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Cordon toll pricing in a multi-modal linear monocentric city with stochastic auto travel time

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This paper proposes an analytical model to determine the optimal cordon toll pricing scheme in a linear monocentric city with a competitive railway/highway system. It is assumed that the daily commuting time by rail mode is deterministic, whilst that by auto is stochastic and location-dependent across the city. The uncertainty in the auto travel time could affect residents' residential location choices and thus the urban spatial structure due to a long-term adjustment. In order to hedge against travel time variability on highway, commuters by auto often consider a travel time budget longer than the expected trip time to avoid late arrivals. It shows that ignorance of auto travel time variation may cause a significant bias in the prediction of the urban spatial structure in terms of residential distribution and city size. The implementation of cordon toll pricing scheme can rationalize the urban residential distribution and promote the efficiency of the urban system in terms of social welfare.

Keywords: travel time uncertainty; travel time budget; mode choice; cordon toll pricing scheme; households' residential location choices.

1. Introduction

It has been widely recognized that there is a strong interaction between design of urban transportation policies and the spatial patterns of urban land use. On one hand, urban land use, which involves the distribution of urban population, and housing demand and supply, can govern the travel demand and the demand distribution across the city, and thus affects the making-decisions on transportation policies for long-term transportation planning purposes. On the other hand, design of urban transportation policies can affect location accessibility and household's annual transportation expenses, which is a key factor to affect housing demand and supply, and thus affects households' residential distribution in the city. For example, Xu et al. (2015) reveals that implementation of private driving restriction in Beijing has raises the demand for subway proximity. It is, thus, essential to explore the interaction between transportation policy and urban land use for long-term planning purposes.

The variation of travel time on highway always exists in practice. In transportation network, the daily travel time of commuters often varies due to various random factors, such as stochastic road capacity and travel demand fluctuation. The former can be caused by events like unpredicted vehicle breakdowns, traffic accidents, and inclement weather conditions. The latter can be spawned by daily and seasonal demand variations, special events, and population characteristics. Owing to the uncertainty in travel time, commuters do not know exactly when they will arrive at the destination. It is necessary to incorporate commuters' responses to daily travel time uncertainty (e.g., reserving a buffer time to reduce the risk of late arrivals) in the transportation policy design.

The effects of travel time variability on travelers' route and mode choice behaviors have been widely explored in previous studies (Abdel-Aty et al., 1995; Sumalee and Watling, 2003; Lo et al., 2006; Shao et al., 2006; Lam et al., 2008; Chen and Zhou, 2010; Du and Wang, 2014). Abdel-Aty et al. (1995) found that travel time reliability played an important role in commuters' route choice behaviors. Sumalee and Watling (2003) investigated the effects of dependent link fails on travel time reliability in a network. Lo et al. (2006) proposed a reliable user equilibrium (RUE) model through introducing the concept of travel time budget to hedge against travel time variability. Based on the research work of Lo et al. (2006), Shao et al. (2006) and Lam et al. (2008) studied the impacts of uncertainties in both demand and supply sides. Chen and Zhou (2010) further proposed a METE model to explicitly consider the unreliability aspects of travel time variability. Du and Wang (2014) developed a multimodal equilibrium model to explore the impacts of travel time reliability on commuters' mode choice. However, they all studied travelers' behaviors for a given transportation network with OD pairs given and fixed, and thus were unable to investigate the effects of travel time variability on households' residential location choices, namely, the choices of origins of trips.

In view of the above, Ioannides (1983) first investigated the impact of random transportation costs on residential location decisions of households. He found out that uncertainty in transportation costs steepens the fall of land prices with distance from the Centre Business District (CBD) and engenders an increase in optimal city size. However, the study assumed that housing space throughout the city is fixed and equal, thus failed to consider the impact of uncertainty on household's housing space consumption. Papageorgiou and Pines (1988) relaxed this assumption and investigated the effects on urban structure of uncertainty in transportation cost, and found that risk would cause the urban area to be diminished under certain condition. However, he implicitly assumed a congestion-free road network and the same variation level of travel time across the city. In reality, congestion is one of the most important reasons that cause uncertainty in auto travel time. For instance, location with higher congestion often implies higher variation level of travel time. Accordingly, it fails to study the effects of traffic congestion and the variation levels of travel time across the city. Besides, the above urban models have assumed a single travel mode, and thus can not estimate the impacts of travel time uncertainty on commuters' mode choices.

This study makes one important step forward by explicitly considering the effects of travel time uncertainty on the equilibrium modal split and urban residential distribution in a competitive railway/highway system, where the uncertainties in auto travel time are location-dependent. The major contributions made in this article are as follows. First, a multi-modal urban system equilibrium model is developed, in which the effects of travel time uncertainty on commuters' mode choices and households' residential location choices are considered. Second, the optimal cordon toll pricing scheme is presented to rationalize commuters' mode choices and household residential distribution across the city so as to alleviate traffic congestion and travel time uncertainty. Third, the impacts of the level of auto travel time uncertainty on the urban system performances are analytically explored, together with that on design of the optimal cordon toll pricing scheme in terms of toll level and location.

This paper is organized as follows. In Section 2, the multi-modal urban system equilibrium with stochastic auto travel time is formulated, together with its relevant properties. It follows with presentation of the social welfare maximization model in Section 3 with determination of the optimal cordon toll pricing scheme in terms of the toll level and location. In Section 4, a numerical example is provided to illustrate the applications of the proposed model. Finally, conclusions and recommendations for further studies are given in Section 5.

2. Urban system equilibrium

2.1. *Assumptions*

To facilitate the presentation of essential ideas without loss of generality, the following basic assumptions are made in this paper.

A1 We consider a linear, closed and monocentric city, which implies that all job opportunities cluster in a highly compact city center or central business district (CBD), and the total number of households in the city is exogenously given and fixed. Also, all the land within the city boundary is supposed to be owned by absentee landlords and the value of land at/beyond the city boundary equals its opportunity cost. These assumptions have been widely adopted in the field of urban economics, such as Alonso (1964), Muth (1969), Solow(1972), Fujita (1989), Li et al. (2012a, 2012b, 2013) and Chen et al. (2016).

A2 Three types of stakeholders are concerned in the urban system: households, property developer and the authority. All households are supposed to be homogenous in terms of their income level and utility function. Each household has a quasi-linear utility function and aims to maximize its own utility by determining the residential location, size of housing space and amount of non-housing goods within the budget constraint (e.g., Song and Zenou, 2006; Kono et al., 2012). The property developer is assumed to maximize its net profit by determining its capital investment intensity. The authority seeks for the cordon toll pricing scheme in terms of toll level and location that maximizes the social welfare of the urban system.

A3 It is assumed that each commuter chooses its preferred travel mode (auto or railway) based on the generalized travel cost which is defined as the monetary cost plus the time cost weighted by the value of time (VOT), leading to a deterministic mode choice equilibrium. Following Wang et al. (2004), Liu et al. (2009) and Li et al. (2012a), it is assumed that under free-flow and no-toll conditions, railway has lower fixed cost but higher variable cost than auto. In addition, the travel cost by auto travelling from the city boundary to the CBD is supposed to be lower than that by railway, which ensures both travel modes are used in the urban system.

A4 Commuters are assumed to have perfect knowledge of the daily travel time distribution. In order to hedge against the risk of late arrivals, commuters are supposed to allow for a travel time budget longer than the expected trip time so as to avoid late arrivals. The concept of travel time budget (TTB) is proposed by Lo et al. (2006), and widely adopted by following studies thereafter (see Shao et al., 2006; Siu and Lo, 2008; Zhou and Chen, 2008; Chen and Zhou, 2010).

2.2. *Continuum equilibrium of travel mode choice*

2.2.1. *Generalized travel cost by rail mode*

In a linear monocentric city as assumed in A1, each location across the corridor is characterized by its distance from the CBD. Denote *x* as the distance of a location from the CBD. Let $C^{\mathrm{r}}(x)$ be the round-trip travel cost by railway between location *x* and the CBD, which is defined as the monetary cost plus the travel time weighted by the value of time (VOT), i.e.,

$$C^{\mathrm{r}}(x) = 2\left(\delta\left(a_0^{\mathrm{r}} + t_0^{\mathrm{r}}x\right) + f_0^{\mathrm{r}} + f^{\mathrm{r}}x\right), \tag{1}$$

where the "2" denotes a round-trip journey, δ is the VOT of commuters, which is used to convert the travel time into equivalent monetary units, a_0^{r} is the average access/egress time of passengers from their home/workplace to the railway station, which is supposed to be independent of location *x*, t_0^{r} is the average movement time of trains per unit distance, and f_0^{r} and f^{r} are, respectively, the fixed and variable components of the fare paid for the rail ride.

