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Non-Concave Utility Maximization with Portfolio Bounds

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The problems of non-concave utility maximization appear in many areas of finance and economics, such as in behavioral economics, incentive schemes, aspiration utility, and goal-reaching problems. Existing literature solves these problems using the concavification principle. We provide a framework for solving non-concave utility maximization problems, where the concavification principle may not hold and the utility functions can be discontinuous. We find that adding portfolio bounds can offer distinct economic insights and implications consistent with existing empirical findings. Theoretically, by introducing a new definition of viscosity solution, we show that a monotone, stable, and consistent finite difference scheme converges to the value functions of the non-concave utility maximization problems.

Key words: Portfolio Constraints, Behavioral Economics, Incentive schemes, Concavification Principle

1. Introduction

Although the objective functions are concave in traditional utility maximization problems, there is also a large body of literature on non-concave utility maximization in economics and finance. Examples include the S-shaped utility function in behavioral economics (e.g., Kahneman and Tversky (1979), Berkelaar, Kouwenberg and Post (2004), and Jin and Zhou (2008)), the goal-reaching problem (e.g., Browne (1999a) and Spivak and Cvitanić (1999)), delegated portfolio choices with non-concave compensation schemes (e.g., Carpenter (2000), Basak, Pavlova and Shapiro (2007), and He and Kou (2018)), and the aspiration utility maximization (e.g., Diecidue and van de Ven (2008) and Lee, Zapatero and Giga (2018)). Almost all models in the above literature rely on the concavification principle, namely, replacing a non-concave utility with its concave envelope and thus reducing the non-concave utility maximization problem to a concave one.

In this paper, we attempt to provide a general framework for solving non-concave utility maximization problems where the concavification principle may not hold. We find that adding portfolio bounds, which makes the concavification principle invalid, can offer distinct economic insights.

Motivation. From Figure 1, which depicts the optimal portfolio weights of non-concave utility maximization from six models in the literature, one can immediately see high leverage ratios, from 300% to 4000% or even infinity. This naturally leads us to consider portfolio constraints. In particular, we will focus on one-side and two-side portfolio bounds.[1]

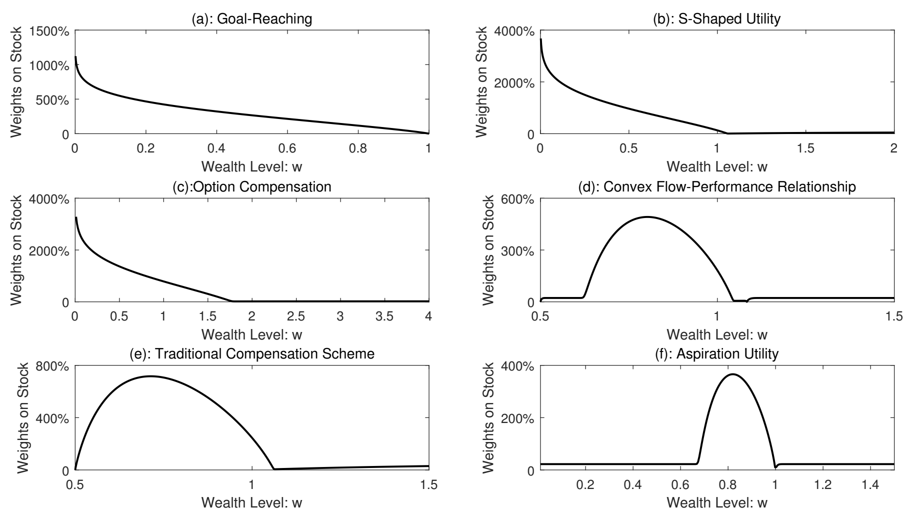

Figure 1 The unconstrained optimal fraction of total wealth invested in the stock at time zero in six economic models. The six sub-figures (a)-(f) correspond to the models of Browne (1999a), Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova and Shapiro (2007), He and Kou (2018), and Lee, Zapatero and Giga (2018), respectively. A compulsory liquidation at $w = 0.5$ is imposed for the models of Basak, Pavlova and Shapiro (2007) and He and Kou (2018).

However, when borrowing constraints are imposed, the optimal strategies (as we will prove later) may involve short-selling to the highest extent, even if the original optimal strategies without borrowing constraints do not involve short-selling at all. More precisely, Figure 2 uses the same models and parameters as in Figure 1, except that the constraint $[-500\%, 300\%]$ (i.e., 300% borrowing constraint and 500% short-selling constraint) is imposed; the figure shows that the optimal strategies may borrow or short-sell as much as permitted, as the portfolio weights in all six sub-figures

[1] We will not distinguish between portfolio constraints and portfolio bounds in this paper. They are exchangeable in most places, both referring to the latter.

reach both bounds (i.e., -500% and 300%). Indeed, if one relaxes the constraint to $[-1000\%, 300\%]$, the optimal strategies for all six sub-figures reach both of new bounds (i.e., -1000% and 300%).

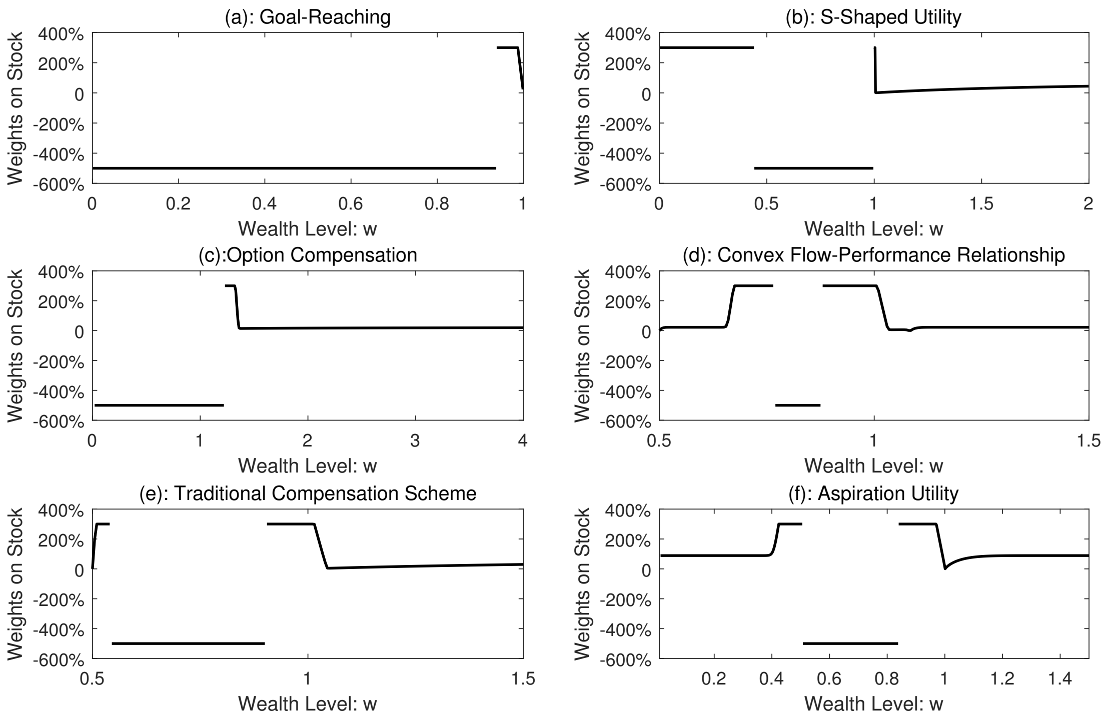

Figure 2 The optimal fraction of total wealth invested in the stock at time zero with mild portfolio constraints in six economic models. The portfolio constraints are 300% for borrowing and 500% for short-selling constraints, i.e., $d = -5$ and $u = 3$. The six sub-figures (a)-(f) correspond to the models of Browne (1999a), Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova and Shapiro (2007), He and Kou (2018), and Lee, Zapatero and Giga (2018), respectively. A compulsory liquidation at $w = 0.5$ is imposed for the models of Basak, Pavlova and Shapiro (2007) and He and Kou (2018).

This motivates us to study two-side constraints and treat things carefully, as intuition can go wrong with non-concave utilities. This paper provides a general framework for solving non-concave utility maximization with portfolio bounds where the concavification principle does not hold. By applying our methodology to the aforementioned six models in the literature of non-concave utility maximization, we find that adding portfolio bounds, which makes the concavification principle invalid, can offer distinct economic insights and implications. Theoretically, in view of the difficulties incurred by non-concave value functions, we do three things: (a) We introduce a new definition of viscosity solution. (b) Based on the new definition, we establish a novel comparison principle, which is used to prove that the value function of the non-concave utility maximization problem is the unique viscosity solution (in terms of the new definition) of the Hamilton-Jacobi-Bellman

(HJB) equation. (c) We then show that a monotone, stable, and consistent finite difference scheme converges to the solution of the utility maximization problem.

Economic Insights and Implications. There are three general findings for the non-concave utility maximization with portfolio bounds. First, the concavification principle may no longer hold, because the resulting value function may not be globally concave before maturity in general. Intuitively, convex incentives would induce investors to borrow large amounts of the risk-free asset or to take large short-sale positions, which is prohibited due to portfolio bounds. Consequently, it is impossible to obtain a hypothetical value function that is the concave envelope of the original non-concave utility, yielding a non-concave value function. Indeed, we find that the non-concavity often occurs locally, depending on the time to maturity and wealth level. This implies that facing convex incentives, investors may choose to gamble or not, depending on scenarios.

Second, investors may gamble by short-selling (borrowing) a stock even with a positive (negative) risk premium as their target is yet to be reached at a time sufficiently close to maturity. For example, consider a fund manager with a convex compensation scheme who suffers a large loss near maturity and faces the no-short-selling constraint but no severe restriction on borrowing. Given the convex incentive and the short-selling constraint, the manager would be induced to leverage on long positions, even with a negative risk premium of the stock.

Third, investors may not be myopic with respect to portfolio constraints in the sense that they may act before portfolio constraints are binding. Intuitively, due to portfolio constraints, investors may raise (reduce) their current stock investment to compensate for the potential limited borrowing (short-selling) in the future, compared to the case without portfolio constraints.

We show that these general economic insights still hold under more general market environments, such as the time-varying Gaussian mean return model, the stochastic volatility model, and the case with multiple stocks. Besides, induced by convex incentives, investors may take advantage of the stochastic investment opportunity set to get a more volatile portfolio through hedging demands.

Our study also has two empirical implications. First, Brown, Harlow, and Starks (1996) find that fund managers who are underperforming (outperforming) in the first-half-of-the-year increases (reduces) their risk level in the second-half-of-the-year. This phenomenon, known as the "risk-increasing" tournament behavior, seems to be a rational response of fund managers due to the convex incentives caused by the convex relationship of fund flow to annual relative performance. However, the results of later empirical studies on the tournament behavior are mixed and conflicting (see, e.g., Chevalier and Ellison (1997), Busse (2001), Kempf, Ruenzi and Thiele (2009), and Schwarz (2012)). Our finding provides an alternative explanation for the seemingly conflicting results. Indeed, we complement the results in Basak, Pavlova and Shapiro (2007) by showing that when a fund is significantly underperforming a benchmark, the fund manager may not alter her risk preference; only should the underperformance be within a certain range (e.g., [0.74, 0.96] as in Figure EC.5) would the fund manager alter her risk preference. In other words, the tournament behavior may be local.

Second, Agarwal, Boyson, and Naik (2009) and Chen, Desai and Krishnamurthy (2013) empirically find that mutual funds that use both buying and short-selling (i.e., long-short) strategies outperform other mutual funds that do not use short-selling (i.e., long-only), in terms of portfolio returns (a risk-neutral evaluation criterion). Because mutual fund managers may have their own risk profiles and face convex incentives, it is unclear why the managers of long-short funds should do better in terms of the (risk-neutral) return criterion, which is not their objective. Our model indicates that although there may be a small utility gain from using the optimal strategy for the long-short fund, the optimal strategy is almost identical to the myopic one in terms of the return criterion. Moreover, no matter whether the optimal strategy or the myopic strategy is adopted, the long-short fund can yield a significantly higher return than the long-only fund, consistent with the empirical finding.

The remainder of the paper is organized as follows. After a literature review in Section 2, Section 3 presents several non-concave utility functions involved in the non-concave portfolio optimization problem. Section 4 is devoted to theoretical analysis. An extensive numerical analysis is given in Section 5. We conclude in Section 6. The new definition of the viscosity solution and a numerical algorithm are presented in Appendices A and B, respectively. An extension to a more general market setting, such as multiple stocks and a stochastic investment opportunity set, is presented in Appendix EC.1. All proofs and additional numerical analysis are relegated to E-Companion.

2. Literature Review

This paper is related to the intersection of two strands of research: the impact of portfolio constraints on dynamic portfolio choice, and risk-shifting incentives derived from the non-concave utility.

DeMiguel et al. (2009) provide a general framework for combining portfolio constraints and estimation for the mean-variance investment problem, and show excellent out-of-sample performance in the presence of estimation error. Karatzas et al. (1991) and Cvitanić and Karatzas (1992) develop a duality method for a concave utility optimization problem with convex portfolio constraints. In particular, for the power utility with portfolio constraints, the optimal investment strategy is myopic in the sense that no additional action would be made before the portfolio constraints are binding (see, e.g., Vila and Zariphopoulou (1997)). We complement this stream of the existing literature by showing that in non-concave utility maximization, the optimal investment strategy may no longer be myopic in the sense that actions may be taken before portfolio constraints are binding. Note that an optimal investment strategy in an incomplete market is often non-myopic concerning portfolio constraints; see, e.g., Dai, Jin and Liu (2011) for CRRA utility maximization with transaction costs and Dai et al. (2021) for dynamic mean-variance analysis in incomplete markets. In contrast, the non-myopic phenomenon occurs even in a complete market for non-concave utility maximization.

There is a large body of literature on portfolio optimization with a stochastic investment opportunity set[2]; see, e.g., Kim and Omberg (1996), Liu (2007), Basak and Chabakauri (2010), and Dai et al. (2021). However, all of the above literature is restricted to concave utilities, and we will examine optimal portfolio choice with non-concave utilities.

Due to the presence of non-concavity, portfolio constraints, and possible discontinuities in the objective functions, the value function may be locally non-concave and singular at the terminal time. This poses a great challenge for the partial differential equation method. For example, it is not easy to obtain the comparison principle and the uniqueness of viscosity solution to the HJB equation, because the standard comparison principle for the HJB equation in Crandall, Ishii and Lions (1992), which guarantees the uniqueness of viscosity solution, requires the continuity of viscosity solution. Facing these challenges, we introduce a new definition of viscosity solution for discontinuous value functions and show that the new definition satisfies some asymptotic conditions at the terminal time. Then we can prove that a novel comparison principle holds for the new definition of the viscosity solutions.

The finite difference method has been widely used to solve the HJB equations arising from continuous-time portfolio optimization problems (see, e.g., Barles and Souganidis (1991), Fleming and Soner (2006), Forsyth and Labahn (2007), and Wang and Forsyth (2008)). Using the new definition of viscosity solution and the novel comparison principle, we show that a monotone, stable, and consistent finite difference scheme still converges to the corresponding value function even with discontinuous utility and portfolio bounds. We then employ the finite difference scheme to conduct an extensive numerical analysis.

Bian, Chen and Xu (2019) investigate the non-concave utility optimization problem with one-side portfolio bounds. They find that the concavification principle still holds and the standard comparison principle remains valid. We complement their results by studying the two-side portfolio bounds and discontinuous utility functions. In our study, neither the concavification principle nor the classical comparison principle may hold, and the viscosity solution may be discontinuous.

[2] Theoretically, there are at least two methods to solve utility maximization for concave utility functions with portfolio constraints. One is the martingale duality method; see, e.g., Karatzas et al. (1991) and Cvitanić and Karatzas (1992). Using the martingale duality method, Haugh, Kogan and Wang (2006) derive an effective way to check whether a given control policy can lead to a good approximation to the optimal control policy. Due to the non-concave utility functions, it is generally difficult to use the martingale duality method. One exception is Spivak and Cvitanić (1999), who study the goal-reaching problem without constraints via the martingale duality method. That is why we study the second method, which is based on partial differential equations.

3. Model Formulation

We consider a financial market with one risky stock and one riskless bond with a constant risk-free rate r. The dynamic of the risky stock follows the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\mathcal{B}_t, \tag{1}$$

where $\mathcal{B}_t$ is the standard Brownian motion, and the drift μ and volatility σ are assumed to be constant. Consider a self-financing portfolio strategy that invests Π_t dollars in the stock at time t. Then the value of the fund, $\tilde{W}_t$, $t \geq 0$, evolves according to $d\tilde{W}_t = r\tilde{W}_t dt + \Pi_t[(\mu - r)dt + \sigma d\mathcal{B}_t]$. As in the standard literature, we have a liquidation constraint, i.e. $\tilde{W}_t \geq Be^{-r(T-t)},\ 0 \leq t \leq T$, for some non-negative constant B. Let $\pi_t := \Pi_t/\tilde{W}_t$ be the proportion of wealth invested in the stock, and $W_t = \tilde{W}_t e^{r(T-t)},\ 0 \leq t \leq T$, be the (forward) wealth at time t. It follows that

$$dW_t = W_t \pi_t (\eta dt + \sigma d\mathcal{B}_t), \tag{2}$$

where $\eta = \mu - r$ is the excess rate of return, and the liquidation constraint is simplified as

$$W_t \geq B,\ 0 \leq t \leq T. \tag{3}$$

3.1. The Non-Concave Utility Optimization Problem

Let $U(\cdot)$ be a utility that an agent will receive at a finite horizon T. The utility function $U(\cdot)$ is not necessarily concave or continuous but is assumed to satisfy the following assumption.

Assumption 3.1. The utility function $U(w),\ w \geq B$ is nondecreasing, right-continuous, and bounded from above by a function $w^{\hat{p}}$ when w is sufficiently large for some $\hat{p} \in (0,1)$, and satisfies either of the following two conditions at $w = B$: (1) $U(B) > -\infty$; (2) $B = 0$ and there are constants $\epsilon > 0$, $\widetilde{p} \leq 0$, $A_1 > 0$, and $A_2 \in \mathbb{R}$ such that $U(w) = A_1 \frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$ for $0 < w < \epsilon$ and $U(w) \geq A_1 \frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$ for all $w > 0$. Note that if $\widetilde{p} = 0$, then $U(w) = A_1 \ln(w) + A_2$ for $0 < w < \epsilon$.

To understand Assumption 3.1, we illustrate it by using the classical CRRA utility function $(w^p - 1)/p$, $p < 1$. The upper bound condition in Assumption 3.1 applies to the case $p \in (0,1)$ such that the utility function would not grow too fast when w is large, and the condition is satisfied for $p \leq 0$. The lower bound condition in Assumption 3.1 applies to the case $p \leq 0$ such that the utility function would not go to $-\infty$ too fast when w tends to zero, and the condition is satisfied for $p \in (0,1)$.

The agent attempts to choose an optimal portfolio strategy π to achieve the following objective

$$\sup_{d \leq \pi \leq u} E[U(W_T)], \tag{4}$$

where the wealth W follows (2) subject to liquidation constraint (3). Here, d and u are the constant lower and upper portfolio bounds. Without loss of generality, we assume that $d \leq 0$ and $u \geq 0$ throughout the paper. Unbounded portfolio constraints are covered by setting $d = -\infty$ and/or $u = \infty$. A finite upper bound $u \geq 1$ implies a borrowing constraint that limits the leverage ratio. In particular, $u = 1$ means that no borrowing is permitted. On the other side, a finite lower bound $d \leq 0$ implies the degree of short-sale constraints. In particular, $d = 0$ means that short-sale is prohibited.

3.2. Examples of Non-Concave Utility Functions

3.2.1. A Discontinuous Utility of the Goal-Reaching Problem. Many portfolio managers are interested in achieving some performance goal, such as beating a stock index. Browne (1999a) studies the optimal investment strategy of a fund manager who aims to maximize the probability of beating a benchmark by a given finite horizon, where the corresponding utility function is expressed as an indicator function

$$U(w) = 1_{w \geq 1}. \tag{5}$$

Here w refers to the value of the fund under management normalized by the benchmark. The utility function indicates that if the fund manager fails to reach the goal, then he will be indifferent to any outcomes incurred. Note that the utility function is discontinuous and non-concave.

Due to its simplicity and tractability for calibrating investors' risk profiles, the goal-based objective has been widely used by robo-advising firms (e.g., Betterment) and preeminent fund managers when they formulate portfolio optimization problems; see the book by Sironi (2016).

3.2.2. The S-shaped Utility of Prospect Theory. Kahneman and Tversky (1979) propose the following S-shaped utility function:

$$U(w) = \begin{cases} (w - W_0)^p & \text{for} \quad w > W_0 \\ -\lambda(W_0 - w)^p & \text{for} \quad w \leq W_0, \end{cases} \tag{6}$$

where W_0 is the reference point that distinguishes gains from losses, $0 < p < 1$ measures the degree of risk aversion over gains, and the loss aversion coefficient $\lambda > 1$ indicates that the pain from one dollar loss is higher than the pleasure from one dollar gain. Berkelaar, Kouwenberg and Post (2004) study the optimal investment strategy with the S-shaped utility function (6)[3].

[3] Jin and Zhou (2008) incorporate probability distortion into Berkelaar, Kouwenberg and Post (2004). Note that probability distortion invalidates the dynamic programming principle and creates an issue of time inconsistency. We do not consider probability distortion as we use an approach based on the HJB equation.

3.2.3. The Delegated Portfolio Choice with Convex Compensation Schemes. Delegated fund managers are usually paid by convex compensation schemes, such as an option compensation, management fee proportional to the asset under management which is convex on the performance of the fund, or a performance fee linked to the profit which has limited liability.

Option Compensation. Carpenter (2000) considers a risk-averse manager compensated with a call option. The manager's payoff at time T is α shares of call option over the fund that matures at T with strike K, plus a constant base C. Then, the payoff function f_{Car} is given by

$$f_{Car}(w) = \alpha \max\{w - K, 0\} + C, \tag{7}$$

where w is the terminal wealth level of the fund. The utility function of the manager is taken as $U(w) = [(f_{Car}(w))^p - 1]/p$, where $p < 1$. The convex structure of the option payoff makes the utility function U non-concave over the terminal wealth level of the fund.

Convex Flow-Performance Relationship. Basak, Pavlova and Shapiro (2007) study a portfolio choice model that the fund manager's compensation is proportional to the assets under management. The assets under management depend on the fund's performance: in general, outperformance attracts cash inflow, and underperformance leads to money redemption. For example, the flow rate of the fund, f_{BPS}, can be specified as follows:

$$f_{BPS}(w) = \begin{cases} f_L & \text{for} \quad \ln(w/W_0) < \eta_L \\ f_L + \psi(\ln(w/W_0) - \eta_L) & \text{for} \quad \eta_L \le \ln(w/W_0) < \eta_H \\ f_H & \text{for} \quad \ln(w/W_0) \ge \eta_H, \end{cases} \tag{8}$$

where W_0 is the initial asset under management,[4] η_L, η_H are the lower and upper performance thresholds, f_L, f_H are the flow rates in case of bad and good performance, and $\psi = (f_H - f_L)/(\eta_H - \eta_L)$ such that the function $f_{BPS}(w)$ is continuous. The utility function of the manager can be taken as $U(w) = [(w f_{BPS}(w))^p - 1]/p$, where $p < 1$. The utility is non-concave over w due to the convex flow-performance $f_{BPS}(w)$.

Convex Performance Fee Schemes. He and Kou (2018) study the optimal investment strategy of a fund manager with two kinds of performance fee schemes: the traditional scheme and the first-loss scheme. By assuming that a proportion γ of the fund belongs to the manager and the manager can charge a proportion α of the profit, the manager's net profit-or-loss function under the traditional scheme is given by

$$f_T(w) = \begin{cases} (\gamma + \alpha(1-\gamma))(w - W_0) & \text{for} \quad w > W_0 \\ \gamma(w - W_0) & \text{for} \quad 0 \le w \le W_0. \end{cases} \tag{9}$$

[4] Strictly speaking, W_t represents the time t relative performance, i.e., the ratio of the fund value to the benchmark.

In contrast, under the first-loss scheme, the manager will firstly use his money in the fund to cover the loss. Thus, the net profit-or-loss structure is changed into

$$f_{FL}(w) = \begin{cases} (\gamma + \alpha(1-\gamma))(w - W_0) & \text{for} \quad w > W_0 \\ w - W_0 & \text{for} \quad (1-\gamma)W_0 \leq w \leq W_0 \\ -\gamma W_0 & \text{for} \quad 0 \leq w \leq (1-\gamma)W_0. \end{cases} \tag{10}$$

Assume the fund manager is risk averse over the profit and risk seeking over the loss, then the utility function of the fund manager is given by $U(w) = g(f_T(w))$ or $U(w) = g(f_{FL}(w))$, where $g(\cdot)$ is an S-shaped function defined by

$$g(z) = \begin{cases} z^p & \text{for} \quad z > 0 \\ -\lambda(-z)^p & \text{for} \quad z \leq 0, \end{cases}$$

with $0 < p < 1$. The non-concave feature of the utility function originates from both the S-shaped function and the performance fee scheme.

3.2.4. Aspiration Utility. Lee, Zapatero and Giga (2018) analyze the demand for skewness that results from an aspiration utility similar to Diecidue and van de Ven (2008). In Lee, Zapatero and Giga (2018), the economic agent cares not only about the normal consumption but also about the status which is conveyed through the consumption of non-divisible goods, such as a luxury car or a house. So, the utility of the agent will jump when his wealth reaches the level from which he can consume the non-divisible good. The utility function could be given by

$$U(w) = \begin{cases} u_1(w) := \frac{w^p - 1}{p} & \text{if} \quad w < R \\ u_2(w) := c_1 \frac{w^p - 1}{p} + c_2 & \text{if} \quad w \geq R, \end{cases} \tag{11}$$

where w is the terminal wealth level, R is the aspiration level, $p < 1$, $c_1 > 0$, and c_1, c_2 are constants such that $U(R-) < U(R)$. Since the utility function jumps at the aspiration level R, it is non-concave.

4. Theoretical Analysis

This section is devoted to theoretical analysis for the portfolio optimization problem (4). Denote by $V(t,w)$ the value function of the optimization problem (4) conditional on $W_t = w$. At the terminal time T, the value function equals the utility function by definition, i.e.

$$V(T,w) = U(w), \text{ for all } \; w \geq B. \tag{12}$$

When $W_t = B$ for some $t < T$, liquidation is necessary, which implies a boundary condition

$$\begin{cases} \lim\limits_{(s,w)\to(t,B)} V(s,w) = U(B) & \text{if} \quad U(B) > -\infty \\ \lim\limits_{(s,w)\to(t,0)} V(s,w) - V_{CRRA}(s,w) = 0 & \text{if} \quad B = 0 \text{ and } \; U(0) = -\infty, \end{cases} \tag{13}$$

where V_{CRRA} is the value function with utility $U(w) = A_1 \frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$, $\widetilde{p} < 1$ and constraint $d \leq \pi \leq u$.[5] The value function formally satisfies the following HJB equation

$$\frac{\partial V(t,w)}{\partial t} + \sup_{d\leq \pi \leq u} \left\{ \frac{1}{2}\pi^2 w^2 \sigma^2 \frac{\partial^2 V(t,w)}{\partial w^2} + \pi w \eta \frac{\partial V(t,w)}{\partial w} \right\} = 0, \quad t < T, \quad w > B; \tag{14}$$

we will justify this rigorously by introducing a new definition of viscosity solution.

4.1. A New Definition of Viscosity Solution

Facing the challenges of non-concavity, portfolio bounds, and possible discontinuity together, we introduce a new definition of viscosity solution to the HJB equation in Appendix A, which adds a special treatment at the terminal time T. To give an intuition, consider two cases.

(i) The portfolio set $[d,u]$ is bounded. In this case, we later show that the value function satisfies the following asymptotic property (see Proposition EC.3.3):

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) - U(w-) - 2\Phi\left(\frac{\min\{0, \log \zeta / w\}}{L\sqrt{T-t}}\right)(U(w) - U(w-)) = 0, \tag{15}$$

where $U(w-)$ is the left limit of U at w, $U(B-) = U(B)$, $L = \sigma \max\{u, -d\}$, and $\Phi(x)$ is the standard normal cumulative distribution function. When the utility function $U(\cdot)$ is continuous, the asymptotic condition (15) is simplified as

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) = U(w) = V(T,w), \tag{16}$$

which implies the continuity of the value function at maturity. However, when the utility function $U(\cdot)$ is discontinuous, e.g., at some w_0, the value function has a singularity at $(T-, w_0)$.

To elaborate the singularity, let us take the goal-reaching utility (5) as an example which is discontinuous at $w = 1$. Then the asymptotic condition (15) reduces to

$$\lim_{(t,\zeta)\to(T-,1-)} V(t,\zeta) - 2\Phi\left(\frac{\log \zeta}{L\sqrt{T-t}}\right) = 0. \tag{17}$$

On the left-hand side of (17), both terms have a singularity at $(T-, 1-)$ but their difference vanishes. In fact, the latter term proves to be the value function of an alternative goal-reaching problem that only concerns the diffusion term of the dynamic process (cf. Lemma EC.3.1). The intuition behind (17) is as follows: If the goal is yet to be reached ($\zeta < 1$) at a time sufficiently close to maturity, fund managers are inclined to *use as much leverage or short-selling as permitted to raise the likelihood of achieving the goal* (i.e., $\pi = u$ or $\pi = d$).

[5] It can be verified that $V_{CRRA}(t,w) = A_1 \frac{e^{\widetilde{p}\Lambda(T-t)}w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$ where $\Lambda = \sup_{\pi\in[d,u]} \left\{ \eta\pi - \frac{1-\widetilde{p}}{2}\sigma^2\pi^2 \right\} < +\infty$.

(ii) The portfolio set $[d,u]$ is unbounded. In this case, the value function converges to the concave envelope of the utility function, i.e.,

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta)=\hat{U}(w), \tag{18}$$

where $\hat{U}$ is the concave envelope of U; see Proposition EC.3.3 or Bian, Chen and Xu (2019). Thus, for the one-side constraints, the value function is concave and the concavification principle holds. The new definition of the viscosity solution is not needed for this case.

4.2. Comparison Principle, Uniqueness of Viscosity Solution, and Convergence of a Numerical Algorithm

The singularity of the value function at the terminal time brings a great challenge on the comparison principle and uniqueness of viscosity solution to the HJB equation. This is because the standard comparison principle, which guarantees the uniqueness of viscosity solution, requires the continuity of viscosity solution. The following theorem shows that a new comparison principle holds for the new definition of the viscosity solutions satisfying the asymptotic property (15) or (18) (see Definition A.1 in Appendix A for the new definitions of the viscosity subsolution/supersolution/solution).

THEOREM 4.1 **(A New Comparison Principle)**. *Assume the portfolio set* $[d,u]$ *is bounded (or unbounded). Let* $\bar{v}$ *and* $\underline{v}$ *be a viscosity subsolution and supersolution of the HJB equation* (14)*, respectively, with the boundary condition* (13) *and the asymptotic condition* (15) *(or* (18)*). Suppose* $\bar{v}$ *and* $\underline{v}$ *are bounded from above by* $C_1 w^{\hat{p}}+C_2$ *for some* $C_1,C_2>0$*, where* $\hat{p}$ *is as given in Assumption 3.1. Then* $\bar{v}\le\underline{v}$ *for all* $w\ge B$ *and* $0<t<T$*.*

With a bounded portfolio set, $\bar{v}$ and $\underline{v}$ above are likely discontinuous at $t=T$ since the utility function is allowed to be discontinuous (e.g., in the goal-reaching problem). In contrast, Bian, Chen and Xu (2019) focus on the case of unbounded portfolio constraints, where the concavification principle still holds and the effective utility function must be Hölder continuous; thus, their sub-/super-solutions must be continuous at $t=T$. Based on Theorem 4.1, we can prove the following theorem that the value function is the unique (new) viscosity solution of the HJB equation.

THEOREM 4.2 **(Uniqueness and a Link to the Value Function)**. *When the set* $[d,u]$ *is bounded (or unbounded), the value function* V *of* (4) *is the unique viscosity solution of the HJB equation* (14) *with the boundary condition* (13) *and the asymptotic condition* (15) *(or* (18)*). Besides,* V *is continuous for* $t<T$*.*

Since analytical solutions are usually unavailable for non-concave portfolio optimization with portfolio bounds, we resort to the finite difference method to numerically solve for the value function and the optimal portfolios. The following theorem gives the convergence of monotone, stable, and consistent finite difference schemes (see Barles and Souganidis(1991) for the definitions of monotonicity, stability, and consistency).

THEOREM 4.3 **(Convergence of a Numerical Algorithm)**. *A monotone, stable, and consistent finite difference scheme in Appendix B for the HJB equation* (14) *with the boundary condition* (13) *and the asymptotic condition* (15) *(or* (18)*) converges to the value function as the discretization size tends to zero.*

We can extend Theorems 4.1, 4.2, and 4.3 to a more general multivariate setting with the portfolio constraint set $\boldsymbol{C}$ being closed in $\mathbb{R}^n$; see Theorems EC.1.1, EC.1.2, and EC.1.3 and their proofs in E-Companion. The proof of Theorem 4.3 is nontrivial due to the non-concavity and possible discontinuity, and the new comparison principle for viscosity solutions, as given in Theorem 4.1, again plays a critical role. In contrast to the standard fully implicit finite difference scheme with upwind treatment for the first-order derivatives (e.g., the one in IX.3.13 of Fleming and Soner, 2006), we follow Wang and Forsyth (2008) to maximize the use of the central difference approximation to improve accuracy while still attempting to maintain the monotonicity, stability, and consistency.

5. Numerical Analysis

We shall provide numerical evidence of the findings for the six models with portfolio bounds.

5.1. General Findings

To demonstrate the general findings with a non-concave utility, we employ the goal-reaching model and the aspiration utility maximization model, by incorporating portfolio bounds. Both models have discontinuous utility functions. The cases without portfolio bounds have been studied by Browne (1999a) and Lee, Zapatero and Giga (2018), respectively. Numerical evidence for other models with portfolio bounds is similar and is given in E-Companion.

5.1.1. The Goal-Reaching Problem with Portfolio Bounds. Take the goal-reaching problem as in (5). We first consider the no-borrowing and no-short-sale constraints, i.e., $[d,u] = [0,1]$. The default parameter values come from Browne (1999b): $\mu = 0.15$, $r = 0.07$, $\sigma = 0.3$, and $B = 0$ (no liquidation).

Figure 3 gives a 3-D plot of the value function against the wealth and the time to maturity for the constrained case (left panel) and unconstrained case (right panel), respectively. First, the value function is globally concave in wealth for the unconstrained case at any fixed time horizon but is not concave for the constrained case. This shows that a non-concave optimization problem with portfolio bounds cannot be reduced to a concave optimization problem in general. It is worth pointing out that short-selling is never optimal even in the unconstrained case (cf. Figure 1a). Hence, if only the no-short-selling constraint is imposed, the corresponding value function must be the same as the unconstrained value function that is concave, consistent with Theorem 4.2 (see

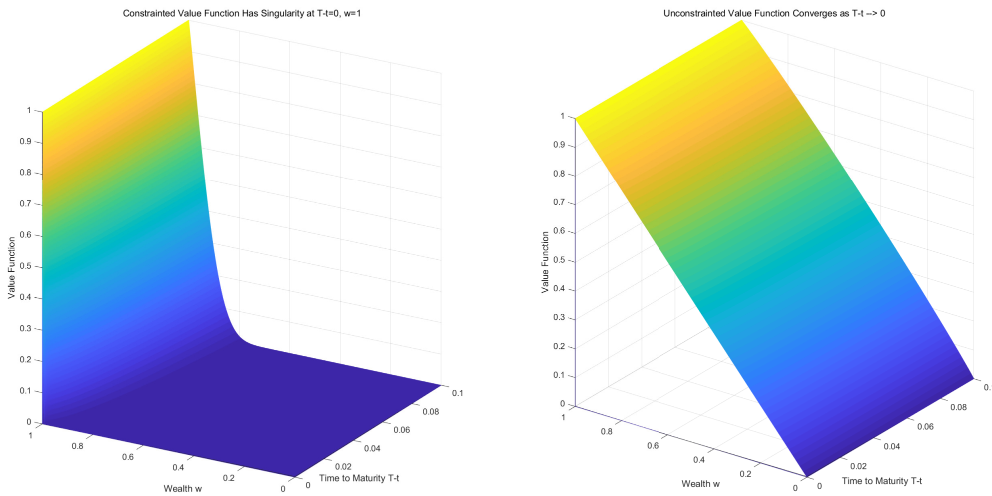

Figure 3 Discontinuous value function at $w=1$ and $t=T-$ in the constrained case, and continuous value function in the unconstrained case. The constrained(left panel) and unconstrained (right panel) value functions are associated with the goal-reaching problem. The parameters are: $r=0.07$, $\mu=0.15$, $\sigma=0.3$, $T=1$, $B=0$, and $[d,u]=[0,1]$.

also Bian, Chen and Xu (2019)). Second, it can be seen that the value function in the constrained case has a singularity at $w=1$ and $t=T-$, consistent with the asymptotic condition (15) or (17), while the unconstrained (or one-side portfolio constrained) value function converges to the concave envelope of the utility in the goal-reaching problem as $t\to T-$, consistent with the asymptotic condition (18).

