

A New Characterization of Comonotonicity and its Application in Behavioral Finance

Zuo Quan Xu*

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Abstract

It is well-known that an \mathbb{R}^n -valued random vector (X_1, X_2, \dots, X_n) is comonotonic if and only if (X_1, X_2, \dots, X_n) and $(Q_1(U), Q_2(U), \dots, Q_n(U))$ coincide *in distribution*, for *any* random variable U uniformly distributed on the unit interval $(0, 1)$, where $Q_k(\cdot)$ are the quantile functions of X_k , $k = 1, 2, \dots, n$. It is natural to ask whether (X_1, X_2, \dots, X_n) and $(Q_1(U), Q_2(U), \dots, Q_n(U))$ can coincide *almost surely* for *some* special U . In this paper, we give a positive answer to this question by construction. We then apply this result to a general behavioral investment model with a law-invariant preference measure and develop a universal framework to link the problem to its quantile formulation. We show that any optimal investment output should be anti-comonotonic with the market pricing kernel. Unlike previous studies, our approach avoids making the assumption that the pricing kernel is atomless, and consequently, we overcome one of the major difficulties encountered when one considers behavioral economic equilibrium models in which the pricing kernel is a yet-to-be-determined unknown random variable. The method is applicable to many other models such as risk sharing model.

Keywords: Comonotonicity, atomic, atomless/non-atomic, behavioral finance, quantile formulation, pricing kernel, cumulative prospect theory, rank-dependent utility theory, economic equilibrium model

1 Introduction

The concept of comonotonicity has wide applications in actuarial science and financial risk management, see e.g., Deelstra, Dhaene, and Vanmaele [4], Dhaene, Denuit, Goovaerts, Kaas, and Vyncke [5, 6], Di Nunno and Øksendal [7]. It mainly refers to the perfect positive dependence between the components of a random vector, essentially saying that they can be represented as increasing functions of a single random variable. In this paper, we will show that these functions can be specified as their individual's quantile functions

*Email: maxu@polyu.edu.hk. Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong.

and this random variable specified as a random variable uniformly distributed on the unit interval $(0, 1)$ that is comonotonic with the components of the random vector.

This paper consists of two parts. The first part is dedicated to studying a new characterization of comonotonic random vectors. In the second part, we apply this characterization to a general behavioral investment problem with a law-invariant preference measure and develop a universal framework to link the problem to its quantile formulation, which overcomes one of the major difficulties encountered when one considers a behavioral economic equilibrium model with a law-invariant preference measure, where the classical dynamic programming and probabilistic approaches do not work.

We start by introducing the concept of comonotonicity for a random vector. We first recall the definition and the characterizations of a comonotonic random vector. For the rest part of this paper, all the random variables/vectors are in the same given probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 1 *An \mathbb{R}^n -valued random vector (X_1, X_2, \dots, X_n) is comonotonic if there exists a set $\widehat{\Omega} \times \widehat{\Omega} \subseteq \Omega \times \Omega$ with full measure, such that*

$$(X_i(\omega') - X_i(\omega))(X_j(\omega') - X_j(\omega)) \geq 0, \text{ for all } (\omega', \omega) \in \widehat{\Omega} \times \widehat{\Omega}, i, j \in \{1, 2, \dots, n\}.$$

Denote by \mathbb{U} the set of all random variables uniformly distributed on the unit interval $(0, 1)$ in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In order to get a new characterization of comonotonic random vector, it is necessary to assume, throughout this paper, that

Assumption 1 *The set \mathbb{U} is not empty.*

Define the quantile function $Q_X(\cdot)$ of an \mathbb{R} -valued random variable X as the right-continuous inverse function of its cumulative distribution function (cdf) $F_X(\cdot)$, that is,

$$Q_X(x) = \sup\{t \in \mathbb{R} : F_X(t) \leq x\}, \quad x \in (0, 1),$$

with convention $\sup \emptyset = -\infty$.

The following well-known result characterizes the comonotonic random vectors (see e.g., Dhaene, Denuit, Goovaerts, Kaas, and Vyncke, [5, 6]).

Theorem 1 *An \mathbb{R}^n -valued random vector $\mathcal{X} = (X_1, X_2, \dots, X_n)$ is comonotonic if and only if one of the following conditions holds:*

1. *For any vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we have*

$$F_{\mathcal{X}}(\mathbf{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\},$$

where $F_{\mathcal{X}}(\cdot)$ and $F_{X_k}(\cdot)$, $k = 1, 2, \dots, n$, are cdfs of \mathcal{X} and X_k , $k = 1, 2, \dots, n$, respectively;

2. *For any $U \in \mathbb{U}$, we have*

$$\mathcal{X} \stackrel{d}{=} (Q_1(U), Q_2(U), \dots, Q_n(U)),^1$$

where $Q_k(\cdot)$ are the quantile functions of X_k , $k = 1, 2, \dots, n$, respectively;

¹We write $X \stackrel{d}{=} Y$ if random variables/vectors X and Y are identically distributed.

