An Optimal Consumption-Investment Model with Constraint on Consumption

Zuo Quan Xu* and Fahuai Yi†

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Abstract

A continuous-time consumption-investment model with constraint is considered for a small investor whose decisions are the consumption rate and the allocation of wealth to a risk-free and a risky asset with logarithmic Brownian motion fluctuations. The consumption rate is subject to an upper bound constraint which linearly depends on the investor's wealth and bankruptcy is prohibited. The investor's objective is to maximize the total expected discounted utility of consumption over an infinite trading horizon. It is shown that the value function is (second order) smooth everywhere but a unique possibility of a known exception point and the optimal consumption-investment strategy is provided in a closed feedback form of wealth. According to this model, an investor should take the similar investment strategy as in Merton's model regardless his financial situation. By contrast, the optimal consumption strategy does depend on the investor's financial situation: he should use a similar consumption strategy as in Merton's model when he is in a bad situation, and consume as much as possible when he is in a good situation.

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1 Introduction

The publication of the monumental 1952 article *Portfolio Selection* and the 1959 book of the same title by Harry M. Markowitz (1952, 1959) heralded the beginning of modern

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong. This author acknowledges financial supports from the Hong Kong Early Career Scheme (No.533112), the Hong Kong General Research Fund (No.529711), NNSF of China (No.11471276), and the Hong Kong Polytechnic University. Email: maxu@polyu.edu.hk.

†School of Finance, Guangdong University of Foreign Studies, Guangzhou, China. The project is supported by NNSF of China (No.11271143, No.11371155, No.11471276 and No.71471045), University Special Research Fund for Ph.D. Program of China (20124407110001), NSF of Guangdong Province of China (No.2015A030313574) and The Humanities and Social Science Research Foundation of the National Ministry of Education of China (No.15YJAZH051). Email: fhyi@scnu.edu.cn.

finance. To develop a general theory of portfolio choice, Samuelson (1969) and Merton (1969, 1971) initiated the study of dynamic optimal consumption-investment problems. The problem concerning optimal consumption-investment decisions involves the decisions of an investor endowed with some initial wealth who seeks to maximize the expected (discounted) utility of consumption over time. The decisions (called *consumption-investment strategy*) are the consumption rate and the allocation of wealth to risk-free and risky assets over time. According to Merton (1975), studying this type of problems is the natural starting point for the development of a theory of finance.

Samuelson and Merton's pioneering papers prompted researchers to contribute a considerable volume of new work on the subject in various directions. The literature has extensively covered the optimal consumption-investment problems in the financial markets that are subject to constraints and market imperfections. For example, the book authored by Sethi (1997) summarized the research conducted by Sethi and his collaborators on the optimal consumption-investment problems under various constraints such as bankruptcy prohibited, subsistence consumption requirement, borrowing prohibited, and random coefficients market. Fleming and Zariphopoulou (1991) considered the optimal consumption-investment problem with borrowing constraints. Cvitani and Karatzas (1992, 1993) considered the scenario in which the investment strategy of an investor is restricted to take values in a given closed convex set. Zariphopoulou (1994) considered the problem under the constraint that the amount of money invested in a risky asset must not exceed an exogenous function of the wealth, and bankruptcy is prohibited at any time. Elie and Touzi (2008) considered the optimal consumption-investment problem with the constraint that the wealth process never falls below a fixed fraction of its running maximum. Davis and Norman (1990), Zariphopoulou (1992), Shreve and Soner (1994), Akian, Menaldi, and Sulem (1996), and Dai and Yi (2009) considered proportional transaction costs in the study of optimal consumption-investment problems. These optimal consumption-investment models focus on the constraints on the wealth process and the investment strategy.

Bardhan (1994) considered the optimal consumption-investment problem with constraint on the consumption rate and the wealth. The constraint is that the investor must consume at a minimal (constant) rate throughout the investment period, which is known as the subsistence consumption requirement, and must maintain their wealth over a low bound at all times. However, in financial practice, an upper boundary constraint on the consumption rate typically exists in addition to the subsistence consumption requirement. An example of such scenario is an investment firm with cash flow commitments that is subject to regulatory capital constraints. No study in the extant literature has considered an upper boundary constraint on the consumption rate in the theory of optimal consumption-investment in intertemporal economies.

Harry Markowitz, a Nobel laureate in economics, stated, "It remains to be seen whether the introduction of realistic investor constraints is an impenetrable barrier to analysis, or a golden opportunity for someone with a novel approach; and whether progress in this direction will come first from discrete or from continuous-time models," in the foreword of the book by Sethi (1997). Research on the optimal consumption-investment problem that considers the upper constraint on the consumption rate is scant, although extensive research has been conducted on the problem involving other constraints, such as no bankruptcy or limits on the amount of money borrowed. Consequently, this research topic has not been sufficiently explored. This motivated us to investigate the optimal consumption-investment problems with constraint on consumption rate.

In this paper, we consider a continuous-time consumption-investment model with an upper bound constraint on the consumption rate, which linearly depends on the amount of wealth of an investor at any time. The problem is considered in a standard Black-Scholes market with a risk-free and a risky asset over an infinite trading horizon. We make the usual assumption that shorting is allowed but bankruptcy is prohibited in the market. We will primarily use techniques derived from the theories of free boundary and viscosity solution in the field of differential equations to solve the problem (See e.g., Crandall and Lions (1983), Lions (1983), Fleming and Soner (1992), Dai, Xu and Zhou (2010), Dai and Xu (2011), Chen and Yi (2012)). As is well-known, the value function is the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. Using this fact, we first prove that the viscosity solution of the equation is smooth everywhere but a unique possibility of (known) exception point. The detailed descriptions of an unconstrained and a constrained trading regions are then provided. Finally, we derive the optimal consumption-investment strategy in a closed feedback form of wealth. In particular, the optimal consumption strategy is explicitly derived and its expression does not involve the value function. The results also show that an investor should use a similar optimal consumption-investment strategy as in the unconstrained case when his financial situation is bad and should consume at the maximum possible rate when his situation is good.

The paper is organized as follows. We formulate a continuous-time optimal consumption-investment model with constraint on the consumption rate in Section 2. A case without constraint is studied in Section 3. In Section 4, the associated Hamilton-Jacobi-Bellman equation to the problem is introduced and a case with homogeneous constraint is investigated. Using the techniques in the theory of viscosity solution, we show some properties of the value function of the problem in Section 5. The descriptions of an unconstrained and a constrained trading regions are provided in Section 6. Finally, we derive the optimal consumption-investment strategy in a closed feedback form of wealth in Section 7. We conclude the paper in Section 8.

2 Model Formulation

We consider a standard Black-Scholes financial market with two assets: a *bond* and a *stock*. The price of the bond is driven by an ordinary differential equation (ODE):

$$\mathrm{d}P_t = rP_t\,\mathrm{d}t,$$

where r is the risk-free interest rate. While the price of the stock is driven by a stochastic differential equation (SDE):

$$\mathrm{d}S_t = \alpha S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where α is the mean return rate of the stock, σ is the volatility of the stock, and $W(\cdot)$ is a standard one-dimensional Brownian motion on a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $\{\mathcal{F}_t = \sigma(W_s, s \leqslant t), t > 0\}$ the filtration generated by the Brownian motion. The interest rate r, the mean rate of return α, and the volatility σ are constants satisfying $r > 0$, $\sigma > 0$, and $\mu := \alpha - r > 0$. There are no transaction fees or taxes and shorting is also allowed in the market.