In Figure 4, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case (dotted line) and Browne's unconstrained case (dashed line). The portfolio bounds are $\pi\in[0,1]$ for the upper panel and $\pi\in[-2,1]$ for the lower panel, respectively. For a lower wealth level, Browne's strategy requires a large leverage ratio that exceeds the scope of the figure and is not displayed (see Figure 1a for a complete picture). Observe that our constrained optimal strategy is not myopic with respect to portfolio bounds. For example, the upper panel of Figure 4 shows that our constrained optimal portfolio (with portfolio bounds $\pi\in[0,1]$) takes more weight on the stock than Browne's strategy does when the wealth is close to the target level $w=1$. This is because fund managers who face portfolio bounds would like to raise risk exposure in advance to compensate for the potential binding of portfolio bounds.

The lower panel of Figure 4 presents a surprising result which is, however, consistent with the implication of (15): given the positive risk premium $\mu-r=0.08$, the constrained optimal strategy is to short sell the stock when the current wealth level is away from the target level $w=1$. This is because in this case, a restricted short-sale is permitted, which induces fund managers to gamble

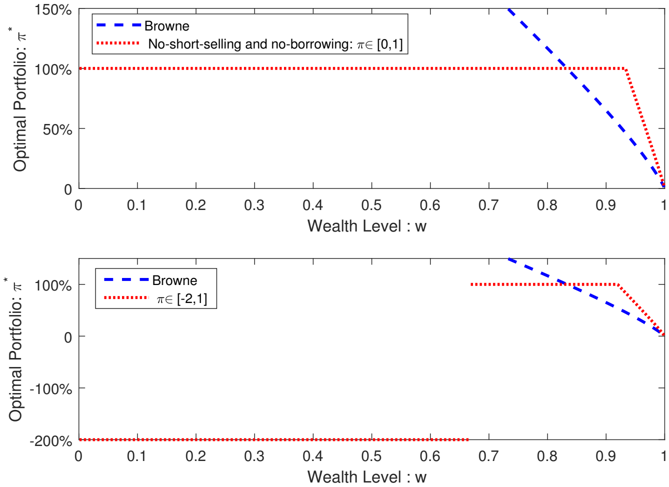

Figure 4 Non-myopic behavior and short-selling behavior (even if the risk premium is positive) of the constrained portfolio strategy. The portfolio bounds are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The dashed line stands for Browne's unconstrained case (the part that exceeds the scope of the figure is not displayed). Parameter values: $r = 0.07,\ \mu = 0.15,\ \sigma = 0.3,\ T = 1$, and $B = 0$.

by taking the largest short-selling ratio $\pi = -2$, even in the presence of positive risk premium, as the current wealth level is far below the target level $w = 1$.

Next, we show that shorting the risky asset to bet on volatility can perform much better than a sub-optimal strategy that is a modification of the optimal strategy using a portfolio weight of 100% when the optimal portfolio weight $\pi^* = -200\%$. Indeed, we define the equivalent wealth loss λ, caused by the sub-optimal strategy, as follows:

$$V_1(0, W) = V(0, (1-\lambda)W), \tag{19}$$

where V_1 refers to the value function associated with the sub-optimal strategy. Figure 5 shows that the equivalent wealth loss is pretty large, especially at a low wealth level.

Table 1 presents the length of time needed for a strategy to beat the benchmark strategy (All Cash or All Stock) by 10% with a probability of 95% or 99% for the unconstrained case and the constrained case (no-short-selling and no-borrowing, i.e., $\pi \in [0,1]$), respectively. The "All Cash" ("All Stock") strategy means a strategy putting all money in the riskless asset (stock). "Browne" refers to Browne's optimal strategy for the unconstrained case studied in Browne (1999a). "Our Strategy" refers to the optimal strategy for the constrained case. "Myopic" refers to the strategy that follows Browne's strategy before the constraints are binding. "Kelly" refers to Kelly's strategy, namely $\pi^* = (\mu - r)/\sigma^2 = 88.9\%$. Note that Kelly's strategy does not incur short-selling or borrowing for the given parameter values.

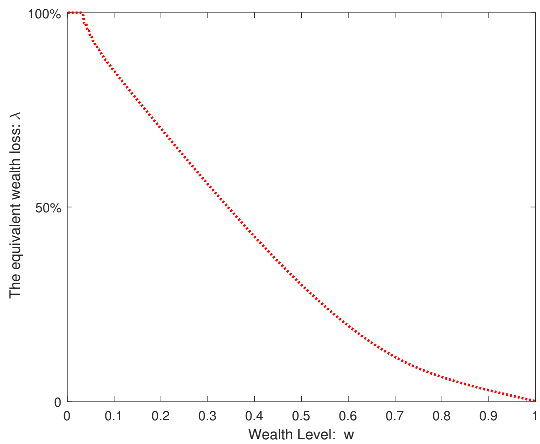

Figure 5 The equivalent wealth loss caused by a sub-optimal strategy at time zero for the goal-reaching problem. The difference between the sub-optimal strategy and the optimal strategy is that the sub-optimal strategy would use a portfolio weight of 100% when the optimal portfolio weight is -200%. The portfolio bounds are $\pi \in [-2,1]$, and default parameter values are $r=0.07,\ \mu=0.15,\ \sigma=0.3,\ T=1$, and $B=0$.

Time (in years) need to beat All Cash by 10% with high probability				
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$		
Probability	Browne	Myopic	Our Strategy	Kelly
95%	1.3	8.9	7.4	10.8
99%	13.8	26.9	22.6	49.9
Time (in years) need to beat All Stock by 10% with high probability				
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$		
Probability	Browne	Myopic	Our Strategy	Kelly
95%	86.3	201.5	131.8	693.5
99%	884.2	1193.1	943.6	3190.7

Table 1 The length of time needed to beat the benchmark strategy (All Cash or All Stock) by 10% with a probability of 95% or 99%. "All Cash" ("All Stock") means a strategy putting all money in the riskless bond (stock). "Browne" refers to the optimal strategy for the unconstrained case studied in Browne (1999a). "Our Strategy" refers to the optimal strategy for the constrained case (no-short-selling and no-borrowing, i.e., $\pi \in [0,1]$). "Myopic" refers to the strategy that follows the "Browne" strategy before the constraints are binding. "Kelly" refers to Kelly's strategy, namely $\pi^* = (\mu - r)/\sigma^2 = 88.9\%$. Default parameter values: $r=0.07,\ \mu=0.15,\ \sigma=0.3,\ T=1,\ B=0$.

From Table 1, we can see that Browne's strategy is better than Kelly's strategy for the unconstrained case. However, as shown in Figure 1a, Browne's strategy may incur an unlimited leverage ratio. Under the no-borrowing and no-short-selling constraint, our constrained optimal strategy outperforms both Kelly's strategy and the myopic strategy. For example, to beat the All Cash benchmark by 10% with a 95% probability, our strategy needs 7.4 years, compared to 10.8 years needed by Kelly's strategy and 8.9 years by the myopic strategy. This is consistent with our previous result that the constrained optimal strategy is not myopic.

A key practical difference between the optimal strategies for one-side and two-side portfolio bounds is that the optimal strategies for the one-side case may lead to unbounded leverage, but those for the two-side case do not have this drawback. For example, if only the no-borrowing constraint is imposed, i.e., $\pi \in (-\infty, 1]$, the concavification principle applies, and the problem can be solved by replacing the non-concave goal-reaching utility (5) with its concave envelope $\hat{U}(w) = w * 1_{0\le w\le 1} + 1_{w>1}$. Before maturity, the value function V is concave and satisfies the HJB equation (14) where $\pi \in (-\infty, 1]$. Thus, when the risk premium is positive, i.e., $\eta > 0$, the optimal portfolio is

$$\pi^* = \min\left\{1, -\frac{\frac{\partial V(t,w)}{\partial w}}{w\frac{\partial^2 V(t,w)}{\partial w^2}}\frac{\eta}{\sigma^2}\right\},$$

for $t < T$ and $w > 0$, which implies that it is never optimal to short a stock with a positive risk premium before maturity. Thus, for $t < T$, the goal-reaching problem of Bian, Chen and Xu (2019) for $\pi \in (-\infty, 1]$ is equivalent to the portfolio optimization problem for the concave utility $\hat{U}(w)$ with $\pi \in [0, 1]$. But when $t = T$, the optimal strategy for this one-side constraint may have to short an unlimited amount of stock; otherwise, the resulting value function at $t = T$ may differ from the concave envelope.

There are other differences between the one-side and two-side constraints. Figure 6 presents the value functions and the optimal portfolios at one year to maturity for the goal-reaching problem with the two-side constraint ($\pi \in [0, 1]$) and the one-side constraint $\pi \in (-\infty, 1]$. First, as in the unconstrained case, shorting the risky asset never occurs with the one-side constraint, and the corresponding value function is concave. Second, in contrast to the two-side constrained case, one may put less weight on risky asset in the one-side constrained case since the resulting effective utility $\hat{U}(w)$ is concave.

Figure 7 depicts the average return rate under three investment strategies. The dotted (long-short) and dashed (long-only) lines refer to the optimal constrained strategy with portfolio bounds $\pi \in [-2, 2]$ and $\pi \in [0, 1]$, respectively; the dot-dashed (myopic) line refers to the myopic strategy with portfolio bounds $\pi \in [-2, 2]$, where the myopic strategy means following the unconstrained optimal strategy until the constraints are binding. The rate of return is defined as $W_T/W_0 - 1$, where W_0 is the initial wealth and W_T is the wealth level at maturity T. The average is taken over 5000 samples where the excess return rate η is generated from a uniform distribution on the interval $[-10\%, 10\%]$. First, the average return rate under both the long-short and myopic cases is much better than that under the long-only case. This is consistent with the empirical findings of Agarwal, Boyson, and Naik (2009) and Chen, Desai and Krishnamurthy (2013) that mutual funds

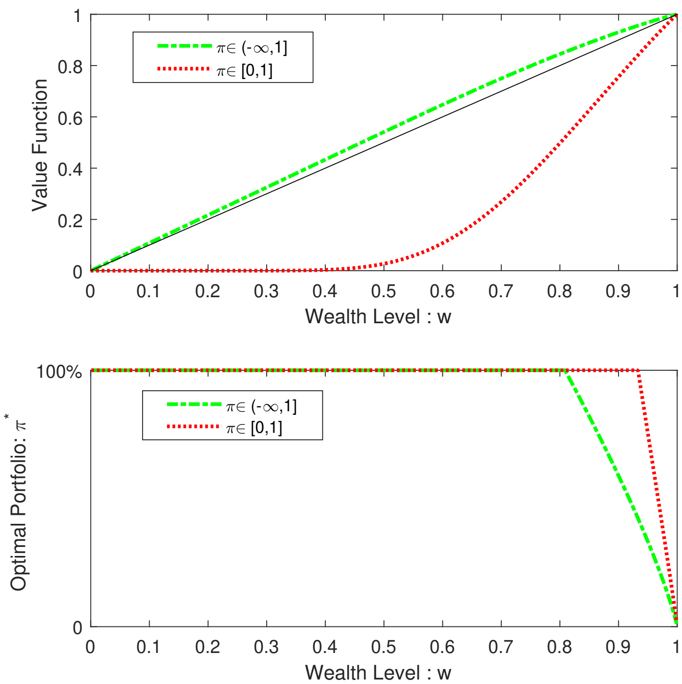

Figure 6 **A comparison between the goal-reaching problems with one-side and two-side constraints. The parameters are:** $r = 0.07$**,** $\mu = 0.15$**,** $\sigma = 0.3$**,** $T = 1$**,** $B = 0$**,** $[d,u] = [0,1]$ **and** $[d,u] = [-\infty, 1]$ **for two-side and one-side constrained cases, respectively.**

that use short-sales outperform those without using short-sales. Second, no matter whether fund managers use the optimal strategy or the myopic strategy, the difference between their returns is small (although there might be a small utility gain for the optimal strategy), and both are consistent with the empirical facts, which demonstrates the robustness of their empirical conclusion.

5.1.2. The Aspiration Utility Maximization with Portfolio Bounds. Now we study how portfolio bounds affect the portfolio choice under the aspiration utility given in (11). The unconstrained case with a discrete-time setting has been discussed in Lee, Zapatero and Giga (2018).

In Figure 8, for the non-concave utility optimization problem discussed in Lee, Zapatero and Giga (2018) where the utility is discontinuous at the aspiration level R, we give a 3-D plot of the value functions against the wealth and the time to maturity for the constrained case (left panel) and unconstrained case (right panel), respectively. As in the goal-reaching problem, it can be seen that: (i) For each fixed time horizon, the value function is globally concave in the unconstrained case but is not concave in the constrained case; (ii) the constrained value function has a singularity at the aspiration level $w = R-$ and the terminal time $t = T-$, consistent with the asymptotic con-

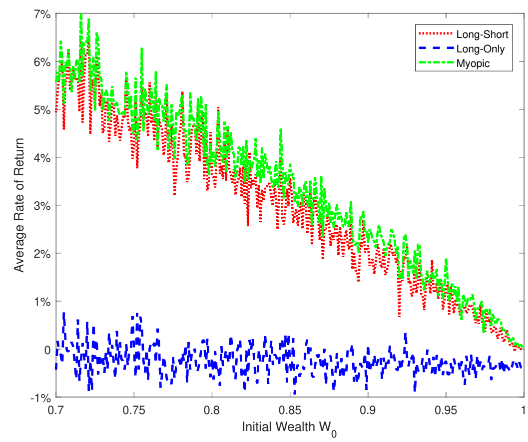

Figure 7 A comparison of returns for bounded constrained problems associated with the goal-reaching problem with and without short-selling constraints. The rate of return is defined as $W_T/W_0 - 1$, where W_0 is the initial wealth and W_T is the wealth level at the terminal time T under various strategies: the dotted (long-short) and dashed (long-only) lines refer to the optimal constrained strategy with portfolio constraints $\pi \in [-2, 2]$ and $\pi \in [0, 1]$, respectively; the dot-dashed (myopic) line refers to the myopic strategy with portfolio bounds $\pi \in [-2, 2]$, where the myopic strategy means following the unconstrained optimal strategy before the constraints are binding. The average is taken over 5000 samples where the excess return rate η is generated from a uniform distribution on the interval $[-10\%, 10\%]$. The other parameters are: $r = 0.07$, $\sigma = 0.3$, $T = 1$, $B = 0$.

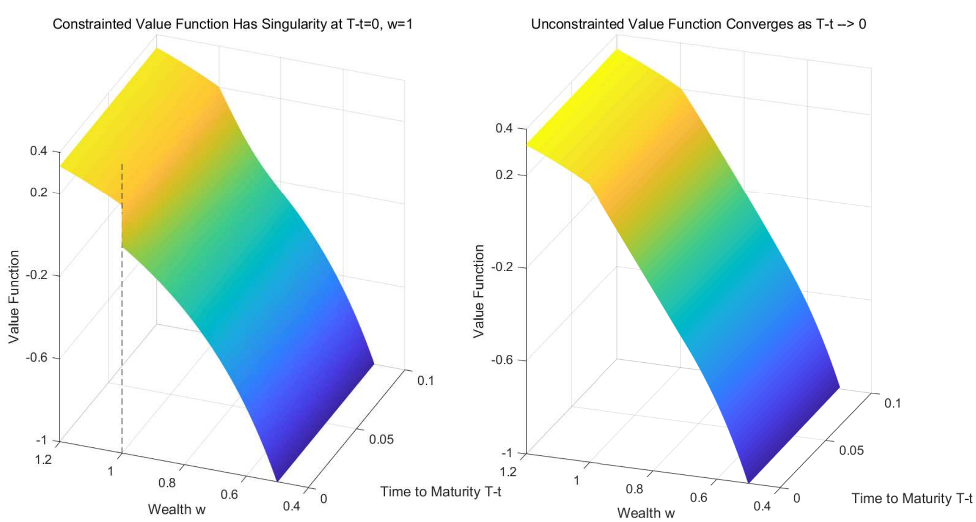

Figure 8 Discontinuous value function at $w = R-$ and $t = T-$ for the constrained case and continuous value function for the unconstrained case. The constrained (left panel) and unconstrained (right panel) value functions are associated with the non-concave utility optimization problem in Lee, Zapatero and Giga (2018). The parameter values are: $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, $B = 0$, and $[d, u] = [0, 1]$.

dition (15). As a comparison, the unconstrained (or one-side portfolio constrained) value function converges to the concave envelope of the aspiration utility at $t \to T-$, consistent with (18).

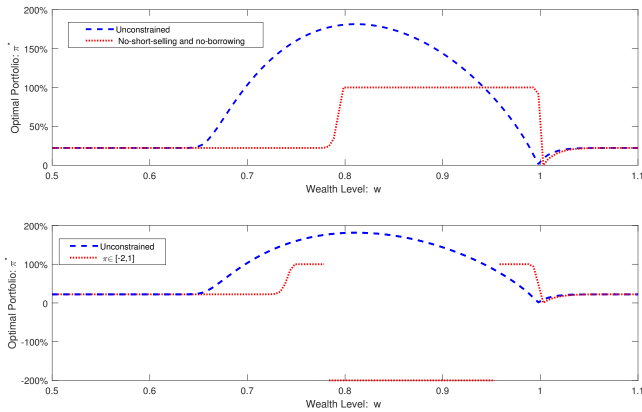

Figure 9 A comparison between our constrained strategy and the unconstrained strategy for the non-concave utility optimization problem discussed in Lee, Zapatero and Giga (2018). The portfolio bounds in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The parameter values are: $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, $T = 1/12$, and $B = 0$.

In Figure 9, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. The portfolio bounds are $\pi \in [0,1]$ in the upper panel and $\pi \in [-2,1]$ in the lower panel, respectively. Since the utility is risk averse for extremely low or high wealth levels, the fraction of total wealth invested in the stock tends to Merton's line as the wealth level approaches infinity or zero. It can be observed that, similar to Figure 4, investors are non-myopic with respect to portfolio bounds such that early action is needed before portfolio constraints are binding.

Similar to the lower panel of Figure 4, the lower panel of Figure 9 reveals that as the wealth level moves far away from the aspiration level, short-selling is likely optimal even with a positive risk premium, provided that a large short-selling ratio ($d = -2$) is permitted. The intuition is the same as before: a large short-selling ratio induces investors to gamble. Even if short-selling is optimal only for a small range of wealth level (i.e., $0.78 < w < 0.95$), abandoning it may lead to a certain wealth loss. As in (19), we similarly define the equivalent wealth loss λ caused by a sub-optimal strategy that uses portfolio weight of 100% when the optimal portfolio weight $\pi^* = -200\%$. Figure 10 reports the equivalent wealth loss against initial wealth level. It can be seen that the maximum loss can be approximately 2%.

It is worthwhile highlighting that the above "gambling" is local, resulting in local non-concavity of the value function. This result may help explain the mixed and conflicting empirical results

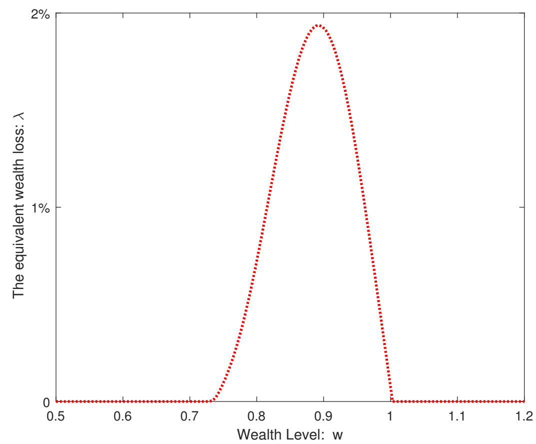

Figure 10 The equivalent wealth loss caused by a sub-optimal strategy at time zero for the aspiration utility optimization problem in Lee, Zapatero and Giga (2018). In particular, the sub-optimal strategy would use a portfolio weight of 100% when the optimal portfolio weight is -200%. The portfolio bounds are $\pi \in [-2, 1]$, and the parameter values are $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, $T = 1/12$, and $B = 0$.

about fund tournament behavior (see Brown, Harlow, and Starks (1996), Chevalier and Ellison (1997), Busse (2001), Kempf, Ruenzi and Thiele (2009), and Schwarz (2012)). Indeed, when a fund is significantly underperforming a benchmark, the fund manager may not alter her risk preference; only when the underperformance is within a reasonable range, e.g., [0.78, 0.95] as in Figure 9, would the fund manager alter her risk preference.

5.2. Model-Specific Findings

5.2.1. The Goal-Reaching Problem with Portfolio Bounds. Without portfolio bounds, Browne (1999a) proves the equivalence between the unconstrained optimal strategy and the replication strategy of a specific digital option under the Black-Scholes market.[6] As a result, the unconstrained optimal strategy must be independent of μ, the drift of the stock, when there is only one stock. However, the constrained optimal strategy usually depends on μ, as revealed by Figure 11. Thus, it is no longer equivalent to the replicating strategy of a digital option.

5.2.2. The S-Shaped Utilities with Portfolio Bounds. Now we study how portfolio bounds affect the portfolio choice of a behavioral investor with the S-shaped utility given in (6). Berkelaar, Kouwenberg and Post (2004) consider the unconstrained case and show that compared to Merton's strategy, a loss averse investor considerably reduces the weight on stocks around the reference point $W_0 = 1$, yielding an explanation to the equity premium puzzle since the wealth levels of most investors are around $W_0 = 1$. This is confirmed by Figure 12, where the portfolio weight of the unconstrained strategy is below Merton's line for wealth level around $W_0 = 1$. Comparing

[6] The result holds for general unconstrained markets; see Föllmer and Leukert (1999) and Spivak and Cvitanić (1999).

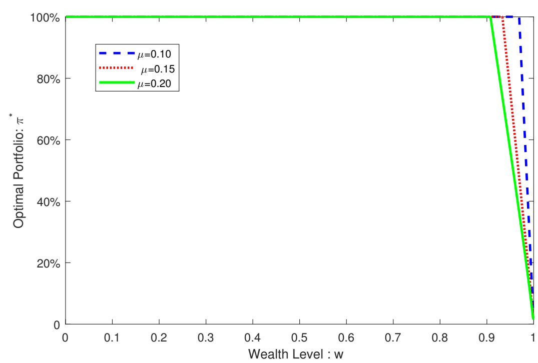

Figure 11 **The dependence of constrained strategies on the stock return μ. Borrowing and short-selling are not permitted, i.e., $u = 1$ and $d = 0$. Default parameter values: $r = 0.07, \mu = 0.15, \sigma = 0.3, T = 1$, and $B = 0$.**

to the unconstrained strategy, Figure 12 shows that the constrained optimal strategy may further reduce weights on stocks around the reference point $W_0 = 1$. The intuition is that the constrained investor is more loss averse since a potential unlimited risk-seeking strategy in the loss region is prohibited by portfolio bounds.

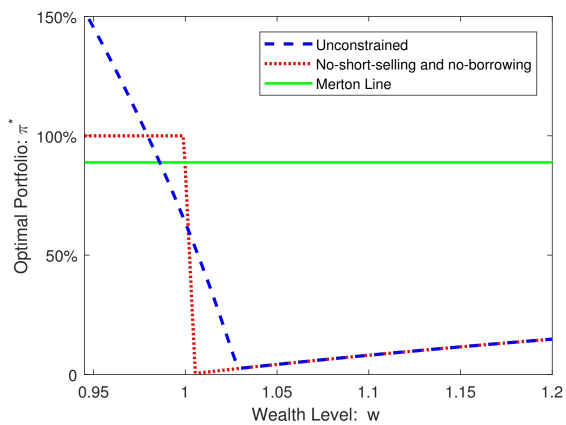

Figure 12 **The unconstrained and constrained strategies for the non-concave portfolio optimization problem in Berkelaar, Kouwenberg and Post (2004). Default parameters values are $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $\lambda = 2.25$, $W_0 = 1$, $T = 1/12$, $B = 0.5$, $\pi \in [0, 1]$ and Merton line $\pi^* = (\mu - r)/((1 - p)\sigma^2) = 88.9\%$.**

5.2.3. The Delegated Portfolio Choice with Portfolio Bounds.

Option Compensation. Now we study how portfolio bounds affect the risk incentive of a risk averse manager compensated with a call option (7). In Figure 13, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for Merton's strategy (solid line), our constrained strategy (dotted line), and the unconstrained strategy (dashed line), respectively. When the option is out of the money, the unconstrained strategy requires a large leverage ratio that exceeds the scope of the figure. Assuming no portfolio bounds, Carpenter (2000) shows that the option compensation may lead to less risk taking, compared with Merton's strategy; this is confirmed by Figure 13, where the portfolio weight of the unconstrained strategy is below Merton's line when the option is deeply in the money (e.g., $w > 1.2$). Interestingly, Figure 13 shows that for the constrained optimal strategy, the risk reduction induced by the option compensation becomes more significant in a larger range (e.g., $w > 1.1$). This is because the fear that the option ends up out of the money leads to risk averse, while portfolio bounds further reduce risk-seeking.

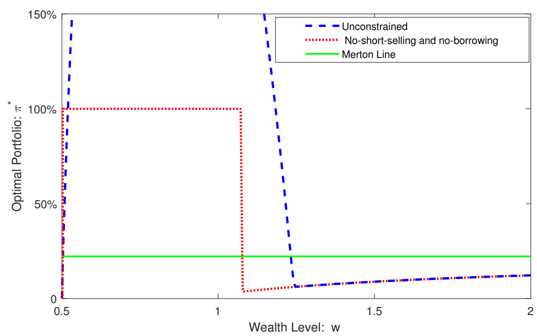

Figure 13 The unconstrained and constrained strategies for the portfolio optimization with the option compensation scheme in Carpenter (2000). The parameters are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = -1$, $K = 1$, $\alpha = 0.2$, $C = 0.02$, $W_0 = 1$, $T = 1$, $T - t = 1/12$, $B = 0.5$, $\pi \in [0,1]$, **and Merton line** $\pi^* = (\mu - r)/((1-p)\sigma^2) = 22.2\%$.

Convex Flow-Performance Relationship. Now we incorporate portfolio bounds into the delegated portfolio optimization with the convex flow-performance relationship and study the cost of delegation induced by fund managers' convex incentive. Basak, Pavlova and Shapiro (2007) point out that there are two types of delegation cost: the first derives from the fact that the manager's attitude towards risk could be different from the investor's; the second derives from the incentive induced by the convex flow-performance relationship. We only consider the second type of delegation cost. In particular, we examine how portfolio bounds affect the cost of delegation, which is the utility loss to the investor due to the manager's deviating from the investor's optimal policy.

Assume that the investor is equipped with a power utility with the same risk-averse coefficient as that of the manager. Then, the investor's optimal policy is to follow Merton's strategy. The cost of delegation, λ, is defined by $V_I(0, W_0) = \tilde{V}_I(0, (1-\lambda)W_0)$, where $V_I(0, W_0)$ is the value function of the investor with initial fund value W_0 at time 0 under the manager's optimal policy accounting for the local convex incentives, and $\tilde{V}_I(0, (1-\lambda)W_0)$ is the value function of the investor with initial fund value $(1-\lambda)W_0$ at time 0 under Merton's policy.

Table 2 exhibits the cost of delegation with and without portfolio bounds. It can be seen that the portfolio bounds reduce the cost of delegation, especially when the time to maturity is short.

Intuitively, this is because portfolio bounds effectively prohibit unlimited leverage that the fund manager would otherwise adopt as the deadline approaches.

Time to Maturity T	Cost of Delegation λ (%)	
	Unconstrained Case	Constrained Case: $\pi \in [0,1]$
Stock with Lower Sharpe Ratio		
1/12	2.06	0.02
1/2	2.06	0.69
1	2.04	1.16
Stock with Higher Sharpe Ratio		
1/12	0.39	0.14
1/2	0.91	0.60
1	1.38	1.09

Table 2 A comparison of the cost of delegation between our constrained strategy and the unconstrained strategy in Basak, Pavlova and Shapiro (2007). The parameter values for the upper panel are: $\mu - r = 0.04$**,** $\sigma = 0.3$**,** $p = -1$**, and the parameter values for the lower panel are:** $\mu - r = 0.08$**,** $\sigma = 0.2$**,** $p = -2$**, The other common parameter values are:** $\eta_L = -0.08$**,** $\eta_H = 0.08$**,** $f_L = 0.8$**,** $f_H = 1.5$**,** $B = 0.5$**, and** $W_0 = 1$**.**

Convex Performance Fee Schemes. In Table 3, we show the effect of portfolio bounds on the value functions (utilities) of the managers and the investors when the traditional scheme is replaced by a first-loss scheme. The third and fourth columns list the utilities of the managers and the investors under the managers' optimal unconstrained strategies and constrained strategies, respectively. The '*' in the superscript means that the utility is improved when replacing the traditional scheme with the first-loss scheme. The third and fourth columns show that, without portfolio bounds, both the managers and the investors are better off when the traditional scheme (20% performance fee) is replaced by the first-loss scheme (30% performance fee). However, if the performance fee in the first-loss scheme is 40% or above, such substitution renders investors worse off. These results are consistent with the general findings of He and Kou (2018). However, the fifth and sixth columns show that, with portfolio bounds $\pi \in [0,1]$, the performance fee will be as high as 50% for the upper panel (60% for the lower panel) so that both the utilities of the managers and investors are better off when the traditional scheme is replaced with the first-loss schemes. We believe that it is again because portfolio bounds prohibit unlimited risk-taking such that the managers with the first-loss scheme demand a high-performance fee as compensation.

5.2.4. The Aspiration Utilities with Portfolio Bounds. In Figure 14, we plot the skewness of the optimal terminal wealth against the wealth level for the constrained strategy (dotted line) and the unconstrained strategy (dashed line) under the aspiration utility given in (11). For the unconstrained case, the skewness is negative when the wealth level is slightly smaller than the aspiration level $R = 1$, and the skewness turns positive when the wealth level moves far away from

		Utilities for Unconstrained Case		Utilities for Constrained Case	
Scheme	Performance Fee	Manager	Investor	Manager	Investor
Stock with Lower Sharpe Ratio					
Traditional	$\alpha = 0.2$	0.11	0.041	0.057	0.036
First-Loss	$\alpha = 0.3$	0.12^*	0.043^*	0.046	0.060^*
First-Loss	$\alpha = 0.4$	0.13^*	0.036	0.056	0.051^*
First-Loss	$\alpha = 0.5$	0.15^*	0.027	0.065^*	0.039^*
Stock with Higher Sharpe Ratio					
Traditional	$\alpha = 0.2$	0.14	0.038	0.056	0.023
First-Loss	$\alpha = 0.3$	0.15^*	0.043^*	0.031	0.060^*
First-Loss	$\alpha = 0.4$	0.18^*	0.032	0.040	0.055^*
First-Loss	$\alpha = 0.5$	0.21^*	0.020	0.050	0.047^*
First-Loss	$\alpha = 0.6$	0.24^*	0.005	0.060^*	0.035^*

Table 3 A comparison of the utilities improvement between our constrained strategy and the unconstrained strategy of He and Kou (2018). The first column is the type of compensation scheme (either the traditional scheme or the first-loss scheme). The second column is the performance fee percentage. The '*' in the superscript means that the utility is improved when replacing the traditional scheme with the first-loss scheme. The parameter values for the upper panel are: $\mu - r = 0.04$, $\sigma = 0.3$, $p = 0.5$, **and the parameter values for the lower panel are:** $\mu - r = 0.08$, $\sigma = 0.2$, $p = 0.6$. **The other common parameter values are:** $\gamma = 0.1$, $T = 1/12$, $B = 0.5$, $W_0 = 1$, $\pi \in [0, 1]$.

the aspiration level, consistent with the finding of Lee, Zapatero and Giga (2018) that investors' demand for skewness is endogenous in the aspiration utility model.

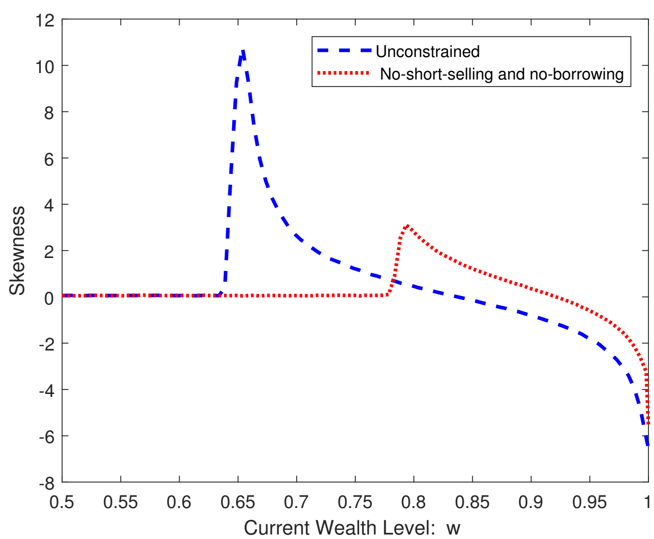

Figure 14 A comparison of the skewness of the terminal wealth between our constrained strategy and the unconstrained strategy in Lee, Zapatero and Giga (2018). Here $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, $T = 1/12$, $B = 0$, **and the portfolio bounds are** $\pi \in [0, 1]$.

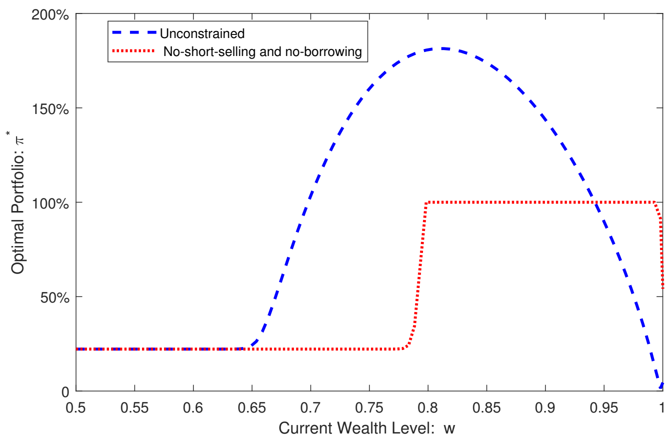

Figure 15 **A comparison between our constrained strategy and the unconstrained strategy in Lee, Zapatero and Giga (2018). Here $\mu = 0.07$, $r = 0.03$, $\sigma = 0.3$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, $T = 1/12$, $B = 0$, and the portfolio bounds are $\pi \in [0,1]$.**

We can observe that with portfolio bounds, the demand for the positive skewness is less. To investigate the reason, we plot in Figure 15 (see also the upper panel of Figure 9) the time-0 optimal fraction of total wealth invested in the stock π^* against the wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. Observe that the optimal strategy requires high leverage for the unconstrained case as the wealth level is far from the aspiration level $R = 1$, which incurs a positive skewness. However, such a gambling strategy is prohibited due to portfolio bounds so that the demand for the positive skewness is much less. Figure 14 shows that the constrained strategy may still induce a significant demand for negative skewness for the wealth level being close to the aspiration level $R = 1$. This is because the investors only need a small gain to consume the status goods and thus considerably reduce the fraction of wealth invested in the stock.