3. There exist a random variable Y and non-decreasing functions $f_k(\cdot)$, $k = 1, 2, \dots, n$, such that

$$\mathcal{X} \stackrel{d}{=} (f_1(Y), f_2(Y), \dots, f_n(Y));$$

4. Let $Y = X_1 + X_2 + \dots + X_n$. There exist non-decreasing functions $f_k(\cdot)$, $k = 1, 2, \dots, n$, such that

$$\mathcal{X} = (f_1(Y), f_2(Y), \dots, f_n(Y)).^2$$

In particular, two \mathbb{R} -valued random variables X_1 and X_2 are comonotonic if there exists a non-decreasing function $f(\cdot)$ such that $X_1 = f(X_2)$ or $X_2 = f(X_1)$. We will use this fact frequently in the following analysis.

The last characterization in Theorem 1 gives a way to express the components of a comonotonic random vector by a single random variable. However, this expression is not easy to implement in practice because neither the summation of the components nor the functions $f_k(\cdot)$, $k = 1, 2, \dots, n$, are easy to estimate. In this paper, we give an easy-to-implement characterization.

Our starting point is the second characterization in Theorem 1. This characterization says that \mathcal{X} and $(Q_1(U), Q_2(U), \dots, Q_n(U))$ coincide *in distribution* for any $U \in \mathbb{U}$. Naturally one may ask whether these two random vectors can coincide *almost surely*. Of course, they may not coincide for *every* $U \in \mathbb{U}$, but may for *some* of them. The main result in the first part of this paper is to confirm this fact by constructing such a random variable. We first construct it for \mathbb{R} -valued random variable and then apply this and the last characterization in Theorem 1 to \mathbb{R}^n -valued random vector. This new characterization has the advantage over the last characterization in Theorem 1 for that the functions $Q_k(\cdot)$, $k = 1, 2, \dots, n$, are independent of each other and easy to estimate.

In the second part of this paper, we consider an application of the new characterization in a general continuous-time investment problem with a law-invariant preference measure in a complete financial market setting. Continuous-time investment problems in complete market are investigated in the literature under various economic theories such as expected utility theory (EUT), rank-dependent utility theory (RDUT) (see e.g., Xia and Zhou [16]), cumulative prospect theory (CPT) (see e.g., He and Zhou [8], Jin, Zhang, and Zhou [9], Jin and Zhou [10], Xu [17], Xu and Zhou [18]). These investment problems are typically treated as follows. They first boil down to some solvable static optimization problems over all possible final outputs. Then, one simply replicates the optimal outputs by certain hedging strategies as the markets are assumed to be complete. One key idea for reformulating these problems into the static optimization problems in the first step is to show that the optimal candidates must be anti-comonotonic³ with the market pricing kernel⁴. In the literature, this is done under the assumption that the pricing kernel is

²For any two random variables/vectors X and Y , we write $X = Y$ if $\mathbf{P}(X = Y) = 1$.

³An \mathbb{R}^2 -valued random vector (X, Y) is called anti-comonotonic if $(X, -Y)$ is comonotonic.

⁴The pricing kernel is sometimes referred to as the stochastic discount factor or the state-price density.

atomless^{5,6} One may regard this assumption as a technical matter when considering investment problems in which the pricing kernel is set a priori. However, if one studies a general economic equilibrium model, then the pricing kernel is a part of the solution, so one indeed cannot make any a priori assumption about its distribution.⁷ In this paper, with the help of a new characterization of comonotonicity, we prove that the optimal candidates must be anti-comonotonic with the pricing kernel without making any assumption about the pricing kernel's distribution. This overcomes one of the major difficulties encountered when one considers general economic equilibrium models with law-invariant preference measures. The method is also applicable to many other models such as risk sharing model.

The remainder of this paper is organized as follows. We introduce a new characterization of the comonotonic random vector in Section 2. An application of the new characterization in a general financial investment problem is considered in Section 3. We finally conclude the paper in Section 4. A technical proof is placed in an Appendix.

2 A New Characterization of Comonotonicity

In this section, we first express any \mathbb{R} -valued random variable by its quantile function and a random variable in \mathbb{U} and then use this result to give a new characterization for \mathbb{R}^n -valued comonotonic random vector.

Let us start with two simple facts.

Lemma 2 *If an \mathbb{R} -valued random variable X is atomless, then*

$$F_X(X) \in \mathbb{U}.$$

PROOF. For any $0 < t < 1$, define

$$x_t = \sup\{x : F_X(x) \leq t\}.$$

Because X is atomless, its cdf $F_X(\cdot)$ is continuous. Hence $F_X(x_t) = t$, and consequently,

$$\mathbf{P}(F_X(X) \leq t) = \mathbf{P}(X \leq x_t) = F_X(x_t) = t.$$

Therefore, $F_X(X) \in \mathbb{U}$. □

By this lemma, Assumption 1 is equivalent to there existing at least one atomless \mathbb{R} -valued random variable in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

⁵A random variable is called atomless or non-atomic if its cdf is continuous, and called atomic otherwise. A random variable is atomless if and only if its quantile function is strictly increasing.

