses to bus routes to minimize waiting times and to reduce crowding levels of passengers; we allocate bus routes to boarding locations to minimize waiting times. Passenger waiting time plays an important role in the efficient utilization of bus resources (Chakroborty, 2003; Zolfaghari et al., 2004; Niu et al., 2015). Besides, passengers' choice on different bus routes to their destination is widely concerned; in this study, passengers departing from the focused bus station should select a boarding location with at least one bus route heading to the destination.

Frequency-based transit assignment approach is commonly used in planning bus services and is a major precondition in the BRCP. This approach considers that buses are operated with constant frequencies but with no trustworthy schedule; they arrive at the station every few minutes (i.e., the service is frequent). A passenger does not worry about the schedule even though no reliable bus schedule is provided (Chriqui and Robillard, 1975; Spiess and Florian, 1989; Oliker and Bekhor, 2018). The frequency-based approach is suitable for the BRCP, which occurs in a busy and congested transit network where bus services are frequent, and passengers are not concerned about the schedule. In the BRCP, we define the "connecting lines" as the bus routes heading to a passenger's destination. The passenger does not predetermine his path; instead, he boards the first arriving bus among a set of connecting lines.

From another aspect, the BRCP can be considered as a variant of the generalized assignment problem (GAP). We refer to a comprehensive literature review of GAP and its applications studied by Öncan (2007). The GAP is to find the optimal assignment of certain items into several knapsacks, each of which has a fixed capacity availability. The GAP is known to be NP-hard (Sahni and Gonzalez, 1976), and heuristic or metaheuristic solution approaches are constantly developed for the large instances in the literature (Yagiura and Ibaraki, 2004, 2007). Meanwhile, many variants and applications (Öncan, 2007) of the GAP are developed, in which the BRCP has not been mentioned so far. As a variant of the GAP, the BRCP has also been proved to be NP-hard in this study by reducing it to the well-known NP-hard maximum clique problem (Bomze et al., 1999; Östergård, 2002; Konc and Janezic, 2007). Objectives are various in different variants of the GAP, while in the BRCP, the aim is to minimize the waiting time of passengers. The objective of the BRCP involves the nonlinear interaction in the model formulation (i.e., bus frequencies of different bus routes are clustered, which has a nonlinear relationship with waiting times of passengers in each cluster). A related study of nonlinear constraint refers to Mazzola (1989) who have discussed the nonlinear capacity constrained GAP. Besides, the underlining rules of clustering are also different in each variant. Mulvey and Beck (1984) solved a capacity clustering problem, clustering entities into several groups where the "size" of each group is restricted, with the objective to minimize the sum of the distance between each entity and a designated group median. Osman and Christofides (1994) defined a capacity clustering problem (CCP) in which a given set of weighted objects is partitioned into clusters so that the total weight of objects in each cluster is less than a given value (i.e., cluster's capacity), while the objective is to minimize the total scatter of objects from the "center" of the cluster to which they have been allocated. The clustering problem discussed by De La Vega et al. (2003) also aims at minimizing the sum of all intra-cluster distances, while in the BRCP, the underlining rule of the clustering gathers the bus frequencies to the same destination to shorten passenger waiting times.

To solve the BRCP, an MISOCP is proposed in this study. A second-order cone programming (SOCP) problem is defined as a problem where a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of second-order cones (Alizadeh and Goldfarb, 2003). Theoretical findings and industrial applications related to SOCP have been discussed in the past decades (Ben-Tal and Nemirovski, 2000; Ndambuki et al., 2000; Bertsimas et al., 2004; Chen et al., 2007; Frangioni and Gentile, 2009; Du et al., 2011). For instance, Ben-Tal and Nemirovski (2000) proposed a robust formulation in which an ellipsoid uncertainty set is utilized to restrict uncertain parameters and obtain the worst values. Under the ellipsoidal uncertainty set, the robust formulation becomes an SOCP. Du et al. (2011) cast a mixed-integer nonlinear programming model as an MISOCP model to overcome the nonlinear intractability introduced by the consideration of fuel consumption. Many off-the-shelf solvers are available to solve SOCP models to optimality, such as CPLEX and MOSEK (Mittelmann, 2003).

3 Model Formulation

In this section, an MISOCP model is formulated for the BRCP. The model includes decisions of assigning bus routes to different boarding locations of a bus station. Some properties of the proposed model are also discussed in this section.

3.1 Assumptions

Before addressing the model, the underlying assumptions are clarified as follows.

(i) The origin-destination demand from the focused bus station to any destination (i.e., downstream bus stop) in each bus route is fixed and constant over the study period.

(ii) Passengers do not have preferences among connecting lines (i.e., connecting lines denote the bus routes heading to the passenger’s destination); they also do not have preferences between buses, and board the first arriving bus of the connecting lines.

(iii) Passengers’ choice of bus routes does not affect the buses’ dwell time at boarding locations. This assumption will then be relaxed in Remark 6 in Section 3.3.

(iv) Passengers select the boarding location with the highest frequency of buses that head to their destinations.

(v) All passengers can board the buses when the first bus heading to their destination arrives at the boarding location, considering the arriving bus has sufficient capacity after servicing upstream stops. This assumption will then be relaxed in Remark 6 in Section 3.3.
(vi) Bus arrivals of each bus route follow a Poisson process. This assumption will then be relaxed in Section 4 as a model extension, in which the arrival rate of buses can follow any random distributions.
(vii) The arrival times of passengers are uniformly distributed over the study period.

Considerations and limitations of the BRCP are explained as follows. Assumption (ii) considers passengers take the first bus of the connecting lines, which might be limited from the perspective of express services. In reality, some lines may be more attractive than others, depending on which other lines are available at the bus cluster. In the BRCP, we mainly concern the waiting time, while in-vehicle travel times are not measured in the objective function. If a serious detour of a route exists, we rule the route out from the start. We consider that the BRCP faces an efficient bus route network with reasonable and acceptable in-vehicle times, as a downstream study after the bus route network design. Certain limitations exist when some routes are less express compared with other available connecting lines of a bus cluster.

3.2 Notation

The notation is defined as follows.

Sets

R: Set of bus routes;

W: Set of boarding locations;

S: Set of all destinations (i.e., downstream bus stops) of the focused bus station that is visited by all bus routes in R;

Indices

r: A bus route;

w: A boarding location;

s: A destination (i.e., downstream bus stop);

Parameters

f_r: Frequency of bus route $r \in$ R (number of departures per hour);

δ_{rs}: A binary indicator equals 1 if and only if bus route $r \in$ R visits destination $s \in$ S;

c_w: Capacity of boarding location $w \in \mathrm{W}$, which is measured in bus frequencies;

q_s: Travel demand from the focused bus station to the destination $s \in \mathrm{S}$ (number of passengers per hour);

M_s: A large positive number used to linearize the model that corresponds to the destination $s \in \mathrm{S}$, where $M_s := \sum_{r \in \mathrm{R}} f_r \delta_{rs}$;

Main decision variables

z_{wr}: A binary variable equals 1 if and only if bus route $r \in \mathrm{R}$ is assigned to cluster $w \in \mathrm{W}$;

z: Matrix defined as $\mathbf{z} \coloneqq (z_{wr}, w \in \mathrm{W}, r \in \mathrm{R})$;

Auxiliary decision variables

x_{ws}: Total frequency (number of departures per hour) of all bus routes that use boarding location $w \in \mathrm{W}$ and visit destination $s \in \mathrm{S}$;

y_{ws}: A binary variable equals 1 if and only if passengers heading to destination $s \in \mathrm{S}$ use boarding location $w \in \mathrm{W}$;

x_s: Total frequency (number of departures per hour) available to destination $s \in \mathrm{S}$, defined as $x_s := \max\{x_{ws} | w \in \mathrm{W}\}$, $s \in \mathrm{S}$, since passengers select the boarding location with the highest frequency.