6. Conclusion

We provide a general framework for non-concave portfolio optimization with possible two-side portfolio bounds, and utility functions are allowed to be discontinuous. We show that portfolio bounds significantly affect investors' non-concave portfolio choice. More precisely, we find that in general: (1) With two-side portfolio bounds, the concavification principle does not hold and the value function associated with non-concave portfolio optimization is generally non-concave. (2) Investors are not myopic with respect to portfolio bounds in the sense that they take actions in anticipation of portfolio bounds being potentially binding. (3) A large short-selling or leverage ratio may induce investors to gamble in the case of underperformance. Theoretically, we prove a new comparison principle for discontinuous viscosity solutions associated with general non-concave portfolio optimization problems. Using the new comparison principle, we show that a monotone, stable, and consistent finite difference scheme is still applicable and convergent for the general problems. Moreover, we extend the results to a stochastic investment opportunity set or the case of multiple risky assets.

Appendix A: A New Definition of Viscosity Solution Used

Compared with the standard definition of viscosity solution, e.g., Definition 7.4 of Crandall, Ishii and Lions (1992) or Definition 1.1 of Barles and Souganidis (1991), the new definition pays special attention to the asymptotic property (15). Define the lower semicontinuous envelope and upper semicontinuous envelope of the value function V as

$$V_*(t,w) = \liminf_{(t_1,w_1)\to(t,w)} V(t_1,w_1), \text{ and } V^*(t,w) = \limsup_{(t_1,w_1)\to(t,w)} V(t_1,w_1), \tag{20}$$

and define the Hamiltonian:

$$H(w,p,A) := \sup_{d\le\pi\le u} \{\frac{1}{2}\pi^2 w^2\sigma^2 A + \pi w\eta p\}, \quad w>0. \tag{21}$$

Then, we can rewrite the HJB equation (14) as

$$-\frac{\partial V(t,w)}{\partial t} - H\left(w, \frac{\partial V(t,w)}{\partial w}, \frac{\partial^2 V(t,w)}{\partial w^2}\right) = 0. \tag{22}$$

Note that $H(w,p,A)$ is continuous except at $A=0$ at which it is likely lower semicontinuous for unbounded portfolio set $[d,u]$ [7]. Thus, we need to use its lower semicontinuous envelope H_* and upper semicontinuous envelope H^* (cf. (20)) to define the viscosity supersolution and subsolution as follows:

Definition A.1. For $w \ge B$, let $K(w) = U(w)$ if the portfolio set $[d,u]$ is bounded, and $K(w) = \hat{U}(w)$ if the portfolio set $[d,u]$ is unbounded, where $\hat{U}$ is the concave envelope of U. Define $K(B-) = K(B)$ and $L = \sigma\max\{-d,u\}$. Let V be a locally bounded function.

(i). We say that V is a viscosity subsolution of the HJB equation (14) with the boundary condition (13) and the asymptotic property (15) (or (18) for unbounded portfolio set) if it satisfies the following conditions:

a) For all smooth ψ such that $V^* \le \psi$ and $V^*(\bar{t},\bar{w}) = \psi(\bar{t},\bar{w})$ for some $(\bar{t},\bar{w}) \in [0,T)\times(B,+\infty)$,

$$-\frac{\partial\psi(\bar{t},\bar{w})}{\partial t} - H^*\left(\bar{w}, \frac{\partial\psi(\bar{t},\bar{w})}{\partial w}, \frac{\partial^2\psi(\bar{t},\bar{w})}{\partial w^2}\right) \le 0. \tag{23}$$

b) For all $0 \le t < T$,

$$\begin{cases} \limsup\limits_{(s,w)\to(t,B)} V(s,w) - U(B) \le 0 & \text{if } U(B) > -\infty \\ \limsup\limits_{(s,w)\to(t,0)} V(s,w) - V_{CRRA}(s,w) \le 0 & \text{if } B=0 \text{ and } U(0) = -\infty. \end{cases} \tag{24}$$

c) For all $w \ge B$,

$$\limsup_{(t,\zeta)\to(T-,w)} V(t,\zeta) - K(w-) - 2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L\sqrt{T-t}}\right)(K(w) - K(w-)) \le 0. \tag{25}$$

[7] When the portfolio set $[d,u]$ is bounded, the Hamiltonian H is continuous. When the portfolio set $[d,u]$ is unbounded, the Hamiltonian H is infinite for $A>0$, finite and continuous for $A<0$, and either finite or infinite at $A=0$; if it is finite at $A=0$, it is left continuous at $A=0$, and thus, it is lower-semicontinuous at $A=0$.

(ii). We say that V is a viscosity supersolution of the HJB equation (14) with the boundary condition (13) and the asymptotic condition (15) (or (18) for unbounded portfolio set) if it satisfies the following conditions:
a) For all smooth φ such that $V_* \geq \varphi$ and $V_*(\bar{t},\bar{w}) = \varphi(\bar{t},\bar{w})$ for some $(\bar{t},\bar{w}) \in [0,T) \times (B,+\infty)$,

$$-\frac{\partial\varphi(\bar{t},\bar{w})}{\partial t} - H_*\left(\bar{w}, \frac{\partial\varphi(\bar{t},\bar{w})}{\partial w}, \frac{\partial^2\varphi(\bar{t},\bar{w})}{\partial w^2}\right) \geq 0. \tag{26}$$

b) For all $0 \leq t < T$,

$$\begin{cases} \liminf\limits_{(s,w)\to(t,B)} V(s,w) - U(B) \geq 0 & \text{if } U(B) > -\infty \\ \liminf\limits_{(s,w)\to(t,0)} V(s,w) - V_{CRRA}(s,w) \geq 0 & \text{if } B = 0 \text{ and } U(0) = -\infty. \end{cases} \tag{27}$$

c) For all $w \geq B$,

$$\liminf_{(t,\zeta)\to(T-,w)} V(t,\zeta) - K(w-) - 2\Phi\left(\frac{\min\{0, \log\zeta/w\}}{L\sqrt{T-t}}\right)(K(w) - K(w-)) \geq 0. \tag{28}$$

(iii). We say that V is a viscosity solution if it is both a viscosity supersolution and subsolution.

Theorem 4.2 characterizes the value function $V(t,w)$ as the unique viscosity solution of the HJB equation (14). The value function is continuous at the liquidation boundary $w = B$ if $U(B) > -\infty$, i.e. $\lim\limits_{(t',\zeta)\to(t,B)} V(t',\zeta) = U(B) = V(t,B)$. However, at the terminal time T, there are three cases: (1) If the portfolio set $[d,u]$ is bounded and the utility function U is continuous, Theorem 4.2 indicates that the value function is continuous at the terminal condition, that is, $\lim\limits_{(t,\zeta)\to(T-,w)} V(t,\zeta) = U(w) = V(T,w)$. (2) If the portfolio set $[d,u]$ is unbounded, Theorem 4.2 indicates that the value function converges to the concave envelope of the utility function (see the right panel of Figures 3 and 8) and thus is in general discontinuous, that is, $\lim\limits_{(t,\zeta)\to(T-,w)} V(t,\zeta) = \hat{U}(w) \neq U(w)$ in general. (3) If the portfolio set $[d,u]$ is bounded and the utility function U is discontinuous, Theorem 4.2 shows that the value function is discontinuous at (T,w) where w denotes a discontinuous point of $U(\cdot)$. Different from case (2), the left limit of the value function even does not exist (see the left panel of Figures 3 and 8), that is, $\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta)$ does not exist, where w is the discontinuous point of $U(\cdot)$.

In the special case of one-side constraints, Theorem 4.1 and Theorem 4.2 are related to Bian, Chen and Xu (2019). As one-side portfolio constraints are unbounded, the corresponding non-concave utility can be replaced by its concave envelope; thus, the non-concave portfolio optimization problem with one-side portfolio bounds studied in Bian, Chen and Xu (2019) is essentially a concave one. This is very different from the case of two-side portfolio bounds (i.e., with bounded portfolio sets), in which the conconvafication principle fails. Besides, Bian, Chen and Xu (2019) restrict attention to a geometric Brownian motion market; in Theorems EC.1.1 and EC.1.2, we extend their results to a stochastic investment environment.

Appendix B: The Finite Difference Scheme Used

Given a big enough upper bound $A > B$. Let $\Sigma_\Delta = \{(t_n, w_i) : 0 \leq n \leq N_t, 0 \leq i \leq N_w\}$ be a discretization mesh of the domain $[0,T] \times [B,A]$ if $U(B) > -\infty$ or $[0,T] \times [\delta, A]$ if $B = 0$ and $U(0) = -\infty$ with fixed step size Δt and Δw, where $\delta > 0$ is a constant. Let V_i^n be the solution at grid (t_n, w_i) of the following discretization version of the HJB equation (14):

$$-\frac{V_i^{n+1} - V_i^n}{\Delta t} - \sup_{d\leq\pi\leq u}\left\{\frac{\pi^2 w_i^2\sigma^2}{2}\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta w)^2} + \pi w_i \eta \frac{\Delta V_i^n(\pi)}{\Delta w}\right\} = 0, \tag{29}$$

with the boundary and terminal conditions [8]

$$V_0^n = \begin{cases} U(B) & \text{if } U(B) > -\infty \\ V_{CRRA}(\delta) & \text{if } B = 0 \text{ and } U(0) = -\infty \end{cases}, \quad n = 0, 1, 2, \ldots, N_t, \tag{30}$$

$$V_{N_w}^n = U(A), \quad n = 0, 1, 2, \ldots, N_t, \tag{31}$$

$$V_i^{N_t} = U(w_i), \quad i = 1, 2 \ldots, N_w - 1. \tag{32}$$

The first order difference $\Delta V_i^n(\pi)$ is defined for $d \le \pi \le u$ as following:

$$\Delta V_i^n(\pi) = \begin{cases} (V_{i+1}^n - V_{i-1}^n)/2 & \text{if } |\pi| > \pi_i \\ V_{i+1}^n - V_i^n & \text{if } |\pi| < \pi_i \text{ and } \pi\eta > 0 \\ V_i^n - V_{i-1}^n & \text{if } |\pi| < \pi_i \text{ and } \pi\eta < 0, \end{cases} \tag{33}$$

where $\pi_i = |\eta|\Delta w/(\sigma^2 w_i)$, and the difference method is chosen such that the objective function of the optimization problem in (29) is upper semicontinuous. The difference form in (33) is to maximize the use of central differencing as in Wang and Forsyth(2008). By Lemma EC.4.6, the scheme is monotone, consistent and stable. (29) is a nonlinear equation, which can be solved by the following iterative procedure: $V_i^{n,0} = V_i^{n+1}$, and given $V_i^{n,k}$, let

$$\pi_i^{n,k} = \arg \sup_{d \le \pi \le u} \left\{ \frac{\pi^2 w_i^2 \sigma^2}{2} \frac{V_{i+1}^{n,k} - 2V_i^{n,k} + V_{i-1}^{n,k}}{(\Delta w)^2} + \pi w_i \eta \frac{\Delta V_i^{n,k}(\pi)}{\Delta w} \right\}, \tag{34}$$

where $\Delta V_i^{n,k}(\pi)$ is defined in (33) replacing all $\{n\}$ in superscript by $\{n,k\}$, and $V_i^{n,k+1}$ solves the following linear system of equations:

$$\frac{V_i^{n+1} - V_i^{n,k+1}}{\Delta t} + \frac{(\pi_i^{n,k})^2 w_i^2 \sigma^2}{2} \frac{V_{i+1}^{n,k+1} - 2V_i^{n,k+1} + V_{i-1}^{n,k+1}}{(\Delta w)^2} + \pi_i^{n,k} w_i \eta \frac{\Delta V_i^{n,k+1}(\pi_i^{n,k})}{\Delta w} = 0.$$

In matrix form, the above system together with boundary conditions (30) and (31) can be rewritten as

$$\frac{V^{n+1} - V^{n,k+1}}{\Delta t} - A(\pi^{n,k}) V^{n,k+1} = F^{n,k},$$

where $V^{n+1} = \{V_1^{n+1}, V_2^{n+1}, \ldots, V_{N_w-1}^{n+1}\}^\top$, $V^{n,k+1} = \{V_1^{n,k+1}, V_2^{n,k+1}, \ldots, V_{N_w-1}^{n,k+1}\}^\top$, $\pi^{n,k} = \{\pi_1^{n,k}, \pi_2^{n,k}, \ldots, \pi_{N_w-1}^{n,k}\}^\top$, $F^{n,k}(\pi^{n,k}) = \{-\alpha_1(\pi_1^{n,k}) V_0^{n,k}, 0, \ldots, 0, -\beta_{N_w-1}(\pi_{N_w-1}^{n,k}) V_{N_w}^{n,k}\}^\top$, and the components of matrix $A(\pi^{n,k})$ are zeros except

$$\begin{cases} A_{i,i-1}(\pi^{n,k}) = -\alpha_i(\pi_i^{n,k}) & \text{for } 2 \le i \le N_z - 1 \\ A_{i,i}(\pi^{n,k}) = \alpha_i(\pi_i^{n,k}) + \beta_i(\pi_i^{n,k}) & \text{for } 1 \le i \le N_z - 1 \\ A_{i,i+1}(\pi^{n,k}) = -\beta_i(\pi_i^{n,k}) & \text{for } 1 \le i \le N_z - 2 \end{cases}$$

with $\alpha_i(\pi)$ and $\beta_i(\pi)$, $i = 1, \ldots, N_w$ defined below:

$$\begin{cases} \alpha_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2} - \frac{\pi w_i \eta}{2\Delta w}, \ \beta_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2} + \frac{\pi w_i \eta}{2\Delta w} & \text{if } |\pi| > \pi_i \\ \alpha_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2}, \ \beta_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2} + \frac{\pi w_i \eta}{\Delta w} & \text{if } |\pi| < \pi_i \text{ and } \pi\eta > 0 \\ \alpha_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2} - \frac{\pi w_i \eta}{\Delta w}, \ \beta_i(\pi) = \frac{\pi^2 w_i^2 \sigma^2}{2(\Delta w)^2} & \text{if } |\pi| < \pi_i \text{ and } \pi\eta < 0. \end{cases}$$

Our program code for the goal-reaching problem under the Black-Scholes market is available at https://www.dropbox.com/s/00csp2xup5k85qe/CodeGoal.rar?dl=0.

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[8] If the original portfolio set $[d,u]$ is unbounded, we can replace U by its concave envelope $\hat{U}$. By Lemma EC.4.3, we can replace the unbounded portfolio constraint set $[d,u]$ by a bounded set $[-C,C] \cap [d,u]$ for some large C.

References

Agarwal, V., N. Boyson, and N.Y. Naik (2009) Hedge funds for retail investors? An examination of hedged mutual funds. *Journal of Financial and Quantitative Analysis*, 44, 273–305.

Barles, G. and P.E. Souganidis (1991) Convergence of approximation schemes for fully nonlinear equations. *Asymptotic Analysis*, 4, 271–283.

Basak, S. and G. Chabakauri (2010) Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23, 2970–3016.

Basak, S., A. Pavlova, and A. Shapiro (2007) Optimal asset allocation and risk shifting in money management, *Review of Financial Studies*, 20, 1583–1621.

Berkelaar, A.B., R. Kouwenberg, and T. Post (2004) Optimal portfolio choice under loss aversion, *The Review of Economics and Statistics*, 86(4), 973–987.

Bian, B.J., X.F. Chen, and Z.Q. Xu (2019) Utility maximization under trading constraints with discontinuous utility, *SIAM Journal on Financial Mathematics* 10 (1), 243–260.

Browne, S. (1999a) Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Advances in Applied Probability*, 31(2), 551–577.

Browne, S. (1999b) The risk and rewards of minimizing shortfall probability. *Journal of Portfolio Management*, 25(4), 76–85.

Brown, K.C., W.V. Harlow, and L.T. Starks (1996) Of tournaments and temptations: An analysis of managerial incentives in the mutual fund industry. *Journal of Finance*, 51, 85-110.

Busse, J.A. (2001) Another look at mutual fund tournaments. *Journal of Financial and Quantitative Analysis*, 36, 53–73.

Carpenter, J. (2000) Does option compensation increase managerial risk appetite? *Journal of Finance*, 55(5), 2311–2331.

Chen, H., H. Desai, and S. Krishnamurthy (2013) A first look at mutual funds that use short sales. *Journal of Financial and Quantitative Analysis*, 48(3), 761–787.

Chevalier, J. and G. Ellison (1997) Risk taking by mutual funds as a response to incentives. *Journal of Political Economy*, 105, 1167–200.

Crandall, M.G., H. Ishii, and P. L. Lions (1992) User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27, 1–67.

Cvitanić, J. and I. Karatzas (1992) Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2(4), 767–818.

Dai, M., H. Jin, and H. Liu (2011) Illiquidity, position limits, and optimal investment for mutual funds, *Journal of Economic Theory*, 146(4), 1598–1630.

Dai, M., H. Jin, S. Kou, and Y. Xu (2021) A dynamic mean-variance analysis for log returns, *Management Science*, 67(2), 1093–1108.

DeMiguel, V., L. Garlappi, F.J. Nogales, and R. Uppal (2009) A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms, *Management Science*, 55(5), 798–812.

Diecidue, E. and J. van de Ven (2008) Aspiration level, probability of success and failure, and expected Utility. *International Economic Review*, 49, 683–700.

Fleming, W.H. and H.M. Soner (2006) *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York.

Föllmer, H. and P. Leukert (1999) Quantile hedging, *Finance and Stochastics* 3, 251–273.

Forsyth, P.A. and G. Labahn (2007) Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance. *Journal of Computational Finance*, 11(2), 1–43.

Haugh, M.B., L. Kogan, and J. Wang (2006) Evaluating portfolio policies: A duality approach. *Operations Research*, 54, 405-418.

He, X.D. and S. Kou (2018) Profit sharing in hedge funds. *Mathematical Finance*, 28(1), 50–81.

Jin, H. and X. Y. Zhou (2008) Behavioral portfolio selection in continuous time, *Mathematical Finance*, 18(3), 385–426.

Kahneman, D. and A. Tversky (1979) Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2), 263–291.

Karatzas, I, J.P. Lehoczky, S.E. Shreve, and G.L. Xu (1991) Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization* 29, 702–730.

Kempf, A., S. Ruenzi, and T. Thiele (2009) Employment risk and compensation incentives as determinants of managerial risk taking. *Journal of Financial Economics*, 92, 92–108.

Kim, T. S. and E. Omberg (1996) Dynamic nonmyopic portfolio behavior. *Review of Financial Studies*, 9, 141–61.

Lee, S., F. Zapatero, and A. Giga (2018) Rolling the skewed die: Economic foundations of the demand for skewness. Available at SSRN:3168944.

Liu, J. (2007) Portfolio selection in stochastic environments. *Review of Financial Studies*, 20, 1–39.

Sironi, P. (2016) *FinTech Innovation: From Robo-advisors to Goal Based Investing and Gamification.* The Wiley Finance Series, Chichester.

Schwarz, S.G. (2012) Mutual fund tournaments: The sorting bias and new evidence, *Review of Financial Studies*, 25(3), 913–936.

Spivak, G. and J. Cvitanić (1999) Maximizing the probability of a perfect hedge. *Annals of Applied Probability*, 9(4), 1303–1328.

Vila, J.L. and T. Zariphopoulou (1997) Optimal consumption and portfolio choice with borrowing constraints, *Journal of Economic Theory*, 77(2), 402–431.

Wang, J. and P.A. Forsyth (2008) Maximal use of central differencing for Hamilton-Jacobi-Bellman PDEs in finance, *SIAM Journal on Numerical Analysis*, 46(3), 1580–1601.

Online Supplement

Non-Concave Utility Maximization with Portfolio Bounds

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Appendix EC.1: Model Extension

In this section, we extend our model to a more general market setting, including a stochastic investment opportunity set and multiple risky stocks. We assume that the financial market has one riskless bond and N risky stocks. The risk-free interest rate is r, and the dynamics of the stock prices $S_i, i=1,\dots,N$, follow the following diffusion processes

$$dS_{it}/S_{it} = \mu_i(t,\boldsymbol{x}_t)dt + \sum_{j=1}^{N} \sigma_{ij}(t,\boldsymbol{x}_t)d\mathcal{B}_{jt}, \tag{EC.1}$$

where $\boldsymbol{\mathcal{B}}_t = (\mathcal{B}_{1t},\dots,\mathcal{B}_{Nt})^\top$ is a standard N-dimensional Brownian motion, the drift $\boldsymbol{\mu}_t = (\mu_{1t},\dots,\mu_{Nt})^\top$ and volatility $\boldsymbol{\sigma}_t = \{\sigma_{ij}\}_{N\times N}$ are assumed to be deterministic functions of time t and a K-dimensional stochastic state variable $\boldsymbol{x}_t$, namely, $\boldsymbol{\mu}_t = \boldsymbol{\mu}(t,\boldsymbol{x}_t)$, $\boldsymbol{\sigma}_t = \boldsymbol{\sigma}(t,\boldsymbol{x}_t)$. Assume that $\boldsymbol{x}_t$ follows the dynamics:

$$d\boldsymbol{x}_t = \boldsymbol{m}(t,\boldsymbol{x}_t)dt + \boldsymbol{\nu}(t,\boldsymbol{x}_t)d\boldsymbol{\mathcal{B}}^{\boldsymbol{x}}_t, \tag{EC.2}$$

where $\boldsymbol{\mathcal{B}}^x_t = (\mathcal{B}^x_{1t},\dots,\mathcal{B}^x_{Kt})^\top$ is another standard K-dimensional Brownian motion with $E[d\boldsymbol{\mathcal{B}}_{it}d\boldsymbol{\mathcal{B}}^x_{jt}] = \rho_{ij}dt$, for $i=1\dots,N$ and $j=1,\dots,K$, and the K-dimensional column vector $\boldsymbol{m}(\cdot,\cdot)$ and the K-by-K matrix $\boldsymbol{\nu}(\cdot,\cdot)$ also depend on t and $\boldsymbol{x}_t$. Suppose that all of $\boldsymbol{\mu}(\cdot,\cdot)$, $\boldsymbol{\sigma}(\cdot,\cdot)$, $\boldsymbol{m}(\cdot,\cdot)$, and $\boldsymbol{\nu}(\cdot,\cdot)$ are continuous, and r and ρ_{ij} are constants.

Consider a self-financing portfolio strategy that invests $\Pi_{i,t}$ dollars in the risky asset S_i at time t, $1\le i\le N$, and $\boldsymbol{\Pi}_t = (\Pi_{1,t},\dots,\Pi_{N,t})^\top$, resulting in the portfolio value $\tilde{W}_t$, subject to a liquidation constraint $\tilde{W}_t \ge Be^{-r(T-t)}$, $0\le t\le T$, for some non-negative constant B. For convenience, let $\boldsymbol{\pi}_t := \boldsymbol{\Pi}_t/\tilde{W}_t$ be the proportion of wealth invested in the stocks, and let $W_t = \tilde{W}_t e^{r(T-t)}$, $0\le t\le T$ be the (forward) portfolio value at time t. It follows that

$$dW_t = W_t(\boldsymbol{\pi}_t^\top \boldsymbol{\eta}(t,\boldsymbol{x}_t)dt + \boldsymbol{\pi}_t^\top \boldsymbol{\sigma}(t,\boldsymbol{x}_t)d\boldsymbol{\mathcal{B}}_t), \tag{EC.3}$$

where $\boldsymbol{\eta}(t,x) = \boldsymbol{\mu}(t,x) - r$ is the excess rate of return of the risky assets, and the liquidation constraint is simplified as

$$W_t \ge B, \quad 0\le t\le T. \tag{EC.4}$$

Let $U(\cdot)$ be a utility function that an agent uses to evaluate the portfolio at a finite maturity T. The agent's portfolio maximization problem could be stated as following:

$$\sup_{\boldsymbol{\pi}\in \boldsymbol{C}} E[U(W_T)], \tag{EC.5}$$

where W_t follows (EC.3) subject to liquidation constraint (EC.4), and the portfolio constraint set $\boldsymbol{C}$ is a closed set in $\mathbb{R}^N$ containing the origin and $\boldsymbol{C}\backslash\{\mathbf{0}\}$ is not empty.

Let $V(t,w,\boldsymbol{x})$ be the time t value function of problem (EC.5) conditional on $W_t = w, \boldsymbol{x}_t = \boldsymbol{x}$. At the terminal time T, the value function is the utility function by definition, i.e.

$$V(T,w,\boldsymbol{x}) = U(w), \text{ for all } \ w \geq B. \tag{EC.6}$$

When $W_t = B$ for some $t < T$, liquidation is necessary, which implies the following boundary condition (See Proposition EC.3.2)

$$\begin{cases} \lim\limits_{(s,w,\boldsymbol{\xi})\to(t,B,\boldsymbol{x})} V(s,w,\boldsymbol{\xi}) = U(B) & \text{if } U(B) > -\infty \\ \lim\limits_{(s,w,\boldsymbol{\xi})\to(t,0,\boldsymbol{x})} V(s,w,\boldsymbol{\xi}) - V_{CRRA}(s,w,\boldsymbol{\xi}) = 0 & \text{if } B = 0 \text{ and } \ U(0) = -\infty, \end{cases} \tag{EC.7}$$

where V_{CRRA} is the value function with utility $U(w) = A_1\frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$, $\widetilde{p} < 1$ and constraint $\boldsymbol{\pi} \in \boldsymbol{C}$.

The associated HJB equation is

$$\begin{aligned} &\frac{\partial V}{\partial t} + \sup_{\boldsymbol{\pi}_t\in \boldsymbol{C}} \left\{ \frac{1}{2} w^2 \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi}_t \frac{\partial^2 V}{\partial w^2} + w\boldsymbol{\pi}_t^\top \boldsymbol{\eta}_t \frac{\partial V}{\partial w} + w\boldsymbol{\pi}_t^\top \boldsymbol{\sigma}_t \boldsymbol{\rho} \boldsymbol{\nu}_t^\top \nabla_{\boldsymbol{x}} (\frac{\partial V}{\partial w}) \right\} \\ &\quad + \frac{1}{2} Tr(\boldsymbol{\nu}_t \boldsymbol{\nu}_t^\top \nabla^2_{\boldsymbol{x}} V) + \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} V = 0, \end{aligned} \tag{EC.8}$$

where $\boldsymbol{\Sigma}_{t,\boldsymbol{x}} := \boldsymbol{\sigma}(t,\boldsymbol{x})\boldsymbol{\sigma}(t,\boldsymbol{x})^\top$ is the variance-covariance matrix of the stocks, $\nabla_{\boldsymbol{x}} f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_K)^\top$ and $\nabla^2_{\boldsymbol{x}} f = (\partial^2 f/\partial x_i x_j)_{K\times K}$ are the gradient and Hessian matrix of function f, respectively, $Tr(A)$ is the trace of matrix A, and to simplify notations we denote $\boldsymbol{\eta}_t = \boldsymbol{\eta}(t,\boldsymbol{x}),\ \boldsymbol{m}_t = \boldsymbol{m}(t,\boldsymbol{x}),\ \boldsymbol{\nu}_t = \boldsymbol{\nu}(t,\boldsymbol{x})$.

Let us first specify some technique assumptions needed for our proof.

Assumption EC.1.1.

(1) The utility function $U(w)$ satisfies Assumption 3.1.

(2) $\boldsymbol{\sigma}(t,\boldsymbol{x})$, $\boldsymbol{\nu}(t,\boldsymbol{x})$ are invertible, and $\boldsymbol{\mu}(t,\boldsymbol{x})$, $\boldsymbol{\sigma}(t,\boldsymbol{x})$, $\boldsymbol{m}(t,\boldsymbol{x})$, and $\boldsymbol{\nu}(t,\boldsymbol{x})$ are $C^{1,2}$ in $(t,\boldsymbol{x})$.

(3) For any $\delta > 0$, $\lim\limits_{t\to T} P[\sup\limits_{t\leq s\leq T} |\boldsymbol{x}_s - \boldsymbol{x}| \geq \delta | \boldsymbol{x}_t = \boldsymbol{x}] \to 0$ local uniformly in $\boldsymbol{x}$. Besides, $\lim\limits_{n\to+\infty} P[\sup\limits_{t\leq s\leq T} |\boldsymbol{x}_s| \geq n | \boldsymbol{x}_t = \boldsymbol{x}] \to 0$.

(4) For the unconstrained problem, there exists a classical solution for $U(w) = \frac{w^{\hat{p}}}{\hat{p}}$, $0 < \hat{p} < 1$, in a time interval $(t_{\hat{p}}, T]$, where $t_{\hat{p}}$ is a continuous and increasing function w.r.t. $\hat{p}$ for $0 < \hat{p} < 1$. Moreover, we assume the value function has the form [9]

$$V(t,w,\boldsymbol{x}) = \frac{w^{\hat{p}}}{\hat{p}} F^{\hat{p}}(t,\boldsymbol{x}). \tag{EC.9}$$

[9] Kim and Omberg (1996) give closed-form solutions for a Gaussian mean return model, where the value function may be infinite for a finite time horizon, named *nirvana solution* in their Section 2.3.

Under Assumption EC.1.1, we can derive asymptotic conditions of the value function $V(t,w,\boldsymbol{x})$ for the above more general market setting as following (see Proposition EC.3.3 for a rigorous proof): (i). When the portfolio set $\boldsymbol{C}$ is bounded,

$$\lim_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) - U(w-) - 2\Phi\left(\frac{\min\{0,\log\zeta/w\}}{L_{T,\boldsymbol{x}}\sqrt{T-t}}\right)(U(w)-U(w-)) = 0, \tag{EC.10}$$

where $U(w-)$ is the left limit of U at w, $U(B-)=U(B)$, $L_{t,\boldsymbol{x}} = \sqrt{\max_{\boldsymbol{\pi}_t\in\boldsymbol{C}} \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi}_t}$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable. (ii). When the portfolio set $\boldsymbol{C}$ is unbounded,

$$\lim_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) = \hat{U}(w), \tag{EC.11}$$

where $\hat{U}$ is the concave envelope of U.

Thus, when the portfolio set $\boldsymbol{C}$ is unbounded, the value function is concave and the concavification principle holds. When the portfolio set C is bounded and the utility function U(·) is not concave, we infer by (EC.10) that the value function is unlikely concave as time approaches the deadline T, which implies that the concavification principle does not hold.

We extend the new definition of viscosity solution in Appendix A by replacing asymptotic conditions (15) and (18) with (EC.10) and (EC.11) for the general market setting. As a preparation, define the Hamiltonian associated with the HJB equation (EC.8):

$$H(t,w,\boldsymbol{x},\boldsymbol{p},\boldsymbol{A}) := \sup_{\boldsymbol{\pi}_t\in\boldsymbol{C}} \{\frac{1}{2}Tr\left(\boldsymbol{A}\Sigma_{t,w,\boldsymbol{x}}\right) + \left(w\boldsymbol{\pi}_t^\top\boldsymbol{\eta}_t, \boldsymbol{m}_t^\top\right)\boldsymbol{p}\}, \tag{EC.12}$$

where $\boldsymbol{A}$ is a $(K+1)\times(K+1)$ matrix, $\boldsymbol{p}$ is a $K+1$-dimensional column vector, $Tr(\boldsymbol{A}\Sigma_{t,w,\boldsymbol{x}})$ is the trace of the matrix $\boldsymbol{A}\Sigma_{t,w,\boldsymbol{x}}$, and $\Sigma_{t,w,\boldsymbol{x}}$ is the variance-covariance matrix function of joint process $(W_t,\boldsymbol{x}_t)$ (cf. (EC.3) and (EC.2)) at time t and $W_t=w$ and $\boldsymbol{x}_t=\boldsymbol{x}$, given by

$$\Sigma_{t,w,\boldsymbol{x}} = \begin{pmatrix} w^2\boldsymbol{\pi}_t^\top\boldsymbol{\Sigma}_{t,\boldsymbol{x}}\boldsymbol{\pi}_t & w\boldsymbol{\pi}_t^\top\boldsymbol{\sigma}_t\boldsymbol{\rho}\boldsymbol{\nu}_t^\top \\ w\boldsymbol{\pi}_t^\top\boldsymbol{\sigma}_t\boldsymbol{\rho}\boldsymbol{\nu}_t^\top & \boldsymbol{\nu}_t\boldsymbol{\nu}_t^\top \end{pmatrix}. \tag{EC.13}$$

Then, we can rewrite the HJB equation (EC.8) as

$$-\frac{\partial V}{\partial t}(t,w,\boldsymbol{x}) - H\left(t,w,\boldsymbol{x},\nabla_{w,\boldsymbol{x}}V,\nabla^2_{w,\boldsymbol{x}}V\right) = 0.. \tag{EC.14}$$

Note that $H(t,w,\boldsymbol{x},\boldsymbol{p},\boldsymbol{A})$ is continuous except at $A_{11}=0$ at which it is likely lower semicontinuous for unbounded portfolio set $\boldsymbol{C}$.[10] Thus, we need to use its lower semicontinuous envelope H_* and upper semicontinuous envelope H^* (cf. (20)) to define the viscosity supersolution and subsolution as following:

[10] A_{11} is the first row and first column element of matrix $\boldsymbol{A}$. When the portfolio set $\boldsymbol{C}$ is bounded, the Hamiltonian H is continuous. When the portfolio set $\boldsymbol{C}$ is unbounded, the function is infinite for $A_{11}>0$, finite and continuous for $A_{11}<0$, and either finite or infinite at $A_{11}=0$; if it is finite at $A_{11}=0$, it is left continuous at $A_{11}=0$, and if it is infinite at $A_{11}=0$, the left limit at $A_{11}=0$ is infinite. So, it would be lower-semicontinuous at $A_{11}=0$.

Definition EC.1.1. For $w \geq B$, let $K(w) = U(w)$ if the portfolio set $\boldsymbol{C}$ is bounded, and $K(w) = \hat{U}(w)$ if the portfolio set $\boldsymbol{C}$ is unbounded, where $\hat{U}$ is the concave envelope of U. Define $K(B-) = K(B)$ and $L_{t,\boldsymbol{x}} = \sqrt{\max_{\boldsymbol{\pi}_t \in \boldsymbol{C}} \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi}_t}$. Let V be a locally bounded function.

(i) We say that V is a viscosity subsolution of the HJB equation (EC.8) with the boundary condition (EC.7) and the asymptotic property (EC.10) (or (EC.11) for unbounded portfolio set) if it satisfies the following conditions:

(a) For all smooth ψ such that $V^* \leq \psi$ and $V^*(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) = \psi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}})$ for some $(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) \in [0, T) \times (B, +\infty) \times \mathbb{R}^K$,

$$-\frac{\partial \psi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}})}{\partial t} - H^* \left(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}, \nabla_{w,\boldsymbol{x}} \psi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}), \nabla^2_{w,\boldsymbol{x}} \psi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) \right) \leq 0. \tag{EC.15}$$

(b) For all $0 \leq t < T$,

$$\begin{cases} \limsup_{(s,w,\boldsymbol{\xi}) \to (t,B,\boldsymbol{x})} V(s, w, \boldsymbol{\xi}) \leq U(B) & \text{if } U(B) > -\infty \\ \limsup_{(s,w,\boldsymbol{\xi}) \to (t,0,\boldsymbol{x})} V(s, w, \boldsymbol{\xi}) - V_{CRRA}(s, w, \boldsymbol{\xi}) \leq 0 & \text{if } B = 0 \text{ and } U(0) = -\infty. \end{cases} \tag{EC.16}$$

(c) For all $w \geq B$,

$$\limsup_{(t,\zeta,\boldsymbol{\xi}) \to (T-,w,\boldsymbol{x})} V(t, \zeta, \boldsymbol{\xi}) - K(w-) - 2\Phi\left(\frac{\min\{0, \log \zeta / w\}}{L_{T,\boldsymbol{x}} \sqrt{T - t}} \right) (K(w) - K(w-)) \leq 0. \tag{EC.17}$$

(ii) We say that V is a viscosity supersolution of the HJB equation (EC.8) with the boundary condition (EC.7) and the asymptotic condition (EC.10) (or (EC.11) for unbounded portfolio set) if it satisfies the following conditions:

(a) For all smooth φ such that $V_* \geq \varphi$ and $V_*(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) = \varphi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}})$ for some $(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) \in [0, T) \times (B, +\infty) \times \mathbb{R}^K$,

$$-\frac{\partial \varphi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}})}{\partial t} - H_* \left(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}, \nabla_{w,\boldsymbol{x}} \varphi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}), \nabla^2_{w,\boldsymbol{x}} \varphi(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}) \right) \geq 0. \tag{EC.18}$$

(b) For all $0 \leq t < T$,

$$\begin{cases} \liminf_{(s,w,\boldsymbol{\xi}) \to (t,B,\boldsymbol{x})} V(s, w, \boldsymbol{\xi}) \geq U(B) & \text{if } U(B) > -\infty \\ \liminf_{(s,w,\boldsymbol{\xi}) \to (t,0,\boldsymbol{x})} V(s, w, \boldsymbol{\xi}) - V_{CRRA}(s, w, \boldsymbol{\xi}) \geq 0 & \text{if } B = 0 \text{ and } U(0) = -\infty. \end{cases} \tag{EC.19}$$

(c) For all $w \geq B$,

$$\liminf_{(t,\zeta,\boldsymbol{\xi}) \to (T-,w,\boldsymbol{x})} V(t, \zeta, \boldsymbol{\xi}) - K(w-) - 2\Phi\left(\frac{\min\{0, \log \zeta / w\}}{L_{T,\boldsymbol{x}} \sqrt{T - t}} \right) (K(w) - K(w-)) \leq 0. \tag{EC.20}$$

(iii) We say that V is a viscosity solution if it is both a viscosity supersolution and subsolution.