Let us consider a small investor in the market. The investor's trading will not affect the market prices of the two assets. His trading strategy is self-financing meaning that there is no incoming or outgoing cash flow during the whole investment period. Then it is well-known that the wealth process of the investor is driven by an SDE:

$$\begin{cases} \mathrm{d}X_t = (rX_t + \pi_t\mu - c_t)\,\mathrm{d}t + \pi_t\sigma\,\mathrm{d}W_t, \\ X_0 = x, \end{cases} \tag{1}$$

where $x > 0$ is the initial endowment of the investor, π_t is the amount of money invested in the stock at time t, $c_t \geqslant 0$ is the consumption rate at time t. In this paper, we assume that no bankruptcy is allowed, namely

$$X_t \geqslant 0, \quad \forall\, t > 0, \tag{2}$$

almost surely (a.s.). The target of the investor is to choose the best consumption-investment strategy $(c(\cdot), \pi(\cdot))$, which is subject to certain constraints specified below, to maximize the total expected (discounted) utility from consumption over an infinite trading horizon

$$\mathbf{E}\left[\int_0^\infty e^{-\beta t} U(c_t)\,\mathrm{d}t\right], \tag{3}$$

where $U : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is the utility function of the investor, which is strictly increasing, and $\beta > 0$ is a constant discounting factor. In this paper, we consider risk-averse investor only, this is equivalent to that $U(\cdot)$ is concave.

The consumption-investment strategy $(c(\cdot), \pi(\cdot))$ is required to satisfy the following integrability constraint

$$\mathbf{E}\left[\int_0^T e^{-\beta t}(\pi_t^2 + c_t)\,\mathrm{d}t\right] < \infty, \quad \forall\, T > 0, \tag{4}$$

in which case, SDE (1) admits a unique solution $X(\cdot)$ satisfying

$$\mathbf{E}\left[\int_0^T e^{-\beta t}|X_t|\,\mathrm{d}t\right] < \infty, \quad \forall\, T > 0.$$

If no other constraint on the consumption rate and investment strategy exists, the problem (3) becomes the classical Merton (1971) consumption-investment problem. However, in practice, constraint on the consumption rate always exists; for example, the consumption rate cannot be too low because an investor has basic needs, which are the minimal amount of resources necessary required for long-term physical well-being; this is the so-called subsistence consumption requirement. Another practical example is when the manager of a fund requests a fixed salary and a proportion of the managed wealth as a bonus. However, most of the wealth still belongs to the owner, and consequently, the manager cannot take excessive amounts from the total wealth. These scenarios motivated us to consider an upper constraint on the consumption rate.

In this paper, specifically, we assume that the consumption rate is upper bounded by a time-invariant linear function of wealth X_t at any time:

$$0 \leqslant c_t \leqslant kX_t + \ell, \quad t \geqslant 0, \tag{5}$$

where k and ℓ are nonnegative constants, at least one of which is positive.

Our problem is summarized as

$$\sup_{(c(\cdot),\pi(\cdot))} \mathbf{E}\left[\int_0^\infty e^{-\beta t} U(c_t)\,\mathrm{d}t\right]. \tag{6}$$

where the wealth process is subject to (1), and the consumption-investment strategy $(c(\cdot), \pi(\cdot))$ is subject to the constraints (2), (4) and (5). Denote by $V(x)$ the optimal value of the problem (6).

Same as Merton (1971)'s model, we focus on the constant relative risk aversion (CRRA) type utility function

$$U(x) = \frac{x^p}{p}, \quad x \geqslant 0 \tag{7}$$

for a constant $0 < p < 1$. It is well-known that the logarithmic utility function can be treated as a limit case of CRRA type utility function as $\log(x) = \lim\limits_{p\to 0} \frac{x^p - 1}{p}$, so the results of this paper can be extended to covering the case of logarithmic utility function.

3 Merton's Model: A Case without Constraint

We first recall the well-known result of Merton (1971) for the scenario without constraint. Let

$$\theta := \frac{\mu^2}{2\sigma^2(1-p)} > 0,$$

and

$$\kappa := \frac{\beta - p(\theta + r)}{1-p}.$$

Theorem 3.1 *If $\kappa > 0$ and there is no constraint on the consumption rate, i.e., $k + \ell = +\infty$, then the optimal consumption-investment strategy for the problem (6) is given by*

$$(c_t, \pi_t) = \left(\kappa X_t, \frac{\mu}{\sigma^2(1-p)} X_t \right), \quad t \geqslant 0,$$

and the optimal value is

$$V^\infty(x) = \frac{1}{p}\kappa^{p-1}x^p. \tag{8}$$

The optimal value $V^\infty(x) = \frac{1}{p}\kappa^{p-1}x^p$ will serve as an upper bound for the optimal value in scenarios with constraint.

4 Hamilton-Jacobi-Bellman Equation

We adopt the viscosity solution approach in differential equations to solve the problem (6). To this end, we need to prove some basic properties of the value function.

Proposition 4.1 *If $\kappa > 0$, then the value function $V(\cdot)$ of the problem (6) satisfies*

$$V(x) \leqslant \frac{1}{p}\kappa^{p-1}x^p, \quad x > 0. \tag{9}$$

Moreover, $V(\cdot)$ is continuous, increasing, and concave on $[0, +\infty)$ with $V(0) = 0$.

Proof. Both the set of admissible controls and the optimal value of the problem (6) are increasing in ℓ. And consequently, an upper bound of the optimal value is given by the scenario $\ell = +\infty$. So the inequality (9) follows from (8).

If the initial endowment of the problem (6) is 0, then the unique admissible consumption-investment strategy is $(c(\cdot), \pi(\cdot)) \equiv (0, 0)$, so $V(0) = 0$; and consequently, $V(\cdot)$ is continuous at 0 from the right by (9). By the definition of $V(\cdot)$, it is not hard to prove its the concavity and monotonicity. The continuity of $V(\cdot)$ on $(0, +\infty)$ follows from its finiteness and concavity. □

With this proposition, using the theory of viscosity solution in differential equations (See Crandall and Lions (1983), Lions (1983), Fleming and Soner (1992)), one can prove

Theorem 4.2 *If $\kappa > 0$, then the value function $V(\cdot)$ of the problem (6) is the unique viscosity solution of its associated HJB equation*

$$\begin{aligned} \beta V(x) - \sup_{\pi} \left(\tfrac{1}{2}\sigma^2\pi^2 V_{xx}(x) + \pi\mu V_x(x) \right) - \sup_{0 \leqslant c \leqslant kx+\ell} \Big(U(c) - cV_x(x) \Big) - rxV_x(x) \\ = \beta V(x) + \frac{\mu^2}{2\sigma^2}\frac{V_x^2(x)}{V_{xx}(x)} + (c(x) - rx)V_x(x) - \frac{c^p(x)}{p} = 0, \quad x > 0, \end{aligned} \tag{10}$$

in the class of increasing concave functions on $[0, +\infty)$ with $V(0) = 0$, where

$$c(x) := \min\left\{ (V_x(x))^{\frac{1}{p-1}}, kx + \ell \right\}, \quad x > 0.$$

Proof. The proof is standard (See e.g., Zariphopoulou (1992, 1994)). We leave the details to the interested readers. □

4.1 A Case with Homogeneous Constraint

We first consider the scenario with a homogeneous constraint on the consumption rate. The results will be useful in studying general scenarios in the following sections.