3.3 Mathematical Model

On the basis of the parameters and decision variables, the BRCP can be formulated as the following Model **M1**.

$$[\mathbf{M1}] \quad \min \sum_{s \in \mathrm{S}} q_s \frac{1}{x_s}, \tag{1}$$

subject to

$$\sum_{w \in \mathrm{W}} z_{wr} = 1, r \in \mathrm{R}, \tag{2}$$

$$\sum_{r \in \mathrm{R}} f_r z_{wr} \le c_w, w \in \mathrm{W}, \tag{3}$$

$$x_{ws} = \sum_{r \in \mathrm{R}} f_r \delta_{rs} z_{wr}, w \in \mathrm{W}, s \in \mathrm{S}, \tag{4}$$

$$x_s + M_s(y_{ws} - 1) \le x_{ws}, w \in \mathrm{W}, s \in \mathrm{S}, \tag{5}$$

$$x_{ws} \le x_s, w \in \mathrm{W}, s \in \mathrm{S}, \tag{6}$$

$$\sum_{w \in \mathrm{W}} y_{ws} = 1, s \in \mathrm{S}, \tag{7}$$

$$\frac{\sum_{r \in \mathrm{R}} f_r \delta_{rs}}{|\mathrm{W}|} \leq x_s, s \in \mathrm{S}, \tag{8}$$

$$x_{ws} \geq 0, w \in \mathrm{W}, s \in \mathrm{S}, \tag{9}$$

$$y_{ws} \in \{0,1\}, w \in \mathrm{W}, s \in \mathrm{S}, \tag{10}$$

$$z_{wr} \in \{0,1\}, w \in \mathrm{W}, r \in \mathrm{R}. \tag{11}$$

The objective function (1) minimizes the total waiting time of all the passengers. In consideration of Assumption (vi), that is, bus arrivals follow a Poisson process, and Assumption (vii), that is, passenger arrivals follow a uniform distribution, the waiting time of passengers can be regarded as a negative exponential distribution with an average waiting time of $1/x_s$, which is shown in the objective function. Constraint (2) guarantees that each bus route is designated to dwell at only one boarding location. Constraint (3) states the capacity limitation of each boarding location. Constraint (4) ensures that x_{ws} equals the total frequency of all bus routes that use boarding location w and visit destination s. Constraints (5) and (6) indicate that passengers select the boarding location with the highest frequency of buses heading to their destinations, where $M_s := \sum_{r \in \mathrm{R}} f_r \delta_{rs}$, $s \in \mathrm{S}$. The relationship of x_{ws} and y_{ws} follows Assumption (iv). If the boarding location w has the highest bus frequency heading to destination s, then passengers heading to destination s will select boarding location w and y_{ws} equals one; otherwise, y_{ws} equals 0. Constraint (7) ensures that passengers can only select one boarding location to their destinations. Constraint (8) considers that the worst situation of the passengers' waiting time toward destination $s \in \mathrm{S}$ is that all frequencies to $s \in \mathrm{S}$ are distributed on average by $|\mathrm{W}|$ boarding locations. Constraints (9)–(11) define the domain of decision variables.

The aforementioned model is nonlinear with a challenge that the division $1/x_s$ is in the objective function. To overcome this problem, we propose an MISOCP to reformulate the model. We introduce the following notation:

<u>*Auxiliary decision variables*</u>

t_s: Average waiting time of passengers heading to destination $s \in \mathrm{S}$, $t_s := 1/x_s$.

Given the newly added parameters and decision variables, Model **M1** is modified as Model **M2**.

$$[\mathbf{M2}]\quad \min \textstyle\sum_{s\in S} q_s t_s, \tag{12}$$

subject to Constraints (2)–(11), and

$$t_s \ge 1/x_s,\ s \in S. \tag{13}$$

Constraint (13) is nonlinear and causes difficulty for optimization. Thus, we rewrite it as follows:

$$1 \le x_s t_s,\ s \in S. \tag{14}$$

Here, $1 \le x_s t_s$ is equivalent to $4 \le 4x_s t_s$, which can be transformed to $2^2 - 2x_s t_s \le 2x_s t_s$, equivalent to $2^2 - 2x_s t_s + {x_s}^2 + {t_s}^2 \le 2x_s t_s + {x_s}^2 + {t_s}^2$, further equivalent to $2^2 + (x_s - t_s)^2 \le (x_s + t_s)^2$. Thus, constraint (14) can be transformed to:

$$\sqrt{2^2 + (x_s - t_s)^2} \le x_s + t_s,\ s \in S. \tag{15}$$

Constraint (15) denotes a typical second-order cone programming (SOCP) constraint. The SOCP model can be efficiently solved to optimality by interior point algorithms, and therefore many commercial solvers, such as CPLEX, are capable of solving SOCP problems (Alizadeh and Goldfarb, 2003). To enhance the practical usage of SOCP, integer variables are involved in the SOCP, denoted as a mixed-integer second-order cone program (MISOCP). Substituting Constraint (13) to Constraint (15), Model **M2** can be transformed into an MISOCP formulation denoted by Model **M3,** which can be solved by off-the-shelf solvers, such as CPLEX.

$$[\mathbf{M3}]\quad \min \textstyle\sum_{s\in S} q_s t_s, \tag{16}$$

subject to Constraints (2)–(11) and (15).

In the following, we provide several remarks for different practical considerations of the BRCP, e.g., concerning the detour of bus routes and passengers' preference of one particular bus route.

Remark 1. If a particular destination $s' \in S$ is included by only one particular route $r' \in R$, but not included by the others $r \in R\backslash\{r'\}$; that is, $\delta_{r's'} = 1$ and $\delta_{rs'} = 0$, $r \in R\backslash\{r'\}$, then the component $q_{s'}/x_{s'}$ in Objective function (1) is a constant value which equals $q_{s'}/f_{r'}$. Thus, we can exclude the destination s' from the optimization model without changing optimal solution $\mathbf{z}^*$.

Remark 2. If all the routes $r \in R$ either include both destinations $s_1 \in S$ and $s_2 \in S$, or exclude both of them, that is $\delta_{rs_1} = \delta_{rs_2}$, $r \in R$, then we can set the travel demands as $q_{s_1} \leftarrow q_{s_1} + q_{s_2}$ and exclude destination s_2 from the optimization model without changing the optimal solution $\mathbf{z}^*$.

Remark 3. Some passengers heading to destination s may constantly select a particular route, although more than one route visits destination s. For example, buses that are deployed on a particular route are possibly wheelchair friendly and passengers in wheelchairs constantly select this bus route. Suppose that all q_s passengers are heading to destination s and a constant number of passengers p_{sr} only select route $r \in \mathrm{R}$ as their preference, then we can set $q_s \leftarrow q_s - \sum_{r\in\mathrm{R}} p_{sr}$ and exclude these passengers $\sum_{r\in\mathrm{R}} p_{sr}$ from the optimization model without changing the optimal solution $\mathbf{z}^*$.

Remark 4. Suppose that each passenger only takes one particular bus route as a preference, which indicates $q_s - \sum_{r\in\mathrm{R}} p_{sr} = 0$, then the objective function is ineffective because every passenger has a fixed waiting time and this model will aim at finding a feasible solution. In this situation, the objective function can be set as max $\min_{w\in\mathrm{W}}(c_w - \sum_{r\in\mathrm{R}} f_r z_{wr})/c_w$ to maximize the smallest relative capacity buffer among all the boarding locations.

Remark 5. The set of all connecting lines of a passenger waiting at boarding location $w \in \mathrm{W}$ towards the destination $s \in \mathrm{S}$ can be denoted as $\{r \in \mathrm{R} | \delta_{rs} z_{wr} = 1\}$. If all the routes belong to the same cluster, the connecting lines of the passenger can be simplified as $\{r \in \mathrm{R} | \delta_{rs} = 1\}$. If the bus route $r' \in \mathrm{R}$ takes a serious detour for passengers heading to the destination $s' \in \mathrm{S}$ and no passengers toward s' choose route r', then destination s' can be removed from the route r' by setting $\delta_{r's'} = 0$ in the analysis of the BRCP.

Remark 6. In Assumption (iii), we assume that buses' dwell times are fixed at boarding locations. In reality, the dwell time is dependent on the number of boarding/alighting passengers and can be formulated as $d_r = t_r^0 + \theta \sum_{s\in\mathrm{S}} u_{rs}$ for route $r \in \mathrm{R}$, where (i) t_r^0 denotes the fixed dwell time (e.g., launching, braking time, etc.), (ii) θ is the average time required for one passenger to board, and (iii) u_{rs} is a decision variable that denotes the number of passengers to destination $s \in \mathrm{S}$ boarding the arriving bus of route $r \in \mathrm{R}$ instead of the other available connecting lines.

The relation between d_r, u_{rs} and other decision variables (i.e., bus route clustering decisions z_{wr} and passengers' choice decisions y_{ws}) is $u_{rs} = q_s \times \frac{\frac{60 - \sum_{r\in\mathrm{R}} d_r f_r \delta_{rs}(\sum_{w\in\mathrm{W}} y_{ws} z_{wr})}{\sum_{r\in\mathrm{R}} f_r \delta_{rs}(\sum_{w\in\mathrm{W}} y_{ws} z_{wr})} + d_r \delta_{rs}}{60}$, $r \in \mathrm{R}$, $s \in \mathrm{S}$. In detailed explanation, if a bus to destination s (i.e., $\delta_{rs} = 1$) has occupied the station within its dwell time d_r and a passenger arrives the station during this period, the passenger will board the dwelling bus instead of the others. Aside from passengers arriving within period d_r, the bus also services passengers previously waiting at the station within period $\frac{60 - \sum_{r\in\mathrm{R}} d_r f_r \delta_{rs}(\sum_{w\in\mathrm{W}} y_{ws} z_{wr})}{\sum_{r\in\mathrm{R}} f_r \delta_{rs}(\sum_{w\in\mathrm{W}} y_{ws} z_{wr})}$ as an average, where $\sum_{r\in\mathrm{R}} d_r f_r\, \delta_{rs}(\sum_{w\in\mathrm{W}} y_{ws}\, z_{wr})$ denotes the total dwell time, and $\sum_{r\in \mathrm{R}} f_r\, \delta_{rs} (\sum_{w\in \mathrm{W}} y_{ws}\, z_{wr})$ denotes the total frequency, of all connecting lines to destination s within 60 minutes. Overall, each arriving bus takes up $\frac{60-\sum_{r\in \mathrm{R}} d_r f_r \delta_{rs} (\sum_{w\in \mathrm{W}} y_{ws} z_{wr})}{\sum_{r\in \mathrm{R}} f_r \delta_{rs} (\sum_{w\in \mathrm{W}} y_{ws} z_{wr})} + d_r \delta_{rs}$ in the total of 60 minutes. With q_s passengers arriving at the station each hour follow a uniform distribution, the bus load of the arriving bus of route $r \in \mathrm{R}$ to destination $s \in \mathrm{S}$ is $u_{rs} = q_s \times \frac{\frac{60-\sum_{r\in \mathrm{R}} d_r f_r \delta_{rs} (\sum_{w\in \mathrm{W}} y_{ws} z_{wr})}{\sum_{r\in \mathrm{R}} f_r \delta_{rs} (\sum_{w\in \mathrm{W}} y_{ws} z_{wr})} + d_r \delta_{rs}}{60}$.