⁶In financial risk sharing literature, it is almost standard to assume that the market is atomless, see e.g., Ludkovski and Rüschendorf (2008), Carlier and Dana (2013); the real market is, however, the other way around.

⁷For instance, Xia and Zhou (2012) consider an equilibrium model under RDUT under the assumption that the pricing kernel is atomless.

Lemma 3 *For any \mathbb{R} -valued random variable X , we have*

$$X \stackrel{d}{=} Q(U)$$

for any $U \in \mathbb{U}$, where $Q(\cdot)$ is the quantile function of X .

PROOF. Recall the definition of quantile function,

$$Q(x) = \sup\{t \in \mathbb{R} : F_X(t) \leq x\}, \quad x \in (0, 1),$$

where $F_X(\cdot)$ is the cdf of X . Hence, for any $t \in \mathbb{R}$, we have

$$\{U < F_X(t)\} \subseteq \{Q(U) \leq t\} \subseteq \{U \leq F_X(t)\},$$

and consequently,

$$F_X(t) = \mathbf{P}(U < F_X(t)) \leq \mathbf{P}(Q(U) \leq t) \leq \mathbf{P}(U \leq F_X(t)) = F_X(t),$$

which follows

$$\mathbf{P}(Q(U) \leq t) = F_X(t).$$

The proof is complete. □

The next result gives a characterization of two identical random variables.

Lemma 4 *Two \mathbb{R} -valued random variables are equal if and only if they are comonotonic and identically distributed. In particular, $X = Q(U)$ if (X, U) is comonotonic, where $U \in \mathbb{U}$ and $Q(\cdot)$ is the quantile function of X .*

PROOF. The “only if” part of the lemma is evident, so let us examine the “if” direction. Suppose two \mathbb{R} -valued random variables X and Y are comonotonic and identically distributed. Then by the second characterization in Theorem 1, we have

$$(X, Y) \stackrel{d}{=} (Q(U), Q(U))$$

for any $U \in \mathbb{U}$, where $Q(\cdot)$ is their common quantile function. Therefore,

$$\mathbf{P}(X = Y) = \mathbf{P}(Q(U) = Q(U)) = 1.$$

This confirms the first part of the claim.

If (X, U) is comonotonic, then so is $(X, Q(U))$. By Lemma 3, we also have $X \stackrel{d}{=} Q(U)$. Hence we deduce that $X = Q(U)$ by the first part of the claim. □

By this result, expressing an \mathbb{R} -valued random variable by its quantile function and a random variable in \mathbb{U} boils down to finding a random variable in \mathbb{U} that is comonotonic with the given random variable. The following result states a way to construct such a random variable.

Theorem 5 *For any \mathbb{R} -valued random variable X , there exists a $U \in \mathbb{U}$ such that (X, U) is comonotonic, and such U is unique if X is atomless and not unique otherwise. Moreover, $X = Q(U)$ for any $U \in \mathbb{U}$ such that (X, U) is comonotonic, where $Q(\cdot)$ is the quantile function of X .*

PROOF. If X is atomless, then by Lemma 2, $U := F_X(X) \in \mathbb{U}$. As $(X, U) = (X, F_X(X))$ is comonotonic, by Lemma 4, we have $X = Q(U)$. As X is atomless, its quantile function $Q(\cdot)$ is strictly increasing. Therefore, $U = Q^{-1}(X)$ is unique.

Now suppose that X is not atomless. Denote all of its atoms by $\{x_i, i \in I\}$, where I is a non-empty subset of positive integers. Define its atomic sets

$$\mathcal{A}_i = \{\omega \in \Omega : X(\omega) = x_i\}, \quad i \in I,$$

and the complement atomless set

$$\overline{\mathcal{A}} = \Omega - \cup_{i \in I} \mathcal{A}_i.$$

Our key idea for construction is as follows. We will construct a random variable by adding a continuous random variable at every atom of X to make it to be atomless and comonotonic with X . However, such construction can only be done when X has a *jump* at every atom of it (see claim (i) below). As X may not have such property, we first construct a random variable Y that is comonotonic with X and having the desired property. This is equivalent to finding an increasing function having a jump at every atom of X .

Define an $\mathbb{R} \mapsto \mathbb{R}$ function as follows

$$f(x) = x + \sum_{i \in I} 2^{-i} \mathbb{1}_{(x_i, +\infty)}(x),$$

where $\mathbb{1}_{(x_i, +\infty)}$ is the indicator function of set $(x_i, +\infty)$. As $f(\cdot)$ is strictly increasing, so is its inverse function denoted by $f^{-1}(\cdot)$.

Define $Y = f(X)$ and $y_i = f(x_i)$, $i \in I$. As $f(\cdot)$ is strictly increasing, (X, Y) is comonotonic. We will show that the random variable Y always has a jump at every atom, while, the random variable X may not. For a random variable, having a jump at every atom is the key property we need for constructing its atomless comonotonic counterpart.