2.2.2. *Generalized travel cost by auto mode with travel time uncertainty*

Let $n(x)$ be the household residential density at location x (i.e., number of households per unit of land area), $q(x)$ be the hourly density of travel demand (i.e., number of commuters per unit of land area) at location x, and υ be peak-hour factor (i.e., the ratio of peak-hour flow to the average daily flow), which is used to convert the travel demand from a daily to an hourly basis. $q(x)$ can be defined as

$$q(x) = \upsilon\eta n(x) = \xi n(x), \tag{2}$$

where η is the average daily number of trips to the CBD per household, and $\xi(=\upsilon\eta)$ is the peak-hour trip generation rate (i.e., average number of peak-hour trips to the CBD per household). Let $q^{\mathrm{a}}(x)$ and $q^{\mathrm{r}}(x)$ respectively be the hourly density of travel demand for auto and rail modes at location x. Travel demand conservation at each location requires that

$$q(x) = q^{\mathrm{a}}(x) + q^{\mathrm{r}}(x). \tag{3}$$

Denote $Q^{\mathrm{a}}(x)$ as the total traffic volume on highway at location x, which can be determined by

$$Q^{\mathrm{a}}(x)=\int_x^B q^{\mathrm{a}}(w)dw, \tag{4}$$

where B is the distance from the urban boundary to the CBD (i.e., the length of the corridor). Let $t^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)$ represent the travel time on highway per unit of distance around location x, where the traffic volume $Q^{\mathrm{a}}(x)$ is determined by Equation (4). To capture the travel time variability on highway due to uncertain traffic conditions and commuter characteristics, $t^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)$ is assumed to be the combination of the mean travel time $\bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)$ and a stochastic component $\varepsilon(x)$. Then, we have

$$t^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)=\bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)+\varepsilon(x), \tag{5}$$

where $\varepsilon(x)$ is supposed to follow normal distribution with zero mean and variance $\sigma^2(x)$. In Equation (5), the mean travel time $\bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)$ is assumed to be a strictly increasing function with regard to traffic volume $Q^{\mathrm{a}}(x)$ on highway at location x and can be estimated using the Bureau of Public Roads (BPR) function, as follows:

$$\bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)=t_0^{\mathrm{a}}\left[1.0+\mu_1\left(\frac{Q^{\mathrm{a}}(x)}{K}\right)^{\mu_2}\right], \tag{6}$$

where t_0^{a} is the free-flow travel time per unit of distance on highway, μ_1 and μ_2 are positive parameters, and K is the travel capacity of the corridor.

Denote $T^{\mathrm{a}}(x)$ as the travel time by auto from location x to the CBD, which can be obtained by integrating Equation (5) over the interval $(0,x)$. The mean of $T^{\mathrm{a}}(x)$ can thus be given by

$$E\left(T^{\mathrm{a}}(x)\right)=\int_0^x \bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(w)\right)dw, \tag{7}$$

where $E(\cdot)$ represents the expectation operator. For simplicity, the travel time at each location is assumed to be independent. Allowing for travel time correlation among different locations would significantly complicate theoretical analysis in a continuous model, which is left for our future study. Accordingly, the variance of $T^{\mathrm{a}}(x)$ can be estimated from the variances of the traversed areas, as follows:

$$Var\left(T^{\mathrm{a}}(x)\right)=\int_0^x \sigma^2(w)dw, \tag{8}$$

where $Var(\cdot)$ represents the variance operator. Under the assumption of normal distributed travel time of each location, the total travel time $T^{\mathrm{a}}(x)$ follows a normal distribution, with mean and variance respectively given by Equations (7) and (8).

Although travellers are assumed to have perfect knowledge of daily travel time distribution, they do not know the exact prior travel time. According to A4, when faced with travel time variability, commuters would depart earlier and reserve a buffer time to ensure more frequent on-time arrivals. That is, travellers allow for a longer travel time budget (TTB) to hedge against travel time variability. Denote $\pi^{\mathrm{a}}(x;\lambda)$ as the travel time budget of commuters from location *x* to the CBD by auto with parameter λ reflecting travellers' requirement on punctual arrivals, which can be derived as

$$\pi^{\mathrm{a}}(x;\lambda) = E\left(T^{\mathrm{a}}(x)\right) + \lambda\sqrt{Var\left(T^{\mathrm{a}}(x)\right)}\,, \tag{9}$$

where $E\left(T^{\mathrm{a}}(x)\right)$ and $Var\left(T^{\mathrm{a}}(x)\right)$ are, respectively, the mean and variance of $T^{\mathrm{a}}(x)$, which are given by Equations (7) and (8). The second term on the right-hand side of Equation (9) indicates the buffer time (or reservation time) to hedge against travel time uncertainty. Denote $\zeta^{\mathrm{a}}(x)$ as such buffer time of travellers residing at location *x*, which is given by

$$\zeta^{\mathrm{a}}(x) = \lambda\left(\int_0^x \sigma^2(w)dw\right)^{1/2}. \tag{10}$$

Following Lo et al. (2006), the value of λ can be derived mathematically from the probability that a trip arrives within the travel time budget, specified as

$$\Pr\left(T^{\mathrm{a}}(x) \le \pi^{\mathrm{a}}(x)\right) = \Pr\left(T^{\mathrm{a}}(x) \le E\left(T^{\mathrm{a}}(x)\right) + \lambda\sqrt{Var\left(T^{\mathrm{a}}(x)\right)}\right) = \gamma\,, \tag{11}$$

where γ is the probability that the actual trip time is within the travel time budget, indicating a pre-specified confidence level of travel time reliability. Rearranging Equation (11) yields

$$\Pr\left(\frac{T^{\mathrm{a}}(x) - E\left(T^{\mathrm{a}}(x)\right)}{\sqrt{Var\left(T^{\mathrm{a}}(x)\right)}} \le \lambda\right) = \gamma\,. \tag{12}$$

Note that the travel time $T^{\mathrm{a}}(x)$ in Equation (12) follows a normal distribution. The left-hand side in Equation (12) is thus $T^{\mathrm{a}}(x)$'s standard normal variate. Let $\Phi(\cdot)$ be the cumulative distribution function (CDF) of standard normal variate. Equation (12) can thus be written as

$$\Phi\left(\lambda\right)=\gamma\ . \tag{13}$$

Taking the inverse of Equation (13) yields

$$\lambda=\Phi^{-1}\left(\gamma\right), \tag{14}$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF of standard normal variate.

Previous studies have shown that the standard deviations of the travel time are proportional to the square root of the mean travel time (Richardson and Taylor, 1978; Taylor, 1982). Following these studies, we assume that

$$\sigma(x)=\rho\sqrt{\bar{t}^{\mathrm{a}}\left(Q^{\mathrm{a}}(x)\right)}, \tag{15}$$

where ρ is a positive parameter associated with the degree of travel time variability or uncertainty. The larger the value of ρ, the more stochastic the travel time, and vice versa.

Remark 1. The buffer time $\zeta^{\mathrm{a}}(x)$ is an increasing function with regard to the distance x from the CBD, capturing the characteristic that households residing at the suburb suffer higher level of travel time uncertainty and have to reserve more time to hedge against the risk of late arrivals, in comparison with those living at the CBD area. It can be proved by taking derivation of $\zeta^{\mathrm{a}}(x)$ with respect to x, as follows:

$$\frac{d\zeta^{\mathrm{a}}(x)}{dx}=\frac{1}{2}\lambda\left(\int_0^x\sigma^2(w)dw\right)^{-1/2}\sigma^2(x)\geq 0\,. \tag{16}$$

Let $C^{\mathrm{a}}\left(x\right)$ be the round-trip travel cost by auto between location x and the CBD. Denote λ_1 and λ_2 respectively be the punctual arrival requirement in the morning peak and evening peak with $\lambda_1\geq\lambda_2$, allowing for risk aversion heterogeneity between home-to-work trip and work-to-home trip. Following Wang et al. (2004) and Liu et al. (2009), the travel cost consists of both travel time budget and monetary cost, which can be expressed as

$$C^{\mathrm{a}}(x)=\delta\left(\pi^{\mathrm{a}}(x;\lambda_1)+\pi^{\mathrm{a}}(x;\lambda_2)\right)+2\left(v_0^{\mathrm{a}}+v^{\mathrm{a}}x\right), \tag{17}$$

where δ is the referred value of travel time, the "2" denotes a round-trip journey between location x and the CBD, v_0^{a} is the fixed monetary travel cost (e.g., the parking charge), and v^{a} is the variable monetary travel cost (e.g., auto fuel cost per unit of distance). The first term on the right-hand side of Equation (17) indicates the round-trip (i.e., home-to-work trip and work-to-home trip) travel time budgets, which are the combination of the average travel time cost and the buffer time cost. The second term implies the round-trip monetary cost.