In Assumption (v), we assume that the arriving bus has sufficient capacity. If the capacity of the arriving bus is not sufficient, constraint $\sum_{s\in \mathrm{S}} u_{rs} \le Q_r$ for $r \in \mathrm{R}$ should be considered, where Q_r denotes the remaining capacity of the arriving bus of route $r \in \mathrm{R}$ after servicing the upstream stops, i.e., the maximum number of passengers who can board the bus at the station. Generally, by introducing auxiliary decision variables d_r and u_{rs}, and beforementioned three constraints, Assumption (iii) and (v) can be relaxed.

3.4 Hardness of the BRCP

The following theorem of the BRCP is investigated in this subsection.

Theorem. The BRCP is NP-hard.

Proof. We prove the theorem by showing that if the BRCP can be solved in polynomial time, then the maximum clique problem can also be solved in polynomial time, while it is well known that the maximum clique problem is NP-hard (Karp, 1972).

The maximum clique problem can be stated as follows. We consider an undirected graph $\mathrm{G}(\mathrm{V}, \mathrm{E})$ formed by a finite set of vertices V and a set of unordered pairs of vertices E, which are called edges. A clique $\mathrm{G}'(\mathrm{V}', \mathrm{E}')$ is a complete subgraph of $\mathrm{G}(\mathrm{V}, \mathrm{E})$, consisting of a set of vertices $\mathrm{V}' \subseteq \mathrm{V}$, a set of edges $\mathrm{E}' \subseteq \mathrm{E}$, and an edge $(i, j) \in \mathrm{E}'$ is between every two vertices $\forall i, j \in \mathrm{V}', i \neq j$. A maximum clique of the graph includes the largest possible number of vertices, and the maximum clique problem aims to find such a maximum clique. The number of vertices in the maximum clique, denoted as v, should be an integer between 1 and $|\mathrm{V}|$. Hence, so long as we can check whether there is a clique with $v = 1,2, \dots, |\mathrm{V}|$ vertices in polynomial time, we can solve the maximum clique problem in polynomial time.

We consider a specification of the BRCP and depict it as a maximum clique problem (i.e., with a given value of $v = 1,2, \dots, |\mathrm{V}|$, whether a clique with v vertices exists in $\mathrm{G}(\mathrm{V}, \mathrm{E})$). In the BRCP, we consider a total of $|\mathrm{R}| = |\mathrm{V}|$ bus routes with the same frequency $f_r = 1$; a total of

$|\mathrm{W}| = |\mathrm{V}| + 1 - v$ boarding locations, with one boarding location has the capacity v and each of the other has the capacity 1; a total of $|\mathrm{S}| = |\mathrm{E}|$ destinations with the same travel demand $q_s = 2$, each of which is visited by exactly two bus routes. With the capacity limitation of boarding locations, the feasible solutions of the BRCP consist of v routes using the same cluster and each of the other routes using exclusively one cluster. To solve the BRCP to optimality, we need to determine which v of the total $|\mathrm{V}|$ bus routes are assigned together.

The above-mentioned BRCP can be depicted as a maximum clique problem. Each bus route corresponds to a vertex in the graph; an edge connecting two vertices corresponds to a common stop of the two bus routes. In other words, two bus routes may have a common destination (if there is an edge between the vertices) or not; if two bus routes have a common destination $s \in \mathrm{S}$ and are assigned to the same cluster, then the total waiting time reduction for passengers heading to s is $\frac{q_s}{f_s} - \frac{q_s}{2f_s} = 1$ as a constant. If v bus routes are assigned together and any two of the v routes have a common destination, the total waiting time reduction equals $v(v-1)/2$, then a clique with v vertices exists; otherwise, such a clique does not exist. ■

4 Model Extension

The model in the previous section is formulated based on the assumption that the arrival rate of buses follows a Poisson distribution. In this section, we extend the model in consideration of a general situation that the arrival rate of buses can follow any random distributions. Note that the arrival rate refers to buses deployed on the bus routes assigned to each boarding location $w \in \mathrm{W}$ heading to each destination $s \in \mathrm{S}$ since these bus routes are the connecting line for passengers toward destination s when they wait at the boarding location w.

In different bus route clustering schemes, the bus arrival rate should be different (i.e., with more bus routes toward destination $s \in \mathrm{S}$ assigned to the same cluster, the frequencies to destination s should be higher; thus, the waiting time to destination s correspondingly lower). This general situation means that the waiting times of buses toward $s \in \mathrm{S}$ at cluster $w \in \mathrm{W}$ can be obtained by historical data, with given possible frequencies f_r of connecting lines.

In the following, the index $k_{ws} \in \mathrm{K}_{ws}$ is defined as a possible bus arrival frequency at boarding location w heading to the destination s. If we assume that the frequency f_r of each bus route r is an integer, then k_{ws} is also an integer with an upper bound denoted by $|\mathrm{K}_{ws}|$:= min $(\sum_{r\in \mathrm{R}, \delta_{rs}=1} f_r, c_w)$, and $\mathrm{K}_{ws} = \{1,2,\dots,|\mathrm{K}_{ws}|\}$. The waiting time is infinite when $k_{ws} =$

0. Let parameter $\tau_{ws}^{k_{ws}}$, derived from the historical data, represent the average waiting time of a passenger using boarding location w for heading to destination s if the possible frequency of all bus routes is k_{ws}.

Table 1. Example of the model extension for any bus arrival rate

k_{ws}	$\tau_{ws}^{k_{ws}}$	$x_{ws}^{k_{ws}}$	y_{ws}	x_{ws}
1	57	0		
2	34	0		
3	19	1	1	3
…	…	…		
$\lvert \mathrm{K}_{ws} \rvert$	5	0		

Table 1 provides an example which represents a random distribution. As shown in the first two columns, for each k_{ws}, its corresponding average waiting time $\tau_{ws}^{k_{ws}}$ can be calibrated based on historical data from the boarding location or similar boarding locations (i.e., the third row means the average waiting time of passengers is 19 min when the bus arrival frequency is three departures per hour).

Then, the proposed model can be extended to handle any distributions of bus arrival rate by introducing a new auxiliary decision variable ($x_{ws}^{k_{ws}} \in \{0,1\}$), a binary variable which equals one, if and only if passengers heading to destination $s \in \mathrm{S}$ use boarding location $w \in \mathrm{W}$ with the total bus frequency of connecting lines $k_{ws} \in \mathrm{K}_{ws}$. The last three columns of Table 1 are used to illustrate the concept. In Model **M1**, we denote $y_{ws} = 1$ (column 4) if passengers use boarding location w to the destination s, and denote x_{ws} (three departures per hour $x_{ws} = 3$ in column 5) as the total boarding location frequency of corresponding connecting lines. By setting newly introduced auxiliary decision variable $x_{ws}^{k_{ws}=3} = 1$ when $k_{ws} = 3$ (column 3), the average waiting time of passengers can be obtained as 19 minutes ($\tau_{ws}^{k_{ws}} = 19$ in column 2). Hence, Model **M4** can be formulated as follows:

$$[\mathbf{M4}] \quad \min \sum_{s\in \mathrm{S}} q_s \sum_{w\in \mathrm{W}} y_{ws} \sum_{k_{ws}\in \mathrm{K}_{ws}} \tau_{ws}^{k_{ws}} x_{ws}^{k_{ws}}, \tag{17}$$

subject to Constraints (2)–(4), (7), (9)–(11) and

$$x_{w's} + M_s(y_{ws} - 1) \le x_{ws}, w \in \mathrm{W}, w' \in \mathrm{W}, w \neq w', s \in \mathrm{S}, \tag{18}$$

$$x_{ws}^{k_{ws}} \in \{0,1\}, w \in \mathrm{W}, s \in \mathrm{S},\ k_{ws} \in \mathrm{K}_{ws}, \tag{19}$$

$$\sum_{k_{ws} \in \mathrm{K}_{ws}} x_{ws}^{k_{ws}} = y_{ws}, w \in \mathrm{W}, s \in \mathrm{S}, \tag{20}$$

$$x_{ws} - M_s(1 - y_{ws}) \leq \sum_{k_{ws} \in \mathrm{K}_{ws}} k_{ws} x_{ws}^{k_{ws}} \leq x_{ws} + M_s(1 - y_{ws}), w \in \mathrm{W}, s \in \mathrm{S}, k_{ws} \in \mathrm{K}_{ws}. \tag{21}$$