As will be shown below, Y always has a jump at every atom, so we can construct an atomless random variable that is comonotonic with Y , and consequently, with X as well. The random variable is constructed as follows:

$$Z(\omega) = \begin{cases} Y(\omega) + 2^{-i}U_i(\omega), & \text{if } \omega \in \mathcal{A}_i \text{ for some } i \in I; \\ Y(\omega), & \text{if } \omega \in \overline{\mathcal{A}}; \end{cases}$$

where $U_i \in \mathbb{U}$, $i \in I$, are arbitrarily chosen and may be the same. Because U_i , $i \in I$, can be arbitrarily chosen and arranged,⁸ one can define many such Z 's with an identical cdf.

Now we turn to study the properties of Y and Z . We claim that

⁸For instance, we can replace U_i by $1 - U_i$, $i \in I$.

- (i). The random variable Y has a jump at every atom, that is, if $Y(\omega) > y_n$ for some $\omega \in \Omega$ and $n \in I$, then $Y(\omega) > y_n + 2^{-n}$;
- (ii). The random vector (Y, Z) is comonotonic; and
- (iii). The random variable Z is atomless.

Let us prove these claims one by one.

Proof of claim (i). Suppose $Y(\omega) > y_n$ for some $\omega \in \Omega$ and $n \in I$. Then

$$X(\omega) = f^{-1}(Y(\omega)) > f^{-1}(y_n) = x_n.$$

There are only two possible cases.

- There exists some $m \in I$ such that $X(\omega) \geq x_m > x_n$. Then

$$Y(\omega) = f(X(\omega)) \geq f(x_m) = y_m,$$

and consequently,

$$\begin{aligned} Y(\omega) - y_n &\geq y_m - y_n = f(x_m) - f(x_n) \\ &= x_m - x_n + \sum_{i \in I} 2^{-i} (\mathbb{1}_{(x_i, +\infty)}(x_m) - \mathbb{1}_{(x_i, +\infty)}(x_n)) \\ &\geq x_m - x_n + 2^{-n} (\mathbb{1}_{(x_n, +\infty)}(x_m) - \mathbb{1}_{(x_n, +\infty)}(x_n)) \\ &= x_m - x_n + 2^{-n} > 2^{-n}. \end{aligned}$$

- There does not exist $m \in I$ such that $X(\omega) \geq x_m > x_n$. Then for each fixed $i \in I$, we have that $X(\omega) > x_i$ if and only if $x_n \geq x_i$. Therefore,

$$\mathbb{1}_{(x_i, +\infty)}(X(\omega)) = \mathbb{1}_{[x_i, +\infty)}(x_n), \quad i \in I,$$

and consequently,

$$\begin{aligned} Y(\omega) - y_n &= f(X(\omega)) - f(x_n) \\ &= X(\omega) - x_n + \sum_{i \in I} 2^{-i} (\mathbb{1}_{(x_i, +\infty)}(X(\omega)) - \mathbb{1}_{(x_i, +\infty)}(x_n)) \\ &= X(\omega) - x_n + \sum_{i \in I} 2^{-i} (\mathbb{1}_{[x_i, +\infty)}(x_n) - \mathbb{1}_{(x_i, +\infty)}(x_n)) \\ &= X(\omega) - x_n + \sum_{i \in I} 2^{-i} \mathbb{1}_{\{x_i\}}(x_n) = X(\omega) - x_n + 2^{-n} > 2^{-n}. \end{aligned}$$

The proof of claim (i) is complete.

Proof of claim (ii). Suppose $Y(\omega') > Y(\omega)$. It is sufficient to show that $Z(\omega') > Z(\omega)$.

There are only two possible cases.

- If $\omega \in \mathcal{A}_n$ for some $n \in I$. Then $Y(\omega') > Y(\omega) = y_n$. By claim (i), we have

$$Y(\omega') > y_n + 2^{-n} = Y(\omega) + 2^{-n},$$

and consequently,

$$Z(\omega') \geq Y(\omega') > Y(\omega) + 2^{-n} \geq Y(\omega) + 2^{-n}U_n(\omega) = Z(\omega).$$

- Otherwise, $\omega \in \overline{\mathcal{A}}$. Then

$$Z(\omega') \geq Y(\omega') > Y(\omega) = Z(\omega).$$

The proof of claim (ii) is complete.