Theorem 4.3 *If $k > 0$, $\ell = 0$, and $\kappa > 0$, then the optimal consumption-investment strategy for the problem (6) is given by*

$$(c_t, \pi_t) = \left(\min\{\kappa, k\} X_t, \frac{\mu}{\sigma^2(1-p)} X_t\right), \quad t \geqslant 0, \tag{11}$$

and the optimal value is

$$V(x) = \frac{\min\{\kappa, k\}^p}{p(\kappa(1-p) + \min\{\kappa, k\}p)} x^p = \begin{cases} \frac{k^p}{p(\kappa(1-p)+kp)} x^p, & k < \kappa; \\ \frac{1}{p}\kappa^{p-1} x^p, & k \geqslant \kappa. \end{cases} \tag{12}$$

Proof. Suppose $\kappa > 0$. Let $V(\cdot)$ be defined as in (12). Then

$$\begin{aligned} c(x) = \min\left\{(V_x(x))^{\frac{1}{p-1}}, kx + \ell\right\} &= \min\left\{(V_x(x))^{\frac{1}{p-1}}, kx\right\} \\ &= \min\left\{\frac{\min\{\kappa, k\}^{\frac{p}{p-1}}}{(\kappa(1-p) + \min\{\kappa, k\}p)^{\frac{1}{p-1}}}, k\right\} x = \min\{\kappa, k\} x, \end{aligned}$$

where we used the fact that

$$\frac{\min\{\kappa, k\}^{\frac{p}{p-1}}}{(\kappa(1-p) + \min\{\kappa, k\}p)^{\frac{1}{p-1}}} \geqslant \frac{\min\{\kappa, k\}^{\frac{p}{p-1}}}{(\min\{\kappa, k\}(1-p) + \min\{\kappa, k\}p)^{\frac{1}{p-1}}} = \min\{\kappa, k\} = k,$$

when $k < \kappa$. It is easy to check that $V(\cdot)$ and $c(\cdot)$ satisfy the HJB equation (10). Because $V(\cdot)$ is increasing and concave with $V(0) = 0$, by Theorem 4.2, $V(\cdot)$ is the value function of the problem (6). It is easy to verify that the value (12) is achieved by taking the consumption-investment strategy (11). □

The financial meaning of this result is quite clearly. The optimal consumption rate in the Merton case is $c_t = \kappa X_t$. If this strategy also satisfies the constraint in the problem (6), then it must be optimal for the problem. If it does not satisfy the constraint, then the best strategy is chosen, subject to the constraint, to be as close as possible to the optimal consumption rate in the Merton case, in other words, $c_t = kX_t$.

Corollary 4.4 *If $k \geqslant \kappa > 0$ and $\ell \geqslant 0$, then the optimal consumption-investment strategy for the problem (6) is given by*

$$(c_t, \pi_t) = \left(\kappa X_t, \frac{\mu}{\sigma^2(1-p)} X_t\right), \quad t \geqslant 0, \tag{13}$$

and the optimal value is

$$V(x) = \frac{1}{p}\kappa^{p-1}x^p. \tag{14}$$

If $\kappa \leqslant 0$ and $\ell \geqslant 0$, then the problem (6) is ill-possed, i.e., its optimal value is infinity.

Proof. Both the set of admissible controls and the optimal value are increasing in ℓ; and consequently, the scenario $\ell = +\infty$ gives an upper bound, $V(x) \leqslant V^\infty(x) = \frac{1}{p}\kappa^{p-1}x^p$. It is easy to verify that the upper bound $\frac{1}{p}\kappa^{p-1}x^p$ is achieved by taking the consumption-investment strategy (13).

If κ goes down to 0, then the optimal value $V(x) = \frac{1}{p}\kappa^{p-1}x^p$ goes to infinity. Because the optimal value of the problem (6) is decreasing in β, we conclude that $V(x) = +\infty$ if $\kappa \leqslant 0$ and $\ell \geqslant 0$. □

By Theorem 4.3 and Corollary 4.4, we only need to study the scenario

$$\kappa > k > 0, \quad \ell > 0,$$

which are henceforth assumed unless otherwise specified.

Remark 1 *We will address the scenario $\kappa > k = 0$ and $\ell > 0$ in Remark 2.*

5 The Value Function: Continuity of the First Order Derivative

Theorem 5.1 *The value function $V(\cdot)$ of the problem (6) is in $C[0,+\infty) \cap C^1(0,+\infty)$ if $r \leqslant k$; and in $C[0,+\infty) \cap C^1((0,+\infty)\backslash\{x_e\})$ if $r > k$, where*

$$x_e := \frac{\ell}{r-k}, \tag{15}$$

is the unique possibility of the exception point, in which case, $V_x(x_e-) \leqslant (kx_e + \ell)^{p-1}$ and $V(x_e) = \frac{1}{\beta p}(kx_e + \ell)^p$.

Proof. It is proved that $V(\cdot) \in C[0,+\infty)$ in Proposition 4.1. Note that $V(\cdot)$ is increasing and concave, so we can define the right and left derivatives as

$$V_x(x\pm) := \lim_{\varepsilon \to 0+} \frac{V(x \pm \varepsilon) - V(x)}{\pm\varepsilon} \geqslant 0,$$

for all $x > 0$. Moreover, both $V_x(\cdot\pm)$ are decreasing functions and $0 \leqslant V_x(x+) \leqslant V_x(x-) < +\infty$ for all $x > 0$.

Now we show that $V(\cdot)$ is continuously differentiable on $(0,+\infty)\backslash\{x_e\}$. By Darboux's Theorem, it is sufficient to show that $V(\cdot)$ is differentiable on $(0,+\infty)\backslash\{x_e\}$, which is equivalent to $V_x(x-)=V_x(x+)$ for all positive $x\neq x_e$.