In Model **M4**, Constraints (18) indicates that passengers select the boarding location with the highest frequency of buses heading to their destinations, where $M_s := \sum_{r \in \mathrm{R}} f_r \delta_{rs}$, $s \in \mathrm{S}$. The relationship of x_{ws} and y_{ws} follows Assumption (iv). Constraints (20) and (21) are regarded as two possible situations (i.e., $y_{ws} = 1$ and $y_{ws} = 0$). Given $w \in \mathrm{W}$, $s \in \mathrm{S}$, (i) if $y_{ws} = 1$, then Constraint (20) indicates that one of the binary variables $x_{ws}^{k_{ws}}$ that correspond to possible frequency $k_{ws} \in \mathrm{K}_{ws}$ must be equal to 1 and that Constraint (21) becomes $x_{ws} \leq \sum_{k_{ws} \in \mathrm{K}_{ws}} k_{ws} x_{ws}^{k_{ws}} \leq x_{ws}$. That is to say, $\sum_{k_{ws} \in \mathrm{K}_{ws}} k_{ws} x_{ws}^{k_{ws}} = x_{ws}$, which further implies that $x_{ws}^{k_{ws}}$ with $k_{ws} = x_{ws}$ is set to be 1. (ii) If $y_{ws} = 0$, then Constraint (20) indicates that $x_{ws}^{k_{ws}}$ for all $k_{ws} \in \mathrm{K}_{ws}$ equal 0 and that Constraint (21) is constantly satisfied with M_s, a relatively large number that $M_s := \sum_{r \in \mathrm{R}} f_r \delta_{rs}$, $s \in \mathrm{S}$.

Constraint (22) can substitute Constraint (21) to simplify Model **M4**. (i) If $y_{ws} = 1$, Constraint (22) becomes $k_{ws} x_{ws}^{k_{ws}} \leq x_{ws}$. Since $\tau_{ws}^{k_{ws}}$ is negatively correlated with k_{ws}, indicating a higher bus frequency indicates a lower passenger waiting time of Objective (17). Thus, $x_{ws}^{k_{ws}=x_{ws}}$ will be set to be 1, since $x_{ws}^{k_{ws}>x_{ws}} = 1$ indicating $k_{ws} x_{ws}^{k_{ws}} > x_{ws}$ which violates Constraint (22) and $x_{ws}^{k_{ws}<x_{ws}} = 1$. (ii) If $y_{ws} = 0$, then Constraint (22) is constantly satisfied.

$$k_{ws} x_{ws}^{k_{ws}} \leq x_{ws} + M_s(1 - y_{ws}), w \in \mathrm{W}, s \in \mathrm{S},\ k_{ws} \in \mathrm{K}_{ws}. \tag{22}$$

Besides, Constraint (22) implies that $y_{ws} x_{ws}^{k_{ws}} = x_{ws}^{k_{ws}}$ then Objective function (17) can be reformulated to Objective (23) in Model **M5**. Thus, Model **M5** is reformulated as follows (i.e., Constraint (21) and Objective (17) in Model **M4** is substituted by Constraint (22) and Objective (23), respectively):

$$[\mathbf{M5}] \quad \min \sum_{s \in \mathrm{S}} q_s \sum_{w \in \mathrm{W}} \sum_{k_{ws} \in \mathrm{K}_{ws}} \tau_{ws}^{k_{ws}} x_{ws}^{k_{ws}}, \tag{23}$$

subject to Constraints (2)–(4), (7), (9)–(11), (18)–(20), and (22).

5 Case Study

In this section, we use the MISOCP model (i.e., Model **M3**) to optimize a BRCP in Hong Kong. The investigated bus station is called "Cross-Harbor Tunnel." This tunnel is a significant infrastructure in Hong Kong that is located across the Victoria Harbor, which connects two independent urban logistic networks, financial and commercial districts of both sides, Kowloon and Hong Kong Island. For the investigated bus station from Kowloon to Hong Kong Island, three boarding locations are called "Hong Chong Road outside Hung Hom Station," "Hong Chong Road outside Hung Hom Station (outer bus bay)," and "Hong Chong Road outside Cross-Harbor Tunnel Administration Building" as shown in Figure 2.

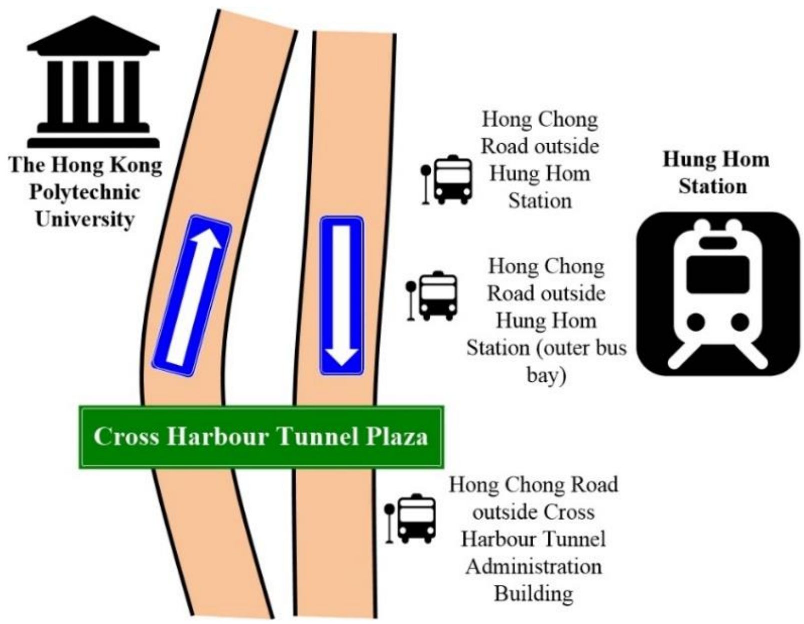

Figure 2. Three bus route clusters of "Cross-Harbor Tunnel."

A total of 17 bus routes pass through "Cross-Harbor Tunnel" and visit 144 destinations. Some routes have common destinations. The numbers of destinations for each bus route in the downstream of the investigated bus station and the headway of each bus route are shown in Table 2, which are based on the New World First Bus Services Limited (2016) and Google Maps (2018). Bus system in Hong Kong is a simple tap-on (without tap-off) system. Passengers only need to tap-on by using a bus card (i.e., Octopus card) when boarding, while they are not required to tap-off when alighting the bus. Since it is hard to access the real travel demands of passengers, travel demands are generated randomly. We generate travel demands in the range of $[50,200]$ from the investigated focused bus station to the destinations. We consider that bus arrivals of each bus route follow a Poisson process. In the results of our experiments, loads of each arriving bus are all within acceptable values, indicating all passengers can board the first arriving bus.

Bus station "Cross-Harbor Tunnel" is considered as one of the busiest stations in Hong Kong with intensive bus routes and downstream bus stations. This case can be considered as a large-scale case in a real application in the BRCP. The problem size information consisting of the numbers of variables and constraints is shown in Table 3. Information of Table 3 is deduced by Model **M3**, according to the number of bus routes, boarding locations, and destinations in this realized case, e.g., Constraint (4) with $w \in \mathrm{W}$ and $s \in \mathrm{S}$ refers to 432 constraints in total, since $|\mathrm{W}| = 3$ and $|\mathrm{S}| = 144$ in this case. All tests are performed by using a PC with 3.40 GHz of Intel Core i7-3770 CPU and 8 GB of RAM. The CPU time is less than five minutes for each test to solve Model **M3** to optimality. Given that the CPU time is short enough for practical applications, the exact one is not reported for each test.

Table 2. Bus route information.

Route	# Destinations	Headway (min)	Frequency (per hour) *
101	19	4	15.00
102	18	6	10.00
103	20	14	4.29
104	20	6	10.00
106	32	8	7.50
107	16	15	4.00
109	9	17	3.53
111	8	6	10.00
112	9	5	12.00
113	21	13	4.62
115	7	14	4.29
116	14	5	12.00
117	6	23	2.61
118	15	6	10.00
170	13	17	3.53
171	13	9	6.67
182	8	14	4.29

Note: * The frequency of a bus route is counted as $60\,\frac{\mathrm{min}}{\mathrm{h}}$ divided by the headway, e.g., $\frac{60\frac{\mathrm{min}}{\mathrm{h}}}{4\mathrm{min}} = 15/\mathrm{h}$ of Route 101.

Table 3. Problem size of the realized case of Cross-Harbor Tunnel.