Proof of claim (iii). For any $t \in \mathbb{R}$,

$$\begin{aligned} \mathbf{P}(Z(\omega) = t) &= \sum_{i \in I} \mathbf{P}(Z(\omega) = t, \omega \in \mathcal{A}_i) + \mathbf{P}(Z(\omega) = t, \omega \in \overline{\mathcal{A}}) \\ &= \sum_{i \in I} \mathbf{P}(Y(\omega) + 2^{-i}U_i(\omega) = t, \omega \in \mathcal{A}_i) + \mathbf{P}(Y(\omega) = t, \omega \in \overline{\mathcal{A}}) \\ &= \sum_{i \in I} \mathbf{P}(y_i + 2^{-i}U_i(\omega) = t, \omega \in \mathcal{A}_i) + \mathbf{P}(Y(\omega) = t, \omega \in \overline{\mathcal{A}}) \\ &\leq \sum_{i \in I} \mathbf{P}(y_i + 2^{-i}U_i(\omega) = t) + \mathbf{P}(Y(\omega) = t, \omega \in \overline{\mathcal{A}}) = 0, \end{aligned}$$

where we used the facts that $\mathbf{P}(y_i + 2^{-i}U_i(\omega) = t) = 0$ as U_i are atomless for all $i \in I$, and $\mathbf{P}(Y(\omega) = t, \omega \in \overline{\mathcal{A}}) = 0$ as Y has no atom on the set $\overline{\mathcal{A}}$. The proof of claim (iii) is complete.

As Z is atomless, by Lemma 2, $U := F_Z(Z) \in \mathbb{U}$. It is easy to check that (X, U) is comonotonic,⁹ so by Lemma 4, we have $X = Q(U)$. The non-uniqueness of U is due to the non-uniqueness of Z . The proof is complete. \square

Now, we can apply the above result and the last characterization in Theorem 1 to characterize any \mathbb{R}^n -valued random vector.

Theorem 6 *An \mathbb{R}^n -valued random vector (X_1, X_2, \dots, X_n) is comonotonic if and only if*

$$(X_1, X_2, \dots, X_n) = (Q_1(U), Q_2(U), \dots, Q_n(U))$$

for some $U \in \mathbb{U}$, where $Q_k(\cdot)$ are the quantile function of X_k , $k = 1, 2, \dots, n$. Moreover, this equality holds for any $U \in \mathbb{U}$ such that $(X_1 + X_2 + \dots + X_n, U)$ is comonotonic.

⁹In fact, we have $U(\omega') > U(\omega)$ if $X(\omega') > X(\omega)$.

PROOF. By Theorem 5, there exists a $U \in \mathbb{U}$ such that

$$(X_1 + X_2 + \cdots + X_n, U)$$

is comonotonic. By the last characterization in Theorem 1, there exist non-decreasing functions $f_k(\cdot)$ such that

$$X_k = f_k(X_1 + X_2 + \cdots + X_n), \quad k = 1, 2, \dots, n.$$

As $(X_1 + X_2 + \cdots + X_n, U)$ is comonotonic, so are

$$(f_k(X_1 + X_2 + \cdots + X_n), U) = (X_k, U), \quad k = 1, 2, \dots, n.$$

We conclude, by Lemma 4, that $X_k = Q_k(U)$, $k = 1, 2, \dots, n$. The proof is complete. \square

3 An Application in Behavioral Finance

There are abundant studies on single-period and continuous-time investment portfolio choice and optimal stopping problems in financial markets under different economic theories, such as behavioral finance.

Let us give some examples of these models that are studied in [8]:

- Merton's portfolio choice model under EUT:

$$\begin{aligned} & \max_X \quad \mathbf{E}[u(X)], \\ & \text{subject to} \quad \mathbf{E}[\rho X] = x, \quad X \geq 0. \end{aligned}$$

- The goal-reaching model, which was proposed by Kulldorff [12] and studied extensively (including various extensions) by Browne [1, 2]:

$$\begin{aligned} & \max_X \quad \mathbf{P}(X \geq b), \\ & \text{subject to} \quad \mathbf{E}[\rho X] = x, \quad X \geq 0. \end{aligned}$$

- Lopes' SP/A Theory [13]:

$$\begin{aligned} & \max_X \quad \int_0^\infty w(\mathbf{P}(X > t)) dt, \\ & \text{subject to} \quad \mathbf{E}[\rho X] = x, \quad X \geq 0, \quad \mathbf{P}(X \geq b) \geq \alpha. \end{aligned}$$

- Kahneman and Tversky's CPT Model [11, 15], which was studied by Jin and Zhou [10]:

$$\begin{aligned} & \max_X \quad \int_0^\infty w_+(\mathbf{P}(u_+(X - B)^+ > t)) dt - \int_0^\infty w_-(\mathbf{P}(u_-(X - B)^- > t)) dt, \\ & \text{subject to} \quad \mathbf{E}[\rho X] = x. \end{aligned}$$

These models except Merton's model have a common feature: They cannot be studied using only classical dynamic programming or martingale approaches. The main difficulty comes from the nonlinear expectations involved in the targets of these models. One key idea to solve these models is to show that the optimal output X must be anti-comonotonic with the market pricing kernel ρ . In the literature, this is done under the assumption that the pricing kernel is atomless. However, if one studies a general economic equilibrium model, then the pricing kernel is a part of the solution, so one cannot make any a priori assumption about its distribution. We will demonstrate how to use the new characterization of comonotonicity to overcome this major difficulty in a universal framework.