Per absurdum, suppose $V_x(x_0+)<V_x(x_0-)$ for some $x_0>0$. Let ξ be any number satisfying $V_x(x_0+)<\xi<V_x(x_0-)$. Define

$$\phi(x)=V(x_0)+\xi(x-x_0)-N(x-x_0)^2,$$

where N is any large positive number. Then by the concavity of $V(\cdot)$,

$$\begin{aligned}V(x)&\leqslant V(x_0)+V_x(x_0-)(x-x_0)=\phi(x)+(V_x(x_0-)-\xi)(x-x_0)+N(x-x_0)^2\\&<\phi(x),\quad \text{if } 0<x_0-x<\frac{1}{N}(V_x(x_0-)-\xi);\end{aligned}$$

and

$$\begin{aligned}V(x)&\leqslant V(x_0)+V_x(x_0+)(x-x_0)=\phi(x)+(V_x(x_0+)-\xi)(x-x_0)+N(x-x_0)^2\\&<\phi(x),\quad \text{if } 0<x-x_0<\frac{1}{N}(\xi-V_x(x_0+)).\end{aligned}$$

Therefore, $V(x_0)=\phi(x_0)$ and $V(x)<\phi(x)$ in a neighbourhood of x_0. By Theorem 4.2, $V(\cdot)$ is a viscosity solution of the HJB (10), noting $\phi(\cdot)\in C^\infty(0,+\infty)$, so

$$\begin{aligned}0&\geqslant\beta\phi(x_0)-\sup_{\pi}\left(\tfrac{1}{2}\sigma^2\pi^2\phi_{xx}(x_0)+\pi\mu\phi_x(x_0)\right)-\sup_{0\leqslant c\leqslant kx_0+\ell}(U(c)-c\phi_x(x_0))-rx_0\phi_x(x_0)\\&=\beta V(x_0)-\frac{\mu^2\xi^2}{4\sigma^2N}-\sup_{0\leqslant c\leqslant kx_0+\ell}(U(c)-c\xi)-rx_0\xi=\beta V(x_0)-\frac{\mu^2\xi^2}{4\sigma^2N}-g(\xi),\end{aligned}$$

where

$$g(\xi):=\sup_{0\leqslant c\leqslant kx_0+\ell}(U(c)-c\xi)+rx_0\xi,\quad 0<\xi<+\infty.$$

Letting $N\to+\infty$, we get

$$g(\xi)\geqslant\beta V(x_0), \tag{16}$$

for all $\xi\in(V_x(x_0+),V_x(x_0-))$.

On the other hand, because $V(\cdot)$ is concave, it is second order differentiable almost everywhere. Thus, there exists a sequence $\{x_n:n\geqslant 1\}$ going up to x_0 such that $V(\cdot)$ is second order differentiable at each x_n. By Theorem 4.2,

$$\begin{aligned}0&=\beta V(x_n)-\sup_{\pi}\left(\tfrac{1}{2}\sigma^2\pi^2V_{xx}(x_n)+\pi\mu V_x(x_n)\right)-\sup_{0\leqslant c\leqslant kx_n+\ell}(U(c)-cV_x(x_n))-rx_nV_x(x_n)\\&\leqslant\beta V(x_n)-\sup_{0\leqslant c\leqslant kx_n+\ell}(U(c)-cV_x(x_n))-rx_nV_x(x_n)\\&=\beta V(x_n)-g(V_x(x_n))+r(x_0-x_n)V_x(x_n).\end{aligned}$$

So

$$g(V_x(x_n)) \leqslant \beta V(x_n) + r(x_0 - x_n)V_x(x_n).$$

Note that $g(\cdot)$ is convex on $(0, +\infty)$, so it is continuous on $(0, +\infty)$. Hence

$$g(V_x(x_0-)) = \lim_{n\to+\infty} g(V_x(x_n)) \leqslant \lim_{n\to+\infty} (\beta V(x_n) + r(x_0 - x_n)V_x(x_n)) = \beta V(x_0). \quad (17)$$

Similarly, we have

$$g(V_x(x_0+)) \leqslant \beta V(x_0). \quad (18)$$

Noting that $g(\cdot)$ is convex on $(0, +\infty)$ and (16), we obtain

$$\max\{g(V_x(x_0-)), g(V_x(x_0+))\} \geqslant g(\xi) \geqslant \beta V(x_0), \quad V_x(x_0+) < \xi < V_x(x_0-). \quad (19)$$

By (17), (18), and (19), we conclude that $g(\xi) = \beta V(x_0)$ for all $\xi \in [V_x(x_0+), V_x(x_0-)]$. Note

$$\begin{aligned} g(\xi) &= \sup_{0\leqslant c\leqslant kx_0+\ell} (U(c) - c\xi) + rx_0\xi \\ &= \begin{cases} U(kx_0+\ell) - (kx_0+\ell - rx_0)\xi, & \text{if } \xi \leqslant (kx_0+\ell)^{p-1}; \\ \left(\frac{1}{p} - 1\right)\xi^{\frac{p}{p-1}} + rx_0\xi, & \text{if } \xi > (kx_0+\ell)^{p-1}. \end{cases} \end{aligned}$$

Therefore, $g(\cdot)$ is a constant on $[V_x(x_0+), V_x(x_0-)]$ if and only if $kx_0 + \ell - rx_0 = 0$ and $V_x(x_0-) \leqslant (kx_0+\ell)^{p-1}$. It could only happen in the scenario $r > k$, $x_0 = x_e$ and $V_x(x_e-) \leqslant (kx_e+\ell)^{p-1}$, in which case, $\beta V(x_e) = g(\xi) = \frac{1}{p}(kx_e+\ell)^p$. The proof is complete. □

Although $V(\cdot)$ may not be differentiable at x_e when $r > k$, we can still define $V_x(x_e-)$. From now on, we denote $V_x(x_e) := V_x(x_e-)$ unless otherwise specified.

6 The Value Function: Properties

Proposition 6.1 *The value function $V(\cdot)$ of the problem (6) has the following properties:*

(a). $V(x)/x^p$ is decreasing, and hence

$$xV_x(x) \leqslant pV(x), \quad x > 0;$$

(b). we have

$$\frac{k^p}{p(\kappa(1-p)+kp)}x^p \leqslant V(x) \leqslant \frac{1}{p}\kappa^{p-1}x^p, \quad x > 0;$$

(c). $V(\cdot)$ is strictly concave on $(0,+\infty)$ and $V_x(\cdot)$ is strictly decreasing on $(0,+\infty)$;

(d). we have

$$\frac{k}{\kappa(1-p)+kp}\kappa^{p-1}x^{p-1} \leqslant V_x(x) \leqslant \kappa^{p-1}x^{p-1}, \quad x>0.$$

Proof. We first consider the scenario $x \neq x_e$.

(a). Let $V(x,\ell)$ denote the value function $V(x)$ with constraint (5). Given the form of CRRA type utility function (7), the dynamics (1) and constraint (5), a standard argument can show that $V(\cdot,\cdot)$ is homogeneous of degree p, i.e.,

$$V(\lambda x, \lambda \ell) = \lambda^p V(x,\ell), \quad \lambda>0.$$

Letting $\lambda = x^{-1}$,

$$V(1, x^{-1}\ell) = x^{-p}V(x,\ell),$$

the property (a) follows from $V(1,x^{-1}\ell)$ is decreasing in x.

(b). The upper bound is given by (9). The lower bound given by the scenario $\ell = 0$ is (12).