2.2.3. *Continuum equilibrium of mode choice*

Travellers are assumed to choose a travel mode that minimizes their individual travel cost. As a result, a Wardrop-type user equilibrium of mode choice can be obtained when no traveller can reduce its travel cost by switching mode.

Definition 1. (*Continuum equilibrium of mode choice*). At equilibrium, the travel cost of used mode at any location of the corridor is at its minimum, and the travel cost of unused mode at any location is greater than or equal to the minimum.

Definition 1 defines the equilibrium conditions of mode choice in a linear continuum corridor, which can be mathematically expressed as

$$q^{\mathrm{a}}(x)\left(C^{\mathrm{a}}(x)-C(x)\right)=0,\ C^{\mathrm{a}}(x)-C(x)\geq 0, q^{\mathrm{a}}(x)\geq 0, \tag{18}$$

$$q^{\mathrm{r}}(x)\left(C^{\mathrm{r}}(x)-C(x)\right)=0,\ C^{\mathrm{r}}(x)-C(x)\geq 0, q^{\mathrm{r}}(x)\geq 0, \tag{19}$$

where $x\in(0,B)$, the generalized travel cost by auto and railway are given by Equations (1) and (17), and $C(x)$ is the minimal round-trip travel cost between location x and the CBD. The annual travel cost (denoted by $\varphi(x)$) for each household residing at location x can be easily estimated by

$$\varphi(x)=\varpi C(x), \tag{20}$$

where ϖ is the average annual number of trips to the CBD per household, which can be estimated using survey data.

Proposition 1. *There must exist some critical location $\bar{x}$ (i.e., referred to as modal boundary in the following), such that the travel cost by auto is equal to that by railway at location $\bar{x}$, and the travel cost by auto is higher (or lower) than that by railway inside (or outside) the modal boundary $\bar{x}$.*

The proof of Proposition 1 is provided in Appendix A. Proposition 1 implies that only railway is used inside the modal boundary $\bar{x}$, and only auto is used outside location $\bar{x}$. As a result, the hourly density of travel demand for auto $q^{\mathrm{a}}(x)$ across the corridor can be determined by

$$q^{\mathrm{a}}(x)=\begin{cases}0, & \forall x\in(0,\bar{x}),\\ \xi n(x), & \forall x\in(\bar{x},B).\end{cases} \tag{21}$$

The traffic volume on highway (4) can thus be rewritten as

$$Q^{\mathrm{a}}(x)=\begin{cases}\int_{\bar{x}}^{B}\xi n(w)dw, & \forall x\in(0,\bar{x}),\\ \int_{x}^{B}\xi n(w)dw, & \forall x\in(\bar{x},B).\end{cases} \tag{22}$$

At the modal boundary, the two modes are indifferent for commuters. Thus, the value of the modal boundary $\bar{x}$ can be determined by

$$C^{\mathrm{a}}(\bar{x})=C^{\mathrm{r}}(\bar{x}). \tag{23}$$

Let N^{r} and N^{a}, respectively, be the total numbers of households using rail and auto, which can be determined by

$$N^{\mathrm{r}}=\int_{0}^{\bar{x}}n(x)dx\text{, and} \tag{24}$$

$$N^{\mathrm{a}}=N-N^{\mathrm{r}}. \tag{25}$$

2.3. *Households' residential location choices*

According to A2, each household is assumed to choose a residential location to maximize its own utility subject to a budget constraint. Following Song and Zenou (2006) and Kono et al. (2012), a quasi-linear utility function is adopted in this paper to facilitate social welfare analysis, with which the social welfare can be easily calculated by adding the utility level to the land rents and toll revenue. The quasi-linear utility function can be specified as

$$U\left(x\right)=z(x)+\alpha\log g(x),\alpha>0, \tag{26}$$

where $U\left(x\right)$ is the annual utility of households at location x, $z(x)$ is the annual composite non-housing goods each household consumed at location x, for which the price is normalized to 1, α is the housing parameter, which implies the part of income spent on housing, and $g(x)$ is the average annual consumption of housing per household at location x, which is measured in square meters of floor space. Each household' income is spent on travel cost, housing consumption and non-housing goods consumption. Thus, the household utility maximization problem can be represented as

$$\max_{z,g} \quad U(x) = z(x) + \alpha \log g(x), \tag{27}$$

$$\text{s.t.} \quad Y - \varphi(x) - z(x) - p(x) g(x) = 0, \tag{28}$$

where Y is the annual wage of each household, $p(x)$ is the average annual rental price per unit of housing floor area at location *x*, and $\varphi(x)$ is determined by Equation (20) indicating the average annual travel cost for residents living at location *x* with consideration of travel time uncertainty.

We define bid rent, $\psi(Y - \varphi(x), u)$, as the maximal rent that a household can pay for residing at location *x* while enjoying a generalized utility level *u*, which can be mathematically expressed as

$$\psi(Y - \varphi(x), u) = \max_{z,g} \left\{ \frac{Y - \varphi(x) - z}{g} \middle| U(z,g) = u \right\} = \max_{g} \left\{ \frac{Y - \varphi(x) - Z(g,u)}{g} \right\}, \tag{29}$$

where $Z(g,u)$ is the solution to $U(z,g) = u$ for *z*, which is a function of annual housing consumption *g*, and annual utility level *u*. Accordingly, $g(\cdot)$ and $p(\cdot)$ can be derived as functions with respect to common utility level *u*, as follows:

$$g(x,u) = \exp\left(\frac{u + \varphi(x) + \alpha - Y}{\alpha} \right), \text{ and} \tag{30}$$

$$p(x,u) = \alpha \exp\left(\frac{Y - \varphi(x) - \alpha - u}{\alpha} \right). \tag{31}$$

Substituting Equations (30) and (31) into Equation (28) yields

$$z(x) = Y - \varphi(x) - \alpha. \tag{32}$$

2.4. *Property developers' housing production behaviour*

Following Song and Zenou (2006) and Li et al. (2013), property developers are supposed to behave in keeping with a Cobb-Douglas form of the housing production function, as follows:

$$h(S(x)) = \theta \left(S(x) \right)^b, 0 < b < 1, \tag{33}$$

where $h(S(x))$ is the housing output per unit of land area at location x, $S(x)$ is the capital investment intensity at location x (i.e., the capital input per unit of land), and θ and b are positive parameters.

Denote $r(x)$ as the rent per unit of land area at location x, and k as the price of capital (i.e., the interest rate). The net profit per unit of land area, $\Omega(x)$, at location *x* can then be given by

$$\Omega(x) = p(x)h\left(S(x)\right) - \left(r(x) + kS(x)\right), \tag{34}$$

where the price per unit of housing floor space $p(x)$ is given by Equation (31). The first term on the right-hand side of Equation (34) is the total revenue from housing supply, and the final two terms in the parentheses are the land rent cost and the capital cost respectively.