In fact, the aforementioned models appear quite different in mathematical formulations and economical interpretations; yet the commonalities among them lead to a universal framework. All these models can be formulated as

$$\max_{X \in \mathfrak{A}_x} J(X), \quad (1)$$

where

$$\mathfrak{A}_x = \{X : \mathbf{E}[\rho X] = x, Q_X(\cdot) \in \mathcal{Q} \cap \mathcal{C}\}, \quad (2)$$

$Q_X(\cdot)$ is the quantile function of the \mathbb{R} -valued decision random variable X , $\rho > 0$ is the pricing kernel in the portfolio choice model or $\rho \equiv 1$ in the optimal stopping model [18], \mathcal{Q} denotes the set of all quantile functions:

$$\mathcal{Q} = \{Q(\cdot) : (0, 1) \mapsto \mathbb{R} : \text{right-continuous and non-decreasing}\},$$

and the set \mathcal{C} specifies some other constraints on the quantile functions. For instance, the no-bankruptcy constraint $X \geq 0$ can be translated into

$$\mathcal{C} = \{Q(\cdot) : Q(0) \geq 0\},$$

and $\mathbf{P}(X \geq b) \geq \alpha$ into

$$\mathcal{C} = \{Q(\cdot) : Q(1 - \alpha) \geq b\}.$$

Without loss of generality, we assume that

Assumption 2 *The set \mathcal{C} is increasing.*

Here “increasing” means that $Q_2(\cdot) \in \mathcal{C}$ whenever $Q_2(\cdot) \geq Q_1(\cdot)$ for some $Q_1(\cdot) \in \mathcal{C}$. This says that if one output X is acceptable, so is any bigger one.

In all the aforementioned models, the objective functions depend only on cdf of the investment payoff X . Such objective functions are called law-invariant.

Assumption 3 *The objective function $J(\cdot)$ is law-invariant and increasing.*¹⁰

¹⁰Increasing here means $J(X + \varepsilon) > J(X)$ for any X and $\varepsilon > 0$.

This assumption is satisfied by all the aforementioned models and should be satisfied by any other investment models with law-invariant preference measures as people always prefer a higher output.

Our target is maximizing the preference (1) under the constraint (2), and in an intuitive sense this is equivalent to minimizing the initial resource while keeping the preference unchanged. As the preference (1) is law-invariant, we can minimize the initial resource while keeping the law of output. The following result is a consequence of this financial intuition.

Theorem 7 *If X^* is an optimal solution to problem (1), then X^* must be anti-comonotonic with the pricing kernel ρ .*

In the literature, this result is typically obtained under the assumption that ρ is atomless. Our result, which makes no assumption on ρ , is new to the best of our knowledge. Most of the time, the assumption that ρ is atomless may be regarded as a technical matter when one considers investment problems where ρ is set a priori. However, if one studies general economic equilibrium models with law-invariant preference measures such as RDUT or CPT, then the pricing kernel ρ is a part of the solution, so that one cannot make any a priori assumption about its distribution. With the help of the new characterization of comonotonicity Theorem 5, we now can prove Theorem 7 without making any assumptions about ρ . This makes it possible to consider general economic equilibrium models with law-invariant preference measures.

To prove Theorem 7, we need a stronger version of the Hardy-Littlewood inequality. Its proof is given in Appendix.

Lemma 8 (Hardy-Littlewood Inequality) *Let (X^*, Y) be comonotonic, and $X \stackrel{d}{=} X^*$. Then*

$$\mathbf{E}[XY] \leq \mathbf{E}[X^*Y],$$

whenever both sides are integrable. Moreover, the equality holds if and only if (X, Y) is comonotonic.

PROOF OF THEOREM 7. Suppose X^* is an optimal solution to problem (1). Denote by $Q^*(\cdot)$ the quantile function of X^* . By Theorem 5, there exists a $U \in \mathbb{U}$ such that (U, ρ) is comonotonic. Then $(Q^*(1 - U), \rho)$ is anti-comonotonic. Noting $1 - U \in \mathbb{U}$, by Lemma 3, we have

$$Q^*(1 - U) \stackrel{d}{=} X^*.$$

Suppose X^* is not anti-comonotonic with ρ . Then by the Hardy-Littlewood Inequality,

$$\mathbf{E}[\rho Q^*(1 - U)] < \mathbf{E}[\rho X^*] = x.$$

Set

$$\delta = \frac{x - \mathbf{E}[\rho Q^*(1 - U)]}{\mathbf{E}[\rho]} > 0,$$

$$\hat{X} = Q^*(1 - U) + \delta.$$

Then $\mathbf{E}[\rho\hat{X}] = x$. It is easy to check that $\hat{X} \in \mathfrak{A}_x$. However,

$$J(\hat{X}) = J(Q^*(1 - U) + \delta) > J(Q^*(1 - U)) = J(X^*),$$

which contradicts the optimality of X^* . \square

By this result, we only need to seek the optimal solution to problem (1) among those random variables in \mathfrak{A}_x that are anti-comonotonic with ρ . However, by Theorem 5, such random variables should be of the form $Q(1 - U)$ where $Q(\cdot)$ is a quantile function and $U \in \mathbb{U}$ is comonotonic with ρ . Therefore, finding an optimal random variable to solve problem (1) boils down to two subproblems: finding an optimal quantile function and finding a random variable that is comonotonic with ρ . As the latter is solved by the new characterization of comonotonicity Theorem 5, let us focus on the first one.