Assumption EC.1.2. *The function $F^{(\hat{p})}(t,\boldsymbol{x})$ of Assumption EC.1.1 has the form*

$$F^{(\hat{p})}(t,\boldsymbol{x}) := e^{\boldsymbol{x}^\top A_{\hat{p}}(t)\boldsymbol{x}+B_{\hat{p}}(t)\cdot\boldsymbol{x}+C_{\hat{p}}(t)}, \tag{EC.21}$$

where $F^{(\hat{p})}$ is finite for $t_{\hat{p}} < t \le T$, $A_{\hat{p}} > 0$ and $A_{\hat{p}}$, $F^{(\hat{p})}$ are strictly decreasing in t, and

$$\begin{cases} \lim\limits_{q\to\hat{p}+} A_q(t) = A_{\hat{p}}(t), \\ \lim\limits_{q\to\hat{p}+} B_q(t) = B_{\hat{p}}(t), \\ \lim\limits_{q\to\hat{p}+} C_q(t) = C_{\hat{p}}(t), \\ A_q(t) > A_{\hat{p}}(t), \qquad \text{when } q > \hat{p},\ t_q < t \le T. \end{cases}$$

According to Liu (2007), the value function satisfies (EC.9) and (EC.21) when the underlying process follows quadratic diffusion processes, covering many affine models. See also Kim and Omberg (1996) for closed-form solutions of a Gaussian mean return model.

The following theorem shows that a novel comparison principle holds for the new definition of the viscosity solution satisfying the asymptotic property (EC.10) or (EC.11).

Theorem EC.1.1 **(A New Comparison Principle)**. *Let Assumptions EC.1.1 and EC.1.2 hold. Then, for any viscosity subsolution u and supersolution v of Definition EC.1.1 that are bounded from above by $(w^{\hat{p}} + C_{\hat{p}})F^{(\hat{p})}(t,\boldsymbol{x})$, we have $u \le v$ for $w \ge B$ and $t_{\hat{p}} < t < T$.*

Based on the new comparison principle in Theorem EC.1.1, we can prove the following theorem that the value function is the unique (new) viscosity solution of the HJB equation under the general multivariate market setting.

Theorem EC.1.2. *Let Assumptions EC.1.1 and EC.1.2 hold. When the portfolio set $\boldsymbol{C}$ is bounded (resp. unbounded), the value function V of* (EC.5) *is the unique viscosity solution of the HJB equation* (EC.8) *with the boundary condition* (EC.7) *and the asymptotic property* (EC.10) *(resp.* (EC.11)*). Furthermore, V is continuous for $t < T$.*

We can also resort to the finite difference method to numerically solve for the value function and the optimal portfolios. The following theorem gives the convergence of monotone, stable, and consistent finite difference schemes (see Barles and Souganidis(1991) for the definitions of monotonicity, stability, and consistency).

Theorem EC.1.3. *Let Assumptions EC.1.1 and EC.1.2 hold. A monotone, stable, and consistent finite difference scheme for the HJB equation* (EC.8) *with the boundary condition* (EC.7) *and the asymptotic condition* (EC.10) *(resp.* (EC.11)*) for bounded (resp. unbounded) portfolio constraint set $\boldsymbol{C}$ converges to the value function as the discretization size tends to zero.*

For the general case, including a stochastic investment opportunity set and multiple risky stocks with a general portfolio constraint $\boldsymbol{\pi} \in \boldsymbol{C}$, we can apply the finite difference scheme of Fleming and Soner (2016) to give a discretization version for the multi-dimensional HJB equation (EC.8). See IX.3.28 of Fleming and Soner (2016) for the details of the differencing method. Similar to (29), the corresponding discretization HJB equation is a nonlinear equation, which can be solved by an iterative procedure where a supermum over $\boldsymbol{\pi} \in \boldsymbol{C}$ is computed repeatedly. The iteration does not increase too much computational burden even for a general set $\boldsymbol{C}$, mainly because the objective function is a quadratic function of $\boldsymbol{\pi}$ (see (34) as an example).

Note that the dimensionality of the HJB equation (EC.8) increases only when the dimensionality of the stochastic state variable $\boldsymbol{x}$ increases. Thus, increasing the number of stocks does not necessarily lead to an increase in the dimensionality of the problem. We can solve the HJB equation (EC.8) effectively by the finite difference scheme for the multiple stocks case when the investment opportunity set is deterministic or when the investment opportunity set is characterized by risk factors such as the stochastic volatility or the stochastic risk premium.

Next, we consider a special incomplete market model with a stochastic investment opportunity set, i.e., the time-varying Gaussian mean return model, and relegate the stochastic volatility model and the model with two stocks to Appendices EC.5.3 and EC.5.4. We demonstrate that the general economic insights as documented in Section 5.1 still hold for the general market setting. Moreover, induced by convex incentives, investors may take advantage of the stochastic investment opportunity set to get a more volatile portfolio through hedging demands.

Consider a market that has one stock and one bank account. The stock price follows a time-varying Gaussian mean return process as given by

$$dS_t = (r + \sigma X_t) S_t dt + \sigma S_t d\mathcal{B}_t, \tag{EC.22}$$

and the state variable X_t, which represents the market price of risk, is governed by an OU process:

$$dX_t = \kappa(\bar{X} - X_t)dt + \nu d\mathcal{B}_t^X, \tag{EC.23}$$

where $E[d\mathcal{B}_t d\mathcal{B}_t^X] = \rho dt$, and $r,\ \sigma,\ \kappa > 0,\ \bar{X} > 0,\ \nu > 0$, and ρ are all constants. Dynamic portfolio choice under this market setting or its special case has been widely studied either under concave utility maximization framework or mean-variance framework; see, e.g., Merton (1971), Kim and Omberg (1996), Campbell and Viceira (1999), Wachter (2002), Basak and Chabakauri (2010), and Dai et al. (2021).

We take the goal-reaching utility (5) as an example. Consider the portfolio bounds $\pi \in [-1, 1]$, and take the parameter values estimated from the historical data by Wachter (2002): $r = 0.017$,

$\sigma = 0.15,\ \kappa = 0.27,\ \bar{X} = 0.273,\ \nu = 0.065,\ \rho = -0.93,\ B = 0,\ T = 1$, and $X_0 = 0.273$. The resulting numerical results for the value function and optimal portfolios are similar to those under the Black-Scholes market and support our three general findings. We put most of the numerical results in Appendix EC.5.2, and examine hedging demand induced by the stochastic investment opportunity set below.

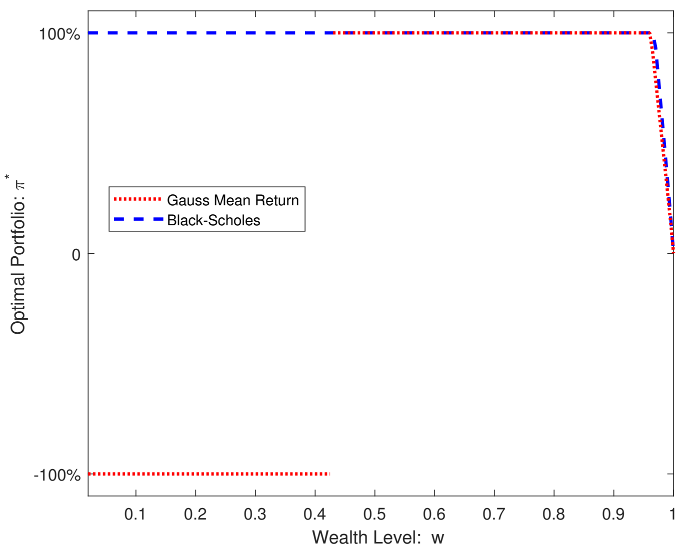

Figure EC.1 Optimal portfolios for the goal-reaching problem under the time-varying Gaussian mean return model (the dotted line) and the Black-Scholes model (the dashed line). The portfolio bounds are $\pi \in [-1, 1]$, and the parameters for time-varying Gaussian mean return model are: $r = 0.017$, $\sigma = 0.15$, $\kappa = 0.27$, $\bar{X} = 0.273$, $\nu = 0.065$, $\rho = -0.93$, $B = 0$, $x = X_0 = 0.273$, $T = 1$, and the parameters for the Black-Scholes model are: $r = 0.017$, $\mu = 0.017 + 0.15 * 0.273 = 0.058$, $\sigma = 0.15$, $B = 0$, $T = 1$.

In Figure EC.1, we plot the time-0 optimal fraction of total wealth invested in the stock π^* (i.e., optimal portfolio) against wealth level for the goal-reaching problem with portfolio bounds $\pi \in [-1, 1]$ under the time-varying Gaussian mean return model (the dotted line) and the Black-Scholes model (the dashed line), respectively. It can be seen that under the Black-Scholes model, short-selling is never optimal because (i) the stock risk premium is positive, and (ii) the portfolio bounds are symmetric, i.e., $u = -d = 1$. However, under the time-varying Gaussian mean return model, short-selling is optimal when the wealth level is far below the goal $w = 1$. The financial interpretation is given as follows: (i) the investor has a strong incentive to gamble when the wealth level is far below the goal; (ii) given negative ρ, short-selling could make the portfolio return and X_t move toward the same direction, raising the probability of reaching goal upon gambling as a result.

Note that the difference between the optimal portfolios of the two models can be regarded as a hedging demand due to the stochastic investment opportunity set associated with the time-varying Gaussian mean return model. Kim and Omberg (1996) find that for the CRRA utility maximization problem with negative ρ, hedging demand would switch from positive to negative when we decrease the investor's risk aversion level, as the hedging demand and $\rho(1-\gamma')$ share the same sign, where γ' is the investor's risk aversion level. Thus, it is reasonable to postulate that given negative ρ, the hedging demand would be negative for a risk-seeking investor. The negative hedging demand for a sufficiently low wealth level as shown in Figure EC.1 is actually consistent with the finding of Kim and Omberg (1995), because the convex incentive implied by the goal-reaching problem leads the investor to be more risk seeking in that case. Hence, induced by convex incentives, investors may construct a higher volatile portfolio through negative hedging demand and short-selling to gamble.

Appendix EC.2: Proof of Theorem EC.1.1

Before proceeding to the proof of the theorem, we present the closed-form classic solutions for the constrained portfolio optimization problem under the power utility in the following lemma. The results are guaranteed by a direct verification and thus the proof is omitted.

Lemma EC.2.1. *If the utility function $U(\cdot)$ in (4) is given by the power function:*

$$Q^{(q)}(T,w)=\begin{cases} w^q/q & q<1 \text{ and } q\neq 0 \\ \ln(w) & q=0, \end{cases} \tag{EC.24}$$

where $w>0$, and there is no liquidation constraint, that is $B=0$, letting $\Lambda=\sup_{d\le\pi\le u}\left\{\eta\pi-\frac{1-q}{2}\sigma^2\pi^2\right\}=\eta\pi_*-\frac{1-q}{2}\sigma^2\pi_*^2$, then the value function is given by

$$Q^{(q)}(t,w)=\begin{cases} e^{q\Lambda(T-t)}w^q/q & q<1 \text{ and } q\neq 0 \\ \ln(w)+\Lambda(T-t) & q=0, \quad w>0,\ 0\le t\le T, \end{cases} \tag{EC.25}$$

Proof of Theorem EC.1.1: We prove the theorem by contradiction. To help to derive a contradiction, for any $\beta>0$, set $\hat{u}=e^{\beta(t-T)}u$, $\hat{v}=e^{\beta(t-T)}v$, then $\hat{u}$ ($\hat{v}$) is a subsolution (supersolution) to

$$-\frac{\partial V(t,w,\boldsymbol{x})}{\partial t}-H(t,w,\boldsymbol{x},\nabla_{w,\boldsymbol{x}}V,\nabla^2_{w,\boldsymbol{x}}V)+\beta V(t,w,\boldsymbol{x})=0.$$

Assume $\hat{u}(\bar{t},\bar{w},\bar{\boldsymbol{x}})-\hat{v}(\bar{t},\bar{w},\bar{\boldsymbol{x}})=2\delta>0$ for some $t_{\hat{p}}<\bar{t}<T$ and $\bar{w}>B$. Consider

$$M_\alpha(t,w,\boldsymbol{x},s,\zeta,\boldsymbol{\xi})=\hat{u}(t,w,\boldsymbol{x})-\hat{v}(s,\zeta,\boldsymbol{\xi})-\varphi(t,w,\boldsymbol{x},s,\zeta,\boldsymbol{\xi}),$$

where $1>q>\hat{p}$ s.t. $F^{(q)}$ is finite in $[\bar{t}-(q-\hat{p}),T]$, $C>C_{\hat{p}}$ and

$$\begin{aligned}\varphi(t,w,\boldsymbol{x},s,\zeta,\boldsymbol{\xi})=&\epsilon_3 G(t,w,\boldsymbol{x},s,\zeta,\boldsymbol{\xi})\\ &+\frac{\epsilon_1}{t-t_{\hat{p}}}+\frac{\epsilon_2}{T-t}+\frac{\alpha}{2}(|t-s|^2+|w-\zeta|^2+|\boldsymbol{x}-\boldsymbol{\xi}|^2)\end{aligned} \tag{EC.26}$$

with

$$G(t,w,\boldsymbol{x},s,\zeta,\boldsymbol{\xi}) := \left((w^q + C)F^{(q)}(t-(q-\hat{p}),\boldsymbol{x}) + (\zeta^q + C)F^{(q)}(s-(q-\hat{p}),\boldsymbol{\xi})\right). \tag{EC.27}$$

Here, $\epsilon_1, \epsilon_2, \epsilon_3$ are positive, and are sufficiently small such that

$$M_\alpha(\bar{t},\bar{w},\bar{\boldsymbol{x}},\bar{t},\bar{w},\bar{\boldsymbol{x}}) > \delta.$$

When μ and Σ are constants, $G(t,w,s,\zeta) = Q^{(q)}(t,w) + Q^{(q)}(s,\zeta)$ and $Q^{(q)}(t,w)$ is given by (EC.25) in Lemma EC.2.1 for some q, such that $1 > q > \hat{p}$. The proof is analogous.

Set $(t_\alpha, w_\alpha, \boldsymbol{x}_\alpha, s_\alpha, \zeta_\alpha, \boldsymbol{\xi}_\alpha)$ to be an interior point which makes M_α take maximum. This point can be achieved because when w is sufficiently large,

$$(w^{\hat{p}} + C_{\hat{p}})F^{(\hat{p})}(t,\boldsymbol{x}) \le \epsilon_3(w^q + C)F^{(\hat{p})}(t,\boldsymbol{x}) \le \epsilon_3(w^q + C)F^{(q)}(t-(q-\hat{p}),\boldsymbol{x}).$$

On the other side,

$$\epsilon_d := \min_{\|\boldsymbol{\pi}\|_2 = 1, t_{\hat{p}} \le t \le T} \{\boldsymbol{\pi}^\top [A_q(t-(q-\hat{p})) - A_{\hat{p}}(t)]\boldsymbol{\pi}\} > 0.$$

For the remaining set in which w is bounded, if $|\boldsymbol{x}|$ is sufficiently large,

$$(w^{\hat{p}} + C_{\hat{p}})F^{(\hat{p})}(t,\boldsymbol{x}) \le (w^{\hat{p}} + C_{\hat{p}})e^{-\frac{\epsilon_d}{2}|\boldsymbol{x}|^2}F^{(q)}(t-(q-\hat{p}),\boldsymbol{x}) \le \epsilon_3(w^q + C)F^{(q)}(t-(q-\hat{p}),\boldsymbol{x}).$$

Notice that $\hat{u}$ is upper semicontinuous and $\hat{v}$ is lower semicontinuous. Combined with the singularities at $t = t_{\hat{p}}, T$, we can find $(t_\alpha, w_\alpha, x_\alpha), (s_\alpha, \zeta_\alpha, \xi_\alpha)$ in the interior at which M_α takes its maximum. Sending $\alpha \to \infty$, there exists a subsequence such that

$$\lim_{\alpha \to +\infty} \alpha(|t_\alpha - s_\alpha|^2 + |w_\alpha - \zeta_\alpha|^2 + |\boldsymbol{x}_\alpha - \boldsymbol{\xi}_\alpha|^2) = 0, \tag{EC.28}$$

and both $(t_\alpha, w_\alpha, \boldsymbol{x}_\alpha)$ and $(s_\alpha, \zeta_\alpha, \boldsymbol{\xi}_\alpha)$ converge to an interior point $(\hat{t}, \hat{w}, \hat{\boldsymbol{x}})$. The point depends on the choice of $\epsilon_1, \epsilon_2, \epsilon_3$. Let $(\hat{t}_0, \hat{w}_0, \hat{\boldsymbol{x}}_0)$ be a limit of $(\hat{t}, \hat{w}, \hat{\boldsymbol{x}})$ as $\epsilon_2 \to 0$. We assert that $(\hat{t}_0, \hat{w}_0, \hat{\boldsymbol{x}}_0)$ is an interior point. For this assertion, we only need to show that $\hat{t}_0 < T$. If $\hat{t}_0 = T$, we have

$$\begin{aligned}
&\limsup_{(\hat{t},\hat{w},\hat{\boldsymbol{x}})\to(T-,\hat{w}_0,\hat{\boldsymbol{x}}_0)} (\hat{u}(\hat{t},\hat{w},\hat{\boldsymbol{x}}) - \hat{v}(\hat{t},\hat{w},\hat{\boldsymbol{x}})) \\
\le &\limsup_{(\hat{t},\hat{w},\hat{\boldsymbol{x}})\to(T-,\hat{w}_0,\hat{\boldsymbol{x}}_0)} \hat{u}(\hat{t},\hat{w},\hat{\boldsymbol{x}}) - U(\hat{w}_0-) - 2\Phi\left(\frac{0 \wedge (\ln \hat{w} - \ln \hat{w}_0)}{L_{T,\hat{\boldsymbol{x}}_0}\sqrt{T-\hat{t}}}\right)(U(\hat{w}_0) - U(\hat{w}_0-)) \\
&- \liminf_{(\hat{t},\hat{w},\hat{\boldsymbol{x}})\to(T-,\hat{w}_0,\hat{\boldsymbol{x}}_0)} \hat{v}(\hat{t},\hat{w},\hat{\boldsymbol{x}}) - U(\hat{w}_0-) - 2\Phi\left(\frac{0 \wedge (\ln \hat{w} - \ln \hat{w}_0)}{L_{T,\hat{\boldsymbol{x}}_0}\sqrt{T-\hat{t}}}\right)(U(\hat{w}_0) - U(\hat{w}_0-)) \\
\le &0,
\end{aligned}$$

which contradicts the fact that, for each ϵ_2 and α,

$$\hat{u}(t_\alpha, w_\alpha, \boldsymbol{x}_\alpha) - \hat{v}(s_\alpha, \zeta_\alpha, \boldsymbol{\xi}_\alpha) \geq M_\alpha(t_\alpha, w_\alpha, \boldsymbol{x}_\alpha, s_\alpha, \zeta_\alpha, \boldsymbol{\xi}_\alpha) \geq M_\alpha(\bar{t}, \bar{w}, \bar{\boldsymbol{x}}, \bar{t}, \bar{w}, \bar{\boldsymbol{x}}) > \delta > 0. \tag{EC.29}$$

Thus, we can take α sufficiently large and ϵ_2 small enough such that

$$\frac{\epsilon_1}{(t_\alpha - t_{\hat{p}})^2} \geq \frac{\epsilon_2}{(T - t_\alpha)^2}, \text{ and } \quad w_\alpha \geq \frac{\hat{w}_0}{2} > 0. \tag{EC.30}$$

For simplification, in the following, we use $(t, w, \boldsymbol{x}, s, \zeta, \boldsymbol{\xi})$ to represent $(t_\alpha, w_\alpha, \boldsymbol{x}_\alpha, s_\alpha, \zeta_\alpha, \boldsymbol{\xi}_\alpha)$. Denote $G(t, w, \boldsymbol{x}) = (w^q + C)F^{(q)}(t - (q - \hat{p}), \boldsymbol{x})$.

By Ishii lemma, for any $\lambda > 0$, there exist $(K+1) \times (K+1)$ matrices M, N , s.t.

$$-\nabla_t \varphi - H^*(t, w, \boldsymbol{x}, \nabla_{w,\boldsymbol{x}}\varphi, M) + \beta \hat{u} \leq 0, \tag{EC.31}$$

$$\nabla_s \varphi - H_*(s, \zeta, \boldsymbol{\xi}, -\nabla_{\zeta,\boldsymbol{\xi}}\varphi, N) + \beta \hat{v} \geq 0. \tag{EC.32}$$

where φ is given in (EC.26), and

$$\begin{pmatrix} M, & 0 \\ 0, & -N \end{pmatrix} \leq \nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi + \gamma(\nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi)^2. \tag{EC.33}$$

By a direct calculus, we have

$$\begin{aligned}
&\nabla_t \varphi = -\frac{\epsilon_1}{(t - t_{\hat{p}})^2} + \frac{\epsilon_2}{(T - t)^2} + \alpha(t - s) + \epsilon_3 \nabla_t G, \qquad \nabla_s \varphi = -\alpha(t - s) + \epsilon_3 \nabla_s G, \\
&\nabla_w \varphi = \alpha(w - \zeta) + \epsilon_3 \nabla_w G, \qquad \nabla_\zeta \varphi = -\alpha(w - \zeta) + \epsilon_3 \nabla_\zeta G, \\
&\nabla_{\boldsymbol{x}} \varphi = \alpha(\boldsymbol{x} - \boldsymbol{\xi}) + \epsilon_3 \nabla_{\boldsymbol{x}} G, \qquad \nabla_{\boldsymbol{\xi}} \varphi = -\alpha(\boldsymbol{x} - \boldsymbol{\xi}) + \epsilon_3 \nabla_{\boldsymbol{\xi}} G
\end{aligned}$$

and

$$\nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi = \begin{pmatrix} \alpha I + \epsilon_3 \nabla^2_{w,\boldsymbol{x}} G, & -\alpha I \\ -\alpha I, & \alpha I + \epsilon_3 \nabla^2_{\zeta,\boldsymbol{\xi}} G \end{pmatrix},$$

where

$$\nabla^2_{w,\boldsymbol{x}} G = \begin{pmatrix} \nabla^2_w G, & \nabla_{w,\boldsymbol{x}} G \\ \nabla_{w,\boldsymbol{x}} G, & \nabla^2_{\boldsymbol{x}} G \end{pmatrix} = \begin{pmatrix} -q(1-q)w^{q-2}F^{(q)}, & q w^{q-1} \nabla_{\boldsymbol{x}} F^{(q)} \\ q w^{q-1} \nabla_{\boldsymbol{x}} F^{(q)}, & w^q \nabla^2_{\boldsymbol{x}} F^{(q)} \end{pmatrix}$$

and

$$\nabla^2_{\zeta,\boldsymbol{\xi}} G = \begin{pmatrix} \nabla^2_\zeta G, & \nabla_{\zeta,\boldsymbol{\xi}} G \\ \nabla_{\zeta,\boldsymbol{\xi}} G, & \nabla^2_{\boldsymbol{\xi}} G \end{pmatrix} = \begin{pmatrix} -q(1-q)\zeta^{q-2}F^{(q)}, & q \zeta^{q-1} \nabla_{\boldsymbol{\xi}} F^{(q)} \\ q \zeta^{q-1} p \nabla_{\boldsymbol{\xi}} F^{(q)}, & \zeta^q \nabla^2_{\boldsymbol{\xi}} F^{(q)} \end{pmatrix}.$$

Then, by (EC.33), for any pair of $K+1$ dimensional matrices X, Y, we have

$$\begin{aligned}
Tr(X^T M X - Y^T N Y) =& Tr\left(\left(X^T, Y^T\right) \begin{pmatrix} M, & 0 \\ 0, & -N \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right) \\
\leq& Tr\left(\left(X^T, Y^T\right) (\nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi + \gamma(\nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi)^2 \begin{pmatrix} X \\ Y \end{pmatrix} \right) \\
=& \alpha Tr\left((X-Y)^T(X-Y)\right) + \epsilon_3 Tr\left(X^T \nabla^2_{w,\boldsymbol{x}} G X + Y^T \nabla^2_{\zeta,\boldsymbol{\xi}} G Y\right) \\
&+ \gamma Tr\left(\left(X^T, Y^T\right) (\nabla^2_{w,\boldsymbol{x},\zeta,\boldsymbol{\xi}} \ \varphi)^2 \begin{pmatrix} X \\ Y \end{pmatrix} \right).
\end{aligned}$$

When ϵ_3 is fixed, $(t, w, \boldsymbol{x}, s, \zeta, \boldsymbol{\xi})$ is always in a bounded area. Thus, we can choose γ small enough, such that

$$Tr(X^T MX - Y^T NY) \leq \alpha Tr\left((X-Y)^T(X-Y)\right) + \epsilon_3 Tr\left(X^T \nabla^2_{w,\boldsymbol{x}} GX + Y^T \nabla^2_{\zeta,\boldsymbol{\xi}} GY\right) + o(1). \tag{EC.34}$$

Recall (EC.31) and (EC.32). For the case that $\boldsymbol{C}$ is bounded, we have $H^* = H = H_*$; for the case that $\boldsymbol{C}$ is unbounded, $H_* = H$ and $H^* = H$ when $M_{11} < 0$. In (EC.34), we can take $X = Y = \{\{1, 0\}, \{0, 0\}\}$. Then

$$M_{11} - N_{11} = -\epsilon_3 q(1-q) w^{q-2} F^{(q)} - \epsilon_3 q(1-q)\zeta^{q-2} F^{(q)} + o(1) < 0.$$

That is, $M_{11} < N_{11}$. Furthermore, by (EC.32), $N_{11} \leq 0$, thus $M_{11} < 0$. So, in (EC.31) and (EC.32), we always have $H^* = H = H_*$.

Let $\sigma_{t,w,\boldsymbol{x}}$ be the $K+1$ dimensional volatility matrix function of joint process $(W_t, \boldsymbol{x}_t)$ (cf. (EC.3) and (EC.2)) at time t and $W_t = w$ and $\boldsymbol{x}_t = \boldsymbol{x}$, given by

$$\sigma_{t,w,\boldsymbol{x}} = \begin{pmatrix} w\boldsymbol{\pi}_t^\top \boldsymbol{\sigma}_t & 0 \\ \boldsymbol{\nu}_t \boldsymbol{\rho}^\top & \boldsymbol{\nu}_t \boldsymbol{\rho}_0^\top \end{pmatrix}, \tag{EC.35}$$

where $\boldsymbol{\rho}_0$ is a constant matrix, satisfying $\boldsymbol{\rho}_0^\top \boldsymbol{\rho}_0 = I - \boldsymbol{\rho}^\top \boldsymbol{\rho}$. Then $\sigma_{t,w,\boldsymbol{x}} \sigma_{t,w,\boldsymbol{x}}^\top = \Sigma_{t,w,\boldsymbol{x}}$ (cf. (EC.13)). Let $X = \sigma_{t,w,\boldsymbol{x}}$ and $Y = \sigma_{s,\zeta,\boldsymbol{\xi}}$. Then, by (EC.31) and (EC.32), we have

$$\begin{aligned}
0 \leq & \nabla_t \varphi + H(t, w, \boldsymbol{x}, \nabla_{w,\boldsymbol{x}}\varphi, M) - \beta\hat{u} + \nabla_s \varphi - H(\zeta, \boldsymbol{\xi}, -\nabla_{\zeta,\boldsymbol{\xi}}\varphi, N) + \beta\hat{v} \\
= & \nabla_s \varphi + \varphi_t + \sup_{\boldsymbol{\pi}_t \in \boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(XX^\top M\right) + w\boldsymbol{\pi}_t^\top \boldsymbol{\eta}_t \nabla_w \varphi + \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} \varphi \right\} \\
& - \sup_{\boldsymbol{\pi}_s \in \boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(YY^\top N\right) + \zeta \boldsymbol{\pi}_s^\top \boldsymbol{\eta}_s \nabla_\zeta \varphi + \boldsymbol{m}_s^\top \nabla_{\boldsymbol{\xi}} \varphi \right\} + \beta(\hat{v} - \hat{u}).
\end{aligned}$$

Plugging in the derivatives, we have

$$\begin{aligned}
0 \leq & -\frac{\epsilon_1}{(t - t_{\hat{p}})^2} + \frac{\epsilon_2}{(T-t)^2} + \epsilon_3 \nabla_t G + \epsilon_3 \nabla_s G \\
& + \sup_{\boldsymbol{\pi} \in \boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(X^\top MX\right) + w\boldsymbol{\pi}^\top \boldsymbol{\eta}_t (\alpha(w-\zeta) + \epsilon_3 \nabla_w G) + \boldsymbol{m}_t^\top (\alpha(\boldsymbol{x} - \boldsymbol{\xi}) + \epsilon_3 \nabla_{\boldsymbol{x}} G) \right\} \\
& - \sup_{\boldsymbol{\pi} \in \boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(Y^\top NY\right) + \zeta \boldsymbol{\pi}^\top \boldsymbol{\eta}_s (\alpha(w-\zeta) - \epsilon_3 \nabla_\zeta G) + \boldsymbol{m}_s^\top (\alpha(\boldsymbol{x} - \boldsymbol{\xi}) - \epsilon_3 \nabla_{\boldsymbol{\xi}} G) \right\} + \beta(\hat{v} - \hat{u}).
\end{aligned}$$

Since $\boldsymbol{\eta}(t, \boldsymbol{x})$ and $\boldsymbol{m}(t, \boldsymbol{x})$ are Lipschitz w.r.t. $(t, \boldsymbol{x})$,

$$\alpha(w-\zeta)(w\boldsymbol{\eta}(t,\boldsymbol{x}) - \zeta\boldsymbol{\eta}(s,\boldsymbol{\xi})) \text{ and } \alpha(\boldsymbol{x} - \boldsymbol{\xi})(\boldsymbol{m}(t,\boldsymbol{x}) - \boldsymbol{m}(s,\boldsymbol{\xi}))$$

are of order $O(\alpha(|t-s|^2 + |w-\zeta|^2 + |\boldsymbol{x} - \boldsymbol{\xi}|^2)) = o(1)$, according to (EC.28). Then by (EC.30)

$$\begin{aligned}
0 \leq & \epsilon_3 \nabla_t G + \epsilon_3 \nabla_s G + \sup_{\boldsymbol{\pi} \in \boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(X^\top MX\right) + \epsilon_3 w\boldsymbol{\pi}^\top \boldsymbol{\eta}_t \nabla_w G + \epsilon_3 \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} G \right. \\
& \left. - \frac{1}{2} Tr\left(Y^\top NY\right) + \epsilon_3 \zeta \boldsymbol{\pi}^\top \boldsymbol{\eta}_s \nabla_\zeta G + \epsilon_3 \boldsymbol{m}_s^\top \nabla_{\boldsymbol{\xi}} G + o(1)(1 + ||\boldsymbol{\pi}||) \right\} + \beta(\hat{v} - \hat{u}).
\end{aligned}$$

Since $\boldsymbol{\sigma}(t,\boldsymbol{x})$ and $\boldsymbol{\nu}(t,\boldsymbol{x})$ are Lipschitz w.r.t. $(t,\boldsymbol{x})$, recalling (EC.35), $\alpha Tr\left((X-Y)^T(X-Y)\right)$ are of order $||\boldsymbol{\pi}||^2 O(\alpha(|t-s|^2+|w-\zeta|^2+|\boldsymbol{x}-\boldsymbol{\xi}|^2)) = ||\boldsymbol{\pi}||^2 o(1)$. Using (EC.34), we have

$$\begin{aligned}0 \leq & \epsilon_3 \nabla_t G + \epsilon_3 \nabla_s G + \epsilon_3 \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} \left\{ \frac{1}{2} Tr\left(X^T \nabla^2_{w,\boldsymbol{x}} G X\right) + w\boldsymbol{\pi}^\top \boldsymbol{\eta}_t \nabla_w G + \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} G \right. \\ & \left. + \frac{1}{2} Tr\left(Y^T \nabla^2_{\zeta,\boldsymbol{\xi}} G Y\right) + \zeta \boldsymbol{\pi}^\top \boldsymbol{\eta}_s \nabla_\zeta G + \boldsymbol{m}_s^\top \nabla_{\boldsymbol{\xi}} G + o(1)(1+||\boldsymbol{\pi}||+||\boldsymbol{\pi}||^2) \right\} + \beta(\hat{v}-\hat{u}).\end{aligned}$$

Note that the first diagonal element of $\nabla^2_{w,\boldsymbol{x}} G$ and $\nabla^2_{\zeta,\boldsymbol{\xi}} G$ are both strictly negative, thus $o(1)(1+\boldsymbol{\pi}+\boldsymbol{\pi}^2)$ in brace is negligible even in the case that $\boldsymbol{C}$ is unbounded. Thus, we have

$$\begin{aligned}0 \leq & o(1) + \epsilon_3\left(\nabla_t G + H\left(t,w,\boldsymbol{x},\nabla_{w,\boldsymbol{x}} G, \nabla^2_{w,\boldsymbol{x}} G\right)\right) \\ & + \epsilon_3\left(\nabla_s G + H\left(s,\zeta,\boldsymbol{\xi},\nabla_{\zeta,\boldsymbol{\xi}} G, \nabla^2_{\zeta,\boldsymbol{\xi}} G\right)\right) + \beta(\hat{v}-\hat{u}).\end{aligned} \tag{EC.36}$$

Recall that $G(t,w,\boldsymbol{x}) = w^q F^{(q)}(t-(q-\hat{p}),\boldsymbol{x}) + CF^{(q)}(t-(q-\hat{p}),\boldsymbol{x})$. And $w^q F^{(q)}$ is a classical solution of the unconstrained problem, and thus satisfies (cf. (EC.8))

$$\begin{aligned}& w^q\left(\nabla_t F^{(q)} + \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} F^{(q)} + \frac{1}{2} Tr\left(\boldsymbol{\nu}_t \boldsymbol{\nu}_t^\top \nabla^2_{\boldsymbol{x}} F^{(q)}\right)\right) \\ & \quad + \sup_{\boldsymbol{\pi}} \left\{ \frac{1}{2} w^2 \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi} \nabla^2_w (w^q F^{(q)}) + w\boldsymbol{\pi}^\top \boldsymbol{\eta}_t \nabla_w (w^q F^{(q)}) + w\boldsymbol{\pi}_t^\top \boldsymbol{\sigma}_t \boldsymbol{\rho} \boldsymbol{\nu}_t^\top \nabla_{w,\boldsymbol{x}} (w^q F^{(q)}) \right\} = 0.\end{aligned} \tag{EC.37}$$

Thus, $w^q F^{(q)}$ is a supersolution of the constrained problem, that is,

$$\nabla_t (w^q F^{(q)}) + H\left(t,w,\boldsymbol{x},\nabla_{w,\boldsymbol{x}}(w^q F^{(q)}), \nabla^2_{w,\boldsymbol{x}}(w^q F^{(q)})\right) \leq 0. \tag{EC.38}$$

Noting that $F^{(q)}$ is independent with w, we have (cf. (EC.8))

$$\begin{aligned}& \nabla_t F^{(q)} + H\left(t,w,\boldsymbol{x},\nabla_{w,\boldsymbol{x}} F^{(q)}, \nabla^2_{w,\boldsymbol{x}} F^{(q)}\right) \\ = & \nabla_t F^{(q)} + \boldsymbol{m}_t^\top \nabla_{\boldsymbol{x}} F^{(q)} + \frac{1}{2} Tr\left(\boldsymbol{\nu}_t \boldsymbol{\nu}_t^\top \nabla^2_{\boldsymbol{x}} F^{(q)}\right) \\ = & -\sup_{\boldsymbol{\pi}} \left\{ \frac{1}{2} w^2 \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi} \nabla^2_w (w^q F^{(q)}) + w\boldsymbol{\pi}^\top \boldsymbol{\eta}_t \nabla_w (w^q F^{(q)}) + w\boldsymbol{\pi}_t^\top \boldsymbol{\sigma}_t \boldsymbol{\rho} \boldsymbol{\nu}_t^\top \nabla_{w,\boldsymbol{x}} (w^q F^{(q)}) \right\} / w^q \\ \leq & 0,\end{aligned} \tag{EC.39}$$

where the second equality is due to (EC.37). Combining (EC.38) and (EC.39), the two terms in brackets of (EC.36) are both negative. It follows

$$0 \leq o(1) + \beta(\hat{v}-\hat{u}) < -\beta\delta < 0,$$

by (EC.29), which leads to a contradiction.