(c). Note $V_x(\cdot)$ is decreasing by the concavity of $V(\cdot)$. Suppose it is not strictly decreasing. Then $V_x(x) = A$, $x \in (x_1,x_2)$ for some constant $A \geqslant 0$ and $(x_1,x_2) \subset (0,+\infty)$. It follows that $V_{xx}(x) = 0$, $x \in (x_1,x_2)$. If $A=0$, because $V(\cdot)$ is concave and increasing, $V_x(x) = 0$, $x \in (x_1,+\infty)$ which contradicts the property (b). Suppose $A>0$. Applying the HJB equation (10),

$$\beta V(x) - \sup_{\pi}\left(\tfrac{1}{2}\sigma^2\pi^2V_{xx}(x) + \pi\mu V_x(x)\right) - \sup_{0\leqslant c\leqslant kx+\ell}\left(U(c) - cV_x(x)\right) - rxV_x(x) = 0,$$
$$x_1 < x < x_2,$$

we get

$$\beta V(x) = \sup_{\pi}(\pi\mu A) + \sup_{0\leqslant c\leqslant kx+\ell}(U(c)-cA) + rxA = +\infty, \quad x_1<x<x_2,$$

which contradicts the property (b).

(d). The upper bound follows from the properties (a) and (b). Note $V(\cdot)$ is concave and apply the property (b),

$$\begin{aligned} V_x(x) &\geqslant \frac{V(x+y)-V(x)}{y} \\ &\geqslant \frac{1}{y}\left(\frac{k^p}{p(\kappa(1-p)+kp)}(x+y)^p - \frac{1}{p}\kappa^{p-1}x^p\right), \quad x>0,\ y>0. \end{aligned}$$

Let $y = \frac{\kappa-k}{k}x$ in the above inequality,

$$\begin{aligned}V_x(x) &\geqslant \frac{k}{(\kappa-k)x}\left(\frac{k^p}{p(\kappa(1-p)+kp)}\left(\frac{\kappa}{k}\right)^p x^p - \frac{1}{p}\kappa^{p-1}x^p\right)\\ &= \frac{k}{(\kappa-k)x}\left(\frac{\kappa}{\kappa(1-p)+kp}-1\right)\frac{1}{p}\kappa^{p-1}x^p\\ &= \frac{k}{\kappa-k}\left(\frac{\kappa p-kp}{\kappa(1-p)+kp}\right)\frac{1}{p}\kappa^{p-1}x^{p-1}\\ &= \frac{k}{\kappa(1-p)+kp}\kappa^{p-1}x^{p-1}, \quad x>0.\end{aligned}$$

Thus the property (d) is proved.

For the scenario $x = x_e$, all the properties can be proved by a limit argument. The proof is complete. □

Define an unconstrained trading region $\mathcal{U}$ and a constrained trading region $\mathcal{C}$ as follows:

$$\mathcal{U} := \{x>0 : V_x(x)^{\frac{1}{p-1}} < kx+\ell\},$$
$$\mathcal{C} := \{x>0 : V_x(x)^{\frac{1}{p-1}} \geqslant kx+\ell\}.$$

One of the main results of this paper is providing detailed descriptions of these two regions.

It follows from Theorem 4.2 that

$$\beta V(x) + \frac{\mu^2}{2\sigma^2}\frac{V_x^2(x)}{V_{xx}(x)} - rxV_x(x) + \left(1-\tfrac{1}{p}\right)V_x(x)^{\frac{p}{p-1}} = 0, \quad x\in\mathcal{U}; \tag{20}$$

$$\beta V(x) + \frac{\mu^2}{2\sigma^2}\frac{V_x^2(x)}{V_{xx}(x)} + (kx+\ell-rx)V_x(x) - \tfrac{1}{p}(kx+\ell)^p = 0, \quad x\in\mathcal{C}. \tag{21}$$

Define

$$\eta := \kappa\left(\frac{k}{\kappa(1-p)+kp}\right)^{\frac{1}{p-1}} > \kappa.$$

Proposition 6.2 *We have*

$$\left(\frac{\ell}{\kappa-k},\ +\infty\right) \subseteq \mathcal{C}, \tag{22}$$

and

$$\left(0, \frac{\ell}{\eta-k}\right) \subseteq \mathcal{U}. \tag{23}$$

PROOF. By the property (d) in Proposition 6.1, we have

$$V_x(x) \leqslant \kappa^{p-1}x^{p-1}, \quad x > 0,$$

and hence,

$$V_x(x)^{\frac{1}{p-1}} \geqslant \kappa x > kx + \ell, \quad \text{if } x \in \left(\frac{\ell}{\kappa - k},\ +\infty\right),$$

thus (22) follows.

Similarly, we have

$$V_x(x) \geqslant \frac{k}{\kappa(1-p)+kp}\kappa^{p-1}x^{p-1} = (\eta x)^{p-1}, \quad x > 0,$$

and hence,

$$V_x(x)^{\frac{1}{p-1}} \leqslant \eta x < kx + \ell, \quad \text{if } x \in \left(0,\ \frac{\ell}{\eta - k}\right),$$

thus (23) follows. □

Corollary 6.3 *If $\kappa > r > k$, then $x_e \in \mathcal{C}$. If $r > \eta$, then $x_e \in \mathcal{U}$.*

PROOF. If $\kappa > r > k$, then

$$x_e = \frac{\ell}{r-k} > \frac{\ell}{\kappa - k}.$$

Similarly, notting $\eta > \kappa > k$, if $r > \eta$, then $r > k$, and

$$x_e = \frac{\ell}{r-k} < \frac{\ell}{\eta - k}.$$

The claim follows from the above result. □

Now we are ready to provide the detailed descriptions of the regions $\mathcal{U}$ and $\mathcal{C}$.

Theorem 6.4 *If $\kappa \geqslant k + r$, then there exists a constant*

$$x^* \in \left[\frac{\ell}{\eta - k},\ \frac{\ell}{\kappa - k}\right] \tag{24}$$

such that

$$\mathcal{U} = (0, x^*), \tag{25}$$

and

$$\mathcal{C} = [x^*, +\infty). \tag{26}$$

Proof. In order to prove the claim, we first derive the formula of the solution in the unconstrained region $\mathcal{U}$, although the problem does not admit a closed form solution on $(0,+\infty)$.