In fact, the quantile function that we are searching for is the solution to the following problem:

$$\max_{Q(\cdot) \in \mathcal{Q}_x} J(Q(1 - U)), \quad (3)$$

where

$$\mathcal{Q}_x = \left\{ Q(\cdot) : \int_0^1 Q_\rho(1 - t)Q(t) dt = x, Q(\cdot) \in \mathcal{Q} \cap \mathcal{C} \right\},$$

$U \in \mathbb{U}_\rho$, and

$$\mathbb{U}_\rho = \{U \in \mathbb{U} : (U, \rho) \text{ is comonotonic}\}.$$

According to Theorem 5, \mathbb{U}_ρ is not empty, so the problem is well-defined in the sense that it has a non-empty feasible set. As the cost function $J(\cdot)$ is law-invariant, the optimal solution to problem (3) does not depend on the choice of $U \in \mathbb{U}$. Because problem (3) is highly related to problem (1), it is called the quantile formulation of problem (1).

A probabilistic optimization problem (1) now completely reduces to a functional optimization problem (3). The latter has the advantage over the former for that it may be solved by the classical functional optimization theory such as convex analysis, while, the former may not.

Now, we present the main result in the application part of this paper that states the relationship between problem (1) and its quantile formulation (3).

Theorem 9 *A random variable X^* is an optimal solution to problem (1) if and only if*

$$X^* = Q^*(1 - U),$$

where $Q^(\cdot)$ is an optimal solution to problem (3) and $U \in \mathbb{U}_\rho$.*

PROOF. By Theorem 7, problem (1) is equivalent to problem

$$\max_{X \in \mathfrak{A}_x} J(X),$$

in the sense that they have the same optimal value and optimal solution, if it exists, where

$$\overline{\mathfrak{A}}_x = \{X : \mathbf{E}[\rho X] = x, \quad Q_X(\cdot) \in \mathcal{Q} \cap \mathcal{C}, \quad (X, \rho) \text{ is anti-comonotonic}\}.$$

First, we claim that

- (X, ρ) is anti-comonotonic if and only if $X = Q_X(1 - U)$ for some $U \in \mathbb{U}_\rho$.

Proof of claim. The “ if ” part of our claim is evident, so let us prove the “ only if ” part. As $(-X, \rho)$ is comonotonic, by Theorem 6, we have

$$(-X, \rho) = (Q_{-X}(U), Q_\rho(U))$$

for some $U \in \mathbb{U}$. Note that $(U, \rho) = (U, Q_\rho(U))$ is comonotonic, so $U \in \mathbb{U}_\rho$. As $-X = Q_{-X}(U)$, we have

$$(-X, U) = (Q_{-X}(U), U)$$

is comonotonic, and consequently, so is $(X, 1 - U)$ by definition. As $1 - U \in \mathbb{U}$, by Theorem 5,

$$X = Q_X(1 - U).$$

The claim is proved.

By this, we can express $\overline{\mathfrak{A}}_x$ as

$$\overline{\mathfrak{A}}_x = \{Q_X(1 - U) : \mathbf{E}[\rho Q_X(1 - U)] = x, \quad Q_X(\cdot) \in \mathcal{Q} \cap \mathcal{C}, \quad U \in \mathbb{U}_\rho\}.$$

Note that for any $U \in \mathbb{U}_\rho$, we have, by Theorem 5, that $\rho = Q_\rho(U)$. Therefore, for any $U \in \mathbb{U}_\rho$,

$$\mathbf{E}[\rho Q_X(1 - U)] = \mathbf{E}[Q_\rho(U) Q_X(1 - U)] = \int_0^1 Q_\rho(1 - t) Q_X(t) dt.$$

As a result, we can further express $\overline{\mathfrak{A}}_x$ as

$$\begin{aligned} \overline{\mathfrak{A}}_x &= \left\{ Q_X(1 - U) : \int_0^1 Q_\rho(1 - t) Q_X(t) dt = x, \quad Q_X(\cdot) \in \mathcal{Q} \cap \mathcal{C}, \quad U \in \mathbb{U}_\rho \right\} \\ &= \{Q_X(1 - U) : Q_X(\cdot) \in \mathcal{Q}_x, \quad U \in \mathbb{U}_\rho\}. \end{aligned}$$

Therefore, we conclude that finding an optimal solution X^* to problem (1) is equivalent to finding an optimal $Q^*(\cdot)$ to problem (3) first, and then setting

$$X^* = Q^*(1 - U),$$

where U is any random variable in \mathbb{U}_ρ . □

As mentioned in the introduction, in order to link problem (1) to its quantile formulation (3), it is always assumed, in the literature, that the pricing kernel ρ is atomless. In this case, the set \mathbb{U}_ρ is singleton, that is $\mathbb{U}_\rho = \{F_\rho(\rho)\}$, so Theorem 9 reduces to the well-known result:

Corollary 10 *If ρ is atomless, then a random variable X^* is an optimal solution to problem (1) if and only if*

$$X^* = Q^*(1 - F_\rho(\rho)),$$

where $Q^(\cdot)$ is an optimal solution to problem (3).*

This yields the same result as in He and Zhou [8], Jin, Zhang, and Zhou [9], Jin and Zhou [10], Xu and Zhou [16], among many others. It justifies the financial wisdom that one should have a good output if the state-price density is low and bad if high.