Appendix EC.3: Proof of Theorem EC.1.2

We will take three steps to verify that the value function satisfies the conditions a), b), and c) in Definition EC.1.1. Then, the value function of (EC.5) is a viscosity solution of the HJB equation (EC.8) with the boundary condition (EC.7) and the asymptotic condition (EC.10) (or (EC.11) for unbounded portfolio set). The uniqueness of the viscosity solution and the continuity of the value function before the maturity are guaranteed by the comparison principle in Theorem EC.1.1. The property of the value function at the maturity is fully characterized by the asymptotic condition (EC.10) (resp. (EC.11) for unbounded portfolio set). Thus Theorem EC.1.2 is proved.

Step 1: We verify Condition a) in Definition EC.1.1.

In the standard viscosity solution approach (see, e.g., Chapter V of Fleming and Soner, 2006), a dynamic programming principle based on the continuity of value functions is used to verify Condition a). In our model, the utility function U is upper semicontinuous. Similar to Bouchard and Touzi (2011), we first give a weak version of dynamic programming for the value function V.

Proposition EC.3.1 **(Weak Dynamic Programming)**. *Denote $W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}}$ as the wealth process W_s starting from $W_t = w$, $\boldsymbol{x}_t = \boldsymbol{x}$ under the portfolio $\boldsymbol{\pi}$. For any stopping time τ taking values within $[t,T]$, and $(t,w,\boldsymbol{x}) \in [0,T) \times (B,+\infty) \times \mathbb{R}^K$, we have*

$$V(t,w,\boldsymbol{x}) \le \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[V^*(\tau, W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_\tau)] \tag{EC.40}$$

and

$$V(t,w,\boldsymbol{x}) \ge \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[V_*(\tau, W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_\tau)]. \tag{EC.41}$$

Proof of Proposition EC.3.1: First, the inequality (EC.40) follows from the inequality (3.1) of Bouchard and Touzi (2011), which is a direct consequence of the law of iterated expectations.

Next, we give the proof for the inequality (EC.41). For each portfolio $\boldsymbol{\pi}$, denote

$$J(t,w,\boldsymbol{x};\boldsymbol{\pi}) := E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})]. \tag{EC.42}$$

Bouchard and Touzi (2011) prove the inequality (EC.41) in their Corollary 3.6 under the assumption that $J(t,w,\boldsymbol{x};\boldsymbol{\pi})$ in (EC.42) is lower semicontinuous in $(t,w,\boldsymbol{x})$, for any $\boldsymbol{\pi}\in\boldsymbol{C}$. In our problem, by Assumption 3.1, the utility function U is upper semicontinuous, bounded from above by a power function $w^{\hat{p}}$ for sufficiently large w, and bounded from below by $A_1\frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$. Then, $J(t,w,\boldsymbol{x};\boldsymbol{\pi})$ is bounded from above by the value function in (EC.9) plus a constant and is bounded from below by V_{CRRA}. By the dominated convergence theorem and the continuity of the controlled process $W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, t \le s < T$, the function $J(\cdot,\cdot,\cdot;\boldsymbol{\pi})$ is upper semicontinuous. However, by Lemma 3.5 of

Reny(1999), $J(\cdot,\cdot,\cdot;\boldsymbol{\pi})$ can be approximated from above by a sequence of continuous functions (or lower semicontinuous functions in a wider domain of functions) on each compact domain. So, we can mimic the proof of Bouchard and Touzi (2011) by firstly replacing $J(\cdot,\cdot,\cdot;\boldsymbol{\pi})$ by its continuous approximations, and then take a limit to recover $J(\cdot,\cdot,\cdot;\boldsymbol{\pi})$. We give below the detail of the proof.

By definition, for fixed $\epsilon>0$ and any $(s,\zeta,\boldsymbol{\xi})$, there exists a portfolio $\boldsymbol{\pi}^{(s,\zeta,\boldsymbol{\xi})}\in\boldsymbol{C}$, such that

$$J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s,\zeta,\boldsymbol{\xi})})\geq V(s,\zeta,\boldsymbol{\xi})-\epsilon.$$

Consider any lower semicontinuous function $\psi^{(s,\zeta,\boldsymbol{\xi})}(\cdot)\geq J(\cdot;\boldsymbol{\pi}^{(s,\zeta,\boldsymbol{\xi})})$ and any upper semicontinuous function $\varphi\leq V$ (later, we will send $\psi^{(s,\zeta,\boldsymbol{\xi})}(\cdot)\to J(\cdot;\boldsymbol{\pi}^{(s,\zeta,\boldsymbol{\xi})})$ and $\varphi\to V_*$). By the definition of semicontinuity, there exists $r_{(s,\zeta,\boldsymbol{\xi})}>0$, such that

$$\psi^{(s,\zeta,\boldsymbol{\xi})}(s_1,\zeta_1,\boldsymbol{\xi}_1)\geq\psi^{(s,\zeta,\boldsymbol{\xi})}(s,\zeta,\boldsymbol{\xi})-\epsilon,\varphi(s_1,\zeta_1,\boldsymbol{\xi}_1)\leq\varphi(s,\zeta,\boldsymbol{\xi})+\epsilon,$$
$$\text{for any}\ \ (s_1,\zeta_1,\boldsymbol{\xi}_1)\in B(s,\zeta,\boldsymbol{\xi},r_{(s,\zeta,\boldsymbol{\xi})}),$$

where $B(s,\zeta,\boldsymbol{\xi},r_{(s,\zeta,\boldsymbol{\xi})}):=\{(s_1,\zeta_1,\boldsymbol{\xi}_1):|s_1-s|<r_{(s,\zeta,\boldsymbol{\xi})},|\zeta_1-\zeta|<r_{(s,\zeta,\boldsymbol{\xi})},|\boldsymbol{\xi}_1-\boldsymbol{\xi}|<r_{(s,\zeta,\boldsymbol{\xi})}\}$. Then

$$\begin{aligned}\psi^{(s,\zeta,\boldsymbol{\xi})}(s_1,\zeta_1,\boldsymbol{\xi}_1)&\geq\psi^{(s,\zeta,\boldsymbol{\xi})}(s,\zeta,\boldsymbol{\xi})-\epsilon\geq J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s,\zeta,\boldsymbol{\xi})})-\epsilon\geq V(s,\zeta,\boldsymbol{\xi})-2\epsilon\geq\varphi(s,\zeta,\boldsymbol{\xi})-2\epsilon\\&\geq\varphi(s_1,\zeta_1,\boldsymbol{\xi}_1)-3\epsilon,\ \text{for any}\ \ (s_1,\zeta_1,\boldsymbol{\xi}_1)\in B(s,\zeta,\boldsymbol{\xi},r_{(s,\zeta,\boldsymbol{\xi})}).\end{aligned}\tag{EC.43}$$

$\{B(s,\zeta,\boldsymbol{\xi},r_{(s,\zeta,\boldsymbol{\xi})})\}_{(s,\zeta,\boldsymbol{\xi})}$ forms an open covering of $[0,T]\times(B,\infty)\times\mathbb{R}^K$. Then by the Lindelöf covering theorem, there exists a countable sequence $(s_i,\zeta_i,\boldsymbol{\xi}_i)_{i\geq1}$, such that the sequence of open sets $\{B(s_i,\zeta_i,\boldsymbol{\xi}_i,r_{(s_i,\zeta_i,\boldsymbol{\xi}_i)})\}_{(s_i,\zeta_i,\boldsymbol{\xi}_i),i\geq1}$ forms an open covering of $[0,T]\times(B,\infty)\times\mathbb{R}^K$. Then we can define a disjoint partition $\{A_n\}_{n\geq1}$:

$$A_1:=B(s_1,\zeta_1,\boldsymbol{\xi}_1,r_{(s_1,\zeta_1,\boldsymbol{\xi}_1)}),A_n:=B(s_n,\zeta_n,\boldsymbol{\xi}_n,r_{(s_n,\zeta_n,\boldsymbol{\xi}_n)})\setminus(\cup_{1\leq i\leq n-1}A_i),\ \text{for}\ \ n>1.$$

Recalling (EC.43), for $(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i$, by setting $\pi(s)=\pi^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(s)$ for $\tau\leq s\leq T$, we have

$$\psi^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}\geq\varphi(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}-3\epsilon.$$

Now, we are ready to prove (EC.41). For any portfolio $\boldsymbol{\pi}\in\boldsymbol{C}$, define the portfolio

$$\boldsymbol{\pi}_n(s):=\boldsymbol{\pi}(s)1_{t\leq s\leq\tau}+1_{\tau\leq s\leq T}\left(\sum_{i=1}^n\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}+\boldsymbol{\pi}(s)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\notin\cup_{1\leq i\leq n}A_i}\right).$$

Then

$$\sum_{i=1}^n\psi^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}\geq\sum_{i=1}^n\varphi(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}-3\epsilon,$$

and

$$\begin{aligned}&E[\sum_{i=1}^{n}\psi^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}+J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi})1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\notin\cup_{1\le i\le n}A_i}]\\ \ge&E[\varphi(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in\cup_{1\le i\le n}A_i}+J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi})1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\notin\cup_{1\le i\le n}A_i}]-3\epsilon.\end{aligned}\tag{EC.44}$$

As aforementioned, for each $(s_i,\zeta_i,\boldsymbol{\xi}_i)$, $J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)})$ is upper semicontinuous. By Lemma 3.5 of Reny(1999), there exists a sequence of continuous functions $\{\tilde{\psi}_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}:j\ge 1\}$ such that

$$\tilde{\psi}_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(s,\zeta,\boldsymbol{\xi})\ge J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)})$$

and $\lim_{j\to\infty}\tilde{\psi}_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}(s,\zeta,\boldsymbol{\xi})=J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)})$, for all $(s,\zeta,\boldsymbol{\xi})\in A_i$. Let $\psi_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}:=\max_{j\ge j'}\tilde{\psi}_{j'}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}$, then $\psi_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}$ is non-increasing in j and converges to $J(s,\zeta,\boldsymbol{\xi};\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)})$ as j tends to infinity. Similarly, we can find a nondecreasing sequence of continuous functions $\{\varphi_j:j\ge 1\}$ such that $\varphi_j(s,\zeta,\boldsymbol{\xi})\le V_*(s,\zeta,\boldsymbol{\xi})$ and $\lim_{j\to\infty}\varphi_j(s,\zeta,\boldsymbol{\xi})=V_*(s,\zeta,\boldsymbol{\xi})$ on $\cup_{1\le i\le n}A_i$. Replacing $\psi^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}$ by $\psi_j^{(s_i,\zeta_i,\boldsymbol{\xi}_i)}$ on the left-hand side of (EC.44), and replacing φ by φ_j on the right-hand side of (EC.44), sending $j\to\infty$, by the monotone convergence theorem, we have

$$\begin{aligned}&E[\sum_{i=1}^{n}J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi}^{(s_i,\zeta_i,\boldsymbol{\xi}_i)})1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in A_i}+J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi})1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\notin\cup_{1\le i\le n}A_i}]\\ \ge&E[V_*(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\in\cup_{1\le i\le n}A_i}+J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi})1_{(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)\notin\cup_{1\le i\le n}A_i}]-3\epsilon.\end{aligned}\tag{EC.45}$$

Note that the left-hand side of (EC.45) is $E[J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi}_n)]$. Now, sending $n\to\infty$ on the right side of (EC.45), by the dominated convergence theorem, we have

$$E[J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi}_n)]\ge E[V_*(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)]-3\epsilon.$$

By the law of iterated expectations

$$\begin{aligned}V(t,w,\boldsymbol{x})\ge&E[J(t,w,\boldsymbol{x};\boldsymbol{\pi}_n)]=E[E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}_n})|\mathcal{F}(\tau)]]\\ =&E[J(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau;\boldsymbol{\pi}_n)]\ge E[V_*(\tau,W_\tau^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_\tau)]-3\epsilon.\end{aligned}\tag{EC.46}$$

By the arbitrariness of $\boldsymbol{\pi}$ and ϵ, (EC.41) follows. □

Then, Condition a) is verified by Corollary 5.6 of Bouchard and Touzi (2011).

Step 2: We verify Condition b) in Definition EC.1.1 by the following proposition.

PROPOSITION EC.3.2. *For all $t_{\hat{p}}<t,\bar{t}<T$ and $w>B\ge 0$, we have,*

$$\begin{cases}\lim\limits_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})}V(t,w,\boldsymbol{y})=U(B) & \text{if } U(B)>-\infty\\ \lim\limits_{(t,w,\boldsymbol{y})\to(\bar{t},0,\boldsymbol{x})}V(t,w,\boldsymbol{y})-V_{CRRA}(t,w,\boldsymbol{y})=0 & \text{if } B=0 \text{ and } U(0)=-\infty.\end{cases}\tag{EC.47}$$

Proof of Proposition EC.3.2: For any portfolio $\boldsymbol{\pi}$, define the first passage time $\tau^b_{t,w,\boldsymbol{y},\boldsymbol{\pi}} = \inf\{s \geq t : W_s = b | W_t = w, \boldsymbol{x}_t = \boldsymbol{y}\}$. First, we show that, for any $\zeta > 0$,

$$\lim_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} P\left[\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} \leq \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}\right] = 0. \tag{EC.48}$$

Define $\tau_{t,w,\boldsymbol{y},\boldsymbol{\pi}} = \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, \tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\}$, and let $f(t,w,\boldsymbol{y})$ be the maximum probability that the wealth process W hits $B+\zeta$ before hitting B by maturity without portfolio bounds, provided that the wealth process W starts at $W_t = w$ and $\boldsymbol{x}_t = \boldsymbol{y}$, that is,

$$\begin{aligned} f(t,w,\boldsymbol{y}) &= \sup_{\boldsymbol{\pi}} P\left[\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} \leq \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}\right] \\ &= \sup_{\boldsymbol{\pi}} E[\mathbf{1}_{\{W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_{\tau_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\wedge T} \geq B+\zeta\}}] \\ &= \sup_{\boldsymbol{\pi}} E[\mathbf{1}_{\{W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T \geq B+\zeta\}}] \\ &= \sup_{\boldsymbol{\pi}} E[\mathbf{1}_{\{W^{t,w-B,\boldsymbol{y},\boldsymbol{\pi}}_T \geq \zeta\}}] \\ &\leq \sup_{\boldsymbol{\pi}} E[(W^{t,w-B,\boldsymbol{y},\boldsymbol{\pi}}_T)^p]/\zeta^p, \end{aligned} \tag{EC.49}$$

for any $0 < p < 1$, where $W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T$ is the wealth at time T under portfolio $\boldsymbol{\pi}$ with initial value $W_t = w$ and $\boldsymbol{x}_t = \boldsymbol{y}$. Due to the explicit solution for CRRA utility, we have

$$\begin{aligned} &\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} P\left[\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} \leq \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}\right] \\ \leq &\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} f(t,w,\boldsymbol{y}) \leq \limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} \sup_{\boldsymbol{\pi}} E[(W^{t,w-B,\boldsymbol{y},\boldsymbol{\pi}}_T)^p]/\zeta^p \leq \limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} \frac{(w-B)^p}{\zeta^p} F^{(p)}(t,\boldsymbol{y}) = 0. \end{aligned}$$

On the other hand, the probabilities are always nonnegative. Thus, (EC.48) is proved.

Second, we prove (EC.47). For any $\zeta > 0$,

$$V(t,w,\boldsymbol{y}) \leq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T)\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} \leq \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}}] + \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T)\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} > \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}}]. \tag{EC.50}$$

We first prove

$$\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T)\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}} \leq \min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}}, T\}}] \leq 0. \tag{EC.51}$$

We prove by contradiction. If this inequality does not hold, we can find a subsequence $(t_n, w_n, \boldsymbol{y}_n) \to (\bar{t}, B, \boldsymbol{x})$, such that

$$\limsup_{(t_n,w_n,\boldsymbol{y}_n)\to(\bar{t},B,\boldsymbol{x})} \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t_n,w_n,\boldsymbol{y}_n,\boldsymbol{\pi}}_T)\mathbf{1}_{\tau^{B+\zeta}_{t_n,w_n,\boldsymbol{y}_n,\boldsymbol{\pi}} \leq \min\{\tau^B_{t_n,w_n,\boldsymbol{y}_n,\boldsymbol{\pi}}, T\}}] = \epsilon > 0. \tag{EC.52}$$

That contradicts (EC.48) and the local uniform boundedness of $\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U^q(W^{t,w,\boldsymbol{y},\boldsymbol{\pi}}_T)]$ for sufficiently small $q > 1$ implied by Assumption EC.1.1 (4).

On the other hand,

$$\begin{aligned}\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W_T^{t,w,\boldsymbol{y},\boldsymbol{\pi}})\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\le\min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}},T\}}] =& \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W_T^{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}},B+\zeta,\boldsymbol{\pi}})\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\le\min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}},T\}}]\\ \ge& U(B+\zeta)\sup_{\boldsymbol{\pi}^{(1)}\in\boldsymbol{C}} E[\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}^{(1)}}\le\min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}^{(1)}},T\}}],\end{aligned}$$

where $\boldsymbol{\pi}_s^{(1)}\equiv 0$ for $\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\le s\le T$. By (EC.48), we have

$$\liminf_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})}\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W_T^{t,w,\boldsymbol{y},\boldsymbol{\pi}})\mathbf{1}_{\tau^{B+\zeta}_{t,w,\boldsymbol{y},\boldsymbol{\pi}}\le\min\{\tau^B_{t,w,\boldsymbol{y},\boldsymbol{\pi}},T\}}]\ge 0.$$

Combining with (EC.51), the first term on the right-hand side of (EC.50) goes to zero as $(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})$.

Case 1: $U(B)>-\infty$. The second term on the right-hand side of (EC.50) is bounded by $U(B+\zeta)$. We have

$$\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} V(t,w,\boldsymbol{y})\le U(B+\zeta).$$

Sending ζ to zero, since $U(\cdot)$ is nondecreasing and right-continuous, we get

$$\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} V(t,w,\boldsymbol{y})\le U(B).$$

On the other hand, it is trivial that

$$\liminf_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} V(t,w,\boldsymbol{y})\ge U(B).$$

Thus, (EC.47) holds for the case that $U(B)>-\infty$.

Case 2: $B=0$ and $U(0)=-\infty$. For ζ small enough, the second term on the right-hand side of (EC.50) is bounded by $V_{CRRA}(t,w,\boldsymbol{y})$. We have

$$\limsup_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} V(t,w,\boldsymbol{y})-V_{CRRA}(t,w,\boldsymbol{y})\le 0.$$

On the other hand, taking the optimal strategy $\boldsymbol{\pi}^*$ for CRRA utility $A_1\frac{x^{\widetilde{p}}-1}{\widetilde{p}}+A_2$ and by the dominated convergence theorem, we have

$$\begin{aligned}&\liminf_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} V(t,w,\boldsymbol{y})-V_{CRRA}(t,w,\boldsymbol{y})\\ \ge&\liminf_{(t,w,\boldsymbol{y})\to(\bar{t},B,\boldsymbol{x})} E\left[\left(U(W_T^{t,w,\boldsymbol{y},\boldsymbol{\pi}^*})-A_1\frac{(W_T^{t,w,\boldsymbol{y},\boldsymbol{\pi}^*})^{\widetilde{p}}-1}{\widetilde{p}}-A_2\right)\mathbf{1}_{W_T^{t,w,\boldsymbol{y},\boldsymbol{\pi}^*}\ge\epsilon'}\right]\\ \ge&0.\end{aligned}\tag{EC.53}$$

Thus, (EC.47) holds and the proposition is proved. □

Step 3: We verify Condition c) in Definition EC.1.1 by the following proposition.

PROPOSITION EC.3.3. *Let Assumption EC.1.1 hold. Denote $\hat{U}$ as the concave envelope of U. (i). When the portfolio set $\boldsymbol{C}$ is bounded, denoting $L_{t,\boldsymbol{x}} = \sqrt{\max_{\boldsymbol{\pi}_t \in \boldsymbol{C}} \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi}_t}$ and $U(B-) = U(B)$, for any $w \geq B$, we have*

$$\lim_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) - U(w-) - 2\Phi\left(\frac{0 \wedge (\log\zeta - \log w)}{L_{T,\boldsymbol{x}}\sqrt{T-t}}\right)(U(w) - U(w-)) = 0. \tag{EC.54}$$

Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable, and $U(w-)$ is the left limit of U at w.

(ii). When the portfolio set $\boldsymbol{C}$ is unbounded, for any $w \geq B$, we have

$$\lim_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) = \hat{U}(w).$$

For Part (i), we first consider the case where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are constants, and (EC.54) degenerates to the following one:

$$\lim_{(t,\zeta)\to(T-,w)} V(t,\zeta) - U(w-) - 2\Phi\left(\frac{\min\{0, \log\zeta/w\}}{L\sqrt{T-t}}\right)(U(w) - U(w-)) = 0, \tag{EC.55}$$

where $L = \sqrt{\max_{\boldsymbol{\pi}\in\boldsymbol{C}} \boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}}$. After (EC.55) is proved, we proceed to the proof for the general case.

To prove (EC.55), below we need Proposition EC.3.4 that focuses on the simplest discontinuous case: the goal-reaching problem. We present Proposition EC.3.4 first, while its proof will be given at the end of the proof for Proposition EC.3.3(i).

PROPOSITION EC.3.4. *Assume that L is finite. For $t < T$ and $0 \leq w \leq 1$, let $V(t,w) := \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\mathbf{1}_{W_T \geq 1} | W_t = w]$ be the value function of the goal-reaching problem, then*

$$\limsup_{t\to T^-} \sup_{0\leq w\leq 1} \left| V(t,w) - f\left(\frac{\log w}{\sqrt{T-t}}\right)\right| = 0, \tag{EC.56}$$

where $f(z) := 2\Phi(\frac{\min\{0,z\}}{L})$. More specifically,

$$f\left(\frac{\log w}{\sqrt{T-t}} - M\sqrt{T-t}\right) \leq V(t,w) \leq f\left(\frac{\log w}{\sqrt{T-t}} + M\sqrt{T-t}\right), \tag{EC.57}$$

where $M := \max_{d\leq\pi\leq u} \{|\boldsymbol{\eta}^\top\boldsymbol{\pi} - \frac{1}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi}|\} < +\infty$.

Proof of Proposition EC.3.3(i): (I) We consider the case where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are constants, and the portfolio constraint set $\boldsymbol{C}$ is bounded.

Let a and $f(\cdot)$ be given in Proposition EC.3.4. Define $\tilde{V}(t,y) := V(t,w)$, where $y = \log w$, then $\tilde{V}$ is the value function associated with the utility $\tilde{U}(y) := U(e^y) \le C_1 + C_2 e^{\hat{p}y}$ and the log-wealth process $Y_s^{t,y,\boldsymbol{\pi}} = \log W_s$, satisfying

$$dY_s^{t,y,\boldsymbol{\pi}} = (\boldsymbol{\eta}^\top \boldsymbol{\pi} - \frac{1}{2}\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi})ds + \boldsymbol{\pi}_s^\top \boldsymbol{\sigma} d\boldsymbol{\mathcal{B}}_s, \qquad Y_t^{t,y,\boldsymbol{\pi}} = y. \tag{EC.58}$$

We aim to prove that, for any y_0,

$$\lim_{(t,y)\to(T-,y_0)} \tilde{V}(t,y) - \tilde{U}(y_0-) - f\left(\frac{y-y_0}{\sqrt{T-t}}\right)(\tilde{U}(y_0) - \tilde{U}(y_0-)) = 0. \tag{EC.59}$$

First, we show that with portfolio bounds, the controlled process Y of (EC.58) will not move too far away from its initial point in a short time. To be more specific, for a fixed y_0 and any $\epsilon > 0$, since $\tilde{U}$ is nondecreasing and right-continuous, there exists a $\delta > 0$, s.t.

$$\tilde{U}(y_0-) - \epsilon \le \tilde{U}(y) \le \tilde{U}(y_0-), \text{ when } \ y_0 - \delta \le y < y_0, \tag{EC.60}$$

$$\tilde{U}(y_0) \le \tilde{U}(y) \le \tilde{U}(y_0) + \epsilon, \text{ when } \ y_0 \le y \le y_0 + \delta. \tag{EC.61}$$

Then, for any y such that $|y - y_0| \le \delta/2$, by the second inequality of (EC.57), for any $\boldsymbol{\pi} \in \boldsymbol{C}$, we have

$$P(Y_T^{t,y,\boldsymbol{\pi}} \ge y_0 + \delta) = P(Y_T^{t,y-y_0-\delta,\boldsymbol{\pi}} \ge 0) \le f\left(\frac{y-y_0-\delta}{\sqrt{T-t}} + M\sqrt{T-t}\right). \tag{EC.62}$$

Note that for $|y - y_0| \le \delta/2$, $y - y_0 - \delta < -\delta/2$, thus the right-hand side of (EC.62) goes to zero as $t \to T$. Similarly, considering the process $-Y$ instead of Y, by the second inequality of (EC.57), for any $\boldsymbol{\pi} \in \boldsymbol{C}$, we have

$$P(Y_T^{t,y,\boldsymbol{\pi}} \le y_0 - \delta) = P(-Y_T^{t,y-y_0+\delta,\boldsymbol{\pi}} \ge 0) \le f\left(\frac{y_0-y-\delta}{\sqrt{T-t}} + M\sqrt{T-t}\right). \tag{EC.63}$$

The right-hand side of (EC.63) also goes to zero as $t \to T$, since $y_0 - y - \delta < -\delta/2$ for $|y - y_0| \le \delta/2$. That is, as the control $\boldsymbol{\pi} \in \boldsymbol{C}$ is bounded, the terminal log-wealth $Y_T^{t,y,\boldsymbol{\pi}}$ concentrates on the domain $(y_0 - \delta, y_0 + \delta)$ when $T - t$ is small.

Second, we prove (EC.59). Let $\tilde{U}_1(y) := \tilde{U}(y_0-) + \mathbf{1}_{y \ge y_0}(\tilde{U}(y_0) - \tilde{U}(y_0-))$, which is a linear transformation of the goal-reaching utility. Let $\tilde{V}_1(t,y)$ be the value function for the optimization problem with the utility $\tilde{U}_1(y)$ and the log-wealth process Y of (EC.58). Then, it is a linear transformation of the value function in Proposition EC.3.4. By (EC.56), we have

$$\lim_{(t,y)\to(T-,y_0)} \tilde{V}_1(t,y) - \tilde{U}(y_0-) - f\left(\frac{y-y_0}{\sqrt{T-t}}\right)(\tilde{U}(y_0) - \tilde{U}(y_0-)) = 0. \tag{EC.64}$$

By (EC.60), (EC.61) and the definition of $\tilde{U}_1$, we have

$$E[|\tilde{U}(y) - \tilde{U}_1(y)|] \le \epsilon, \text{ when } \ |y - y_0| \le \delta.$$

Then, for any strategy $\boldsymbol{\pi}$,

$$\begin{aligned}
&E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}_1(Y_T^{t,y,\boldsymbol{\pi}})|] \\
\le&\epsilon\ E[1_{\{|Y_T^{t,y,\boldsymbol{\pi}}-y_0|\le\delta\}}] + E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}(y_0)|1_{\{Y_T^{t,y,\boldsymbol{\pi}}>y_0+\delta\}}] + E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}(y_0-)|1_{\{Y_T^{t,y,\boldsymbol{\pi}}<y_0-\delta\}}] \\
\le&\epsilon + E[(C_1 + C_2 e^{\hat{p}Y_T^{t,y,\boldsymbol{\pi}}} + |\tilde{U}(y_0)|)1_{\{Y_T^{t,y,\boldsymbol{\pi}}>y_0+\delta\}}] + E[(\tilde{C}_1 + \tilde{C}_2 e^{-\bar{p}Y_T^{t,y,\boldsymbol{\pi}}} + |\tilde{U}(y_0-)|)1_{\{Y_T^{t,y,\boldsymbol{\pi}}<y_0-\delta\}}],
\end{aligned}$$

where $\tilde{C}_1, \tilde{C}_2, \bar{p}$ are positive constants related to $C_1, C_2, \hat{p}$ and $A_1, A_2, \widetilde{p}$ (cf. Assumption 3.1). By integration by part, we have

$$\begin{aligned}
&E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}_1(Y_T^{t,y,\boldsymbol{\pi}})|] \\
\le&\epsilon + (C_1 + |\tilde{U}(y_0)| + C_2 e^{\hat{p}(y_0+\delta)})P[Y_T^{t,y,\boldsymbol{\pi}} > y_0+\delta] + (\tilde{C}_1 + |\tilde{U}(y_0-)| + \tilde{C}_2 e^{-\bar{p}(y_0-\delta)})P[Y_T^{t,y,\boldsymbol{\pi}} < y_0-\delta] \\
&+ C_2\int_\delta^\infty \hat{p}e^{\hat{p}(y_0+v)}P(Y_T^{t,y,\boldsymbol{\pi}} \ge y_0+v)dv + \tilde{C}_2\int_\delta^\infty \bar{p}e^{-\bar{p}(y_0-v)}P(Y_T^{t,y,\boldsymbol{\pi}} \le y_0-v)dv.
\end{aligned}$$

By (EC.62) and (EC.63), we have

$$\begin{aligned}
&E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}_1(Y_T^{t,y,\boldsymbol{\pi}})|] \\
\le&\epsilon + (C_1 + |\tilde{U}(y_0)| + C_2 e^{\hat{p}(y_0+\delta)})f(\frac{y-y_0-\delta+M(T-t)}{\sqrt{T-t}}) \\
&+ (\tilde{C}_1 + |\tilde{U}(y_0-)| + \tilde{C}_2 e^{-\bar{p}(y_0-\delta)})f(\frac{y_0-y-\delta+M(T-t)}{\sqrt{T-t}}) \\
&+ C_2\int_\delta^\infty \hat{p}e^{\hat{p}(y_0+v)}f(\frac{y-y_0-v+M(T-t)}{\sqrt{T-t}})dv + \tilde{C}_2\int_\delta^\infty \bar{p}e^{-\bar{p}(y_0-v)}f(\frac{y_0-y-v+M(T-t)}{\sqrt{T-t}})dv.
\end{aligned}$$

The bound is independent with $\boldsymbol{\pi}$. Then taking supremum for $\boldsymbol{\pi} \in \boldsymbol{C}$ and sending $t \to T$, since $y - y_0 - v < -\delta/2$ and $y_0 - y - v < -\delta/2$ for $|y - y_0| \le \delta/2$ and $v \ge \delta$, we have

$$\limsup_{t\to T}\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}_1(Y_T^{t,y,\boldsymbol{\pi}})|] \le \epsilon.$$

Then

$$\limsup_{(t,y)\to(T-,y_0)} |\tilde{V}(t,y) - \tilde{V}_1(t,y)| \le \limsup_{(t,y)\to(T,y_0)}\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[|\tilde{U}(Y_T^{t,y,\boldsymbol{\pi}}) - \tilde{U}_1(Y_T^{t,y,\boldsymbol{\pi}})|] \le \epsilon.$$

Since ϵ is arbitrary, we have $\lim_{(t,y)\to(T-,y_0)} |\tilde{V}(t,y) - \tilde{V}_1(t,y)| = 0$. Then (EC.59) follows by (EC.64). Hence, (EC.55) is proved.

(II) We consider the general case where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are functions on $(t, \boldsymbol{x})$. As in Proposition EC.3.4, define the log-wealth process $Y_s^{\boldsymbol{\pi}} = \log W_s$ satisfying

$$dY_s^{\boldsymbol{\pi}} = (\boldsymbol{\eta}_s \top \boldsymbol{\pi}_s - \frac{1}{2}\boldsymbol{\pi}_s^\top \boldsymbol{\Sigma}_{s,\boldsymbol{x}_s}\boldsymbol{\pi}_s)ds + \boldsymbol{\pi}_s^\top \boldsymbol{\sigma}_s d\boldsymbol{\mathcal{B}}_s, \qquad Y_t^{\boldsymbol{\pi}} = y := \log w. \tag{EC.65}$$

Fix the limit point $(T, w, \boldsymbol{x})$. First, we aim to prove

$$\limsup_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) - U(w-) - 2\Phi\left(\frac{0\wedge(\log\zeta - \log w)}{L_{T,\boldsymbol{x}}\sqrt{T-t}}\right)(U(w) - U(w-)) \le 0. \tag{EC.66}$$

By Assumption EC.1.1 (3) and the assumption that the volatility function $\boldsymbol{\sigma}$ is invertible and continuous, given any $\epsilon > 0$, there exists $\delta > 0$, such that $||\boldsymbol{\sigma}(t,\boldsymbol{\xi})\boldsymbol{\sigma}(T,\boldsymbol{x})^{-1} - I|| \leq \epsilon/2$ for any $(t,\boldsymbol{\xi})$ such that $|T-t| \leq \delta$, $||\boldsymbol{\xi} - \boldsymbol{x}|| \leq \delta$. Denote by $\Omega^{\delta}_{(t,\boldsymbol{\xi})}$ the probability set such that $\sup_{t\leq s\leq T} ||\boldsymbol{x}_s - \boldsymbol{x}|| \leq \delta$, conditional on $\boldsymbol{x}_t = \xi$. By Assumption EC.1.1 (3), $\lim_{(t,\boldsymbol{\xi})\to(T-,\boldsymbol{x})} P[\Omega^{\delta}_{(t,\boldsymbol{\xi})}] = 1$. On $\Omega^{\delta}_{(t,\boldsymbol{\xi})}$, we have

$$M_{t,\boldsymbol{\xi}} := \sup_{t\leq s\leq T} \sup_{\boldsymbol{\pi}_s\in\boldsymbol{C}} |\boldsymbol{\eta}_s \top \boldsymbol{\pi}_s - \frac{1}{2}\boldsymbol{\pi}_s^\top \boldsymbol{\Sigma}_{s,\boldsymbol{x}_s}\boldsymbol{\pi}_s| < +\infty. \tag{EC.67}$$

Let $a = \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} ||\boldsymbol{\pi}||$ and $\boldsymbol{C}^{\epsilon} = \{\bar{\boldsymbol{\pi}} : \inf_{\boldsymbol{\pi}\in\boldsymbol{C}} ||\boldsymbol{\pi} - \bar{\boldsymbol{\pi}}|| < a\epsilon\}$. We consider the following problem:

$$\begin{aligned} \bar{V}^{\epsilon}(t,\zeta) &:= \sup_{\bar{\boldsymbol{\pi}}\in\boldsymbol{C}^{\epsilon}} E[\tilde{U}(\bar{Y}_T^{\bar{\boldsymbol{\pi}}}) | \bar{Y}_t^{\bar{\boldsymbol{\pi}}} = \log\zeta] \\ &\quad \text{s.t.} \quad d\bar{Y}_s^{\bar{\boldsymbol{\pi}}} = \bar{\boldsymbol{\pi}}_s^\top \boldsymbol{\sigma}(T,\boldsymbol{x}) d\boldsymbol{\mathcal{B}}_s, \ t\leq s\leq T. \end{aligned} \tag{EC.68}$$

For each $\boldsymbol{\pi}_s \in \boldsymbol{C}$, let $\bar{\boldsymbol{\pi}}_s := (\sigma(s,\boldsymbol{x}_s)\boldsymbol{\sigma}(T,\boldsymbol{x})^{-1})^\top \boldsymbol{\pi}_s$. Then

$$d\bar{Y}_s^{\bar{\boldsymbol{\pi}}} = \boldsymbol{\pi}_s^\top \sigma(s,\boldsymbol{x}_s) d\boldsymbol{\mathcal{B}}_s. \tag{EC.69}$$

Comparing (EC.65) and (EC.69), on $\Omega^{\delta}_{(t,\boldsymbol{\xi})}$ (cf. (EC.67)), we have

$$Y_s^{\boldsymbol{\pi},(t,\log\zeta)} \leq \bar{Y}_s^{\bar{\boldsymbol{\pi}},(t,\log\zeta+(T-t)M_{t,\boldsymbol{\xi}})},$$

for $t \leq s \leq T$, where the former is the original log-wealth process (EC.65) starting from $Y_t = \log\zeta$, and the latter is the process in (EC.69) starting from $\bar{Y}_t = \log\zeta + (T-t)M_{t,\boldsymbol{\xi}}$.