Let $\mathrm{Z}(\cdot):(c_1,c_2)\mapsto\mathcal{U}$ be uniquely determined by

$$V_x(\mathrm{Z}(c))=c^{p-1},\quad c_1<c<c_2. \tag{27}$$

By Corollary 6.3, $x_e\notin\mathcal{U}$, so $V_x(\cdot)$ is continuous and strictly decreasing on $\mathcal{U}$. Thus $\mathrm{Z}(\cdot)$ is well-defined and strictly increasing. It follows that

$$V_{xx}(\mathrm{Z}(c))\,\mathrm{Z}'(c)=(p-1)c^{p-2},\quad c_1<c<c_2. \tag{28}$$

Applying (27) and (28), equation (20) becomes

$$\beta V(\mathrm{Z}(c))-\theta c^p\,\mathrm{Z}'(c)-rc^{p-1}\,\mathrm{Z}(c)+\frac{p-1}{p}c^p=0,\quad c_1<c<c_2.$$

Differentiate the above equation with respect to c,

$$\begin{aligned}\beta V_x(\mathrm{Z}(c))\,\mathrm{Z}'(c)-\theta(c^p\,\mathrm{Z}''(c)+pc^{p-1}\,\mathrm{Z}'(c))-r(c^{p-1}\,\mathrm{Z}'(c)&\\ +(p-1)c^{p-2}\,\mathrm{Z}(c))+(p-1)c^{p-1}&=0,\end{aligned}$$

and apply (27) again and eliminating c^{p-2},

$$\beta c\,\mathrm{Z}'(c)-\theta(c^2\,\mathrm{Z}''(c)+pc\,\mathrm{Z}'(c))-r(c\,\mathrm{Z}'(c)+(p-1)\,\mathrm{Z}(c))+(p-1)c=0.$$

Now we obtain an ordinary differential equation (ODE) for $\mathrm{Z}(\cdot)$:

$$\mathcal{L}\,\mathrm{Z}=0,\quad c_1<c<c_2, \tag{29}$$

where

$$\mathcal{L}\,\mathrm{Z}:=-\theta c^2\,\mathrm{Z}''(c)+(\beta-\theta p-r)c\,\mathrm{Z}'(c)+r(1-p)\,\mathrm{Z}(c)-(1-p)c.$$

Now we are ready to prove (25). Per absurdum, suppose the unconstrained region $\mathcal{U}$ contains more than two separate intervals. Then by (22) and (23), there exist x_1 and x_2 such that

$$\frac{\ell}{\eta-k}\leqslant x_1<x_2\leqslant\frac{\ell}{\kappa-k},\quad (x_1,x_2)\subseteq\mathcal{U},\quad V_x(x_1)^{\frac{1}{p-1}}=kx_1+\ell,\quad V_x(x_2)^{\frac{1}{p-1}}=kx_2+\ell,$$

where the last two identities are from the continuity of $V_x(\cdot)$ and Corollary 6.3. Let $c_1=\mathrm{Z}^{-1}(x_1)$ and $c_2=\mathrm{Z}^{-1}(x_2)$. Then recalling (27),

$$\mathrm{Z}(c_1)=x_1=\frac{V_x(x_1)^{\frac{1}{p-1}}-\ell}{k}=\frac{V_x(\mathrm{Z}(c_1))^{\frac{1}{p-1}}-\ell}{k}=\frac{c_1-\ell}{k}>0, \tag{30}$$

$$\mathrm{Z}(c_2)=x_2=\frac{V_x(x_2)^{\frac{1}{p-1}}-\ell}{k}=\frac{V_x(\mathrm{Z}(c_2))^{\frac{1}{p-1}}-\ell}{k}=\frac{c_2-\ell}{k}>0. \tag{31}$$

Thus

$$c > c_1 > \ell, \quad \text{if } c_1 < c < c_2.$$

We now confirm $c \mapsto \frac{c-\ell}{k}$ is a supersolution of the ODE (29) with boundary conditions (30) and (31). In fact, $c \mapsto \frac{c-\ell}{k}$ satisfies boundary conditions (30) and (31), thus we only need to confirm $\mathcal{L}\left(\frac{c-\ell}{k}\right) \geqslant 0$. Note

$$\begin{aligned}\mathcal{L}\left(\frac{c-\ell}{k}\right) &= (\beta-\theta p-r)\frac{c}{k} + r(1-p)\frac{c-\ell}{k} - (1-p)c \\ &= \left(\frac{\beta-\theta p-r+r(1-p)}{1-p} - k\right)(1-p)\frac{c}{k} - r(1-p)\frac{\ell}{k} \\ &= \left(\frac{\kappa(1-p)}{1-p} - k\right)(1-p)\frac{c}{k} - r(1-p)\frac{\ell}{k} \\ &= (\kappa-k)(1-p)\frac{c}{k} - r(1-p)\frac{\ell}{k} > (\kappa-k)(1-p)\frac{\ell}{k} - r(1-p)\frac{\ell}{k} \geqslant 0,\end{aligned}$$

where we used the assumption $\kappa \geqslant k + r$ in the last inequality. Thus we proved $\frac{c-\ell}{k}$ is a supersolution of the ODE (29) with boundary conditions (30) and (31). Therefore,

$$\mathrm{Z}(c) \leqslant \frac{c-\ell}{k}, \quad c_1 \leqslant c \leqslant c_2,$$

and consequently, $V_x(\mathrm{Z}(c))^{\frac{1}{p-1}} = c \geqslant k\,\mathrm{Z}(c) + \ell$ that contradicts $\mathrm{Z}(c) \in \mathcal{U}$, $c_1 < c < c_2$. Thus we proved (25). By the definitions of $\mathcal{U}$ and $\mathcal{C}$, (26) follows immediately.

The claim (24) follows from (22) and (23). □

7 The Value Function: Continuity of the Second Order Derivative and the Optimal Strategy

Theorem 7.1 *Suppose $\kappa \geqslant k + r$. If $k \geqslant r$, then $V_{xx}(\cdot) \in C(0,+\infty)$. If $k < r$, then $V_{xx}(\cdot) \in C\big((0,+\infty)\backslash\{x_e\}\big)$, where x_e defined in (15) is the unique possibility of exception point.*

Our main idea to prove the above result is to consider the dual function of the value function $V(\cdot)$. Make dual transformation

$$v(y) := \max_{x>0}(V(x) - xy), \quad y > 0. \tag{32}$$

Then $v(\cdot)$ is a finite decreasing convex function on $(0,+\infty)$. Since $V_x(\cdot)$ is strictly decreasing, we denote the inverse function of $V_x(x) = y$ by

$$I(y) = x. \tag{33}$$

By the property (d) in Proposition 6.1, $I(\cdot)$ is decreasing and mapping $(0,+\infty)$ to itself. From (32),

$$v(y) = [V(x) - xV_x(x)]\Big|_{x=I(y)} = V(I(y)) - yI(y). \tag{34}$$

By differentiating the above equation with respect to y, it follows that

$$v_y(y) = V_x(I(y))I'(y) - yI'(y) - I(y) = -I(y), \tag{35}$$

$$v_{yy}(y) = -I'(y) = -\frac{1}{V_{xx}(I(y))}. \tag{36}$$

Insert (35) into (34),

$$V(I(y)) = v(y) - yv_y(y).$$

Make the transformation (33), apply (34), (35), (36), and $V_x(x) = y$, then the HJB equation (10) becomes

$$\beta(v(y) - yv_y(y)) - \frac{\mu^2}{2\sigma^2}y^2 v_{yy}(y) + yd(y) + ryv_y(y) - \frac{1}{p}d^p(y) = 0, \quad y > 0, \tag{37}$$

where

$$d(y) := \min\left\{y^{\frac{1}{p-1}},\ \ell - kv_y(y)\right\}.$$

The equation (37) is a quasilinear ODE, which degenerates at $y = 0$. It follows that

$$v \in C^2(0,+\infty) \cap C^\infty((0,+\infty)\backslash\{y^*\}),$$

where $y^* = V_x(x^*)$ and x^* is defined in Theorem 6.4.