Routes	Boarding locations	Destinations	Total constraints	Continuous variables		Binary variables			Total variables
R	W	S	1748	t_s	x_s	z_{wr}	x_{ws}	y_{ws}	1203
17	3	144		144	144	51	432	432	

5.1 Optimal solution structure without capacity constraint

In this BRCP, we allocate 17 bus routes on three boarding locations to obtain the minimum total waiting time of passengers toward the 144 destinations. Passengers will select the boarding location with the highest frequency toward the destination; they will not select other boarding locations with lower frequency. Thus, the frequencies related to other boarding locations are all wasted. That is, if all bus routes heading to a destination can be allocated to only one boarding location, then the frequencies toward this destination can be completely used. If the frequencies toward the 144 destinations are completely used, then the total waiting time of passengers is minimal regardless of the capacity constraints of these boarding locations. Therefore, we establish bus routes that share common destinations into one set called an *overlapping bus route set.* Bus routes that belong to different *overlapping bus route sets* are non-overlapping.

In the case of Cross-Harbor Tunnel, 17 bus routes can be distinguished as three *overlapping bus route sets*, that is, $R^1 = \{107,117,170,171\}$, $R^2 = \{102,106,112,116,118\}$, and $R^3 = \{101,103,104,109,111,113,115,182\}$. In the downstream of R^1, R^2, and R^3, 37, 41, and 66 destinations exist called the three *unique destination sets* S^1, S^2 and S^3, which are disjointed with each other. The number of boarding locations is also three in the actual situation, as shown in Figure 2. If the three *overlapping bus route sets* are allocated to three boarding locations accordingly, then the *sum of destinations* shown in Table 4 equals 144, which is exactly the total number of unique destinations in the downstream of the bus station. The frequencies toward the 144 destinations are completely used. Therefore, the objective value, which is 2514.73, cannot be further decreased by adjusting the capacity of boarding locations. If the capacities for the three boarding locations are not determined, then the suitable capacities are 16.80, 51.50, and 56.00. This finding can serve as a guide when the government conducts traffic arrangement or determines the land use of each boarding location.

As above-mentioned findings, passengers would choose the bus cluster with the highest bus frequencies. Passengers may gather together to a bus cluster and wait for the same incoming bus.

The number of waiting passengers to the arriving bus should be in an acceptable load and each arriving bus has sufficient remaining capacity after servicing upstream bus stops (i.e., Assumption (v)). Bus frequencies are high in passengers' chosen bus cluster, while the arrival times of passengers are uniformly distributed based on Assumption (vii) described in Section 3.1. In the results of our experiments, loads of each arriving bus are all within acceptable values, indicating all passengers can board the first coming bus.

The currently applied bus route clustering is presented in Table 5, in which the clustering is not the same as the clustering without capacity constraints (Table 4). The reason for the differences is probably the real capacity limitations of the boarding locations due to the limited land spaces. In our field research, the land space of the boarding location "Hong Chong Road outside Cross Harbor Tunnel Administration Building" is relatively small, which is consistent with the clustering situation that only two routes, $\{115,182\} \subseteq R^3$, are assigned to this boarding location. In the following, we conduct sensitivity analysis on capacity since it is the major component in the BRCP.

Table 4. Optimal solution without capacity constraints.

Boarding location	Assigned bus routes	Used capacity	#Destinations	Sum of destinations	MISOCP objective
1	107/117/170/171	16.80	37	144	2514.73
2	102/106/112/116/118	51.50	41		
3	101/103/104/109/111/113/115/182	56.00	66		

Table 5. Real situation of current bus route clustering in Cross-Harbor Tunnel.

Boarding location	Assigned bus routes			Used capacity	#Destinations	Sum of destinations*
	R^1	R^2	R^3			
Hong Chong Road outside Hung Hom Station (outer bus bay)	107/117/170/171		103/109/113	29.24	81	162
Hong Chong Road outside Hung Hom Station		102/106/112/116/118	101/104/111	86.50	69	
Hong Chong Road outside the Cross Harbor Tunnel Administration Building			115/182	8.57	12	

Note: *Here, the destinations may not be unique.

5.2 Capacity sensitivity analysis

In practice, boarding locations have natural geographic land spaces, and the capacity of each boarding location is known and fixed. In this section, sensitivity analysis in terms of capacity is performed to investigate how the optimal solution changes with the capacity. The same capacity limitation is set for the three boarding locations (Table 6a), and different capacity limitations are set for the three boarding locations (Table 6b). For sufficient capacities of the three boarding locations, the used capacities are 16.80, 51.50, and 56.00, as shown in Table 4. Only if one or more than one boarding location's capacity is insufficient., then the allocation result will be different from that shown in Table 4. As shown in Table 6a, the objective value increases with the decrease in capacity. In comparison with the allocation result without capacity constraints (Table 4), the increase in waiting time provides several insights into the problem.

For Case 1 in Table 6a, route set $\{101,104,109,111,113,182\} \subseteq R^3$ is allocated to boarding location 3 and route set $\{103,115\} \subseteq R^3$ is allocated to boarding location 1 rather than allocating the entire set $R^3 = \{101,103,104,109,111,113,115,182\}$ to only one boarding location. The separation of *overlapping route set* R^3 increases the waiting time of passengers heading to three destinations in *unique destination set* S^3, namely "Central (Macau Ferry)," "Elizabeth House, Gloucester Road," and "Old Wan Chai Police Station, Gloucester Road," which can be reached by boarding locations 1 and 3. In specific, by taking "Central (Macau Ferry)" as an example, boarding location 1 can reach destination "Central (Macau Ferry)" by taking bus route 115 and boarding location 3 can also reach the same destination by taking bus routes 109, 111, and 182. However, all passengers heading to "Central (Macau Ferry)" only select boarding location 3 with a high bus frequency of 17.82, and no passengers select boarding location 1 with a low bus frequency of 4.29. Then, the bus frequency toward "Central (Macau Ferry)" decreases, and the waiting time toward "Central (Macau Ferry)" increases. Similar situations occur in the two other destinations, which are "Elizabeth House, Gloucester Road" and "Old Wan Chai Police Station, Gloucester Road." Therefore, the objective value increases from 2514.73 to 2519.73 when the *sum of destinations* increases from 144 to 147.

For Case 2 in Table 6a, *overlapping bus route set* R^2 is also separated into two subsets that are allocated to two boarding locations aside from the separation of *overlapping route set* R^3 that increases the waiting time of passengers heading to the aforementioned three destinations in S^3. In detail, $\{112\} \subseteq R^2$ is allocated to boarding location 1 and $\{102,106,116,118\} \subseteq R^2$ is allocated to boarding location 2 rather than allocating the entire set $R^2 = \{102,106,112,116,118\}$ to only one boarding location. The separation of R^2 increases the waiting time of passengers heading to eight destinations in S^2. Therefore, the objective value increases from 2519.73 to 2539.31 as the *sum of destinations* increases from 147 to 155. For Case 3 in Table 6a, the separation of all 3 *overlapping bus route sets* R^1, R^2, and R^3 causes the frequencies of 19 (163 minus 144) destinations in S^1, S^2 and S^3 not to be completely used and the objective value becomes 2551.95.

The separation of *overlapping bus route sets* increases the waiting time of passengers. However, the combination of bus routes belonging to different *overlapping bus route sets* does not change (decrease) the total waiting time. We clarify this property by showing that more than one optimal solution exists for Case 2 in Table 6a. To obtain all the optimal solutions, Algorithm 1 is provided.

Algorithm 1. *Obtaining all optimal solutions for the MISOCP model*

Step 0: Let $\mathrm{K} \leftarrow 1$ be the iteration number. Solve Model **M3** and obtain the first optimal solution denoted by $\mathbf{z}^{(\mathrm{K})}$. Let the optimal objective value be Obj^*.

Step 1: Let $\mathrm{K} \leftarrow \mathrm{K}+1$. Solve Model **M3** with the following constraint:

$$\sum_{w\in \mathrm{W}} \sum_{r\in \mathrm{R}} \left[z_{wr}^{(\mathrm{K}')}(1-z_{wr}) + (1-z_{wr}^{(\mathrm{K}')}) z_{wr} \right] \geq 1,\ \mathrm{K}' = 1, \dots, \mathrm{K}-1. \quad (24)$$

The above constraint excludes all the previously generated optimal solutions. If the model is infeasible or if the optimal objective value is larger than Obj^*, then the set of optimal solutions is $\{\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(\mathrm{K}-1)}\}$, and stop. Otherwise, let $\mathbf{z}^{(\mathrm{K})}$ be the optimal solution and go to Step 1.