Furthermore, for $s \in (T-\delta, T)$, in set $\Omega^{\delta}_{(t,\boldsymbol{\xi})}$, we have $||\boldsymbol{x}_s - \boldsymbol{x}|| \leq \delta$ and

$$||\bar{\boldsymbol{\pi}}_s - \boldsymbol{\pi}_s|| = ||(\sigma(s,\boldsymbol{x}_s)\boldsymbol{\sigma}(T,\boldsymbol{x})^{-1} - I)^\top \boldsymbol{\pi}_s|| \leq a||\sigma(s,\boldsymbol{x}_s)\boldsymbol{\sigma}(T,\boldsymbol{x})^{-1} - I|| \leq a\epsilon/2.$$

That is, $\bar{\boldsymbol{\pi}}_s \in \boldsymbol{C}^{\epsilon}$. Denote $\tilde{U}(y) := U(e^u)$, then,

$$\begin{aligned} V(t,\zeta,\boldsymbol{\xi}) &= \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\tilde{U}(Y_T^{\boldsymbol{\pi},(t,\log\zeta)})\mathbf{1}_{\Omega^{\delta}_{(t,\boldsymbol{\xi})}}] + \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\tilde{U}(Y_T^{\boldsymbol{\pi},(t,\log\zeta)})\mathbf{1}_{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c}] \\ &\leq \bar{V}^{\epsilon}(t,\zeta e^{(T-t)M_{t,\boldsymbol{\xi}}}) - \sup_{\bar{\boldsymbol{\pi}}_s\in\boldsymbol{C}^{\epsilon}} E[\tilde{U}(\bar{Y}_T^{\bar{\boldsymbol{\pi}},(t,\log\zeta+(T-t)M_{t,\boldsymbol{\xi}})})\mathbf{1}_{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c}] \\ &\quad + \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\tilde{U}(Y_T^{\boldsymbol{\pi},(t,\log\zeta)})\mathbf{1}_{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c}]. \end{aligned}$$

We will show that the last two terms tend to zero as $(t,\boldsymbol{\xi}) \to (T-,\boldsymbol{x})$. Without loss of generality, we take the last term for example. We write $Y_T^{\boldsymbol{\pi}}$ for $Y_T^{\boldsymbol{\pi},(t,\log\zeta)}$. Given any large positive number A,

$$\begin{aligned} E[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|\mathbf{1}_{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c}] &= E[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|\mathbf{1}_{\{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c, |\tilde{U}(Y_T^{\boldsymbol{\pi}})|>A\}}] + E[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|\mathbf{1}_{\{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c, |\tilde{U}(Y_T^{\boldsymbol{\pi}})|\leq A\}}] \\ &\leq AE[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|/A\mathbf{1}_{\{|\tilde{U}(Y_T^{\boldsymbol{\pi}})|>A\}}] + AE[\mathbf{1}_{\{(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c\}}] \\ &\leq E[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|^q]/A^{q-1} + AP[(\Omega^{\delta}_{(t,\boldsymbol{\xi})})^c], \end{aligned}$$

where $q>1$ is chosen such that $\hat{p}q<1$. Then, $E[|\tilde{U}(Y_T^{\boldsymbol{\pi}})|^q]$ is bounded by Assumption EC.1.1 (4) with $\hat{p}$ replaced by $\hat{p}q$. And the right-hand side converges to 0, as we first take $s\to T$ and then let $A\to\infty$.

Then

$$\limsup_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})}(V(t,\zeta,\boldsymbol{\xi})-\bar{V}^{\epsilon}(t,\zeta e^{(T-t)M_{t,\boldsymbol{\xi}}}))\le 0.$$

When $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are constants, according to (EC.55), we have

$$\begin{aligned}\limsup_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})}&\bar{V}^{\epsilon}(t,\zeta e^{(T-t)M_{t,\boldsymbol{\xi}}})-U(w-)\\&-2\Phi\left(\frac{0\wedge(\log\zeta-\log w+(T-t)M_{t,\boldsymbol{\xi}})}{L^{\epsilon}_{T,\boldsymbol{x}}\sqrt{T-t}}\right)(U(w)-U(w-))\le 0,\end{aligned}$$

where $L^{\epsilon}_{T,\boldsymbol{x}}:=\sqrt{\max_{\boldsymbol{\pi}\in\boldsymbol{C}^{\epsilon}}\boldsymbol{\pi}^{\top}\boldsymbol{\Sigma}_{T,\boldsymbol{x}}\boldsymbol{\pi}}$.

Notice that $\lim_{\lambda\to 1}\sup_{z\in\mathbb{R}}|\Phi(z)-\Phi(\lambda z)|=0$. Denoting $z=\frac{0\wedge(\log\zeta-\log w)}{L_{T,\boldsymbol{x}}\sqrt{T-t}}$, we have

$$\begin{aligned}&\limsup_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})}V(t,\zeta,\boldsymbol{\xi})-U(w-)-2\Phi(z)(U(w)-U(w-))\\&\le 2\sup_{x\in\mathbb{R}}\left|\Phi(z)-\Phi\left(\frac{L_{T,\boldsymbol{x}}}{L^{\epsilon}_{T,\boldsymbol{x}}}z\right)\right|(U(w)-U(w-))\to 0,\ \text{as}\ \ \epsilon\to 0.\end{aligned}$$

By applying the strategy $\boldsymbol{\pi}^*:=\arg\max_{\boldsymbol{\pi}\in\boldsymbol{C}^{\epsilon}}\boldsymbol{\pi}^{\top}\boldsymbol{\Sigma}_{T,\boldsymbol{x}}\boldsymbol{\pi}$ in $[t,T]$, we have

$$\liminf_{(t,\zeta,\boldsymbol{\xi})\to(T-,w,\boldsymbol{x})}V(t,\zeta,\boldsymbol{\xi})-U(w-)-2\Phi(z)(U(w)-U(w-))\ge 0.$$

Thus, Part (i) of Proposition EC.3.3 is proved. □

We now proceed to prove Proposition EC.3.4, which relies on Lemma EC.3.1 below. After the logarithmic transformation $Y_s^{t,y,\boldsymbol{\pi}}=\log W_s, s\ge t$ and $Y_t^{t,y,\boldsymbol{\pi}}=y=\log w$, the optimization problem in Proposition EC.3.4 can be reformulated as

$$\tilde{V}(t,y):=\sup_{\boldsymbol{\pi}\in\boldsymbol{C}}E[\mathbf{1}_{Y_T^{t,y,\boldsymbol{\pi}}\ge 0}],\tag{EC.70}$$

where the log-wealth process Y is given in (EC.58). Then, $\tilde{V}(t,y)=V(t,w)$ with $y=\log w$.

It is worthwhile pointing out that as t approaches T, it is the volatility term (of order $\sqrt{T-t}$) rather than the drift term (of order $T-t$) that plays a dominant role. Thus, we first study a goal-reaching problem without drift term, which turns out to have a closed-form solution.

LEMMA EC.3.1. *For $0\le t\le T, y\le 0$, let $G(t,y)$ be the value function of the following problem:*

$$\begin{aligned}G(t,y):=&\sup_{\boldsymbol{\pi}\in\boldsymbol{C}}E[\mathbf{1}_{\bar{Y}_T^{t,y,\boldsymbol{\pi}}\ge 0}]\\&s.t.\ \ d\bar{Y}_s^{t,y,\boldsymbol{\pi}}=\boldsymbol{\pi}_s^{\top}\boldsymbol{\sigma}d\boldsymbol{\mathcal{B}}_s,\ t<s\le T,\ \ and\ \ \bar{Y}_t^{t,y,\boldsymbol{\pi}}=y.\end{aligned}\tag{EC.71}$$

Then $G(t,y)=f(\frac{y}{\sqrt{T-t}})$, where f is as defined in Proposition EC.3.4.

Note that $f(\frac{y}{\sqrt{T-t}}) = P(\max_{t\le s\le T} \bar{Y}_s^{t,y,\boldsymbol{\pi}^*} \ge 0)$, where $\boldsymbol{\pi}^* \equiv \boldsymbol{l} := \arg\max_{\boldsymbol{\pi}\in\boldsymbol{C}} \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{T,\boldsymbol{x}}\boldsymbol{\pi}$. Thus, the lemma indicates that it is optimal to apply the maximum leverage or short-sale ratio $\boldsymbol{\pi}_s = \boldsymbol{l}$ and to switch to $\boldsymbol{\pi}_s = \mathbf{0}$ once the goal is reached.

Proof of Lemma EC.3.1: First, we show that $G(t,y) = g(\frac{y}{\sqrt{T-t}})$, for some function $g(z)$. Consider any two points (y_1,t_1) and (y_2,t_2) such that $t_t, t_2 < T$, $y_1, y_2 < 0$ and $\frac{y_1}{\sqrt{T-t_1}} = \frac{y_2}{\sqrt{T-t_2}}$. For any adapted strategy $\boldsymbol{\pi}_s$, $t_1 \le s \le T$, and a time-change $C(s) := t_1 + \frac{T-t_1}{T-t_2}(s-t_2)$ for $s\in[t_2,T]$, we can define an adapted strategy $\hat{\boldsymbol{\pi}}_s := \boldsymbol{\pi}_{C(s)}$ and a new Brownian motion $\hat{\mathcal{B}}_s := \sqrt{\frac{T-t_2}{T-t_1}}\left(\mathcal{B}_{C(s)} - \mathcal{B}_{t_1}\right)$ for $s \in [t_2, T]$. By (EC.71),

$$\begin{aligned}\bar{Y}_T^{t_1,y_1,\boldsymbol{\pi}} - y_1 &= \int_{t_1}^T \boldsymbol{\pi}_s^\top \boldsymbol{\sigma} d\boldsymbol{\mathcal{B}}_s = \int_{C(t_2)}^{C(T)} \boldsymbol{\pi}_s^\top \boldsymbol{\sigma} d\boldsymbol{\mathcal{B}}_s = \int_{t_2}^T \boldsymbol{\pi}_{C(s)}^\top \boldsymbol{\sigma} d\mathcal{B}_{C(s)} = \sqrt{\frac{T-t_1}{T-t_2}} \int_{t_2}^T \hat{\boldsymbol{\pi}}_s^\top \boldsymbol{\sigma} d\hat{\mathcal{B}}_s \\ &\overset{d}{=} \sqrt{\frac{T-t_1}{T-t_2}}\left(\bar{Y}_T^{t_2,y_2,\hat{\boldsymbol{\pi}}} - y_2\right), \end{aligned} \tag{EC.72}$$

where $\overset{d}{=}$ represents "equal in distribution". Recalling that $\frac{y_1}{\sqrt{T-t_1}} = \frac{y_2}{\sqrt{T-t_2}}$, we have,

$$\begin{aligned} P(\bar{Y}_T^{t_1,y_1,\boldsymbol{\pi}} \ge 0) =& P(\bar{Y}_T^{t_1,y_1,\boldsymbol{\pi}} - y_1 \ge -y_1) = P(\sqrt{\frac{T-t_1}{T-t_2}}\left(\bar{Y}_T^{t_2,y_2,\hat{\boldsymbol{\pi}}} - y_2\right) \ge -\frac{\sqrt{T-t_1}}{\sqrt{T-t_2}} y_2) \\ =& P(\bar{Y}_T^{t_2,y_2,\hat{\boldsymbol{\pi}}} - y_2 \ge -y_2) = P(\bar{Y}_T^{t_2,y_2,\hat{\boldsymbol{\pi}}} \ge 0). \end{aligned}$$

This means $G(t_1,y_1) \le G(t_2,y_2)$. For the same reason, $G(t_2,y_2) \le G(t_1,y_1)$.

Second, we show that $g(z)$ is the unique viscosity solution to the ODE

$$\begin{cases} -zg'(z) - \sup_{\boldsymbol{\pi}\in\boldsymbol{C}}\{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} g''(z)\} = 0, \ z<0, \\ g(0) = 1, \ \lim_{z\to-\infty} g(z) = 0. \end{cases} \tag{EC.73}$$

The uniqueness holds by a standard approach to proving the comparison principle of this ODE, which we omit here. We only verify that $g(z)$ is a viscosity solution to the above ODE. By Proposition EC.3.1 (Weak Dynamic Programming) and Corollary 5.6 of Bouchard and Touzi (2011), if $h \in C^{1,2}([0,T)\times(-\infty,0))$ and $h - G^*$ attains local minimum 0 at $(\bar{t}, y_0)$, then

$$-\frac{\partial}{\partial t} h(\bar{t},y_0) - \sup_{\boldsymbol{\pi}_t\in\boldsymbol{C}}\left\{\frac{1}{2}\boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}\boldsymbol{\pi}_t \frac{\partial^2}{\partial y^2} h(\bar{t},y_0)\right\} \le 0.$$

Now consider a function $\phi \in C^2((-\infty,0])$, such that $\phi(z) - g^*(z)$ attains local minimum 0 at interior point $-\infty < z_0 < 0$, then $\phi(\frac{y}{\sqrt{T-t}}) \in C^{1,2}([0,T)\times(-\infty,0))$, and $\phi(\frac{y}{\sqrt{T-t}}) - G^*(t,y)$ attains local minimum 0 at point $(t, \sqrt{T-t}\ z_0)$ for any $0 \le t < T$. Then, for any $0\le t<T$, we have

$$-\frac{\partial}{\partial t}\phi(\frac{y}{\sqrt{T-t}})\big|_{\{y=\sqrt{T-t}\ z_0\}} - \sup_{\boldsymbol{\pi}_t\in\boldsymbol{C}}\left\{\frac{1}{2}\boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}\boldsymbol{\pi}_t \frac{\partial^2}{\partial y^2}\phi(\frac{y}{\sqrt{T-t}})\big|_{\{y=\sqrt{T-t}\ z_0\}}\right\} \le 0.$$

By a direct calculation, we have

$$-z_0\phi'(z_0) - \sup_{\boldsymbol{\pi}\in\boldsymbol{C}}\left\{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi}\phi''(z_0)\right\} \le 0.$$

Since $g^*(0)=1$ and $\lim_{z\to-\infty} g^*(w)=0$, we infer that g^* is a viscosity subsolution of (EC.73). A similar argument shows that g_* is a viscosity supersolution. Therefore, g is a viscosity solution of (EC.73).

Third, a direct calculation shows that f is a classical solution to

$$\begin{cases} -zf'(z)-(\boldsymbol{\pi}^*)^\top \boldsymbol{\Sigma}\boldsymbol{\pi}^* f''(z)=0,\ z<0, \\ f(0)=1,\ \lim_{z\to-\infty} f(z)=0. \end{cases}$$

By the convexity of f, f is also a classical solution to

$$\begin{cases} -zf'(z)-\sup_{\boldsymbol{\pi}\in\boldsymbol{C}}\left\{\boldsymbol{\pi}^\top \boldsymbol{\Sigma}\boldsymbol{\pi} f''(z)\right\}=0,\ z<0, \\ f(0)=1,\ \lim_{z\to-\infty} f(z)=0. \end{cases}$$

By the uniqueness of viscosity solution, we prove the lemma. □

Proof of Proposition EC.3.4: For any $(t,y)\in[0,T)\times(-\infty,0]$, let $Y_s^{t,y,\boldsymbol{\pi}}$ be given in (EC.58) with starting point $Y_t^{t,y,\boldsymbol{\pi}}=y$ and portfolio $\boldsymbol{\pi}_s, t\le s\le T$, and let $\bar{Y}_s^{t,y+M(T-t),\boldsymbol{\pi}}$ be given in (EC.71) with starting point $\bar{Y}_s^{t,y+M(T-t),\boldsymbol{\pi}}=y+M(T-t)$ and the same portfolio $\boldsymbol{\pi}_s, t\le s\le T$. Recalling that $M:=\max_{\boldsymbol{\pi}\in\boldsymbol{C}}\{|\boldsymbol{\eta}^\top\boldsymbol{\pi}-\frac{1}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}\boldsymbol{\pi}|\}<+\infty$, by comparison, we have $Y_T^{t,y,\boldsymbol{\pi}}\le \bar{Y}_T^{t,y+M(T-t),\boldsymbol{\pi}}$, a.s. Similarly $Y_T^{t,y,\boldsymbol{\pi}}\ge \bar{Y}_T^{t,y-M(T-t),\boldsymbol{\pi}}$, a.s. Then,

$$G(t,y-M(T-t))\le \tilde{V}(t,y)\le G(t,y+M(T-t)),$$

where $G(t,y)=f(\frac{y}{\sqrt{T-t}})$ is given in Lemma EC.3.1. So,

$$f(\frac{y}{\sqrt{T-t}}-M\sqrt{T-t})\le \tilde{V}(t,y)\le f(\frac{y}{\sqrt{T-t}}+M\sqrt{T-t}).$$

By the uniform continuity of f, the proposition is proved. □

Bian, Chen and Xu (2019) give a proof for Proposition EC.3.3(ii) when $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are constants. Here, we extend their result by allowing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ to be functions on $(t,\boldsymbol{x})$. First, we introduce two lemmas.

LEMMA EC.3.2. *In the case that $\boldsymbol{C}$ is unbounded, we have*

$$V(t,\lambda w_1+(1-\lambda)w_2,\boldsymbol{x})\ge \lambda U(w_1)+(1-\lambda)U(w_2),\quad \textit{for all}\ \ 0<\lambda<1,\ t<T.$$

Proof of Lemma EC.3.2: Since $\boldsymbol{C}$ is unbounded, there is a sequence $\boldsymbol{v}_n$, $n=1,2,3,...$, such that $\|\boldsymbol{v}_n\|_2=1$, $k_n\to+\infty$ and $k_n\boldsymbol{v}_n\in\boldsymbol{C}$. Due to the compactness of the unit ball surface $\{\boldsymbol{u}\in\mathbb{R}^N|\|\boldsymbol{u}\|_2=1\}$, without loss of generality, we assume $\boldsymbol{v}_n\to\boldsymbol{v}$ for some $\boldsymbol{v}$. We focus on the portfolio $k_n\boldsymbol{v}_n^\top\boldsymbol{S}$. For $t<T$, assume the wealth $W_t=w=\lambda w_1+(1-\lambda)w_2$. And let $W_s^n, t\le s\le T$ be the wealth process with $W_t^n=w$, $\boldsymbol{x}_t=\boldsymbol{x}$. Thus, we have

$$dW_s^n/W_s^n = k_n\eta_n(s,\boldsymbol{x}_s)ds+k_n\sigma_n(s,\boldsymbol{x}_s)d\mathcal{B}_s,\quad t\le s\le T,$$

where $\eta_n = \boldsymbol{v}_n^\top(\boldsymbol{\mu} - r)$ and $\sigma_n := \sqrt{\boldsymbol{v}_n^\top \boldsymbol{\Sigma}_{t,x} \boldsymbol{v}_n}$ are continuous. Denote $\tilde{\eta} = \boldsymbol{v}^\top(\boldsymbol{\mu} - r)$ and $\tilde{\sigma} := \sqrt{\boldsymbol{v}^\top \boldsymbol{\Sigma}_{t,x} \boldsymbol{v}}$. By items (2) and (3) in Assumption EC.1.1, for any $\epsilon > 0$, there exists an n_0, such that $n_0\epsilon$ is big enough and

$$P(\min_{t\le s\le T} \tilde{\eta}(s, \boldsymbol{x}_s) > -\frac{n_0}{2}\epsilon) > 1 - \frac{\epsilon}{2}.$$

Then for sufficiently large n,

$$P(\min_{t\le s\le T} \eta_n(s, \boldsymbol{x}_s) > -n_0\epsilon) > 1 - \epsilon.$$

Then, on $\min_{t\le s\le T} \eta_n(s, \boldsymbol{x}_s) > -n_0\epsilon$, we have $W_s^n \ge W_s^{\epsilon,n}$ for $t \le s \le T$, where $W_t^{\epsilon,n} = W_t^n = w$, and for $s > t$,

$$dW_s^{\epsilon,n}/W_s^{\epsilon,n} = -\epsilon n_0 k_n ds + \sigma_n(s, \boldsymbol{x}_s) k_n d\mathcal{B}_s.$$

Define $\tau_1^{(n)}$, $\tau_2^{(n)}$ as the hitting times of w_1, w_2 for the process W_s^n, and $\tau_1^{(\epsilon,n)}$, $\tau_2^{(\epsilon,n)}$ as the hitting times of w_1, w_2 for the process $W_s^{\epsilon,n}$. Next, we show that

$$\lim_{n\to+\infty} P(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) \ge \lambda.$$

Assume $w_1 > w_2$. It follows

$$\begin{aligned}
P(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) \ge& P(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}, \min_{t\le s\le T} \eta_n(s, \boldsymbol{x}_s) > -n_0\epsilon) \\
\ge& P(\tau_1^{(\epsilon,n)} < \min\{\tau_2^{(\epsilon,n)}, T\}, \min_{t\le s\le T} \eta_n(s, \boldsymbol{x}_s) > -n_0\epsilon) \\
\ge& P(\tau_1^{(\epsilon,n)} < \min\{\tau_2^{(\epsilon,n)}, T\}) - \epsilon.
\end{aligned}$$

By rescaling the process, define $\tilde{W}_s^{\epsilon,n} := W_{t+(s-t)/k_n^2}^{\epsilon,n}, t \le s \le t + k_n^2(T - t)$ and $\tilde{W}_t^{\epsilon,n} = w$, then $\tilde{W}_s^{\epsilon,n}$ is governed by

$$d\tilde{W}_s^{\epsilon,n}/\tilde{W}_s^{\epsilon,n} = \frac{-\epsilon n_0}{k_n} ds + \sigma_n(t + \frac{s-t}{k_n^2}, \boldsymbol{x}_{t+\frac{s-t}{k_n^2}}) d\mathcal{B}_s^{(n)}, \quad t \le s \le \tau^1 \wedge \tau^2 \wedge k_n^2(T - t) + t,$$

where $\mathcal{B}_s^{(n)}$ is a standard Brownian motion. Define $\tilde{\tau}_1^{(\epsilon,n)}, \tilde{\tau}_2^{(\epsilon,n)}$ as the hitting times of w_1, w_2 for the process $\tilde{W}_s^{\epsilon,n}$. Then, by definition

$$P(\tau_1^{(\epsilon,n)} < \min\{\tau_2^{(\epsilon,n)}, T\}) = P(\tilde{\tau}_1^{(\epsilon,n)} < \min\{\tilde{\tau}_2^{(\epsilon,n)}, t + k_n^2(T - t)\}).$$

As $n \to \infty$, $\mathbf{1}_{\tilde{\tau}_1^{(\epsilon,n)} < \min\{\tilde{\tau}_2^{(\epsilon,n)}, k_n^2(T-t)+t\}} \to \mathbf{1}_{\tilde{\tau}_1^{(\epsilon,\infty)} < \tilde{\tau}_2^{(\epsilon,\infty)}}$ in distribution, where

$$d\tilde{W}_s^{\epsilon,\infty}/\tilde{W}_s^{\epsilon,\infty} = \tilde{\sigma}(t, \boldsymbol{x}_t) d\mathcal{B}_s, \quad s \ge t,$$

then

$$\lim_{n\to+\infty} P(\tilde{\tau}_1^{(\epsilon,n)} < \min\{\tilde{\tau}_2^{(\epsilon,n)}, k_n^2(T-t)+t\}) = \lambda.$$

Thus,

$$\liminf_{n\to+\infty} P(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) \geq \lambda - \epsilon.$$

Sending $\epsilon \to 0$, we have

$$\liminf_{n\to+\infty} P(\tau_1^{(n)} < \min\{\tau_2^{(n)}, T\}) \geq \lambda.$$

Similarly, we have the other side inequality

$$\liminf_{n\to+\infty} P(\tau_2^{(n)} < \min\{\tau_1^{(n)}, T\}) \geq 1-\lambda.$$

Now, take the following strategy

$$\pi_s^{(n)} = \begin{cases} k_n \boldsymbol{v}_n, & t \leq s \leq \tau_1^{(n)} \wedge \tau_2^{(n)} \wedge T, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$V(t, \lambda w_1 + (1-\lambda) w_2) \geq \limsup_{n\to\infty} E[W_T^{\pi^{(n)}}] \geq \lambda U(w_1) + (1-\lambda) U(w_2).$$

Thus, the lemma is proved. □

LEMMA EC.3.3. *Consider a concave function $f(w)$ on $B < w < +\infty$, which satisfies $\lim_{w\to+\infty} \frac{f(w)}{w^p} = 0$ for some $0 < p < 1$. Then for a fixed point $w_0 > B$ and a fixed number $\epsilon > 0$, there exists a function of type*

$$f_1(w) = C_1 w^q + C_2, \quad p \leq q < 1.$$

s.t. $f_1(w) \geq f(w)$, $\forall w \geq B$, and $f_1(w_0) \leq f(w_0) + \epsilon$.

Proof of Lemma EC.3.3: First, since f is concave, there exists a tangent line $h(w)$ of this function, s.t. $h(w)$ has a finite positive slope, $h(w) \geq f(w)$, and $h(w_0) \leq f(w_0) + \frac{\epsilon}{2}$. What's more, assume $f(w) \leq \frac{h(n)-h(B)}{n^p} w^p + h(B) \leq h(w)$ when $w \geq n$.

Now choose $C_2 = h(B)$ and $C_1 = \frac{h(n)-h(B)}{n^q}$, and let $f_1(w) = C_1 w^q + C_2$. For any $p \leq q < 1$, we have $f_1(B) \geq h(B) \geq f(B)$ and $f_1(n) = h(n) \geq f(n)$. By the concavity of f_1, we have $f_1 \geq h \geq f$ in $[0, n]$, and $f_1(w) \geq \frac{h(n)-h(B)}{n^p} w^p + h(B) \geq f(w)$ when $w \geq n$. That is, $f_1(w) \geq f(w)$, $\forall w \geq B$.

Sending $q \to 1-$ gives $f_1(w_0) \to h(w_0)$. Thus, there exists $q \in [p, 1)$, such that $f_1(w_0) \leq f(w_0) + \epsilon$. □

Proof of Proposition EC.3.3(ii): First, we show that $V(t,w,\boldsymbol{x}) \geq \hat{U}(w)$ for all $t<T, w>B$, $\boldsymbol{x} \in \mathbb{R}^K$. If not, by Lemma EC.3.2, there exists an $\epsilon>0$, s.t.

$$\hat{U}(w) \geq V(t,w,\boldsymbol{x}) + \epsilon \geq \lambda U(w_1) + (1-\lambda)U(w_2) + \epsilon, \quad \forall\ \lambda w_1 + (1-\lambda) w_2 = w,\ \lambda \in (0,1).$$

The region between $\hat{U}$ and $f \equiv U(B)$ is the smallest convex hull of

$$\{(F,w) | U(B) < F \leq U(w), B < w < +\infty\}.$$

By the separation theorem of convex sets, $\hat{U}(w)$ has a positive distance from this convex hull, which leads to a contradiction. Then we have

$$\liminf_{(t,\zeta,\boldsymbol{\xi}) \to (T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) \geq \hat{U}(w).$$

For the other side inequality, consider the case with utility $f_1(w) = C_1 w^q + C_2$ given by Lemma EC.3.3. By Lemma EC.2.1, it has a classical solution $V_1(t,\zeta,\boldsymbol{\xi})$ which is continuous at the terminal time T. Then for any $\epsilon>0$,

$$\limsup_{(t,\zeta,\boldsymbol{\xi}) \to (T-,w,\boldsymbol{x})} V(t,\zeta,\boldsymbol{\xi}) \leq \limsup_{(t,\zeta,\boldsymbol{\xi}) \to (T-,w,\boldsymbol{x})} V_1(t,\zeta,\boldsymbol{\xi}) = f_1(w) \leq \hat{U}(w) + \epsilon.$$

Thus, the desired result follows by sending $\epsilon \to 0$. □

The verification of Condition c) is now finished.

Appendix EC.4: Proof of Theorem EC.1.3

The HJB equation of our non-concave utility maximization problem may involve unbounded wealth level, unbounded stochastic state variable $\boldsymbol{x}_t$, unbounded portfolio constraints, and discontinuous utility function. In this section, we first present several lemmas to show that the value function of the original problem can be approximated by the value function under the case that the wealth processes are controlled in a bounded domain, the stochastic state variable $\boldsymbol{x}_t$ is bounded, the portfolio set is bounded with a positive distance to the origin and the utility function is smooth and bounded. Then, we only need to prove the convergence of the numerical algorithm when (1) the wealth processes are controlled in a bounded domain, (2) the stochastic state variable $\boldsymbol{x}_t$ is bounded, (3) the portfolio set is bounded with a positive distance to the origin, and (4) the utility function is smooth and bounded.

In the following lemma, we show that we only need to consider lower-bounded utility functions.

LEMMA EC.4.1. *If the utility $U(w)$ is unbounded at the lower boundary $w = B$, denote by $V^\delta(t,w,\boldsymbol{x})$ the value function for the optimization problem with the lower boundary condition*

$$V^\delta(s,\delta,\boldsymbol{x}) = V_{CRRA}(s,\delta,\boldsymbol{x}), \quad \forall s \leq T. \tag{EC.74}$$

We have $\lim_{\delta \to 0} V^\delta(t,w,\boldsymbol{x}) = V(t,w,\boldsymbol{x})$ for any $t<T$, $w>0$.

Proof of Lemma EC.4.1: Firstly, we show that for any $s_0 < T$ and $M > 0$,

$$\lim_{\delta\to 0} \sup_{s_0\le s\le T} \sup_{\|\boldsymbol{x}\|_\infty \le M} |V(s,\delta,\boldsymbol{x}) - V_{CRRA}(s,\delta,\boldsymbol{x})| = 0. \tag{EC.75}$$

Let us prove it by contradiction. Suppose not. Then there are an $\epsilon > 0$ and a sequence of t_n, δ_n, $\boldsymbol{x}_n$ $n = 1,2,...$ such that $|V(t_n,\delta_n,\boldsymbol{x}_n) - V_{CRRA}(t_n,\delta_n,\boldsymbol{x}_n)| \ge \epsilon$. Since $t_n \in [s_0, T]$, $\boldsymbol{x}_n$ is bounded and $\lim_{n\to+\infty} \delta_n = 0$, there is a convergent subsequence $(t_{n_k}, \delta_{n_k}, \boldsymbol{x}_{n_k}) \to (t_0, 0, \boldsymbol{x}_{t_0})$, which contradicts the boundary condition (13).

Next, on the one hand, since $U(w) \ge A_1 \frac{w^{\widetilde{p}}-1}{\widetilde{p}} + A_2$, we have $V(t,w,\boldsymbol{x}) \ge V^\delta(t,w,\boldsymbol{x})$. On the other hand, for any admissible strategy $\boldsymbol{\pi}_s$, $t \le s \le T$ of the original problem, we denote $\tau^\delta := \inf\{s \ge t | W_s^{t,w,\boldsymbol{\pi}} \le \delta\}$, $\tau_{\boldsymbol{x}}^M := \inf\{s \ge t | \|\boldsymbol{x}_s\|_\infty \ge M\}$, and have

$$\begin{aligned}
&\limsup_{\delta\to 0} E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})\mathbf{1}_{\tau^\delta \ge T} + V(\tau^\delta, W_{\tau^\delta}^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_{\tau^\delta})\mathbf{1}_{\tau^\delta < T}] \\
&\qquad - E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})\mathbf{1}_{\tau^\delta \ge T} + V_{CRRA}(\tau^\delta, W_{\tau^\delta}^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_{\tau^\delta})\mathbf{1}_{\tau^\delta < T}] \\
&= \limsup_{\delta\to 0} E[V(\tau^\delta, W_{\tau^\delta}^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_{\tau^\delta})\mathbf{1}_{\tau^\delta < T} - V_{CRRA}(\tau^\delta, W_{\tau^\delta}^{t,w,\boldsymbol{x},\boldsymbol{\pi}}, \boldsymbol{x}_{\tau^\delta})\mathbf{1}_{\tau^\delta < T}] \\
&\le \limsup_{\delta\to 0} \Bigg(\sup_{t\le s\le T, \|\boldsymbol{x}\|_\infty \le M} |V(s,\delta,\boldsymbol{x}_{\tau^\delta}) - V_{CRRA}(s,\delta,\boldsymbol{x}_{\tau^\delta})| \\
&\qquad + E\Big[\Big(V(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta}) - V_{CRRA}(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})\Big)\mathbf{1}_{\tau_{\boldsymbol{x}}^M < \tau^\delta < T}\Big]\Bigg).
\end{aligned} \tag{EC.76}$$

In view of (EC.75),

$$\limsup_{\delta\to 0} \sup_{t\le s\le T, \|\boldsymbol{x}\|_\infty \le M} |V(s,\delta,\boldsymbol{x}_{\tau^\delta}) - V_{CRRA}(s,\delta,\boldsymbol{x}_{\tau^\delta})| = 0. \tag{EC.77}$$

For any $q > 1$,

$$\begin{aligned}
&\Big(V(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta}) - V_{CRRA}(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})\Big)^q \\
&\le \Bigg(\sup_{\boldsymbol{\pi}} E\Big[U(W_T^{\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta},\boldsymbol{\pi}}) - \Big(A_1 \frac{(W_T^{\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta},\boldsymbol{\pi}})^{\widetilde{p}} - 1}{\widetilde{p}} + A_2\Big)\Big]\Bigg)^q \\
&\le \Big(\sup_{\boldsymbol{\pi}} E[(W_T^{\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta},\boldsymbol{\pi}})^{\hat{p}} + \hat{C}]\Big)^q \\
&\le \sup_{\boldsymbol{\pi}} E\Big[\Big((W_T^{\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta},\boldsymbol{\pi}})^{\hat{p}} + \hat{C}\Big)^q\Big] \\
&\le \sup_{\boldsymbol{\pi}} 2^q E[(W_T^{\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta},\boldsymbol{\pi}})^{q\hat{p}} + \hat{C}^q].
\end{aligned} \tag{EC.78}$$

for some constant $\hat{C}$ only depending on the utility function $U(w)$. Noticing that the problem with initial position $(t,w,\boldsymbol{x})$ and utility function $\bar{U}(w) := w^{q\hat{p}}$ has finite value function when $q\hat{p} < 1$, then

$$E\Big[\Big(V(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta}) - V_{CRRA}(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})\Big)^q\Big] < +\infty, \quad \text{uniformly with respect to } \delta.$$

That implies the uniform integrability of $V(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})-V_{CRRA}(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})$. Noticing that when $M\to+\infty$, $\mathbf{1}_{\tau_{\boldsymbol{x}}^M<\tau^\delta<T}\to 0$ almost surely, we have

$$\lim_{M\to+\infty}\limsup_{\delta\to 0}E[\left(V(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})-V_{CRRA}(\tau^\delta,\delta,\boldsymbol{x}_{\tau^\delta})\right)\mathbf{1}_{\tau_{\boldsymbol{x}}^N<\tau^\delta<T}]=0. \tag{EC.79}$$

Then we complete the proof. □

The following lemma shows that the value function associated with a discontinuous utility can be approximated by a value function associated with some smooth utility that approximates the discontinuous utility.