Remark 2 *We remark that the problem (6) can be solved in the scenario $\kappa > k = 0$ and $\ell > 0$. In fact, in this scenario,*

$$d(y) = \min\left\{y^{\frac{1}{p-1}},\ \ell\right\} = \begin{cases} \ell, & \text{if } y \leqslant \ell^{p-1}; \\ y^{\frac{1}{p-1}}, & \text{if } y > \ell^{p-1}; \end{cases}$$

and (37) reduces to

$$\begin{cases} \beta(v(y) - yv_y(y)) - \frac{\mu^2}{2\sigma^2}y^2 v_{yy}(y) + \ell y + ryv_y(y) - \frac{1}{p}\ell^p = 0 & \text{if } y \leqslant \ell^{p-1}; \\ \beta(v(y) - yv_y(y)) - \frac{\mu^2}{2\sigma^2}y^2 v_{yy}(y) + y^{\frac{p}{p-1}} + ryv_y(y) - \frac{1}{p}y^{\frac{p}{p-1}} = 0, & \text{if } y > \ell^{p-1}. \end{cases}$$

This ODE can be solved explicitly via applying the smoothing fit condition. So we will not only have an explicit optimal consumption-investment strategy in a feedback form, but also have an explicit expression of the optimal value for the problem (6). We leave the details to the interested readers.

Theorem 7.1 will follow from the following two propositions.

Proposition 7.2 *Suppose $\kappa \geqslant k+r$. Let x^* be defined as in Theorem 6.4. If $k \geqslant r$, then $V_{xx}(\cdot) \in C[x^* + \infty)$. If $k < r$, then $V_{xx}(\cdot) \in C\big([x^* + \infty)\backslash\{x_e\}\big)$, where x_e defined in (15) is the unique possibility of the exception point.*

Proof. By (36), to prove $V_{xx}(\cdot) \in C\big([x^* + \infty)\backslash\{x_e\}\big)$ is equivalent to prove $v_{yy}(y) > 0$ for all $y \in (0, y^*]\backslash\{y_e\}$, where $y_e = V_x(x_e-)$.

Suppose there exists a point $0 < y_0 < y^*$ such that $v_{yy}(y_0) = 0$, which is the minimum value of $v_{yy}(\cdot)$ by the convexity of $v(\cdot)$. It follows that $v_{yyy}(y_0) = 0$. Differentiating (37) with respect to y yields

$$\beta(-yv_{yy}(y)) - \frac{\mu^2}{2\sigma^2}(2yv_{yy}(y) + y^2 v_{yyy}(y)) + \ell - kv_y(y) - kyv_{yy}(y) \\ + r(v_y(y) + yv_{yy}(y)) + k(\ell - kv_y(y))^{\frac{1}{p-1}} v_{yy}(y) = 0, \quad 0 < y < y^*. \quad (38)$$

Applying $v_{yy}(y_0) = 0$ and $v_{yyy}(y_0) = 0$, we get $(k - r)v_y(y_0) = \ell$ which is equivalent to $(r - k)x_0 = \ell$ where $x_0 = I(y_0)$. Hence $x_0 = x_e$ is the unique possibility of exception point which can only happen in the scenario $r > k$.

It remains to show $v_{yy}(y^*) > 0$. If $v_{yy}(y^*) = 0$ which is the minimum value of $v_{yy}(\cdot)$. It follows that $v_{yyy}(y^*-) \leqslant 0$. By (38), it follows $(k - r)v_y(y^*) \geqslant \ell$ which is impossible if $k \geqslant r$ because $v(\cdot)$ is decreasing. If $r > k$, then $(r - k)x^* \geqslant \ell$, $x^* \geqslant x_e$ which is also impossible because x_e is in the interior of $\mathcal{C}$ by the proof of Corollary 6.3. □

Proposition 7.3 *Suppose $\kappa \geqslant k+r$. Let x^* be defined as in Theorem 6.4. Then $V_{xx}(\cdot) \in C(0, x^*]$.*

Proof. It is proved that $v_{yy}(y^*) > 0$ in the proof of Proposition 7.2. Suppose there exists a point $y_0 > y^*$ such that $v_{yy}(y_0) = 0$, which is the minimum value of $v_{yy}(\cdot)$ by the convexity of $v(\cdot)$. It follows that $v_{yyy}(y_0) = 0$. Differentiating (37) with respect to y yields

$$\beta(-yv_{yy}(y)) - \frac{\mu^2}{2\sigma^2}(2yv_{yy}(y) + y^2 v_{yyy}(y)) + r(v_y(y) + yv_{yy}(y)) + y^{\frac{1}{p-1}} = 0, \quad y > y^*.$$

Applying $v_{yy}(y_0) = 0$ and $v_{yyy}(y_0) = 0$, we get $rv_y(y_0) = -y_0^{\frac{1}{p-1}}$ which is equivalent to $V_x(x_0)^{\frac{1}{p-1}} = rx_0$ where $x_0 = I(y_0)$. However, by the property (d) in Proposition 6.1, we have $V_x(x_0)^{\frac{1}{p-1}} \geqslant \kappa x_0 > rx_0$. The proof is complete. □

Before proving the global continuity of the first order derivative of the value function, we recall a result in convex analysis. For the completeness of this paper, a proof is given in the Appendix.

Lemma 7.4 *Let $h(\cdot)$ be a finite concave function on $(0, +\infty)$. Define its convex dual*

$$\widehat{h}(y) := \max_{x>0}(h(x) - xy), \quad y > 0.$$

Let $y_0 = \inf\{y > 0 : \widehat{h}(y) < +\infty\}$. Then the following two statements are equivalent:

1. $\widehat{h}(\cdot)$ *is strictly convex on* $(y_0, +\infty)$.

2. $h(\cdot)$ *is continuous differentiable on* $(0, +\infty)$.

Corollary 7.5 *Suppose* $\kappa \geqslant k + r$. *The value function* $V(\cdot)$ *of the problem* (6) *is in* $C[0, +\infty) \cap C^1(0, +\infty)$.

PROOF. It is proved that $v_{yy}(y) > 0$ if $y \neq V_x(x_e-)$ in the proofs of Proportions 7.2 and 7.3. This implies that $v(\cdot)$ is a strictly convex function on $(0, +\infty)$. Consequently, $V(\cdot)$ is continuous differentiable on $(0, +\infty)$ by Lemma 7.4. □

To give an explicit optimal consumption-investment strategy for the problem (6), we derive the formula of $\mathrm{Z}(\cdot)$ in the unconstrained region $\mathcal{U}$, although we cannot obtain a closed form solution on $(0, +\infty)$, but it is adequate for our purpose.

Proposition 7.6 *Suppose* $\kappa \geqslant k + r$. *Let* $\mathrm{Z}(\cdot)$ *be defined as* (27) *and* x^* *be defined as in Theorem* 6.4, *then*

$$\mathrm{Z}(c) = \frac{1}{\kappa}c - \frac{1}{\kappa}((k-\kappa)x^* + \ell)\left(\frac{c}{kx^* + \ell}\right)^{\lambda}, \quad 0 < c \leqslant kx^* + \ell, \tag{39}$$

where λ *is the unique positive root of the polynomial* $\theta\lambda(\lambda-1) + (r - \beta + p\theta)\lambda + r(p-1)$.