Each property developer in the housing market aims to determine the optimal capital investment intensity so as to maximize the net profit, which is mathematically expressed as

$$\max_{S} \Omega(x) = p(x)\theta S^{b} - \left(r(x) + kS\right). \tag{35}$$

The first-order optimality condition of the maximization problem (35) yields

$$\frac{\partial \Omega}{\partial S} = p(x)\theta b S^{b-1} - k = 0. \tag{36}$$

Substituting $p(x)$ in Equation (31) into Equation (36) produces the optimal capital investment intensity as a function of the common utility level u, namely,

$$S(x,u) = \left(\theta b k^{-1} \alpha\right)^{\frac{1}{1-b}} \exp\left(\frac{Y - \varphi(x) - \alpha - u}{\alpha(1-b)}\right). \tag{37}$$

Substituting Equation (37) into Equation (33), we have

$$h\left(S(x,u)\right) = \left(\theta^{1/b} b k^{-1} \alpha\right)^{\frac{b}{1-b}} \exp\left(\frac{b\left(Y - \varphi(x) - \alpha - u\right)}{\alpha(1-b)}\right). \tag{38}$$

The households residential density, $n(x)$, at location x can thus be obtained by dividing the housing output per unit of land area, $h\left(S(x,u)\right)$, by the housing space consumed by each household, $g(x,u)$, as follows:

$$n(x) = \frac{h\left(S(x,u)\right)}{g(x,u)} = \left(\theta\left(bk^{-1}\alpha\right)^{b}\right)^{\frac{1}{1-b}} \exp\left(\frac{Y - \varphi(x) - \alpha - u}{\alpha(1-b)}\right). \tag{39}$$

Note that the property developers earn zero profit (i.e., $\Omega = 0$) under perfect competition, thus

$$r(x) = p(x)\theta S(x)^{b} - kS(x). \tag{40}$$

Substituting Equations (31) and (37) into Equation (40) yields

$$r(x,u) = \left(\frac{1}{b} - 1\right)\left(\theta b k^{-b} \alpha\right)^{\frac{1}{1-b}} \exp\left(\frac{Y - \varphi(x) - \alpha - u}{\alpha(1-b)}\right). \tag{41}$$

2.5. *Housing market equilibrium*

The urban system equilibrium requires that all households should be within the urban area, which is mathematically expressed by

$$\int_0^B n(x)dx = N , \tag{42}$$

where B is the distance from the city boundary to the CBD as concerned, and N is the total amount of households in the city. Substituting Equation (39) into Equation (42) and solving it for u, we then have

$$u = \alpha(1-b)\log\left(\left(\theta\left(bk^{-1}\alpha\right)^b\right)^{\frac{1}{1-b}}\int_0^B \exp\left(\frac{Y-\varphi(x)-\alpha}{\alpha(1-b)}\right)dx \Big/ N\right). \tag{43}$$

In addition, according to A1, the equilibrium rent of unit land area at the city boundary B equals the agricultural rent or opportunity cost of the land, i.e.,

$$r(B,u) = r_a , \tag{44}$$

where r_a is a constant opportunity cost of land.

Proposition 2. *When the traffic congestion cost for auto can be ignored (i.e., $\mu_1 = 0$), it follows that an increase in the travel time uncertainty (i.e., ρ) or in the punctual arrival requirement (i.e., λ_1 or λ_2),can lead to a decrease in the equilibrium utility level, city length and the total number of auto users, but an increase in the modal boundary $\bar{x}$ between the travel modes, land rent and residential density at the CBD, and the total number of rail passengers as shown in Table 1.*

The proof of this proposition is provided in Appendix B. The general case with considering traffic congestion will be illustrated through numerical simulations in the numerical example section.

2.6. *Calculating the equilibrium solution for the urban system*

In this subsection, we propose a procedure for calculating the equilibrium solution of the urban system. The step-by-step procedure is as follows, where the bolded symbols represent the vectors of the corresponding variables for convenience.

Step 0 Choose minimum and maximum values for the city length, which are respectively indicated by $\underline{B}$ and $\bar{B}$.

Step 1 Set the city size $B=(\underline{B}+\bar{B})/2$. Choose initial values for the corresponding household residential density $\mathbf{n}^{(1)}$ across the city. Set the iteration counter to $i=1$.

Step 2 Calculate the generalized travel cost vector for the rail mode $\boldsymbol{C}^{\boldsymbol{r}(i)}=\left\{C^{\mathrm{r}(i)}(x)\right\}$ in terms of Equation (1) and the value of modal boundary $\bar{x}^{(i)}$ based on Equation (23). Then, determine the values of the traffic volume on highway $\mathbf{Q}^{\mathbf{a}(i)}=\{Q^{\mathrm{a}(i)}(x)\}$ according to Equations (21) and (22), the generalized travel cost on highway $\mathbf{C}^{\mathbf{a}(i)}=\left\{C^{\mathrm{a}(i)}(x)\right\}$ by Equation (17), the minimum travel cost at each location $\mathbf{C}^{(i)}=\left\{C^{(i)}(x)\right\}$, and the annual travel cost $\boldsymbol{\varphi}^{(i)}=\left\{\varphi^{(i)}(x)\right\}$ by Equation (20).

Step 3 Calculate the value of utility $u^{(i)}$ by Equation (43). The values of the vectors $\mathbf{g}^{(i)}$, $\mathbf{p}^{(i)}$, $\mathbf{S}^{(i)}$, $\mathbf{h}^{(i)}$ and $\mathbf{r}^{(i)}$ can then be determined by substituting $u^{(i)}$ into Equations (30), (31), (37), (38) and (41). Meanwhile, the household residential density vector $\tilde{\mathbf{n}}^{(i)}$ can be obtained by Equation (39).

Step 4 Update the household residential density according to $\mathbf{n}^{(i+1)}=\mathbf{n}^{(i)}+\left(\tilde{\mathbf{n}}^{(i)}-\mathbf{n}^{(i)}\right)/i$.

Step 5 If the relative gap $\| n^{(i+1)}-n^{(i)} \| / n^{(i)}$ is less than a pre-specified precision, then go on to Step 6. Otherwise, set $i=i+1$ and go to Step 2.

Step 6 If the value of $r(B)$ obtained is equal to the land opportunity cost r_a (i.e, condition (44) holds), then stop. Otherwise, if $r(B)<r_a$, update $\bar{B}=B$, else if $r(B)>r_a$, update $\underline{B}=B$. Then, go to Step 1.

In Step 1, the initial household residential density can be assumed to be uniform across the city. In Step 2, the modal boundary $\bar{x}$ can be determined by enumerating each location from city boundary B towards the CBD until Equation (23) satisfies.

3. Design of cordon toll pricing scheme

Considerable attention has been paid to cordon toll pricing scheme (see e.g., Mun et al. , 2003, 2005; Ho et al., 2005, 2007; Verhoef, 2005; Chu and Tsai, 2008; Tsekeris and Voβ, 2009; de Palma and Lindsey, 2011; Balijepalli and Shepherd, 2016). Specifically, Meng and Liu (2012) investigated the impact of cordon-based congestion pricing scheme on the modal split in a bimodal transportation network with auto and rail travel modes. Liu et al. (2013) and (2014) respectively designed a speed-based toll and a joint distance-time toll for cordon-based congestion pricing scheme. Cordon toll pricing scheme, requiring all auto users passing through the cordon location towards the CBD to pay a congestion toll, has been widely implemented in practice. For instance, this scheme has been adopted in Singapore, London, Hong Kong, Oslo, Trondheim, and Bergen (see Zhang and Yang, 2004; Rouwendal and Verhoef, 2006; Li et al., 2014).

3.1. *Travel cost under cordon toll pricing scheme*

Cordon toll pricing scheme is defined by the combination of cordon location and toll level. Let τ be the toll level and x_c be the distance of the cordon location from the CBD. The cordon locates between the CBD and the city boundary (i.e., $0 \le x_c \le B$). All the auto users are required to pay the toll when cordon locates at the CBD (i.e., $x_c = 0$), and no commuters need to pay the toll when cordon locates at the city boundary (i.e., $x_c = B$).