We can derive all the optimal solutions by applying Algorithm 1 for the case with the same capacity limitation of 50 (i.e., Case 2 in Table 6a). A total of 12 optimal solutions exist. However, we only show two solutions in Table 6c because the other solutions are permutations of the two solutions. In Solution No. 1 in Table 6c, the combination of $\{112\} \subseteq R^2$ and $R^1 = \{107,117,170,171\}$ is allocated to boarding location 1 and the combination of $\{103,115\} \subseteq R^3$ and $\{102,106,116,118\} \subseteq R^2$ is allocated to boarding location 2. In Solution No. 2, the combination of $R^1 = \{107,117,170,171\}$, $\{112\} \subseteq R^2$, and $\{103,115\} \subseteq R^3$ is allocated to boarding location 1. The two solutions have the same objective value of 2539.31 because the bus routes in the same combination belong to different *overlapping bus route sets* and they do not change the total waiting time. This condition is attributed to *unique destination sets* S^1, S^2 and S^3 being disjointed with one another and no destinations increase their frequencies and further decrease the waiting time of passengers after such a recombination. After all the optimal solutions are identified, decision makers can select one of them based on the factors that are not modeled explicitly (e.g., whether a boarding location is more convenient and more comfortable than the other boarding locations). Moreover, balancing the travel demands of boarding locations can also be a proper selection method, i.e., the total passenger demand for each boarding location can be described as $\sum_{s \in S} q_s y_{ws}$, $w \in W$. Related issues have been explained in Remark 6 in Section 3.3.

We further set different capacity limitations for the three boarding locations. The results are shown in Table 6b, which confirm the findings discussed in the previous paragraphs. First, the separation of the *overlapping bus route set* increases the waiting time of passengers. In Table 6b, the objective values are all 2519.73 when the *sum of destinations* equals 147. Although the allocation results are different in these cases, the reason for the increase in waiting time is the same; $\{103,115\} \subseteq R^3$ and $\{101,104,109,111,113,182\} \subseteq R^3$ are allocated to different boarding locations. Second, the combination of bus routes that belong to different *overlapping bus route sets* does not change the waiting time of passengers, that is, the combination of $\{103,115\} \subseteq R^3$ and $R^1 = \{107,117,170,171\}$ is allocated to boarding location 1 in Cases 1 and 2; the combination of $R^1 = \{107,117,170,171\}$ and $R^2 = \{102,106,112,116,118\}$ is allocated to boarding location 3 in Case 3; and the combination of $R^1 = \{107,117,170,171\}$, $R^2 = \{102,106,112,116,118\}$ and $\{103,115\} \subseteq R^3$ is allocated to boarding location 3 in Case 4.

5.3 Travel demand sensitivity analysis

For the BRCP closely related to daily life, only considering the current situation or one single transportation mode is insufficient for making decisions. For example, the development of a new metro line will certainly cause several influences on the travel demand for people taking buses. Certain destinations may have no passengers due to a new metro line connection. Thus, we conduct 14 cases to determine how the optimal solution changes with the travel demand. In each case, we select 10 out of 144 destinations to let their travel demands become 0.

We set the capacity limitations for the three boarding locations to be 50 in the 14 cases and report the optimal solutions in Table 7. From the table, two different solutions are obtained in the 14 cases. Solution No. 1 is exactly the same as that for the situation when no travel demands of destinations are set to be 0 (i.e., Solution No. 1 in Table 6c). A total of 12 cases have the same optimal as Solution No. 1. The other optimal solution is slightly different from Solution No. 1. These results demonstrate the robustness of the bus route allocation solutions.

Table 6. Bus route allocation results for three boarding locations.

Case	Capacity	Boarding location	Assigned bus routes R^1	Assigned bus routes R^2	Assigned bus routes R^3	Used capacity	# Destinations	Sum of destinations*	MISOCP objective
Table 6a. Allocation results for three boarding locations with the same capacity limitation.									
1	55	1	107/117/170/171		103/115	25.38	62	147	2519.73
	55	2		102/106/112/116/118		51.50	41		
	55	3			101/104/109/111/113/182	47.43	44		
2	50	1	107/117/170/171	112	103/115	37.38	71	155	2539.31
	50	2		102/106/116/118		39.50	40		
	50	3			101/104/109/111/113/182	47.43	44		
3	45	1	107/170/171	112	103/109/113/115	42.91	89	163	2551.95
	45	2		102/106/116/118		39.50	40		
	45	3	117		101/104/111/182	41.89	34		
Table 6b. Allocation results for three boarding locations with different capacity limitations.									
1	40	1	107/117/170/171		103/115	25.38	62	147	2519.73
	50	2			101/104/109/111/113/182	47.43	44		
	60	3		102/106/112/116/118		51.50	41		
2	30	1	107/117/170/171		103/115	25.38	62	147	2519.73
	50	2			101/104/109/111/113/182	47.43	44		
	70	3		102/106/112/116/118		51.50	41		
3	20	1			103/115	8.57	25	147	2519.73
	50	2			101/104/109/111/113/182	47.43	44		
	80	3	107/117/170/171	102/106/112/116/118		68.30	78		
4	10	1				0.00	0	147	2519.73
	50	2			101/104/109/111/113/182	47.43	44		
	90	3	107/117/170/171	102/106/112/116/118	103/115	76.87	103		
Table 6c. Optimal solutions for three boarding locations with the same capacity limitation of 50.									
1	50	1	107/117/170/171	112		25.38	46	155	2539.31
	50	2		102/106/116/118	103/115	51.50	65		
	50	3			101/104/109/111/113/182	47.43	44		
2	50	1	107/117/170/171	112	103/115	37.38	71	155	2539.31
	50	2		102/106/116/118		39.50	40		
	50	3			101/104/109/111/113/182	47.43	44		

Note: *Here, the destinations may not be unique.

Table 7. Allocation results for three boarding locations with the same capacity limitation 50

Solution No.	Capacity	Board-ing loca-tion	Assigned bus routes			Repeat count
			R^1	R^2	R^3	
1	50	1	107/117/170/171	112	103/115	12
	50	2		102/106/116/118		
	50	3			101/104/109/111/113/182	
2	50	1	107/117/170/171	102/106/118		2
	50	2		112/116	103/115	
	50	3			101/104/109/111/113/182	

Subsequently, we decrease the capacity limitation for all the three boarding locations to 45. The results are shown in Table 8. From the table, four different solutions appear in the 14 cases. Solution No. 1, which is repeated 9 times, is the same as the original solution when no travel demands of destinations are set to 0. The other solutions have some changes compared with those in Table 7, where the capacity limitation is set to 50. The changes in the solutions in Table 8 are more intensive than those in Table 7, which are automatically analyzed because the capacity is smaller than the previous cases. Therefore, the government should balance the trade-off in reserving larger capacities for boarding locations to increase the robustness of the bus route assignment decisions or reserve small capacities that require small land spaces.

Table 8. Allocation results for three boarding locations with the same capacity limitation 45

Solution No.	Capacity	Board-ing loca-tion	Assigned bus routes			Repeat count
			R^1	R^2	R^3	
1	45	1		102/106/116/118		9
	45	2	117		101/104/111/182	
	45	3	107/170/171	112	103/109/113/115	
2	45	1		102/106/116/118		1
	45	2			101/104/111/115/182	
	45	3	107/117/170/171	112	103/109/113	
3	45	1	107/117/170/171	102/106/118		3
	45	2			101/104/111/182	
	45	3		112/116	103/109/113/115	
4	45	1		102/106/116/118		1
	45	2		112	101/104/182	
	45	3	107/117/170/171		103/109/111/113/115	

5.4 Comparison with a heuristic algorithm

To test the effectiveness of the proposed MISOCP model, we propose a heuristic algorithm based on the structure of the problem and compare the qualities of the obtained solutions by the model and the heuristic. The principle of the heuristic is described as follows.

As previously analyzed, the increase in the waiting time of passengers is caused by the separation of *overlapping bus route set*. For one destination, separating bus routes that include this destination into different boarding locations wastes the bus frequency of this destination, thereby increasing the waiting time of passengers toward this destination. If the travel demand of this destination is large, then the objective value (total waiting time) is substantially more likely to increase. Therefore, we define a parameter ζ_s for each destination $s \in \mathrm{S}$, as

$$\zeta_s = \sum_{r \in R} f_r \, \delta_{rs} q_s. \tag{25}$$

Then, we rank ζ_s from largest to smallest to prioritize the destination with high bus frequency and travel demand. In the heuristic, we initially take destination s with the largest ζ_s as the *active destination*. The routes that include the *active destination* (i.e., routes r with $\delta_{rs} = 1$) are defined as the *active bus route set*, which is the set of the bus routes that are currently allocated. We allocate all the bus routes in the *active bus route set* to one boarding location.

The second issue is determining a boarding location for route allocation in the *active bus route set*. We define the boarding location used for allocation as the *active boarding location.* For the first allocation, we can take the boarding location with the largest capacity as the *active boarding location* for allocation. However, the *active bus route set* contains several bus routes that are already allocated when we address previous destinations during the allocation. Moreover, these allocated bus routes may be allocated to different boarding locations in advance. Under such situation, we should determine *the most related boarding location* as the *active boarding location* for allocating the rest of unallocated bus routes in this *active bus route set.* The *most related boarding location* is set to be the boarding location with the highest frequency toward the *active destination,* which is the total frequency of the already allocated bus routes in this *active bus route set*.

The third issue is that each boarding location has a limited capacity. Thus, for each allocation, we should compare the total bus frequency of all the unallocated bus routes in the *active bus route set* with the available capacity of the *active boarding location*. If the capacity is sufficient, then we allocate all the unallocated bus routes in the *active bus route set* to this *active boarding location*. However, if the capacity is insufficient, then we mark this *active destination* and do not allocate routes for this *active destination*. Thereafter, we take the next destination with a small ζ_s as the *active destination*.