LEMMA EC.4.2. *There exist an increasing sequence of smooth function $U_{d,k}$ and a decreasing sequence of smooth function $U_{u,k}$, such that for any $k>0$, $U(w-\frac{1}{k})\le U_{d,k}(w)\le U(w)$ and $U(w)\le U_{u,k}(w)\le U(w+\frac{1}{k})$ for all $w\ge B$. Let $V_{u,k}$ and $V_{d,k}$ be the value function associated with the utility $U_{u,k}$ and $U_{d,k}$, respectively. Then, we have*

$$V(t,w,\boldsymbol{x})=\lim_{k\to\infty}V_{d,k}(t,w,\boldsymbol{x})=\lim_{k\to\infty}V_{u,k}(t,w,\boldsymbol{x}),\qquad t<T.$$

Proof of Lemma EC.4.2: For each $k>0$, we can use the classical convolution method to construct smooth utility functions $U_{u,k}$ and $U_{d,k}$. Consider a function ϕ, s.t. ϕ is nonnegative, supported on $[-1,1]$, $\phi\in C_{\mathbb{R}}^\infty$, and $\int_{-1}^1\phi(x)dx=1$. We set $U(x)=U(B)$ when $x\le B$. Then

$$U_{u,k}(w):=\int_{-\infty}^{\infty}2k\phi(2k\zeta)*U(w+\frac{1}{2k}-\zeta)d\zeta,$$

is what we need. Replacing $U(w+\frac{1}{2k}-\zeta)$ by $U(w-\frac{1}{2k}-\zeta)$, we get $U_{d,k}$.

First, it is obvious that $V_{d,k}$ and $V_{u,k}$ are monotonic, and $V_{d,k}\le V\le V_{u,k}$ for any k.

Next we show that for any $\delta>0$ and $t<T-\delta$,

$$\lim_{k\to\infty}V_{d,k}(t,w,\boldsymbol{x})\ge V(t+\delta,w,\boldsymbol{x}). \tag{EC.80}$$

To see this, let $W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}}$ denote the wealth W_s starting from $W_t=x$, $\boldsymbol{x}_t=\boldsymbol{x}$ under the portfolio $\boldsymbol{\pi}$, and note that

$$V_{d,k}(t,w,\boldsymbol{x})=\sup_{\boldsymbol{\pi}\in\boldsymbol{C}}E[U_{d,k}(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})]=\sup_{\boldsymbol{\pi}\in\boldsymbol{C}}E[U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{\pi}})]. \tag{EC.81}$$

For any admissible strategy $\boldsymbol{\pi}_s$, $t+\delta\le s\le T$, let $\tilde{\boldsymbol{\pi}}_t=\boldsymbol{\pi}_s$ for $t+\delta\le s\le T$, and $\tilde{\boldsymbol{\pi}}_t=l$ for $s>T$, where l is any non-zero vector in $\boldsymbol{C}$. Define $\tau^k:=\inf\{s\ge T|W_s^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}}\ge W_T^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}}+\frac{1}{k}\}$ and a strategy $\tilde{\boldsymbol{\pi}}_{\boldsymbol{t}}^{\boldsymbol{k}}$ as following:

$$\tilde{\boldsymbol{\pi}}_s^k=\begin{cases}\boldsymbol{\pi}_s, & t+\delta\le s\le T,\\ l, & T<s\le\tau^k,\\ 0, & \tau^k<s\le T+\delta.\end{cases}$$

Due to the local oscillation of Brownian Motion and strict positiveness of X_T, we have $\lim_{k\to+\infty} \tau^k \to T$, a.s., which means $\liminf_{k\to+\infty} U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}^k}) \geq U(W_T^{t+\delta,w,\boldsymbol{x},\boldsymbol{\pi}})$, a.s. So by Fatou's lemma,

$$\liminf_{k\to+\infty} E[U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}^k})] \geq E[U(W_T^{t+\delta,w,\boldsymbol{x},\boldsymbol{\pi}})],$$

then, by(EC.81),

$$\begin{aligned} V_{d,k}(t,w,\boldsymbol{x}) &= \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}})] \geq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}^k})] \\ &\geq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} \liminf_{k\to+\infty} E[U_{d,k}(W_{T+\delta}^{t+\delta,w,\boldsymbol{x},\tilde{\boldsymbol{\pi}}^k})] \geq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W_T^{t+\delta,w,\boldsymbol{x},\boldsymbol{\pi}})] = V(t+\delta,w,\boldsymbol{x}), \end{aligned}$$

that is, (EC.80) holds.

Recall that V is continuous when $t<T$ by Theorem 4.2. Sending $\delta\to 0$, we have $\lim_{k\to\infty} V_{d,k}(t,w,\boldsymbol{x}) \geq V(t,w,\boldsymbol{x})$, for $t<T$. The converse inequality holds since $U_{d,k}\leq U$. Thus, we have proved the part for $V_{d,k}$. And the part for $V_{u,k}$ follows by similar arguments. □

The following lemma shows that the unbounded portfolio constraint problem can be approximated by a bounded constraint problem.

LEMMA EC.4.3. *Assume that the utility function $U(w)$ is continuous. For any $C>1$, let $V^C(t,w,\boldsymbol{x})$ denote the value function of the optimization problem with portfolio constraint $\boldsymbol{\pi}\in([-C,-\frac{1}{C}]\cup[\frac{1}{C},C])^N\cap\boldsymbol{C}$ under the utility U. Then, we have*

$$\lim_{C\to+\infty} V^C(t,w,\boldsymbol{x}) = V(t,w,\boldsymbol{x}).$$

Proof of Lemma EC.4.3: Note that

$$\limsup_{C\to+\infty} V^C(t,w,\boldsymbol{x}) \leq V(t,w,\boldsymbol{x}).$$

On the other side, given any strategy $\boldsymbol{\pi}$, note that in order to well define the wealth process, we have (see. e.g., Definition 1.2.1 of Karatzas and Shreve, 1998),

$$\int_t^{T\wedge\tau^B} |\boldsymbol{\pi}_s^\top\boldsymbol{\eta}|ds < +\infty, \text{ and } \int_t^{T\wedge\tau^B} \boldsymbol{\pi}_s^\top\boldsymbol{\Sigma}\boldsymbol{\pi} ds < +\infty, \quad \text{almost surely.}$$

where $\tau^B := \inf\{s\geq t | W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}} = B\}$. For each $C>0$, define $\boldsymbol{\pi}_s^{(C)} = \boldsymbol{\pi}_s 1_{\{\boldsymbol{\pi}_s\in([-C,-\frac{1}{C}]\cup[\frac{1}{C},C])^N\}}$, $t\leq s\leq T$. Then, there exists a subsequence C_n, such that $C_n\to\infty$ as $n\to\infty$, and (see. e.g., Problem 3.2.27 and its proof in Karatzas and Shreve, 1991),

$$\begin{aligned} &\int_t^{T\wedge\tau^{B,n}} \boldsymbol{\eta}^\top\boldsymbol{\pi}_s^{(C_n)} ds \to \int_t^{T\wedge\tau^B} \boldsymbol{\eta}^\top\boldsymbol{\pi}_s ds \quad \text{as} \quad n\to\infty, \quad \text{almost surely,} \\ &\int_t^{T\wedge\tau^{B,n}} \boldsymbol{\pi}_s^{(C_n)}\boldsymbol{\sigma} d\boldsymbol{\mathcal{B}}_s \to \int_t^{T\wedge\tau^B} \boldsymbol{\pi}_s\boldsymbol{\sigma} d\boldsymbol{\mathcal{B}}_s \quad \text{as} \quad n\to\infty, \quad \text{almost surely.} \end{aligned}$$

Let $W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}}$ denote the wealth W_s starting from $W_t = w$, $\boldsymbol{x}_t = \boldsymbol{x}$ under the portfolio $\boldsymbol{\pi}$. Recalling (2), we have $W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}} \to W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}}$ as $n \to \infty$ almost surely. Set the boundary condition as $V(t,B,\boldsymbol{x}) = f(t,B,\boldsymbol{x})$ for some continuous function f. By Fatou's lemma, we have

$$\begin{aligned}
&\liminf_{n\to+\infty} E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}})\mathbf{1}_{T<\tau^{B,n}} + f(\tau^{B,n}, B, \boldsymbol{x}_{\tau^{B,n}})\mathbf{1}_{\tau^{B,n}\le T}] \\
\ge & E[\liminf_{n\to+\infty} \left(U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}})\mathbf{1}_{T<\tau^{B,n}} + f(\tau^{B,n}, B, \boldsymbol{x}_{\tau^{B,n}})\mathbf{1}_{\tau^{B,n}\le T} \right)] \\
= & E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})\mathbf{1}_{T<\tau^{B}} + f(\tau^{B}, B, \boldsymbol{x}_{\tau^{B}})\mathbf{1}_{\tau^{B}\le T}], \qquad \text{(EC.82)}
\end{aligned}$$

where $\tau^{B,n} := \inf\{s \ge t | W_s^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}} = B\}$. For any $\boldsymbol{\pi} \in \boldsymbol{C}$,

$$\begin{aligned}
&\limsup_{C\to+\infty} V^C(t,w,\boldsymbol{x}) \\
\ge & \limsup_{n\to+\infty} E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}})\mathbf{1}_{T<\tau^{B,n}} + f(\tau^{B,n}, B, \boldsymbol{x}_{\tau^{B,n}})\mathbf{1}_{\tau^{B,n}\le T}] \\
\ge & \liminf_{n\to+\infty} E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}^{(C_n)}})\mathbf{1}_{T<\tau^{B,n}} + f(\tau^{B,n}, B, \boldsymbol{x}_{\tau^{B,n}})\mathbf{1}_{\tau^{B,n}\le T}] \\
\ge & E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})\mathbf{1}_{T<\tau^{B}} + f(\tau^{B}, B, \boldsymbol{x}_{\tau^{B}})\mathbf{1}_{\tau^{B}\le T}].
\end{aligned}$$

Thus, by (EC.82),

$$\limsup_{C\to+\infty} V^C(t,w,\boldsymbol{x}) \ge \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}})\mathbf{1}_{T<\tau^{B}} + f(\tau^{B}, B, \boldsymbol{x}_{\tau^{B}})\mathbf{1}_{\tau^{B}\le T}] = V(t,w,\boldsymbol{x}).$$

Noting that V^C is increasing in C, the lemma is proved. □

The following lemma shows that we only need to focus on bounded stochastic state variable $\boldsymbol{x}_t$.

LEMMA EC.4.4. *Denote by $V^{C_{sto},\epsilon}(t,w,\boldsymbol{x})$ the value function with an additional boundary condition*

$$V^{C_{sto},\epsilon}(t,w,\boldsymbol{x}) = f(t,w,\boldsymbol{x}) \qquad \text{(EC.83)}$$

on boundary $\|\boldsymbol{x}\|_\infty = C_{sto}$, where $f(t,w,\boldsymbol{x}) = U(w)$, if $U(B) > -\infty$, and if $B = 0$ and $U(B) = -\infty$, $f(t,w,\boldsymbol{x})$ is a smooth function satisfying

$$0 \le \max\{V_{CRRA}(t,w,\boldsymbol{x}), U(w)\} - f(t,w,\boldsymbol{x}) < \epsilon,$$

and the left-hand side holds with equality when $t = T$ or $w = B$. Then for any $t < T$, $w \ge B$ and $\boldsymbol{x} \in \mathbb{R}^K$,

$$\lim_{C_{sto}\to+\infty, \epsilon\to 0} V^{C_{sto},\epsilon}(t,w,\boldsymbol{x}) - V(t,w,\boldsymbol{x}) = 0. \qquad \text{(EC.84)}$$

Proof of Lemma EC.4.4: On the one hand, it is obvious that $V(t,w,\boldsymbol{x}) \geq V^{C_{sto},\epsilon}(t,w,\boldsymbol{x})$ since when $\|\boldsymbol{x}_t\|_\infty = C_{sto}$, the investor can simply liquidate all the stock and keep wealth at w until T or follow Merton's strategy in $[t,T]$. On the other hand,

$$\begin{aligned}
&\limsup_{C_{sto}\to+\infty,\epsilon\to 0} V(t,w,\boldsymbol{x}) - V^{C_{sto},\epsilon}(t,w,\boldsymbol{x}) && \text{(EC.85)}\\
&\leq \limsup_{C_{sto}\to+\infty,\epsilon\to 0} \sup_{\boldsymbol{\pi}} E[\big(V(\tau^{C_{sto}}, W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_{\tau^{C_{sto}}}, \boldsymbol{x}_{\tau^{C_{sto}}}) - U(W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_{\tau^{C_{sto}}})\big)\mathbf{1}_{\tau^{C_{sto}}<\min\{T,\tau^B\}}]\\
&\leq \limsup_{C_{sto}\to+\infty} \sup_{\boldsymbol{\pi}} E[\bigg(V(\tau^{C_{sto}}, W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_{\tau^{C_{sto}}}, \boldsymbol{x}_{\tau^{C_{sto}}}) + |U(B)|\bigg)\mathbf{1}_{\tau^{C_{sto}}<\min\{T,\tau^B\}}]\\
&\leq \limsup_{C_{sto}\to+\infty} \sup_{\boldsymbol{\pi}} E[\bigg(U(W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_{T}) + |U(B)|\bigg)\mathbf{1}_{\tau^{C_{sto}}<\min\{T,\tau^B\}}] && \text{(EC.86)}
\end{aligned}$$

where $\tau^{C_{sto}} := \inf\{s \geq t | \|\boldsymbol{x}_s\|_\infty = C_{sto}\}$. Noticing $\lim\limits_{C_{sto}\to+\infty} P(\tau^{C_{sto}} < \min\{T,\tau^B\}) = 0$, we have (EC.86) $\to 0$ since otherwise $\sup\limits_{\boldsymbol{\pi}} E[|U(W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T)|^q] = +\infty$ for any $q > 1$, which contradicts Assumption EC.1.1 (4). □

The following lemma shows that the portfolio optimization problem with unbounded wealth processes allowed can be approximated by the problem with bounded wealth processes restriction.

LEMMA EC.4.5. *For any $A > 0$, denote $V_{A,\epsilon}$ as the value function with an additional boundary condition*

$$V_{A,\epsilon}(t,A,\boldsymbol{x}) = f(t,A,\boldsymbol{x}), \tag{EC.87}$$

where $f(t,A,\boldsymbol{x})$ is defined as in (EC.83). *Then, we have*

$$\lim_{A\to\infty,\epsilon\to 0} V_{A,\epsilon}(t,w,\boldsymbol{x}) = V(t,w,\boldsymbol{x}). \tag{EC.88}$$

Proof of Lemma EC.4.5: Denote by $W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_s$ the wealth process W_s starting from $W_t = w$, $\boldsymbol{x}_t = \boldsymbol{x}$ under the portfolio $\boldsymbol{\pi}$. We have

$$\begin{aligned}
&|V_{A,\epsilon}(t,w,\boldsymbol{x}) - V(t,w,\boldsymbol{x})|\\
&= \left|\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T)] - \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[U(W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T)\mathbf{1}_{\tau^A>T} + U(A)\mathbf{1}_{\tau^A\leq T}]\right|\\
&= \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\big|f(\tau^A, A, \boldsymbol{x}_{\tau^A}) - V(\tau^A, A, \boldsymbol{x}_{\tau^A})\big|\,\mathbf{1}_{\tau^A\leq T}], && \text{(EC.89)}
\end{aligned}$$

where $\tau^A := \inf\{s \geq t | W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_s = A\}$. Since the portfolio set $\boldsymbol{C}$ is bounded, $\lim\limits_{A\to+\infty} P(\tau^A \leq T) = 0$. Therefore, noticing $\max\limits_{t\leq s\leq T, \|\boldsymbol{x}\|_\infty\leq C_{sto}} f(s,A,\boldsymbol{x}) < +\infty$, we have

$$\begin{aligned}
\text{(EC.89)} &= \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\big|V(\tau^A, A, \boldsymbol{x}_{\tau^A})\big|\,\mathbf{1}_{\tau^A\leq T}]\\
&\leq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[\big|U(T, W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T, \boldsymbol{x}_T)\big|\,\mathbf{1}_{\tau^A\leq T\leq\tau^B} + f(\tau^B, B, \boldsymbol{x}_{\tau^B})\mathbf{1}_{\tau^A\leq T,\tau^B\leq T}]\\
&\leq \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[(|U(T, W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T, \boldsymbol{x}_T)| + M)\mathbf{1}_{\tau^A\leq T}]\\
&= \sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[|U(T, W^{t,w,\boldsymbol{x},\boldsymbol{\pi}}_T, \boldsymbol{x}_T)|\mathbf{1}_{\tau^A\leq T}] + MP(\tau^A \leq T) && \text{(EC.90)}
\end{aligned}$$

where $M \geq \max_{t\leq s\leq T, \|\boldsymbol{x}\|_\infty \leq C_{sto}} f(s,B,\boldsymbol{x})$ is a constant. By Assumption EC.1.1 (4), there is $q>1$, $\sup_{\boldsymbol{\pi}\in\boldsymbol{C}} E[|U(T,W_T^{t,w,\boldsymbol{x},\boldsymbol{\pi}},\boldsymbol{x}_T)|^q] < +\infty$, which implies (EC.90) $\to 0$ as $A\to+\infty$ since $\lim_{A\to+\infty} P(\tau^A \leq T) = 0$. $\square$

With the help of Lemmas EC.4.1, EC.4.2, EC.4.3, EC.4.4 and EC.4.5, we now prove Theorem EC.1.3.

Proof of Theorem EC.1.3: By Lemmas EC.4.1, EC.4.2, EC.4.3, EC.4.4 and EC.4.5, we only need to consider the case that the stochastic state variable $\boldsymbol{x}$ is bounded, the portfolio set $\boldsymbol{C}$ is bounded with a positive distance to the origin, the diffusion is controlled in a bounded domain $[\delta, A]$ for a big enough upper bounded A, a small enough lower bounded δ if $B=0$ and $U(0)=-\infty$; $\delta = B$ if $U(B) > -\infty$, and the utility function U is smooth. Denote by $\bar{V}$ the viscosity solution to this problem. By Theorem IV.4.1 of Fleming and Soner (2006), $\bar{V}$ is indeed $C^{1,2}$ in (t,x).

Let $\Sigma_\Delta = \{(t_n, w_i, \boldsymbol{x}_m) : 0\leq n\leq N_t, 0\leq i\leq N_w, 0\leq m\leq N_{\boldsymbol{x}}\}$ be a discretization mesh of the domain $[t_0,T]\times[\delta,A]\times[-C_{sto},C_{sto}]$ with fixed time and spatial step sizes Δt, Δw and $\Delta \boldsymbol{x}$. Let $V_{i,m}^n$ be the solution at grid $(t_n,w_i,\boldsymbol{x}_m)$ of a monotone, stable, and consistent finite difference scheme for the HJB equation (EC.8) with terminal and boundary conditions (32), (30), (31), and (EC.83). If the discretization solution $V_{i,m}^n$ satisfies the terminal and boundary conditions uniformly, that is,

$$\lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(T-,w,\boldsymbol{x})\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n = U(w), \tag{EC.91}$$

$$\lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,\delta,\boldsymbol{x})\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n = f(t,\delta,\boldsymbol{x}), \text{ and } \lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,A,\boldsymbol{x})\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n = U(A), \tag{EC.92}$$

and

$$\lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,w,\boldsymbol{x})\\ \|\boldsymbol{x}\|_\infty = C_{sto}\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n = f(t,w,\boldsymbol{x}), \tag{EC.93}$$

thanks to our new comparison principle, the solution of the finite difference scheme converges to the unique viscosity solution, i.e., the value function of our problem, via Theorem 2.1 of Barles and Souganidis (1991).

To finish the proof, we show that the discretization solution V_i^n assumes the terminal and boundary conditions (EC.91), (EC.92) and (EC.93). The convergence property (EC.91) is proved by Lemma IX.5.3 of Fleming and Soner (2006). Thus, we only need to show that the discretization solution V_i^n assumes the boundary conditions (EC.92) and (EC.93).

First, we show that

$$\limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,\delta,\boldsymbol{x})\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n \leq f(t,\delta,\boldsymbol{x}), \quad \limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,A,\boldsymbol{x})\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n \leq U(A), \text{ and } \limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,w,\boldsymbol{x})\\ \|\boldsymbol{x}\|_\infty = C_{sto}\\ \Delta t\to0,\Delta w\to0,\Delta\boldsymbol{x}\to0}} V_{i,m}^n \leq f(t,w,\boldsymbol{x}). \tag{EC.94}$$

By Lemma IX.5.2 of Fleming and Soner (2006), we have that, for every $a>0$, there exists $h_0>0$, such that, for $\Delta w<h_0$,

$$V_{i,m}^n \le \overline{V}(t_n,w_i,\boldsymbol{x}_m)+a, \text{ for all } \ n,i,m.$$

By the continuity of $\overline{V}$, we have

$$\limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,\delta,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \le \limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,\delta,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} \overline{V}(t_n,w_i,\boldsymbol{x}_m)+a=\overline{V}(t,\delta,\boldsymbol{x})+a=f(t,\delta,\boldsymbol{x})+a,$$

$$\limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,A,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \le \limsup_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,A,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} \overline{V}(t_n,w_i,\boldsymbol{x}_m)+a=\overline{V}(t,A,\boldsymbol{x})+a=U(A)+a.$$

$$\lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,w,\boldsymbol{x})\\ \|\boldsymbol{x}\|_\infty=C_{sto}\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \le \lim_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,w,\boldsymbol{x})\\ \|\boldsymbol{x}\|_\infty=C_{sto}\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} \overline{V}(t_n,w_i,\boldsymbol{x}_m)=f(t,w,\boldsymbol{x})+a. \tag{EC.95}$$

By the arbitrariness of a, we get (EC.94).

Second, similarly we can show that

$$\liminf_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,\delta,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \ge f(t,\delta,\boldsymbol{x}), \quad \liminf_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,A,\boldsymbol{x})\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \ge U(A), \text{ and } \liminf_{\substack{(t_n,w_i,\boldsymbol{x}_m)\to(t,w,\boldsymbol{x})\\ \|\boldsymbol{x}\|_\infty=C_{sto}\\ \Delta t\to 0,\Delta w\to 0,\Delta \boldsymbol{x}\to 0}} V_{i,m}^n \ge f(t,w,\boldsymbol{x}). \tag{EC.96}$$

That finishes our proof. $\square$

At the end of this section, we show that the finite difference scheme given in Appendix B for a deterministic investment opportunity set is monotone, stable, and consistent [11] . Then, by Theorem 4.3, the discretization solution of the scheme will converge to the value function of our problem. By Lemmas EC.4.1, EC.4.5 and EC.4.3, we only need to consider the case that the portfolio set $[d,u]$ is bounded and the state variable is in a bounded domain, that is, $w\in[\delta,A]$ for a bounded $A>0$.

LEMMA EC.4.6. *Assume that the portfolio set $[d,u]$ is bounded. $\delta=B$ if $U(B)>-\infty$; δ is a small enough lower bounded δ if $B=0$ and $U(0)=-\infty$, and let $A>\delta$ be a finite cutoff, such that the wealth level is restricted in the bounded domain $[\delta,A]$. The finite difference scheme (29-33) given in Appendix B is monotone, stable, and consistent (cf. Barles and Souganidis(1991) for the definition of monotonicity, stability, and consistency).*

[11] For a stochastic investment opportunity set where the HJB equation (EC.8) is essentially multidimensional, the differencing scheme is monotone if the variance matrix is diagonally dominant, see, e.g., the condition IX.3.22 of Fleming and Soner (2006).

Proof of Lemma EC.4.6: Denote the left-hand side of the discretization HJB equation (29) as

$$S\left((\Delta t,\Delta w),(t_n,w_i),V_i^n,\{V_{i-1}^n,V_{i+1}^n,V_i^{n+1}\}\right). \tag{EC.97}$$

Note that in Appendix B S can be rewritten as

$$-\frac{V_i^{n+1}-V_i^n}{\Delta t}-\sup_{d\le\pi\le u}\left\{\alpha_i(\pi)V_{i-1}^n-(\alpha_i(\pi)+\beta_i(\pi))V_i^n+\beta_i(\pi)V_{i+1}^n\right\}. \tag{EC.98}$$

First, we prove the monotonicity. Note that $w_i>0$ and $\Delta w>0$. By definition, we have

$$\alpha_i(\pi)\ge 0,\ \text{and}\ \beta_i(\pi)\ge 0,\ \text{for all}\ \ d\le\pi\le u. \tag{EC.99}$$

Then, S is non-increasing with respective to V_{i-1}^n, V_{i+1}^n and V_i^{n+1}, so the scheme S is monotone.

Second, we prove the consistence. For a smooth function ϕ, by Taylor expansion (cf. the left-hand side of (29)),

$$\begin{aligned}
&S\left((\Delta t,\Delta w),(t,w),\phi(t,w)+\xi,\{\phi(t,w-\Delta w)+\xi,\phi(t,w+\Delta w)+\xi,\phi(t+\Delta t,w)+\xi\}\right)\\
=&S\left((\Delta t,\Delta w),(t,w),\phi(t,w),\{\phi(t,w-\Delta w),\phi(t,w+\Delta w),\phi(t+\Delta t,w)\}\right)\\
=&-\left(\frac{\partial\phi(t,w)}{\partial t}+o(\Delta t)\right)-\sup_{d\le\pi\le u}\left\{w\eta\pi\left(\frac{\partial\phi(t,w)}{\partial w}+o(\Delta w)\right)+\frac{w^2\sigma^2\pi^2}{2}\left(\frac{\partial^2\phi(t,w)}{\partial w^2}+o((\Delta w)^2)\right)\right\}.\\
=&-\left(\frac{\partial\phi(t,w)}{\partial t}+o(\Delta t)\right)-H\left(w,\frac{\partial\phi(t,w)}{\partial w}+o(\Delta w),\frac{\partial^2\phi(t,w)}{\partial w^2}+o((\Delta w)^2)\right),
\end{aligned}$$

where H is the Hamiltonian defined in (21). H is continuous since the portfolio set $[d,u]$ is bounded. Then

$$\begin{aligned}
&\lim_{\Delta t\to0,\Delta w\to0}S\left((\Delta t,\Delta w),(t,w),\phi(t,w)+\xi,\{\phi(t,w-\Delta w)+\xi,\phi(t,w+\Delta w)+\xi,\phi(t+\Delta t,w)+\xi\}\right)\\
=&-\frac{\partial\phi(t,w)}{\partial t}-H\left(w,\frac{\partial\phi(t,w)}{\partial w},\frac{\partial^2\phi(t,w)}{\partial w^2}\right).
\end{aligned}$$

So, the scheme S is consistent.

Third, we prove the stability. Let π_i^n be the optimal of (EC.98). Recalling (EC.99), for $i=2,\dots,N_w-1$, by (EC.98), we have

$$\begin{aligned}
|V_i^n|(1+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n)))=&|V_i^{n+1}+\Delta t(\alpha_i(\pi_i^n)V_{i-1}^n+\beta_i(\pi_i^n)V_{i+1}^n))|\\
\le&||V^{n+1}||_\infty+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))\max\{||V^n||_\infty,D_n\},
\end{aligned}$$

where $||V^n||_\infty=\max\{|V_2^n|,\dots,|V_{N_w-1}^n|\}$ and $D_n=\max\{f((N_t-n)\Delta t,\delta),|U(A)|\}$. Let $i\in\{2,\dots,N_w-1\}$ be choose such that $|V_i^n|=||V^n||_\infty$. Then

$$||V^n||_\infty(1+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n)))\le||V^{n+1}||_\infty+\Delta t(\alpha_i(\pi_i^n)+\beta_i(\pi_i^n))\max\{||V^n||_\infty,D_n\}.$$

It follows that $||V^n||_\infty \le ||V^{n+1}||_\infty$ if $||V^n||_\infty \ge D_n$; otherwise, $||V^n||_\infty$ is bounded by the convex combination of $||V^{n+1}||_\infty$ and D_n. So, for $n = 0, 1, \ldots, N_t - 1$, we have,

$$||V^n||_\infty \le \max\{||V^{n+1}||_\infty, D_n\}.$$

Then, noticing that $D_n \ge D_{n+1} \ge ... \ge D_{N_t}$,

$$||V^n||_\infty \le \max\{\max\{||V^{n+2}||_\infty, D_{n+1}\}, D_n\} = \max\{||V^{n+2}||_\infty, D_n\} \le \cdots \le \max\{||V^{N_t}||_\infty, D_n\} = D_n,$$

where the last equality holds by the terminal condition (32). So, the scheme S is stable. □

Appendix EC.5: Additional Numerical Results

EC.5.1. Black-Scholes Market

This subsection gives more numerical analysis related to Section 5 for the Black-Scholes market.

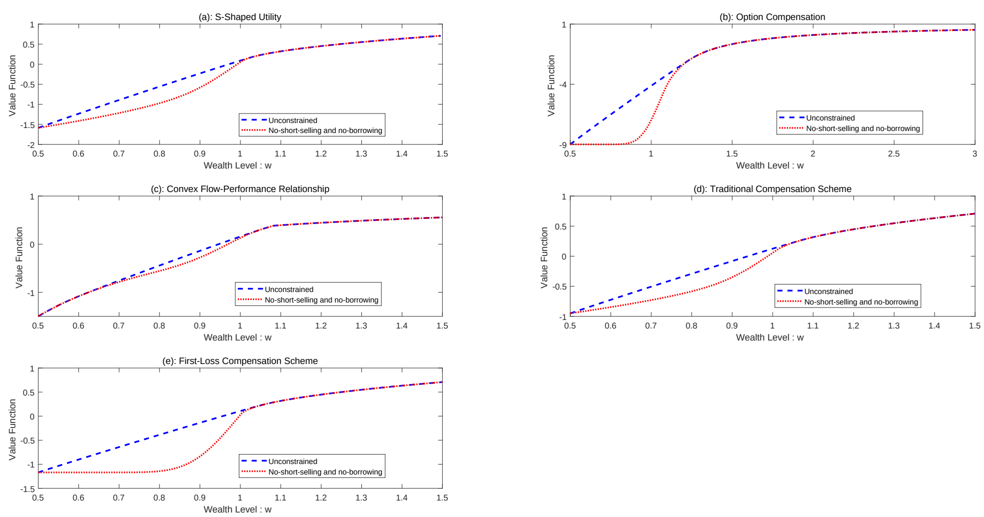

Figure EC.2 A comparison between the constrained value function and the unconstrained value function. The portfolio bounds in the constrained case are no-borrowing and no-short-sale, i.e., $\pi \in [0, 1]$. The five panels (a)-(e) correspond to the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), He and Kou (2018) (the traditional scheme) and He and Kou (2018) (the first-loss scheme). A compulsory liquidation at $w = 0.5$ is imposed.

In Figure EC.2, we plot the time 0 value functions against the wealth level for the constrained case (no-short-selling and no-borrowing: $\pi \in [0, 1]$) and unconstrained case, respectively, for the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), and He and Kou (2018). The dashed line stands for the value function without portfolio bounds, which is globally concave. The dotted line is the value function with portfolio bounds, which turns out to be locally convex and locally concave.

In Figures EC.3-EC.6, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against the wealth level for the constrained case and the unconstrained strategy, respectively, for the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), and He and Kou (2018). The dotted (dashed) line represents the optimal strategy for the constrained (unconstrained) case. In each figure, the portfolio bounds for the constrained case are $\pi \in [0,1]$ for the upper panel and $\pi \in [-2,1]$ for the lower panel, respectively. These figures show that the optimal strategy's non-myopic and gambling properties are common for non-concave optimization problems with portfolio bounds.

Figures EC.7 plots the equivalent wealth loss caused by a sub-optimal strategy at time 0 against the wealth level for the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), and He and Kou (2018), respectively. Here the difference between the sub-optimal strategy and the optimal strategy lies in that the sub-optimal strategy would use a portfolio weight of 100% if the optimal portfolio weight is -200%. The definition of the equivalent loss is the same as that given in (19). Note that the equivalent wealth loss can be large.

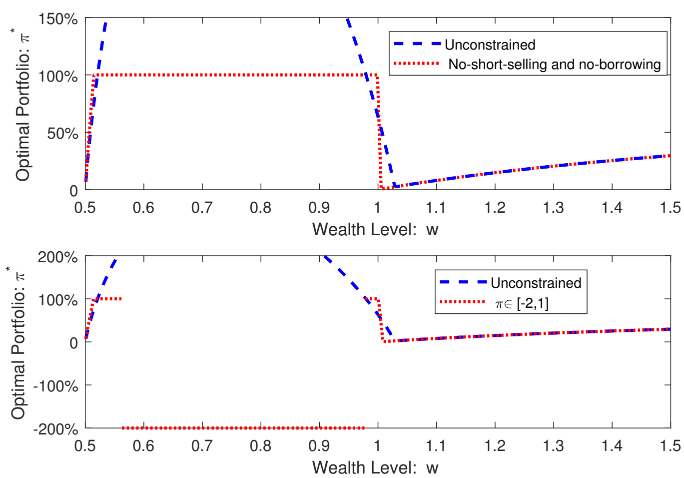

Figure EC.3 The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem in Berkelaar, Kouwenberg and Post (2004). The dotted line is the time 0 optimal fraction of total wealth invested in the stock π^* against the wealth level for our constrained optimal strategy, where the portfolio bounds are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The dashed line stands for the unconstrained optimal strategy (some part that exceeds the scope of the figure is not displayed). Default parameters values are $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $\lambda = 2.25$, $W_0 = 1$, $T = 1/12$, $B = 0.5$.

EC.5.2. Time-Varying Gaussian Mean Return

In this subsection, we numerically demonstrate that our three general findings hold for the time-varying Gaussian mean return model described in Section EC.1.

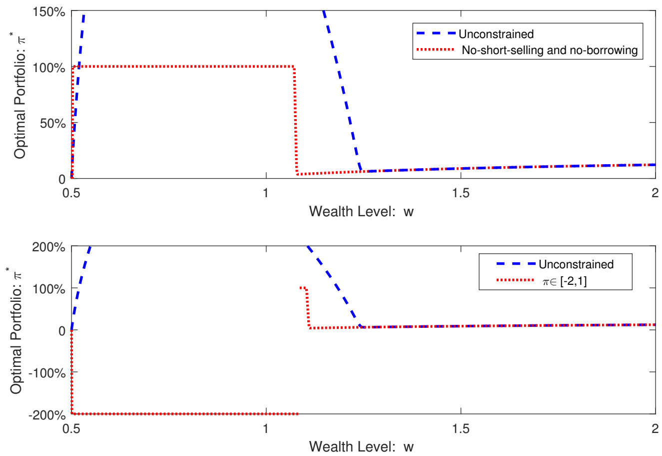

Figure EC.4 The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem in Carpenter (2000). The portfolio bounds in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = -1$, $K = 1$, $\alpha = 0.2$, $C = 0.02$, $W_0 = 1$, $T = 1/12$, $B = 0.5$.

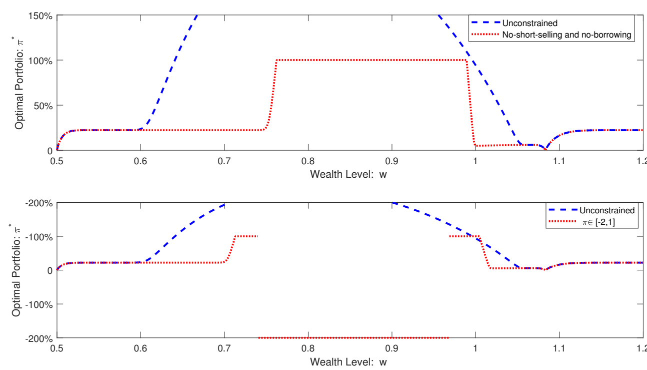

Figure EC.5 The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem in Basak, Pavlova, and Shapiro (2007). The portfolio bounds in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = -1$, $\eta_L = -0.08$, $\eta_H = 0.08$, $f_L = 0.8$, $f_H = 1.5$, $W_0 = 1$, $T = 1/12$, $B = 0.5$.