Denote $q_1^{\mathrm{a}}(x)$ as the hourly density of travel demand for auto at location $x \in (0, x_c)$ and q_2^{a} as the hourly density of travel demand for auto at location $x \in (x_c, B)$. Let $Q_1^{\mathrm{a}}(x)$ and $Q_2^{\mathrm{a}}(x)$ respectively be the traffic volume on highway at location x in sections $(0, x_c)$ and (x_c, B), which can be given by

$$\begin{cases} Q_1^{\mathrm{a}}(x) = \int_x^{x_c} q_1^{\mathrm{a}}(w)dw + \int_{x_c}^{B} q_2^{\mathrm{a}}(w)dw, & \forall x \in (0, x_c) \\ Q_2^{\mathrm{a}}(x) = \int_x^{B} q_2^{\mathrm{a}}(w)dw, & \forall x \in (x_c, B) \end{cases}. \tag{45}$$

The mean travel time on highway per unit of distance around location x, $\bar{t}^{\mathrm{a}}\left(Q_i^{\mathrm{a}}(x)\right)$, can thus be determined by

$$\bar{t}^{\mathrm{a}}\left(Q_i^{\mathrm{a}}(x)\right) = t_0^{\mathrm{a}}\left[1.0 + \mu_1\left(\frac{Q_i^{\mathrm{a}}(x)}{K}\right)^{\mu_2}\right],\ i = 1, x \in (0, x_c); i = 2, x \in (x_c, B). \tag{46}$$

The average travel time cost $\bar{T}^{\mathrm{a}}(x)$ from location x to the CBD can thus be rewritten as

$$\bar{T}^{\mathrm{a}}(x)=\begin{cases}\int_0^x \bar{t}^{\mathrm{a}}\left(Q_1^{\mathrm{a}}(w)\right)dw,\ \forall x\in(0,x_c),\\ \int_0^{x_c} \bar{t}^{\mathrm{a}}\left(Q_1^{\mathrm{a}}(w)\right)dw+\int_{x_c}^{x} \bar{t}^{\mathrm{a}}\left(Q_2^{\mathrm{a}}(w)\right)dw,\ \forall x\in(x_c,B).\end{cases} \tag{47}$$

The travel time budget $\pi^{\mathrm{a}}(x;\lambda)$ with a pre-specified confidence level λ can thus be given by

$$\pi^{\mathrm{a}}(x;\lambda)=\bar{T}^{\mathrm{a}}(x)+\lambda\left(\int_0^x \sigma^2(w)dw\right)^{1/2}. \tag{48}$$

After implementation of the cordon toll pricing scheme, every auto user that originates from outside of the cordon location and heads for the CBD has to pay a congestion toll (i.e., τ). Accordingly, the round-trip travel cost of each commuter by auto between location x and the CBD can be written as

$$C^{\mathrm{a}}(x)=\begin{cases}\delta\left(\pi^{\mathrm{a}}(x;\lambda_1)+\pi^{\mathrm{a}}(x;\lambda_2)\right)+2\left(v_0^{\mathrm{a}}+v^{\mathrm{a}}x\right)\quad,\forall x\in(0,x_c),\\ \delta\left(\pi^{\mathrm{a}}(x;\lambda_1)+\pi^{\mathrm{a}}(x;\lambda_2)\right)+2\left(v_0^{\mathrm{a}}+v^{\mathrm{a}}x\right)+\tau,\ \forall x\in(x_c,B),\end{cases} \tag{49}$$

which means households living beyond the cordon location (i.e., x_c) and commuting by auto have to pay a certain amount of toll (i.e., τ) for entering the congested CBD area, while those living within the cordon location do not need to pay.

The following proposition shows the effects of the cordon toll on the urban system. Its proof is given in Appendix C.

Proposition 3. *When the traffic congestion cost for auto can be ignored (i.e., $\mu_1=0$), it follows that an increase in the toll level (i.e., τ) can lead to a decrease in the equilibrium utility level, city length and the total number of auto users, but an increase in the modal boundary $\bar{x}$ between the travel modes, land rent and residential density at the CBD, and the total number of rail passengers as shown in Table 2.*

3.2. *Social welfare maximization model*

As stated in the previous section, given the cordon location and toll level exogenously, the urban spatial equilibrium can endogenously determine the commuters' mode choices, households' residential location choices, and housing market structure in terms of housing price and space. In this subsection, a social welfare maximization model is proposed to determine the optimal combination of the cordon location and toll level. The social welfare of the urban system (denoted by SW) is defined as the sum of the total utilities of all households and the total net revenue, as follows:

$$\mathrm{SW} = R + Nu, \tag{50}$$

where R denotes the total net revenue, which comprises revenue from land rent and toll charging, i.e.,

$$R = \int_0^B (r(x) - r_a)dx + \tau\varpi\xi^{-1}\int_{x_c}^B q^{\mathrm{a}}(x)dx, \tag{51}$$

where ϖ is the average annual number of trips to the CBD per household referred, and ξ is the peak-hour trip generation rate, which is applied to convert the unit of travel demand from hour-based into day-based. The first term on the right-hand side of Equation (51) represents the aggregate land rent received by the absentee landlords and the second term represents the total revenue from congestion toll.

The authorities aim to maximize the urban efficiency in terms of the total social welfare by determining the optimal combination of cordon location and toll level, which can be mathematically expressed by

$$\max_{\tau, x_c} \ \mathrm{SW} = Nu + \int_0^B (r(x) - r_a)dx + \tau\varpi\xi^{-1}\int_{x_c}^B q^{\mathrm{a}}(x)dx. \tag{52}$$

A simple grid search method is applied to solve the above social welfare maximization problem in the two-dimensional space of cordon location and toll level. Thus, the grid is defined by the cordon location - toll level dimensions. In the grid search, we first establish acceptable ranges of values for both cordon location, x_c, and toll level, τ. Then, both ranges are divided into a set of equal-value intervals by a pre-specified step size, for example, 0.01 km for cordon location x_c and RMB0.1 for toll level τ. For each combination of x_c and τ, one solves the urban system equilibrium for obtaining the associated cordon toll pricing scheme. The combination $\left(x_c^*, \tau^*\right)$ that leads to the maximal social welfare is thus the optimal cordon toll pricing scheme. This procedure is simple and has the merit in its capability to find the global optimal solution, though at the cost of long computational time.

4. Numerical studies

In this section, a numerical example is provided to illustrate the properties of the proposed model and the contributions of this study. The numerical example is intended to compare the performances of urban system under deterministic and stochastic scenarios and ascertain the effects of the level of auto travel time uncertainty on the modal split of travellers. It is also used to investigate the optimal cordon toll pricing scheme and the effects of such pricing scheme on the urban system performance. Additionally, the effect of the level of auto travel time uncertainty on design of the optimal cordon toll pricing scheme is also examined. In the following analyses, unless specifically stated otherwise, the baseline values for the model's input parameters are the same as those shown in Table 3.

4.1. *No toll scenario*

In this section, we focus on the effects of auto travel time uncertainty on the urban system performances without consideration of cordon toll pricing scheme. First, we make a comparison of the urban system performances with and without considering daily travel time uncertainty on highway. Then, a sensitivity analysis is conducted with respect to the level of auto travel time uncertainty.

4.1.1. *Comparison of urban system performances under deterministic and stochastic cases*

We first compare the performances of urban system when the daily travel time on highway is assumed to be deterministic (i.e., $\rho = 0.0$) and stochastic (i.e., $\rho = 0.6$). Figure 1 plots the round-trip generalized travel cost by rail and auto along the corridor under both deterministic and stochastic cases. It can be easily observed that ignorance of uncertainties would underestimate the travel cost on highway, making the intersection between auto and rail cost curves (i.e., modal boundary) moves from location $\bar{x}_s = 8.2$ km to $\bar{x}_d = 5.7$ km. It means the modal boundary moves 2.5 km towards the CBD. Also, it can be seen that for both scenarios the rail cost curve lies below (or lies above) the auto cost curve on the left (or right) of the modal boundary. It implies residents living on the right of the modal boundary choose to use auto due to low generalized travel cost, and those living on the left of the modal boundary take the subway to escape from the severe traffic congestion on highway at central areas. This illustrates the result of Proposition 1.

Figure 2 compares the residential distribution across the corridor under deterministic and stochastic cases. It can be seen that the residential density decreases with the distance further away from the CBD for both cases. However, the decline gradient for stochastic case is more severe than that for deterministic case, and thus leads to a more compact urban spatial structure in comparison with the deterministic case. Specifically, the city lengths under deterministic and stochastic cases are 38.8 and 33.2 km, respectively, which implies the city size is overestimated by 5.6 km when ignoring the daily travel time uncertainty. Notably, the points $\bar{x}_d$ and $\bar{x}_s$ indicate the modal boundary as shown in Figure 1. In addition, it can be observed from Figure 2 that the two curves intersect at point A with distance of 13.0 km from the CBD and residential density of 10.5 thousand households per square kilometres. For the left-hand side of point A, the residential density is underestimated when ignoring auto travel time uncertainty, and it is overestimated for the right-hand side of point A. In view of the above, ignorance of the daily travel time variation will cause a significant bias in the prediction of urban spatial structure.