Two situations may occur based on the above rules. The first situation is that all bus routes can be allocated before going through all the destinations *s* from the largest value of ζ_s to the smallest value of ζ_s, and the algorithm is terminated when all the bus routes are already allocated. The second situation is that bus routes are still unallocated after we go through all the destinations *s* from the largest value of ζ_s to the smallest value of ζ_s. The reason for this situation is that the capacity for the *active boarding location* is insufficient to allocate all the unallocated bus routes in the *active bus route set* to only one boarding location. In this situation, all the marked destinations are re-evaluated from the largest value of ζ_s to the smallest value of ζ_s but the allocation strategy is changed. We allocate each unallocated bus route individually to the most related boarding location, to the second, and the third most related boarding locations rather than allocating all the unallocated bus routes in the *active bus route set* to only one boarding location until we determine a cluster with a sufficient capacity.

Table 9. Comparison of results between the MISOCP model and the heuristic

Case	Boarding location	Capacity	Sum of destinations*	MISOCP objective	Heuristic objective	Relative gap#
1	1	45	163	2551.95	2658.03	4.16%
	2	45				
	3	45				
2	1	50	155	2539.31	2587.24	1.85%
	2	50				
	3	50				
3	1	55	147	2519.73	2531.30	0.46%
	2	55				
	3	55				
4	1	60	144	2514.73	2514.73	0.00%
	2	60				
	3	60				
5	1	40	147	2519.73	2519.73	0.00%
	2	50				
	3	60				
6	1	30	147	2519.73	2519.73	0.00%
	2	50				
	3	70				
7	1	20	147	2519.73	2519.73	0.00%
	2	50				
	3	80				
8	1	10	147	2519.73	2519.73	0.00%
	2	50				
	3	90				

Note: *: Here, the destinations may not be unique.
#: Relative gap = (Heuristic objective – MISOCP objective)/MISOCP objective.

We apply the heuristic to eight cases with different boarding location capacities and compare the results with those of the MISOCP model, as shown in Table 9. For each test, the

CPU time of applying the heuristic is less than five minutes and similar with the CPU time of solving the MISOCP model. Given that the CPU time is short enough for practical applications, the exact one is not reported. However, the heuristic cannot guarantee optimal solutions. The relative gap is larger when the *sum of destinations* is larger, and the relative gap is 0 in some cases when the *sum of destinations* is small (e.g., from Cases 4 to 8). These results demonstrate that the MISOCP model is especially effective when the spaces of boarding locations are limited, and the boarding locations do not have sufficient capacities.

6 Computational Experiments

In this section, we test Model **M3** on more challenging instances, in which the configuration of the bus station is increased to more boarding locations $w \in \mathrm{W}$ with more overlapping bus routes $r \in \mathrm{R}$ and destinations $s \in \mathrm{S}$. Different overlapping scenarios are illustrated as (a), (b), (c) and (d) in Figure 3. We explain Scenario (d) first which is the simplest one, i.e., each circle represents a set of bus routes, bus routes in the same set may overlap each other, while bus routes in different sets cannot overlap each other. Scenario (a) is more complicated, indicating that three sets of bus routes are independent with each other, while each of them may overlap to another set of bus routes (i.e., the circle in the middle). For Scenario (b), neighboring circles have intersection parts, indicating neighboring sets of bus routes may overlap with each other. Scenario (c) is the most complicated one with many intersection parts as shown in the darker areas among circles indicating the overlap. Note that Scenario (a)–(d) in Figure 3 corresponds to (a)–(d) in Table 10, which identifies the computation results for each scenario. In reality, the configuration of the network is not as complex as cases in Table 10, but we still aim at a deeper understanding of the BRCP. Besides, passenger demands are in $[25 + 50\sum_{r \in \mathrm{R}} \delta_{rs}, 100 + 50\sum_{r \in \mathrm{R}} \delta_{rs}]$ for $s \in \mathrm{S}$, considering that the common destinations (i.e., visited by more than one bus route) have more travel demands. Referring to the capacity of each cluster in all tests, $\sum_{w \in \mathrm{W}} c_w - \sum_{r \in \mathrm{R}} f_r < \min\{c_w | w \in \mathrm{W}\}$ should be guaranteed to ensure that all boarding locations are used and that the numbers of boarding locations reported in Table 10 are meaningful. Detailed capacity information of all tests is reported in Table 13 in the Appendix.

For each scenario, different cases are considered corresponding to different configurations of the bus route network. For instance, in Case 1 of Table 10, a total of 20 bus routes and 300 destinations are considered, with 4 boarding locations of the focused bus station, and each bus route visits more than 10 destinations in the downstream; whereas in Case 2, each bus route has more than 15 destinations, indicating bus routes in Case 2 are more likely to overlap with each other. All the cases are tested with the maximum CPU time limit of 900 seconds, to understand the complexity of solving the MISOCP (i.e., Model **M3**) in each configuration. Results of cases in each scenario are reported in the lower part of Table 10, e.g., the optimality gap of Case 2 in Scenario (a) is 0.83%. Detailed results are reported in Table 11, consisting of the best integer result, the best bound (i.e., corresponding to the fractional solution), and the optimality gap between them calculated by CPLEX with a time limitation of 900 seconds. The CPU time of each test is also reported for tests solved to optimality within 900 seconds. For the BRCP, a trivial lower bound exists where every destination $s \in \mathrm{S}$ gets all the available bus route frequencies and can be formulated as $\sum_{s\in \mathrm{S}}(q_s / (\sum_{r\in \mathrm{R}} f_r \delta_{rs}))$. We also report the trivial lower bounds of all tests, from which the trivial gaps can be calculated, as shown in the last two columns of Table 11.

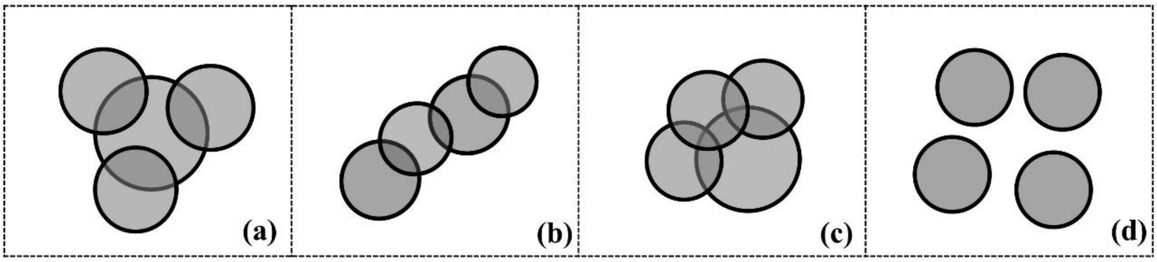

Figure 3. Overlapping scenarios.

Table 10. Computation results of the MISOCP model in different configurations.

Case	1	2	3	4	5	6	7	8	
# Bus routes	20	20	20	20	25	25	30	20	
# Boarding locations	4	4	5	5	4	4	4	5	
# Destinations	300	300	300	300	300	200	200	200	
# Destinations/route	>10	>15	>10	>12	>10	>10	>10	>10	*Average*
(a)	0.00%	0.83%	0.53%	0.87%	0.00%	6.20%	5.51%	0.00%	1.74%
(b)	0.00%	0.00%	0.54%	0.00%	0.00%	0.00%	0.00%	0.00%	0.07%
(c)	0.00%	0.85%	1.22%	1.96%	1.57%	6.59%	5.67%	5.02%	2.86%
(d)	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Average	0.00%	0.42%	0.57%	0.71%	0.39%	3.20%	2.80%	1.25%	

Table 11. Computational results, trivial bounds and gaps.