PROOF. Let $c^* = \mathrm{Z}^{-1}(x^*)$. Because $V_x(\cdot)$ is in $C^1(0, +\infty)$ and (27),

$$c^* = V_x(\mathrm{Z}(c^*))^{\frac{1}{p-1}} = V_x(x^*)^{\frac{1}{p-1}} = kx^* + \ell. \tag{40}$$

Consider the equation (29). The general solution of its corresponding homogeneous equation is $Bc^{\lambda} + \overline{B}c^{\overline{\lambda}}$, where B and $\overline{B}$ are constants, $\overline{\lambda}$ is the unique negative root of $f(\lambda) = \theta\lambda(\lambda-1) + (r - \beta + p\theta)\lambda + r(p-1)$. Observer $f(1) = -\beta + p(\theta + r) < 0$ and $f(+\infty) = +\infty$. It follows that $\lambda > 1$ and $\overline{\lambda} < 0$. Note a particular solution to the inhomogeneous equation (29) is $\frac{1}{\kappa}c$. Thus the general solution to the equation (29) is given by

$$\mathrm{Z}(c) = \frac{1}{\kappa}c - Bc^{\lambda} - \overline{B}c^{\overline{\lambda}}, \quad 0 < c \leqslant c^*.$$

Because $\mathrm{Z}(0+) = 0$ and $\overline{\lambda} < 0$, we conclude that $\overline{B} = 0$, and hence

$$\mathrm{Z}(c) = \frac{1}{\kappa}c - Bc^{\lambda}, \quad 0 < c \leqslant c^*.$$

Using $\mathrm{Z}(c^*) = x^*$ and by (40), we obtain $B = \frac{1}{\kappa}((k-\kappa)x^* + \ell)(kx^* + \ell)^{-\lambda}$; and consequently,

$$\mathrm{Z}(c) = \frac{1}{\kappa}c - \frac{1}{\kappa}((k-\kappa)x^* + \ell)\left(\frac{c}{kx^* + \ell}\right)^{\lambda}, \quad 0 < c \leqslant c^* = kx^* + \ell.$$

The proof is complete. □

The main result of the paper is stated as follows.

Theorem 7.7 *Suppose $\kappa \geqslant k + r$. Let x^* be defined as in Theorem 6.4. The optimal consumption-investment strategy $(c^*(\cdot), \pi^*(\cdot))$ for the problem (6) is given by a closed feedback form of wealth:*

$$(c_t^*, \pi_t^*) = (c^*(X_t), \pi^*(X_t)), \quad t \geqslant 0,$$

where

$$c^*(x) = \begin{cases} \mathrm{Z}^{-1}(x), & 0 < x < x^*; \\ kx + \ell, & x \geqslant x^*, \end{cases} \quad \text{and} \quad \pi^*(x) = -\frac{\mu V_x(x)}{\sigma^2 V_{xx}(x)}, \quad x > 0,$$

and $\mathrm{Z}^{-1}(\cdot)$ is the inverse function of $\mathrm{Z}(\cdot)$ defined in (39).

Proof. It is evident that

$$c^*(x) = \begin{cases} V_x(x)^{\frac{1}{p-1}}, & 0 < x < x^*; \\ kx + \ell, & x \geqslant x^*. \end{cases}$$

We only need to show $V_x(x)^{\frac{1}{p-1}} = \mathrm{Z}^{-1}(x)$ which follows from (27). By the preceding analysis, $(c^*(\cdot), \pi^*(\cdot))$ is indeed an optimal pair for the HJB equation (10), so the value function $V(x)$ is achieved at $(c^*(\cdot), \pi^*(\cdot))$. A standard verification theorem argument can show that the consumption-investment strategy $(c^*(X_t), \pi^*(X_t))$ defined above is optimal for the problem (6). We leave details to the interested readers. □

8 Concluding Remarks

The problem is still open in the scenario $k < \kappa < k + r$. Our proof of Theorem 6.4 needs the condition $\kappa \geqslant k + r$ so as to apply the comparison principle. To handle the scenario $k < \kappa < k + r$, it may require other techniques. We will continuous work on this scenario and hope to fill the gap in the near future.

Appendix

Proof of Lemma 7.4. "1. $\Longrightarrow$ 2.": Per absurdum, suppose $h(\cdot)$ is not differentiable at some $x_0 > 0$, then

$$h(x) - h(x_0) \leqslant y(x - x_0), \quad \forall\, x > 0,$$

for all $y \in [h_x(x_0+), h_x(x_0-)]$. Then it follows

$$h(x) - yx \leqslant h(x_0) - yx_0, \quad \forall\, x > 0,$$

and hence $\widehat{h}(y) = h(x_0) - yx_0$ for all $y \in [h_x(x_0+), h_x(x_0-)]$. This contradicts that $\widehat{h}(\cdot)$ is strictly convex. Therefore, $h(\cdot)$ is differentiable. Because $h_x(\cdot)$ is increasing, by Darboux's Theorem, $h_x(\cdot)$ is also continuous.

$''2. \Longrightarrow 1.''$: Because $h(\cdot)$ is continuous differentiable on $(0, +\infty)$,

$$\widehat{h}(h'(x)) = h(x) - h'(x)x, \quad x > 0.$$

For any $b > a > y_0$, we need to show $2\widehat{h}((a+b)/2) < \widehat{h}(a) + \widehat{h}(b)$. Note that there exists $0 < x_1 < x_2$ such that $b = h'(x_1) > h'(x_2) = a$, and $y \in (x_1, x_2)$ such that $h'(y) = \frac{1}{2}(h'(x_1) + h'(x_2)) = \frac{1}{2}(a+b)$ by the continuity of $h'(\cdot)$. It is sufficient to show

$$2\widehat{h}(h'(y)) < \widehat{h}(h'(x_1)) + \widehat{h}(h'(x_2)),$$

i.e.,

$$2h(y) - 2h'(y)y < h(x_1) - h'(x_1)x_1 + h(x_2) - h'(x_2)x_2. \tag{41}$$

Because $h(\cdot)$ is concave,

$$\begin{aligned} h(y) - h(x_1) &\leqslant (y - x_1)h'(x_1), \\ h(y) - h(x_2) &\leqslant (y - x_2)h'(x_2). \end{aligned}$$

If both of them are identities, then $h(\cdot)$ is linear on $[x_1, x_2]$, and $h'(x_1) = h'(x_2)$, a contradiction. So at least one of them is strict, and consequently,

$$\begin{aligned} 2h(y) - h(x_2) - h(x_1) &< (y - x_1)h'(x_1) + (y - x_2)h'(x_2) \\ &= 2h'(y)y - h'(x_1)x_1 - h'(x_2)x_2, \end{aligned}$$

which is equivalent to the desired inequality (41). □

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