Figures 3 (a-b) respectively exhibit the housing space per household and land price along the linear corridor under determinitic and stochastic cases. It can be observed from Figure 3 (a) that housing space consumed by each household increases with the distance further away from the CBD for both scenarios. But the curve for stochastic case is steeper than that for deterministic case and the two curves intersect at point C with distance of 13.0 km from the CBD. It reveals that the housing space per household is overestimated for central areas (i.e., the left-hand side of point C), and is underestimated for suburban areas (i.e., the right-hand side of point C) when ingoring the dialy auto travel time uncertainty. It’s because under stochastic scenario the level of auto travel time uncertainty increases with the distance further away from the CBD (see Remark 1), making the central area even more attractive than the suburb. Thus, more households are willing to live near the CBD and thus the housing space per household is reduced there. Consequently, the land price per unit of floor space is underestimated for central areas (i.e., left-hand side of point D), and is overestimated for suburban areas (i.e., right-hand side of point D) when ignoring the auto travel time uncertainty, as shown in Figure 3 (b).

4.1.2. *Sensitivity analysis with respect to the level of auto travel time uncertainty*

Figure 4 displays the effects of the level of auto travel time uncertainty on the modal split of auto and rail through changing the value of ρ (see Equation (15)) from 0 to 1.2. It can be observed that as the value of ρ increases, the proportion of auto users decreases, whereas that of rail passengers increases. When the value of ρ is 0.55 (see point E), the whole market is equally shared by the auto and rail modes. It's because an increase in the value of ρ would increse the travel cost on highway, making a number of auto users switch to the rail mode. In addition, as the level of auto travel time uncertainty increases, the city size decreases, whereas the modal boundary moves towards the suburb, as shown in Figure 5. This illustrates the results of Proposition 2.

4.2. *Cordon toll pricing scheme*

In this section, we first determine the optimal cordon toll pricing scheme in terms of cordon location and toll level so as to maximize the annual social welfare. On the basis of this solution, comparison of the urban system performances before and after implementation of the optimal cordon toll pricing scheme is made. Finally, the effects of the level of auto travel time uncertainty on design of the cordon toll pricing scheme are explored.

4.2.1. *Design of the optimal cordon toll pricing scheme*

Figure 6 depicts the variation in the total social welfare with variation in the toll level and location, given that $\rho = 0.6$ and other input parameters are the same as those in Table 3. It can be observed from this figure that the optimal cordon toll pricing scheme occurs at point F_1 with a toll of RMB7.1 and cordon location at 6.6 km from the CBD, leading to the maximal social welfare of 59.571 billion RMB per year. When the cordon toll location moves outward by 2.0 km (i.e., to 8.6 km), the optimal toll level is RMB7.5 (point F_2) and the total social welfare is 59.531 billion RMB per year. When the cordon toll location moves to 10.6 km, the optimal toll level increases to RMB7.8 (point F_3) and the total social welfare is 59.502 billion RMB per year. These observations imply that the shorter is the distance of the cordon toll location from the CBD, the lower is the optimal toll level, and vice versa.

4.2.2. *Comparison of urban system performances with and without cordon pricing*

Given the optimal toll level and location (see point F_1 in Figure 6), Figure 7 plots the round-trip generalized travel cost using auto and rail across the corridor after implementation of this optimal pricing scheme. The pricing scheme requires all auto users passing through the cordon location (i.e., 6.6 km) towards the CBD to pay a toll of RMB7.1. Figure 8 further compares the residential distribution across the linear corridor before and after implementing this optimal pricing scheme. It can be seen that implementation of the cordon toll pricing scheme leads to a higher residential density at urban central area but a lower one at suburban area. This is because implementation of the pricing scheme imposes an additional toll for auto commuters residing outside the cordon location, and thus leads them to switch to rail mode or migrate from the outside to the inside of the cordon so as to evade the extra tolls. Accordingly, the city boundary decreases from 33.2 km to 31.6 km, implying a more compact spatial structure. This result is consistent with that found in Verhoef (2005), in which only auto is considered. Additionally, implementation of the pricing scheme decreases the proportion of auto users, and thus alleviates the travel time uncertainty on highway and further reduces the buffer time cost of auto users as shown in Figure 9. This illustrate the results of Proposition 3.

Table 4 further summarizes the performances of urban system before and after the optimal cordon toll pricing for different values of ρ. It can be seen that regardless of the value of ρ, cordon toll pricing scheme would trigger some auto users to switch to rail mode or migrate to the central area to escape from toll charging, thus leading to a decrease in the city size and further an increase in the average residential density, average housing and land price, and average capital investment intensity. Also, it can be observed that for ρ=0.0, 0.6 and 1.2, the optimal pricing scheme, respectively, results in an increase in the annual social welfare by 430, 140 and 10 million RMB, a save in total buffer time cost of 0, 6.7 and 3.4 million hours. Specifically, when ignoring auto travel time uncertainty (i.e., $\rho = 0.0$), the total buffer time cost equals zero regardless of toll. The ignorance of auto travel time uncertainty will underestimate the travel cost on highway, thus leading to a distorted urban system.

4.2.3. *Effects of the auto travel time uncertainty on the optimal toll level and location*

Figure 10 displays the changes of the optimal toll level and location with the level of auto travel time uncertainty. It can be observed that an increase in the level of auto travel time uncertainty would decrease the toll level and make the cordon location move towards the suburb. This is not surprising because an increase in the travel time uncertainty on highway would trigger some auto users to switch to rail mode, which eases the traffic congestion on highway. Accordingly, the alleviation in the congestion externality on highway decreases the toll level and makes the cordon location move towards the suburb.

5. Conclusions and further studies

This paper proposed an analytical model for design of the cordon toll pricing scheme in a multi-modal urban system with uncertainty in daily travel time on highway. It is assumed that the daily commuting time by rail mode is deterministic, whilst that by auto is stochastic and location-dependent across the city. A comparison of urban system performances was made with and without considering the auto travel time uncertainty. Also, the effects of level of auto travel time uncertainty (i.e., indicated by the value of ρ) on the urban system performance were analytically explored. Then, a social welfare maximization model was formulated to determine the optimal toll level and location simultaneously. In addition, a sensitivity analysis was conducted to investigate the effects of level of auto travel time uncertainty on design of the optimal toll level and location.

The proposed model provides some new insights and important findings. First, ignorance of the daily travel time variation will cause a significant bias in the prediction of the urban spatial structure. Specifically, compared with stochastic case, the assumption of deterministic auto travel time will overestimate the city size and average housing space per household, while underestimate the average residential density, average housing price and average capital investment intensity. Second, implementation of the cordon toll pricing scheme triggers some residents to switch to rail mode or migrate from the outside to the inside of the cordon so as to evade the extra tolls, which thus works to alleviate the traffic congestion on highway and save auto users' buffer time cost associated with travel time uncertainty. In addition, the cordon toll pricing scheme contributes to increasing the social welfare of the urban system through the internalization of traffic congestion externalities. Third, the level of auto travel time uncertainty has a distinct impact on the design of the optimal cordon toll pricing scheme. It's interesting to find that an increase in the level of auto travel time uncertainty will decrease the toll level and make the cordon toll location move towards the suburb due to the alleviation of traffic congestion on highway resulted from modal shift.

Although the proposed model in this paper provides a new avenue for the urban system analysis, further extensions should be made in the following directions. First, it is worthy to extend the model for providing park-and-ride service, as that done in Wang et al., (2004) and Liu et al., (2009) for a given residential distribution pattern. Second, it is challenging to integrate the dynamic traffic patterns of peak-hour commuters into the multi-modal urban model with stochastic auto travel time (Gubins and Verhoef, 2014). Third, the homogenous household assumption is worth being relaxed for enabling the consideration of heterogeneity among residents in terms of their risk-averse attitudes towards travel. Fourth, it is meaningful to extend the monocentric urban structure to a polycentric one in a further study.