4 Concluding Remarks

In this paper, we have proved by construction that every component of a comonotonic random vector can be expressed by its quantile function and a common random variable uniformly distributed on the unit interval $(0, 1)$. This new characterization of comonotonicity is easy to implement in practice. We then apply this result to a general investment problem with a law-invariant preference measure leading to a universal framework covering all mentioned models (and more), without making any assumption on the pricing kernel. This overcomes one of the major difficulties encountered when one considers general economic equilibrium models in which the pricing kernel is a yet-to-be-determined unknown random variable. The result is applicable to many other models such as risk sharing model.

For a general investment model (1), we have proved that the optimal output should be anti-comonotonic with the pricing kernel, regardless whether it is atomless or not. However, it is still open to find the optimal solution to the quantile formulation problem (3). This functional optimization problem is completely solved in Xu (2013) by a change-of-variable and relaxation approach under RDUT framework.

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Appendix

A Proof of Hardy-Littlewood Inequality

Let first show that (X, Y) is comonotonic if and only if

$$\mathbf{P}(X \geq t, Y \geq s) = \min\{\mathbf{P}(X \geq t), \mathbf{P}(Y \geq s)\}$$

for almost everywhere $(t, s) \in \mathbb{R}^2$.

In fact, (X, Y) is comonotonic if and only if $(-X, -Y)$ is comonotonic, if and only if, by Theorem 1,

$$\mathbf{P}(-X \leq t, -Y \leq s) = \min\{\mathbf{P}(-X \leq t), \mathbf{P}(-Y \leq s)\},$$

i.e.,

$$\mathbf{P}(X \geq -t, Y \geq -s) = \min\{\mathbf{P}(X \geq -t), \mathbf{P}(Y \geq -s)\},$$

for all $(t, s) \in \mathbb{R}^2$. By right-continuity of cdfs, it is also equivalent to

$$\mathbf{P}(X \geq t, Y \geq s) = \min\{\mathbf{P}(X \geq t), \mathbf{P}(Y \geq s)\}$$

for almost everywhere $(t, s) \in \mathbb{R}^2$.

Without loss of generality, we assume X^* and Y are both nonnegative, otherwise we may consider their positive and negative parts respectively. In this case, X is also nonnegative as $X \stackrel{d}{=} X^*$, so

$$X(\omega) = \int_0^{+\infty} \mathbb{1}_{X(\omega) \geq t} dt,$$

where $\mathbb{1}_{X(\omega) \geq t}$ is the indicator function of set $\{X(\omega) \geq t\}$. Analogously, we can express X^* and Y . Therefore, applying Fubini's Theorem,

$$\begin{aligned} \mathbf{E}[XY] &= \mathbf{E} \left[\int_0^{+\infty} \mathbb{1}_{X(\omega) \geq t} dt \int_0^{+\infty} \mathbb{1}_{Y(\omega) \geq s} ds \right] = \int_0^{+\infty} \int_0^{+\infty} \mathbf{E} [\mathbb{1}_{X(\omega) \geq t} \mathbb{1}_{Y(\omega) \geq s}] dt ds \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbf{P}(X(\omega) \geq t, Y(\omega) \geq s) dt ds \\ &\leq \int_0^{+\infty} \int_0^{+\infty} \min\{\mathbf{P}(X(\omega) \geq t), \mathbf{P}(Y(\omega) \geq s)\} dt ds \\ &= \int_0^{+\infty} \int_0^{+\infty} \min\{\mathbf{P}(X^*(\omega) \geq t), \mathbf{P}(Y(\omega) \geq s)\} dt ds \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbf{P}(X^*(\omega) \geq t, Y(\omega) \geq s) dt ds \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbf{E} [\mathbb{1}_{X^*(\omega) \geq t} \mathbb{1}_{Y(\omega) \geq s}] dt ds \\ &= \mathbf{E} \left[\int_0^{+\infty} \mathbb{1}_{X^*(\omega) \geq t} dt \int_0^{+\infty} \mathbb{1}_{Y(\omega) \geq s} ds \right] = \mathbf{E}[X^*Y], \end{aligned}$$

where the equality holds if and only if

$$\mathbf{P}(X(\omega) \geq t, Y(\omega) \geq s) = \min\{\mathbf{P}(X(\omega) \geq t), \mathbf{P}(Y(\omega) \geq s)\}$$

for almost everywhere $(t, s) \in \mathbb{R}_+^2$, which is equivalent to (X, Y) being comonotonic.

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