In the upper panel of Figure EC.8, for a fixed market price of risk, we plot the time 0 value function against the wealth level for the constrained case (dotted line) and unconstrained case (dashed line),[12] respectively. As in the Black-Scholes market, the value function is globally concave

[12] There is no closed-form solution for the unconstrained value function and optimal portfolios under a general time-varying Gaussian mean return model. We apply the finite difference method to numerically approximate the unconstrained results by using large leverage and short-sale constraints, e.g., $d = -1000$ and $u = 1000$. For the special case $\rho = -1$ or $\rho = 1$, the market is complete. A Laplace-based analytical result for the unconstrained problem can be derived via the martingale method. Numerical experiments show that the results from the finite difference method and the martingale method are similar, which validates the accuracy of our finite difference method. Similar results are obtained for the stochastic volatility model in the next subsection. To save space, we omit these comparisons.

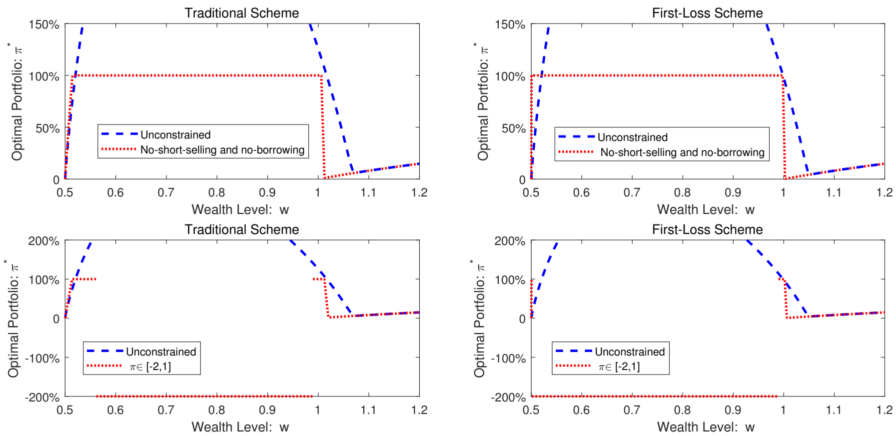

Figure EC.6 **The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem in Basak, Pavlova, and Shapiro (2007). The portfolio bounds in the constrained case are $\pi \in [0,1]$ (upper panel) and $\pi \in [-2,1]$ (lower panel), respectively. The other parameter values are: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $\lambda = 2.25$, $W_0 = 1$, $T = 1/12$, $B = 0.5$.**

in the unconstrained case, but is not concave in the constrained case. In the lower panel of Figure EC.8, we plot the constrained value function against the wealth level for three different levels of the market price of risk. It can be seen that the value function increases as the market price of risk increases.

In the upper panel of Figure EC.9, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against the wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. It can be seen that the optimal portfolios demonstrate the general findings as obtained in the Black-Scholes market: (i) the constrained investors are non-myopic with respect to portfolio constraints, such that early action is taken before portfolio constraints are binding; (ii) given a relatively large loss, short-selling is likely optimal even with a positive risk premium ($x \equiv X_0 > 0$). The lower panel plots the constrained optimal portfolios against the wealth level for three different levels of the market price of risk. The dotted line is for $x \equiv X_0$, the dashed line is for $x \equiv 3X_0$, and the solid line is for $x \equiv -X_0$. As the market price of risk increases from X_0 to $3X_0$, a smaller weight on the risky asset would reach the same goal. Thus, the dashed line is slightly lower than the dotted line when a long position is taken, and turns negative at a lower wealth level. Interestingly, the solid line (for $x \equiv X_0$) is almost symmetric to the dotted line (for $x \equiv -X_0$).

Even if we impose the same borrowing leverage ratio and the short-selling ratio, i.e., $\pi \in [-1,1]$), it is still possible that shorting (longing) the risky asset is optimal when the risk premium is positive (negative), due to the time-varying Gaussian distribution of the risk premium. We consider a sub-optimal strategy that shorts (longs) the risky asset when the risk premium is negative (positive).

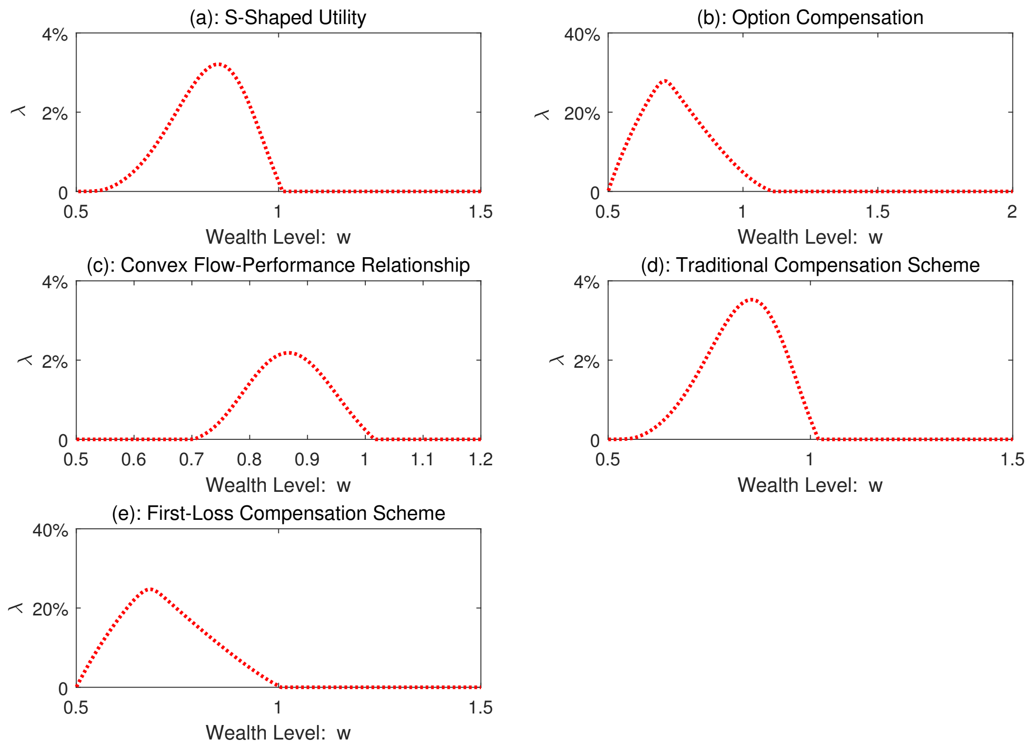

Figure EC.7 The equivalent wealth loss of a sub-optimal strategy at time zero. The difference between the sub-optimal strategy and the optimal strategy lies in that the sub-optimal strategy would use a portfolio weight of 100% when the optimal portfolio weight is -200%. The portfolio bounds are $\pi \in [-2, 1]$. The five panels (a)-(e) correspond to the models of Berkelaar, Kouwenberg and Post (2004), Carpenter (2000), Basak, Pavlova, and Shapiro (2007), He and Kou (2018) (the traditional scheme), and He and Kou (2018) (the first-loss scheme), respectively. A compulsory liquidation at $w = 0.5$ is imposed.

Figure EC.10 shows that the equivalent wealth loss of the sub-optimal strategy may be as high as approximately 100%.

EC.5.3. Stochastic Volatility

In this subsection, we consider a stochastic volatility model with one stock and one bank account. The stock price follows

$$dS_t/S_t = \left(r + \delta X_t^{(1+\beta)/(2\beta)}\right)dt + X_t^{1/(2\beta)} d\mathcal{B}_t, \tag{EC.100}$$

and the one-dimensional state variable X_t is governed by the mean-reverting square-root process:

$$dX_t = \kappa(\bar{X} - X_t)dt + \nu\sqrt{X_t} d\mathcal{B}_t^X, \tag{EC.101}$$

where $E[d\mathcal{B}_t d\mathcal{B}_t^X] = \rho dt$, and $r, \delta, \beta \neq 0, \kappa > 0, \bar{X} > 0, \nu > 0, \rho$ are all constants. Dynamic portfolio choice under this market setting has been widely studied. See, e.g., Liu (2001,2007) for CRRA preferences for the terminal wealth, Chacko and Viceira (2005) for recursive preferences of intermediate

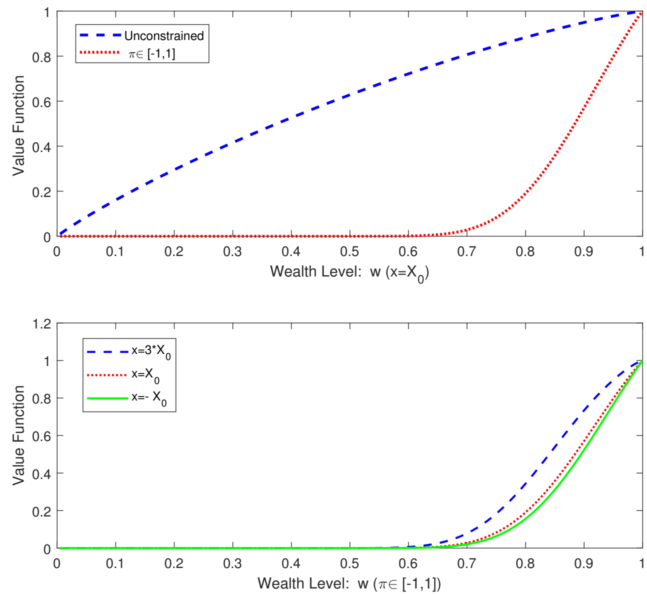

Figure EC.8 A comparison between the constrained value function and the unconstrained value function associated with the goal-reaching problem under the time-varying Gaussian mean return model. The portfolio bounds are $d = -1$ and $u = 1$. The lower panel plots the constrained value function against the wealth level for three different levels of the market price of risk: the dotted line is for $x \equiv X_0$; the dashed line is for $x \equiv 3X_0$; the solid line is for $x \equiv -X_0$. The parameters are: $r = 0.017$, $\sigma = 0.15$, $\kappa = 0.27$, $\bar{X} = 0.273$, $\nu = 0.065$, $\rho = -0.93$, $B = 0$, $X_0 = 0.273$, $T = 1$.

consumption (the case $\beta = -1$), Basak and Chabakauri (2010) for the mean-variance preferences for terminal wealth, and Dai et al. (2021) for the mean-variance preferences for log terminal return. Note that $X_t^{1/\beta}$ is the instantaneous stock return variance and $\delta\sqrt{X_t}$ is the market price of risk. $\beta = 1$ corresponds to the stochastic volatility model of Heston (1993) and X_t is the instantaneous stock return variance.

Consider the portfolio choice problem with the aspiration utility as given in (11). We take the portfolio bounds $\pi \in [-1, 1]$. The default parameter values for the stochastic volatility model (EC.100) and (EC.101) are set as following: $\delta = 1$, $\beta = 1$, $\kappa = 0.3374$, $X_0 = \bar{X} = 0.08$, $\nu = 0.6503$, $\rho = -0.52$, $B = 0$, $T = 1/12$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$.

In the upper panel of Figure EC.11, for a fixed variance, we plot the time 0 value function against the wealth level for the constrained case (dotted line) and unconstrained case (dashed line), respectively. As in the Black-Scholes market, the value function is globally concave in the unconstrained case, but is not concave in the constrained case. In the lower panel of Figure EC.11, we plot the constrained value function against the wealth level for three different levels of the variance. It can be seen that the value function increases as the variance increases. This may be

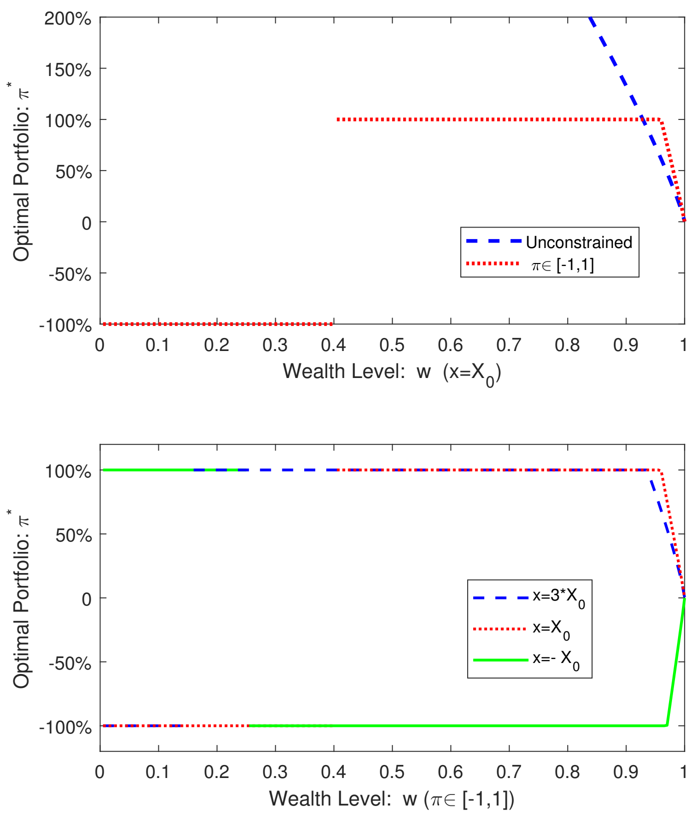

Figure EC.9 **The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem under the time-varying Gaussian mean return model. The dotted (dashed) line represents the time 0 optimal portfolios against the wealth level for fixed market price of risk $x \equiv X_0$ in the constrained (no constrained) case. The lower panel indicates that the optimal portfolios are relatively more extreme when the initial market price of risk is smaller. The lower panel plots the constrained optimal portfolios against the wealth level for three different levels of the market price of risk: the dotted line is for $x \equiv X_0$; the dashed line is for $x \equiv 3X_0$; the solid line is for $x \equiv -X_0$. The parameters are: $r = 0.017$, $\sigma = 0.15$, $\kappa = 0.27$, $\bar{X} = 0.273$, $\nu = 0.065$, $\rho = -0.93$, $B = 0$, $X_0 = 0.273$, $T = 1$, and $\pi \in [-1, 1]$.**

due to the fact that the market price of risk increases as the variance increases, that is, a larger variance means a better investment opportunity.

In the upper panel of Figure EC.12, we plot the time 0 optimal fraction of total wealth invested in the stock π^* against wealth level for the constrained case (dotted line) and the unconstrained case (dashed line), respectively. It can be seen that the optimal portfolios demonstrate the general findings as obtained in the Black-Scholes market: (i) the constrained investors are non-myopic with respect to portfolio constraints, such that early action is taken before portfolio constraints are binding; (ii) given a relatively big distance from the aspiration level, short-selling is likely optimal even with a positive risk premium. The lower panel plots the constrained optimal portfolios against

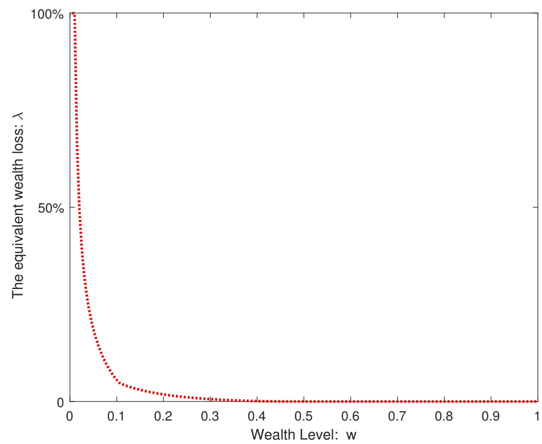

Figure EC.10 The equivalent wealth loss caused by a sub-optimal strategy for the goal-reaching problem under the time-varying Gaussian mean return model. The difference between the sub-optimal strategy and the optimal strategy is that the sub-optimal strategy would short (long) the risky asset when the risk premium is negative (positive). The portfolio bounds are $\pi \in [-1,1]$, and the parameters are: $r = 0.017$, $\sigma = 0.15$, $\kappa = 0.27$, $\bar{X} = 0.273$, $\nu = 0.065$, $\rho = -0.93$, $B = 0$, $T = 1$ and fixed market price of risk $x \equiv X_0 = 0.273$,$X_0 = 0.273$.

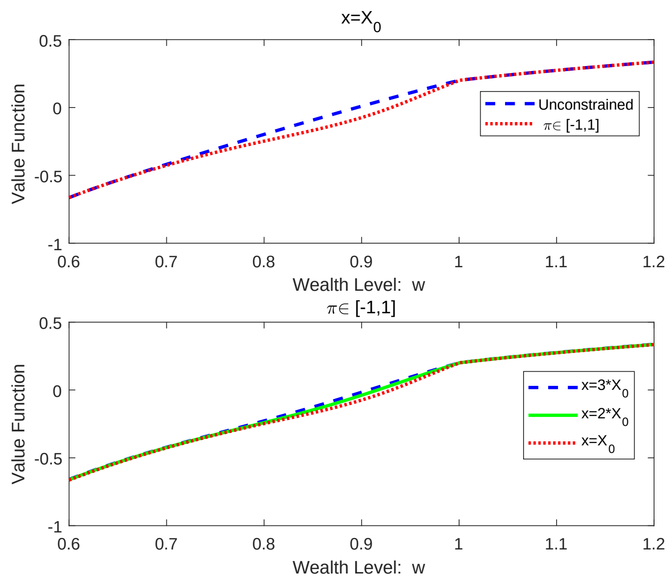

Figure EC.11 A comparison between the constrained and unconstrained value functions associated with the aspiration utility under the stochastic volatility model. In the upper panel, the dotted (dashed) line represents the value function in the constrained (unconstrained) case. The portfolio bounds are $d = -1$ and $u = 1$. The lower panel plots the constrained value function against the wealth level for three different levels of the variance: the dotted line is for $x \equiv X_0$; the solid line is for $x \equiv 2X_0$; the dashed line is for $x \equiv 3X_0$. The parameters are: $\delta = 1$, $\beta = 1$, $\kappa = 0.3374$, $X_0 = \bar{X} = 0.08$, $\nu = 0.6503$, $\rho = -0.52$, $B = 0$, $T = 1/12$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$.

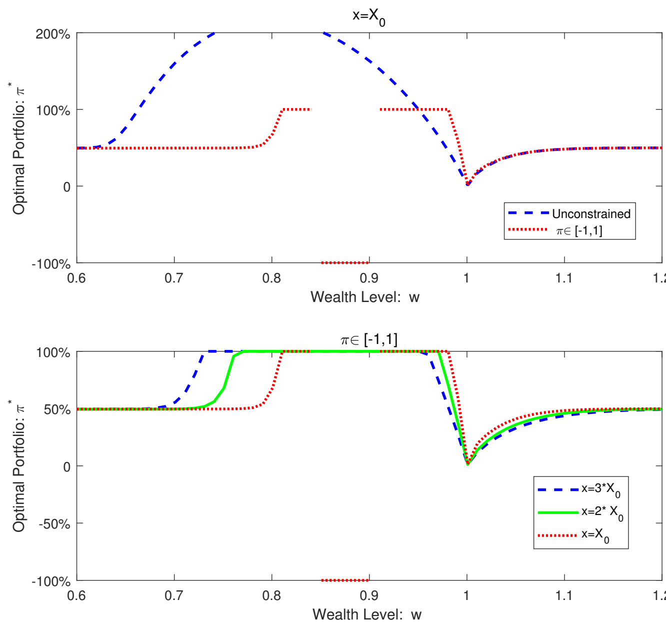

Figure EC.12 The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem under the stochastic volatility model. In the upper panel, the dotted (dashed) line represents the time 0 optimal portfolios against the wealth level for fixed variance $x \equiv X_0$ in the constrained (unconstrained) case. The lower panel indicates that the optimal portfolios are relatively more extreme when the initial variance is smaller. The lower panel plots the constrained optimal portfolios against the wealth level for three different levels of the variance: the dotted line is for $x \equiv X_0$; the dashed line is for $x \equiv 2X_0$; the dash-dotted line is for $x \equiv 3X_0$. The parameters are: $\delta = 1$, $\beta = 1$, $\kappa = 0.3374$, $X_0 = \bar{X} = 0.08$, $\nu = 0.6503$, $\rho = -0.52$, $B = 0$, $T = 1/12$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$, and for the constrained case $\pi \in [-1, 1]$.

wealth level for three different levels of the variance. A smaller initial variance may induce the investor to adopt a relatively more risky strategy: investing a higher proportion of wealth on the stock before portfolio constraints are binding and switching to short-sale earlier as the wealth decreases.

Even if we impose the same borrowing leverage ratio and the short-selling ratio, i.e., $\pi \in [-1, 1]$, it is still possible for a small range of wealth level (e.g., $0.85 < w < 0.9$) that shorting the risky asset is optimal when the risk premium is positive, due to stochastic volatility. Abandoning this short-selling strategy would introduce a wealth loss of the initial wealth. Figure EC.13 shows that the equivalent wealth loss caused by a sub-optimal strategy is 0.25%, where the difference between the sub-optimal strategy and the optimal strategy lies in that the sub-optimal strategy would take a portfolio weight of 100% when the optimal portfolio weight $\pi^* = -100\%$.

In Figure EC.14, we plot the time-0 optimal fraction of total wealth invested in the stock π^* (i.e., optimal portfolio) against wealth level for the aspiration utility maximization problem with portfolio bounds $\pi \in [-1, 1]$ under the stochastic volatility model and the Black-Scholes model,

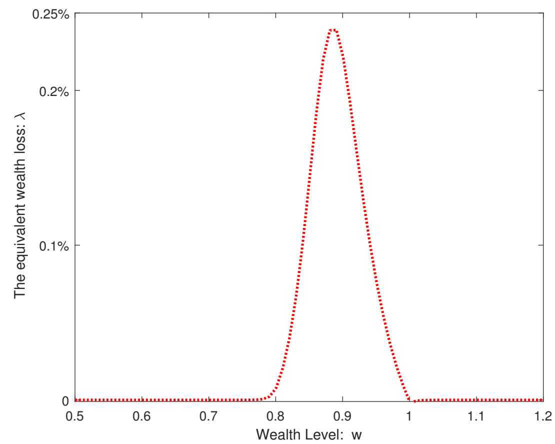

Figure EC.13 The equivalent wealth loss caused by a sub-optimal strategy at time zero under the stochastic volatility model for the aspiration utility maximization problem in Lee, Zapatero and Giga (2018). The difference between the sub-optimal strategy and the optimal strategy is that the sub-optimal strategy would take a portfolio weight of 100% when the optimal portfolio weight $\pi^* = -100\%$. The portfolio bounds are $\pi \in [-1, 1]$, and the parameter values are: $\delta = 1$, $\beta = 1$, $\kappa = 0.3374$, $X_0 = \bar{X} = 0.08$, $\nu = 0.6503$, $\rho = -0.52$, $B = 0$, $T = 1/12$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$.

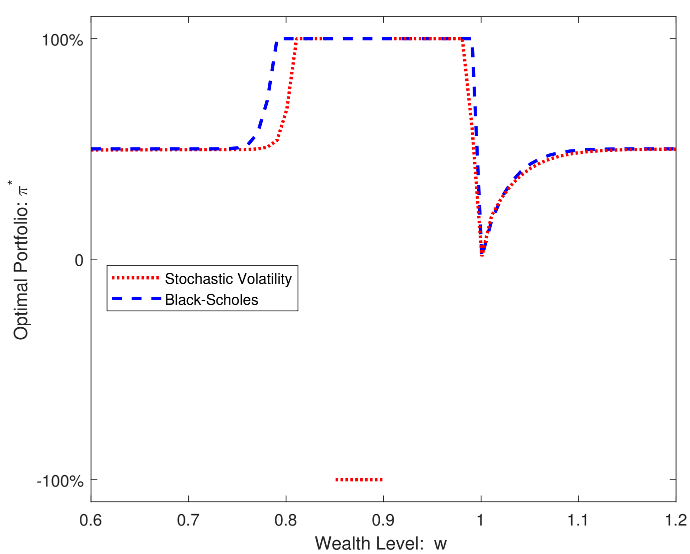

Figure EC.14 The hedging demand. The figure plots the time 0 optimal portfolios against wealth level for fixed variance $x \equiv X_0$ under the stochastic volatility model and the Black-Scholes model, respectively. The portfolio bounds are $\pi \in [-1, 1]$, and the parameters for the aspiration utility are $B = 0$, $T = 1/12$, $p = -1$, $c_1 = 0.8$, $c_2 = 0.2$, $R = 1$. The parameters for the stochastic volatility model are: $\delta = 1$, $\beta = 1$, $\kappa = 0.3374$, $X_0 = \bar{X} = 0.08$, $\nu = 0.6503$, $X_0 = 0.273$, $\rho = -0.52$, and the parameters for the Black-Scholes model are: $\mu - r = 1 * 0.08 = 0.08$, $\sigma = \sqrt{0.08} = 0.2828$.

respectively. Similar to Figure EC.1, we observe the same phenomenon that the hedging demand is negative for negative ρ. The financial interpretation is similar to as given for Figure EC.1: the hedging demand with negative ρ would be negative for a risk-seeking investor. This is consistent with the finding of Liu (2001, 2007) for the stochastic volatility model that is parallel to that of Kim and Omberg (1996): for the CRRA utility maximization problem with negative ρ, the hedging demand would switch from positive to negative as we decrease the investor's risk aversion level.

EC.5.4. Two Risky Assets Following Bivariate Geometric Brownian Motion

In this subsection, we assume that the financial market has two stocks, and the dynamic of the stock prices $\{S_1, S_2\}$ follow the diffusion process

$$\begin{cases} dS_1(t)/S_1(t) = \mu_1 dt + \sigma_1 d\mathcal{B}_{1t}, \\ dS_2(t)/S_2(t) = \mu_2 dt + \sigma_2(\rho d\mathcal{B}_{1t} + \sqrt{1-\rho^2} d\mathcal{B}_{2t}), \end{cases} \tag{EC.102}$$

where $r, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho$ are constant, and $\boldsymbol{\mathcal{B}}_t = (\mathcal{B}_{1t}, \mathcal{B}_{2t})^\top$ is a standard two dimensional Brownian motion.

Consider the goal-reaching utility (5). The default parameter values for the two dimensional geometric Brownian motion (EC.102) are set as following: $r = 0.04$, $\mu_1 = 0.08$, $\sigma_1 = 0.13$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $\rho = 0.3$, $B = 0$, and $T = 1$. In Figures EC.15 and EC.16, as in the single stock case, we demonstrate that our three general findings [13] hold for the two stocks case.

Note that, when the time to maturity is short and the fund is underperforming, it is optimal to gamble on a portfolio that has the maximal local volatility, that is, the portfolio solving $\max_{\boldsymbol{\pi}_t \in \boldsymbol{C}} \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_{t,\boldsymbol{x}} \boldsymbol{\pi}_t$ (see, e.g., the asymptotic condition (EC.10)). As a result, it may still be optimal to short each risky asset despite its positive risk premium. The solution for the portfolio set $\boldsymbol{C} = \{\pi_1 \in [0,1], \pi_2 \in [0,1]\}$ is $\pi_1^* = 100\%$ and $\pi_2^* = 100\%$, and for the portfolio set $\boldsymbol{C} = \{\pi_1 \in [-2,1], \pi_2 \in [-2,1]\}$ is $\pi_1^* = -200\%$ and $\pi_2^* = -200\%$.

Figure EC.17 shows that the equivalent wealth loss caused by a sub-optimal strategy may be approximately 100%, where the difference of the sub-optimal strategy lies in that the sub-optimal strategy would take a portfolio weight of 100% when the optimal portfolio weight $\pi^* = -200\%$.

References

Barles G. and P.E. Souganidis (1991). Convergence of approximation schemes for fully nonlinear equations. *Asymptotic Analysis*, 4, 271–283.

Basak, S. and G. Chabakauri (2010) Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23, 2970–3016.

[13] In the traditional concave utility maximization with two stocks, shorting one of the stocks is likely optimal. However, it is impossible to short both stocks that have a positive risk premium.

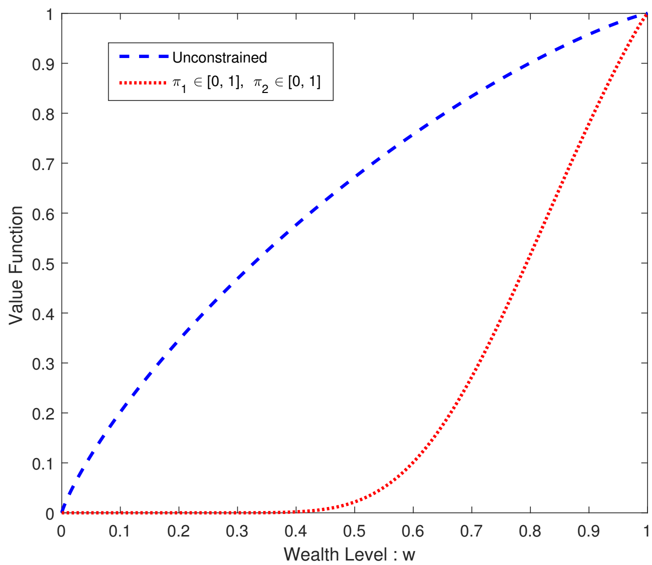

Figure EC.15 A comparison between the constrained and the unconstrained value functions associated with the goal-reaching problem under the two dimensional geometric Brownian motion model. The portfolio bounds are $\pi_1 \in [0,1]$ and $\pi_2 \in [0,1]$. The parameters are: $r = 0.04$, $\mu_1 = 0.08$, $\sigma_1 = 0.13$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $\rho = 0.3$, $B = 0$, and $T = 1$.

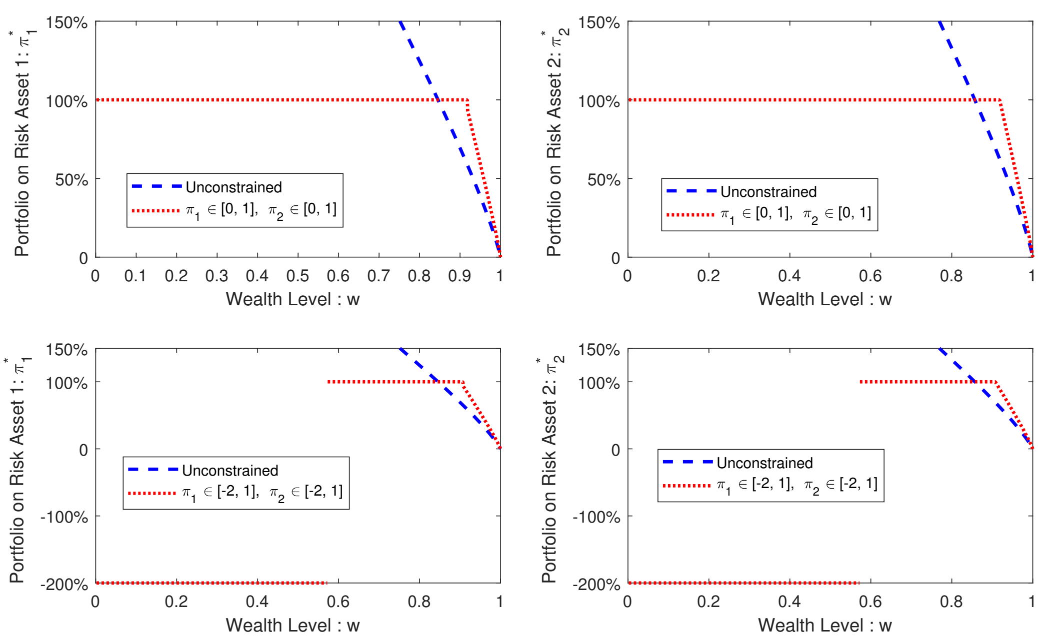

Figure EC.16 The non-myopic behavior and short-selling behavior (even if the risk premium is positive) for the non-concave portfolio optimization problem under two dimensional geometric Brownian motion model. The portfolio bounds are $\pi_1 \in [0,1]$ and $\pi_2 \in [0,1]$ for the upper panel and $\pi_1 \in [-2,1]$ and $\pi_2 \in [-2,1]$ for the lower panel, respectively. The parameters are: $r = 0.04$, $\mu_1 = 0.08$, $\sigma_1 = 0.13$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $\rho = 0.3$, $B = 0$, and $T = 1$.

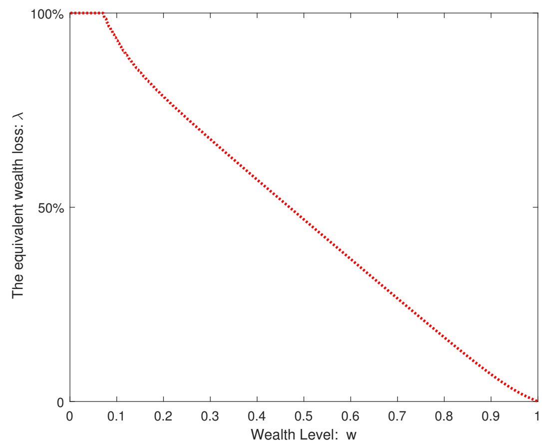

Figure EC.17 The equivalent wealth loss caused by a sub-optimal strategy for the goal-reaching problem with two stocks. The difference between the sub-optimal strategy and the optimal strategy is that the sub-optimal strategy would take a portfolio weight of 100% when the optimal portfolio weight is -200%. The portfolio bounds are $\pi_1 \in [-2,1]$ and $\pi_2 \in [-2,1]$. The parameters are: $r = 0.04$, $\mu_1 = 0.08$, $\sigma_1 = 0.13$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $\rho = 0.3$, $B = 0$, and $T = 1$.

Basak, S., A. Pavlova, and A. Shapiro (2007) Optimal asset allocation and risk shifting in money management, *Review of Financial Studies*, 20, 1583–1621.

Berkelaar, A.B., R. Kouwenberg and T. Post (2004) Optimal portfolio choice under loss aversion, *The Review of Economics and Statistics*, 86(4), 973–987.

Bian, B.J., X.F. Chen and Z.Q. Xu (2019) Utility maximization under trading constraints with discontinuous utility, *SIAM Journal on Financial Mathematics* 10 (1), 243–260.

Bouchard, B. and N. Touzi (2011) Weak dynamic programming principle for viscosity solutions. *SIAM Journal on Control and Optimization*, 49(3), 948–962.

Campbell, J.Y. and L.M. Viceira (1999) Consumption and portfolio decisions when expected returns are time varying. *Quarterly Journal of Economics*, 114, 433–495.

Carpenter, J. (2000) Does option compensation increase managerial risk appetite? *Journal of Finance*, 55(5), 2311–2331.

Chacko, G. and L.M. Viceira (2005) Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies*, 18, 1369–402.

Crandall, M.G., H. Ishii and P. L. Lions (1992) User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27, 1–67.

Dai, M., H. Jin, S. Kou, and Y. Xu (2021) A dynamic mean-variance analysis for log returns, *Management Science*, 67(2), 1093–1108.

Fleming W. H. and H. M. Soner (2006) *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York.

Forsyth,P.A. and G. Labahn (2007) Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance. *Journal of Computational Finance*, 11(2), 1–43.

He. X.D. and S. Kou (2018) Profit sharing in hedge funds. *Mathematical Finance*, 28(1), 50–81.

Karatzas, I. and S.E. Shreve (1991) *Brownian motion and stochastic calculus.* Second Edition, Springer-Verlag, New York.

Karatzas, I. and S.E. Shreve (1998) *Methods of mathematical finance* . Springer-Verlag, New York.

Kim, T. S. and E. Omberg (1996) Dynamic nonmyopic portfolio behavior. *Review of Financial Studies*, 9, 141–61.

Liu, J. (2007) Portfolio selection in stochastic environments. *Review of Financial Studies*, 20, 1–39.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, 3, 373–413.

Reny, P. J. (1999) On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica*, 67, 1029–1056.

Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An explicit solution for complete market. *Journal of Financial and Quantitative Analysis*, 37, 63–91.