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Appendix A: Proof of Proposition 1

We define $F(x)$ as the difference between the travel costs by auto and by railway, as follows:

$$F(x) = C^{\mathrm{a}}(x) - C^{\mathrm{r}}(x),\ \forall x \in [0, B] \tag{A.1}$$

According to A3, railway has lower fixed cost but higher variable cost than auto. We thus have

$$F(0) = C^{\mathrm{a}}(0) - C^{\mathrm{r}}(0) > 0\,. \tag{A.2}$$

In addition, it is assumed that at the city boundary

$$F(B) = C^{\mathrm{a}}(B) - C^{\mathrm{r}}(B) < 0\,. \tag{A.3}$$

By the intermediate value theorem (Smith and Minton, 2012), there must exist some $\bar{x} \in (0, B)$ such that

$$F(\bar{x}) = 0\,. \tag{A.4}$$

Substituting Equations (1) and (17) into Equation (A.1), and then taking the second-order derivative yields

$$F''(x) = \left(2\delta + \frac{(\lambda_1 + \lambda_2)\delta\rho^2}{2\sqrt{\int_0^x \sigma^2(x)dw}} \right) \bar{t}^{\mathrm{a}\prime}\left(Q^{\mathrm{a}}(x)\right) - \frac{\delta\sigma^4(x)\left(\lambda_1 + \lambda_2\right)}{4}\left(\int_0^x \sigma^2(w)dw\right)^{-3/2} < 0\,, \tag{A.5}$$

since $\bar{t}^{\mathrm{a}\prime}\left(Q^{\mathrm{a}}(x)\right) = -t_0^{\mathrm{a}}\mu_1\mu_2 K^{-\mu_2} Q^{\mathrm{a}}(x)^{\mu_2 - 1} q^{\mathrm{a}}(x) \le 0$. Equation (A.5) implies that $F(x)$ is strictly concave. By the properties of concave functions (Niculescu and Persson, 2006), we have

$$\frac{F(x) - F(\bar{x})}{x - \bar{x}} < \frac{F(\bar{x}) - F(0)}{\bar{x} - 0},\ \forall x \in [0, \bar{x}) \text{ or } x \in (\bar{x}, B]\,. \tag{A.6}$$

Combining Equations (A.4) and (A.6), one obtains

$$F(x) \begin{cases} > 0, \ \forall x \in [0, \bar{x}) \\ < 0, \ \forall x \in (\bar{x}, B] \end{cases}, \tag{A.7}$$

which implies the round-trip travel cost by auto is higher (or lower) than that by railway inside (or outside) the modal boundary $\bar{x}$. As a result, only railway is used for any $x \in [0, \bar{x})$, whereas only auto is used for any $x \in (\bar{x}, B]$. This completes the proof of Proposition 1.

Appendix B: Proof of Proposition 2

Taking the derivative of Equations (42) and (44) with respect to ρ yields

$$n(B)\frac{dB}{d\rho}+\int_0^B\left(\frac{\partial n}{\partial u}\frac{du}{d\rho}+\frac{\partial n}{\partial\rho}\right)dx=0\text{, and} \tag{B.1}$$

$$\frac{\partial r}{\partial B}\frac{dB}{d\rho}+\left.\frac{\partial r}{\partial u}\right|_B\frac{du}{d\rho}+\left.\frac{\partial r}{\partial\rho}\right|_B=0. \tag{B.2}$$

Combining Equations (B.1) and (B.2), one obtains

$$\frac{du}{d\rho}=\frac{n(B)\left.\frac{\partial r}{\partial\rho}\right|_B-\frac{\partial r}{\partial B}\int_0^B\frac{\partial n}{\partial\rho}dx}{\frac{\partial r}{\partial B}\int_0^B\frac{\partial n}{\partial u}dx-n(B)\left.\frac{\partial r}{\partial u}\right|_B}\text{, and} \tag{B.3}$$

$$\frac{dB}{d\rho}=\frac{\left.\frac{\partial r}{\partial\rho}\right|_B\int_0^B\frac{\partial n}{\partial u}dx-\left.\frac{\partial r}{\partial u}\right|_B\int_0^B\frac{\partial n}{\partial\rho}dx}{n(B)\left.\frac{\partial r}{\partial u}\right|_B-\frac{\partial r}{\partial B}\int_0^B\frac{\partial n}{\partial u}dx}, \tag{B.4}$$

Then, taking the partial derivatives of $r(x)$ (see Equation (41)) with respect to u, x, and ρ produces

$$\begin{aligned}
&\frac{\partial r}{\partial u}=-\frac{1}{\alpha(1-b)}r(x)<0,\forall x\in[0,B],\\
&\frac{\partial r}{\partial x}=-\frac{1}{\alpha(1-b)}r(x)\frac{\partial\varphi}{\partial x}<0,\forall x\in[0,B],\\
&\frac{\partial r}{\partial\rho}=\begin{cases}-\frac{\varpi}{\alpha(1-b)}r(x)\frac{\partial C^{\mathrm{r}}}{\partial\rho}=0,\forall x\in[0,\bar{x}],\\ -\frac{\varpi}{\alpha(1-b)}r(x)\frac{\partial C^{\mathrm{a}}}{\partial\rho}<0,\forall x\in(\bar{x},B].\end{cases}
\end{aligned} \tag{B.5}$$

Similarly, taking the partial derivatives of $n(x)$ (see Equation (39)) with respect to u, x and ρ yields

$$\begin{aligned}
&\frac{\partial n}{\partial u}=-\frac{1}{\alpha(1-b)}n(x)<0,\forall x\in[0,B],\\
&\frac{\partial n}{\partial x}=-\frac{1}{\alpha(1-b)}n(x)\frac{\partial\varphi}{\partial x}<0,\forall x\in[0,B],\\
&\frac{\partial n}{\partial\rho}=\begin{cases}-\frac{\varpi}{\alpha(1-b)}n(x)\frac{\partial C^{\mathrm{r}}}{\partial\rho}=0,\forall x\in[0,\bar{x}],\\ -\frac{\varpi}{\alpha(1-b)}n(x)\frac{\partial C^{\mathrm{a}}}{\partial\rho}<0,\forall x\in(\bar{x},B].\end{cases}
\end{aligned} \tag{B.6}$$

Applying Equations (B.5) and (B.6), the numerator and denominator of $du/d\rho$ in Equation (B.3) are respectively negative and positive. As a result, the sign of $du/d\rho$ is negative. Similarly, the denominator of $dB/d\rho$ in Equation (B.4) is negative. The sign of $dB/d\rho$ therefore depends on the sign of the numerator (denoted by Λ). Substituting Equations (B.5) and (B.6) into Λ , one obtains

$$\Lambda = \frac{r(B)\varpi}{\alpha^2(1-b)^2}\int_0^B \left(\partial C/\partial\rho|_B - \partial C/\partial\rho\right)n(x)dx > 0 \ , \tag{B.7}$$

because $\frac{\partial C(B)}{\partial\rho} > \frac{\partial C(x)}{\partial\rho}$ holds for any $x \in (0,B)$. The sign of $dB/d\rho$ is thus negative.

At the CBD, Equations (B.5) and (B.6) tell us that both $\partial r(0)/\partial\rho$ and $\partial n(0)/\partial\rho$ equal zero, so that

$$\frac{dr(0)}{d\rho} = \left.\frac{\partial r}{\partial u}\right|_0 \frac{du}{d\rho} + \left.\frac{\partial r}{\partial\rho}\right|_0 = \left.\frac{\partial r}{\partial u}\right|_0 \frac{du}{d\rho} > 0 \ , \tag{B.8}$$

$$\frac{dn(0)}{d\rho} = \left.\frac{\partial n}{\partial u}\right|_0 \frac{du}{d\rho} + \left.\frac{\partial n}{\partial\rho}\right|_0 = \left.\frac{\partial n}{\partial u}\right|_0 \frac{du}{d\rho} > 0 \ . \tag{B.9}$$

Under congestion-free assumption (i.e., $\mu_1 = 0$), $\bar{x}$ can be obtained by solving the following equation:

$$2\left(\delta\left(a_0^{\mathrm{r}} + t_0^{\mathrm{r}}\bar{x}\right) + f_0^{\mathrm{r}} + f^{\mathrm{r}}\bar{x}\right) = 2\delta t_0^{\mathrm{a}}\bar{x} + \left(\lambda_1 + \lambda_2\right)\delta\rho\sqrt{t_0^{\mathrm{a}}\bar{x}} + 2\left(v_0^{\mathrm{a}} + v^{\mathrm{a}}\bar{x}\right). \tag{B.10}$$

Taking the derivative of Equation (B.10) with respect to ρ produces

$$\left(\delta t_0^{\mathrm{r}} + f^{\mathrm{r}} - \delta t_0^{\mathrm{a}} - v^{\mathrm{a}} - \frac{\lambda_1 + \lambda_2}{4\sqrt{\bar{x}}}\delta\rho\sqrt{t_0^{\mathrm{a}}}\right)\frac{d\bar{x}}{d\rho} = \frac{\lambda_1 + \lambda_2}{2}\delta\sqrt{t_0^{\mathrm{a}}\bar{x}} \ . \tag{B.11}$$

The term in the bracket on the left-hand side of Equation (B.11) is positive, since it's assumed that the variable cost of rail is larger than that of auto. Thus, the sign of $d\bar{x}/d\rho$ is positive, which implies as the level of auto travel time uncertainty increases, the modal boundary $\bar{x}$ between the travel modes gets closer to the city boundary until ρ is so large that no one uses auto throughout the city.