Scenario - Case	Best Integer	Best Bound	Optimality Gap *	CPU Time	Trivial Bound	Trivial Gap
(a)-1	4312.76	4312.76	0.00%	37	4256.39	1.31%
(a)-2	4318.74	4282.72	0.83%	900	4137.85	4.19%
(a)-3	4597.53	4573.25	0.53%	900	4488.41	2.37%
(a)-4	4172.76	4136.26	0.87%	900	4066.46	2.55%
(a)-5	4907.92	4907.92	0.00%	583	4820.37	1.78%
(a)-6	2550.88	2392.63	6.20%	900	2339.94	8.27%
(a)-7	2798.20	2644.00	5.51%	900	2577.73	7.88%
(a)-8	2883.02	2883.02	0.00%	605	2813.40	2.42%
(b)-1	4510.90	4510.90	0.00%	40	4452.50	1.29%
(b)-2	4799.67	4799.67	0.00%	243	4646.17	3.20%
(b)-3	4448.69	4424.64	0.54%	900	4350.35	2.21%
(b)-4	4487.10	4487.10	0.00%	484	4412.27	1.67%
(b)-5	4827.87	4827.87	0.00%	42	4773.64	1.12%
(b)-6	2830.15	2830.15	0.00%	679	2732.07	3.47%
(b)-7	3165.19	3165.19	0.00%	800	3066.13	3.13%
(b)-8	2723.63	2723.63	0.00%	898	2646.58	2.83%
(c)-1	4615.26	4615.26	0.00%	116	4532.19	1.80%
(c)-2	4542.50	4503.85	0.85%	900	4375.57	3.67%
(c)-3	4293.12	4240.69	1.22%	900	4157.50	3.16%
(c)-4	4256.21	4172.95	1.96%	900	4089.94	3.91%
(c)-5	4541.37	4470.01	1.57%	900	4376.14	3.64%
(c)-6	2884.81	2694.68	6.59%	900	2603.36	9.76%
(c)-7	2983.52	2814.30	5.67%	900	2747.68	7.90%
(c)-8	2668.92	2535.05	5.02%	900	2462.56	7.73%
(d)-1	4756.99	4756.99	0.00%	6	4739.11	0.38%
(d)-2	4550.33	4550.33	0.00%	24	4496.85	1.18%
(d)-3	4425.59	4425.59	0.00%	12	4404.94	0.47%
(d)-4	4247.74	4247.74	0.00%	48	4206.87	0.96%
(d)-5	4433.70	4433.70	0.00%	5	4428.92	0.11%
(d)-6	3027.86	3027.86	0.00%	15	3008.85	0.63%
(d)-7	3178.77	3178.77	0.00%	2	3176.94	0.06%
(d)-8	2628.75	2628.75	0.00%	15	2591.80	1.41%

Note: *: Optimality Gap = (Best Integer – Best Bound) / Best Integer.
#: Trivial Gap = (Best Integer – Trivial Bound) / Best Integer.

As reported in Table 10, Model **M3** shows its efficiency even for dealing with large-size instances. When reaching the CPU time limitation (i.e., 15 minutes) for termination, the largest optimality gap is only 6.59%, as shown in Case 6 of Scenario (c). Generally, smaller optimality gap indicates the instance is less time-consuming to be solved. Another main advantage of Model **M3** is that increasing the number of boarding locations does not contribute much to the complexity of the BRCP, as the optimality gaps of Case 3 and 4 are just a bit larger than those of Case 1 and 2.

Referring to different performances of scenarios, the optimality gap depends on the overlapping complexity of each scenario, indicated by Figure 3. As shown Table 10, Scenario (c) is most difficult to be solved with the average optimality gap 2.86%, followed by (a) and (b), whose optimality gaps are 1.74% and 0.07%, respectively, while Scenario (d) is solved to optimality. It is noticeable that overlapping degree can also be increased by decreasing the number of total destinations with the other parameters unchanged; that is why the average optimality gap of Case 6 is larger than that of Case 5 (i.e., 3.20% and 0.39%). Similarly, the optimality gap of Case 8 is larger than that of Case 3 (i.e., 1.25% and 0.57%). Besides, increasing the number of destinations visited by each route increases the overlapping degree directly, with the same numbers of total destinations and routes, i.e., comparing Case 1 with 2, or Case 3 with 4. However, the impact of increasing the number of bus routes on the complexity of the BRCP is not obvious, as shown by comparisons between Case 1 and 5, or between Case 6 and 7.

Table 12. Computation results of Model **M5** in different configurations.

Case	1	2	3	4	5	6
#Bus routes	20	20	25	25	30	35
# Boarding locations	4	5	5	6	5	6
# Destinations	300	300	300	300	400	500
# Destinations/route	>15	>15	>15	>15	>20	>25
Optimality gap	2.48%	4.89%	9.60%	14.36%	13.97%	23.80%

Overall, the optimality gap (as well as the computation time) of the BRCP is highly dependent on the overlapping degree of bus networks. Referring to other factors, i.e., the number of bus routes, destinations, and boarding locations, these issues jointly affect the computation time, and the influence highly depends on specific topologies (e.g., topology (a) to (d) in Figure 3) of bus route networks.

In Table 12, we test Model **M5**, the extension model described in Section 4, which can solve the BRCP considering that the arrival rate of buses follows any random distributions. The overlapping scenarios in all cases of Table 12 are according to Scenario (a) of Figure 3, and passenger demands are in $[25 + 50 \sum_{r \in \mathrm{R}} \delta_{rs}, 100 + 50 \sum_{r \in \mathrm{R}} \delta_{rs}]$ for destination $s \in \mathrm{S}$. Still, Model **M5** is tested for different configurations of networks, consisting of different numbers of total bus routes, total destinations, boarding locations, and destinations visited by each route, as shown in the upper part of Table 12. The results are reported in the last row of Table 12, in which all the cases are tested for 900 seconds. Results show that Model **M5** can also deal with large size instances in reasonable times and can be utilized in practice if the historic data is sufficient for measuring passenger's waiting time. When facing insufficient information for the actual relationship between bus frequencies and passenger waiting times of boarding locations with different capacities, the MISOCP (i.e., Model **M3**) is required.

7 Conclusion

The bus route clustering problem (BRCP) is proposed in this study, addressing the problem of assigning bus routes to boarding locations so as to minimize passenger waiting time. The BRCP is proven to be NP-hard by showing that if it can be solved in polynomial time, then the maximum clique problem, which is a well-known NP-hard problem, can also be solved in polynomial time. The BRCP is formulated as a mixed-integer second-order cone program (MISOCP) that can be solved by off-the-shelf solvers. The MISOCP is applied to a major bus station in Hong Kong that has 17 bus routes and 3 boarding locations. The problem size can represent the realistic case, showing that the model is effective in dealing with a real bus route network. The BRCP tackles a highly utilized network with overlapping bus routes and assigns bus routes with common destinations into one boarding location to maximize the bus frequencies of each boarding location and further minimize passenger waiting time. The MISOCP model is demonstrated to be more effective than a greedy heuristic when the capacity of boarding locations is insufficient. The appropriate boarding location allocation avoids long waiting time of passengers and traffic congestion in a significant geographic location with heavy traffic loads and limited land spaces. Larger-size instances with different overlapping scenarios are tested to draw deeper analyses of the BRCP. Results show that the computation time of the MISOCP is highly dependent on the overlapping degree of bus networks, while other factors (i.e., the number of bus routes, destinations, and boarding locations) jointly affect the computation time. More importantly, the influence is instance-specific and highly depends on the specific overlapping scenario of bus route networks. An extension model is also proposed which can be utilized in realistic when sufficient historical data is provided for passengers' waiting times.

The BRCP is practically important to face a highly utilized network where bus routes are overlapping, with determinations on allocating which bus route to which boarding location, concerning limited capacity of each boarding location of the focused bus station. In this work, we try to distribute different bus routes in such a way that each downstream stop can be served by one boarding location if possible. Then, a passenger can access a connecting line with less waiting time, since the passenger has a chance to board all the bus routes heading to the destination. This study may inspire future extensions of the BRCP. For instance, it can be extended to bus rapid transit (BRT) system, e.g., TransMilenio in Bogotá, Colombia, which includes multiple boarding platforms with gates inside each station, and bus services are assigned to these gates. In TransMilenio, multiple boarding locations of each station expand its capacity to that only heavy rail systems had (Wright and Hook, 2007). More issues can be considered for BRT system, such as different types of bus services (i.e., normal or express) that may be assigned to different boarding locations of the BRT station (Peña et al., 2013); besides, the saturation level of passengers can also be concerned to reduce crowding levels and improve service qualities in future study.

Acknowledgment

This study is supported by the National Natural Science Foundation of China (No. 71701178, No. 71771050, No. 71831008, No. 71771062). We appreciate three anonymous reviewers for their valuable comments. All errors are the authors'.

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Appendix.

Table 13. Boarding location capacities (corresponding to results in Table 10).

Scenario - Case	c_1	c_2	c_3	c_4	c_5
(a)-1	57	45	49	33	-
(a)-2	37	49	56	49	-
(a)-3	30	32	37	47	31
(a)-4	35	37	35	47	40
(a)-5	52	61	43	60	-
(a)-6	67	68	68	56	-
(a)-7	65	60	67	67	-
(a)-8	44	31	49	32	31
(b)-1	41	57	30	50	-
(b)-2	35	36	48	47	-
(b)-3	45	30	33	41	35
(b)-4	42	36	33	38	38
(b)-5	61	53	41	46	-
(b)-6	50	44	61	69	-
(b)-7	50	69	67	41	-
(b)-8	42	33	30	30	52
(c)-1	35	59	31	53	-
(c)-2	61	40	48	39	-
(c)-3	43	42	38	34	39
(c)-4	41	45	31	31	47
(c)-5	43	61	63	45	-
(c)-6	64	32	50	64	-
(c)-7	68	65	60	66	-
(c)-8	40	34	46	39	35
(d)-1	30	53	43	42	-
(d)-2	48	33	32	59	-
(d)-3	46	31	41	37	31
(d)-4	32	38	32	42	39
(d)-5	64	62	46	51	-
(d)-6	64	38	53	61	-
(d)-7	52	50	68	68	-
(d)-8	39	44	36	45	45