Taking the derivative of N^{r} (see Equation (24)) with respect to ρ yields

$$\frac{dN^{\mathrm{r}}}{d\rho} = n(\bar{x})\frac{d\bar{x}}{d\rho} + \int_0^{\bar{x}}\left(\frac{\partial n}{\partial u}\frac{du}{d\rho} + \frac{\partial n}{\partial\rho}\right)dx = n(\bar{x})\frac{d\bar{x}}{d\rho} + \int_0^{\bar{x}}\frac{\partial n}{\partial u}\frac{du}{d\rho}dx > 0 \ , \tag{B.12}$$

since $\partial n/\partial\rho$ equals zero within the region $[0,\bar{x}]$ according to Equation (B.6). Applying

Equation (25), one can immediately derive $dN^{\mathrm{a}} \big/ d\rho < 0$.

For the effects of the punctual arrival requirement on the urban system, one needs only to apply the above process to parameters λ_1 and λ_2. This completes the proof of Proposition 2.

Appendix C: Proof of Proposition 3

After introduction of the cordon pricing scheme, the whole corridor is partitioned into multiple sections in each of which only auto or railway is used. To facilitate presentation, we denote A as the area where only auto is used, and $\bar{A}$ as the area where only railway is used. Taking the derivative of $n(x)$ in Equation (39) with respect to τ yields

$$\frac{\partial n}{\partial \tau} = -\frac{\varpi n(x)}{\alpha(1-b)}\frac{\partial C}{\partial \tau}\text{, and} \tag{C.1}$$

$$\frac{\partial r}{\partial \tau} = -\frac{\varpi r(x)}{\alpha(1-b)}\frac{\partial C}{\partial \tau}. \tag{C.2}$$

According to Equation (49), Equations (C.1) and (C.2) are both negative for any location x when $x > x_c$ and $x \in A$, and equal to zero otherwise. Similarly, taking the derivatives of Equations (42) and (44) with respect to τ produces

$$\int_0^B \frac{\partial n}{\partial u}\frac{du}{d\tau}dx + \int_0^B \frac{\partial n}{\partial \tau}dx + n(B)\frac{dB}{d\tau} = 0\text{, and} \tag{C.3}$$

$$\frac{\partial r}{\partial B}\frac{dB}{d\tau} + \left.\frac{\partial r}{\partial u}\right|_B \frac{du}{d\tau} + \left.\frac{\partial r}{\partial \tau}\right|_B = 0, \tag{C.4}$$

Combing Equations (C.3) and (C.4), one immediately obtains

$$\frac{du}{d\tau} = \frac{n(B)\left.\frac{\partial r}{\partial \tau}\right|_B - \frac{\partial r}{\partial B}\int_0^B \frac{\partial n}{\partial \tau}dx}{\frac{\partial r}{\partial B}\int_0^B \frac{\partial n}{\partial u}dx - n(B)\left.\frac{\partial r}{\partial u}\right|_B}\text{, and} \tag{C.5}$$

$$\frac{dB}{d\tau} = \frac{\left.\frac{\partial r}{\partial \tau}\right|_B \int_0^B \frac{\partial n}{\partial u}dx - \left.\frac{\partial r}{\partial u}\right|_B \int_0^B \frac{\partial n}{\partial \tau}dx}{n(B)\left.\frac{\partial r}{\partial u}\right|_B - \frac{\partial r}{\partial B}\int_0^B \frac{\partial n}{\partial u}dx}, \tag{C.6}$$

where $\partial r/\partial u < 0$, $\partial r/\partial B < 0$, $\partial n/\partial u < 0$ and $\partial n/\partial B < 0$ according to Equations (B.5)-(B.6), and $\partial n/\partial \tau \le 0$, $\partial n/\partial \tau \le 0$ according to Equations (C.1)-(C.2). It can thus be easily observed that the denominator in Equation (C.5) is positive, and the numerator is non-positive. Accordingly, we have $du/d\tau \le 0$. Similarly, the denominator of $dB/d\tau$ in Equation (C.6) is negative. Thus, the sign of $dB/d\tau$ is dependent on the sign of the numerator (denoted by Λ_1). Substituting Equations (B.5), (B.6), (C.1) and (C.2) into

Λ_1 yields

$$\Lambda_1 = \frac{\varpi r(B)}{\alpha^2(1-b)^2}\int_0^B \left(\partial C(B)/\partial\tau - \partial C(x)/\partial\tau\right) n(x)dx \geq 0\,, \tag{C.7}$$

because $\dfrac{\partial C(B)}{\partial\tau} \geq \dfrac{\partial C(x)}{\partial\tau} \geq 0$ holds for any $x \in (0,B)$. Thus, we have $dB/d\tau \leq 0$. In addition, all commuters choose to use railway at the CBD, because the fixed cost of railway is assumed to be smaller than that of auto. Thus, the travel cost at the CBD is independent of toll level τ, further implying that $\partial r(0)/\partial\tau$ and $\partial n(0)/\partial\tau$ equal zero. As a result, we have

$$\frac{dr(0)}{d\tau} = \left.\frac{\partial r}{\partial u}\right|_0 \frac{du}{d\tau} + \left.\frac{\partial r}{\partial\tau}\right|_0 = \left.\frac{\partial r}{\partial u}\right|_0 \frac{du}{d\tau} \geq 0\text{, and} \tag{C.8}$$

$$\frac{dn(0)}{d\tau} = \left.\frac{\partial n}{\partial u}\right|_0 \frac{du}{d\tau} + \left.\frac{\partial n}{\partial\tau}\right|_0 = \left.\frac{\partial n}{\partial u}\right|_0 \frac{du}{d\tau} \geq 0\,. \tag{C.9}$$

The cordon location x_c divides the region between the CBD and city boundary into two sections: one is the no-toll area from the CBD to the cordon location (i.e., $x \in (0, x_c)$), and the other is the tolled area from the cordon location to the city boundary (i.e., $x \in (x_c, B)$). At location x_c, the generalized travel cost curve for auto is discontinuous. Within the tolled area, there may be one or no intersection between auto and rail cost curves. For the latter case, only one single mode is used and thus the change of toll level fails to affect commuters' mode choices, which is out of our consideration. Thus, we denote $\bar{x} \in (x_c, B)$ as the intersection between auto and rail cost curves (namely the modal boundary) within the tolled area, which can be determined by

$$2\left(\delta\left(a_0^{\mathrm{r}} + t_0^{\mathrm{r}}\bar{x}\right) + f_0^{\mathrm{r}} + f^{\mathrm{r}}\bar{x}\right) = 2\delta t_0^{\mathrm{a}}\bar{x} + \left(\lambda_1 + \lambda_2\right)\delta\rho\sqrt{t_0^{\mathrm{a}}\bar{x}} + 2\left(v_0^{\mathrm{a}} + v^{\mathrm{a}}\bar{x}\right) + \tau\,. \tag{C.10}$$

Taking the derivative of Equation (C.10) with respect to τ yields

$$\frac{d\bar{x}}{d\tau} = \frac{1}{2\delta t_0^{\mathrm{r}} + 2f^{\mathrm{r}} - 2\delta t_0^{\mathrm{a}} - 2v^{\mathrm{a}} - \dfrac{\lambda_1+\lambda_2}{2\sqrt{\bar{x}}}\delta\rho\sqrt{t_0^{\mathrm{a}}}} > 0\,, \tag{C.11}$$

because the denominator, namely, the difference between the variable cost of railway and auto, is assumed to be positive. Equation (C.11) implies that as the toll level increases, the modal boundary between the travel modes would move close to the suburb until all auto users switch to rail mode.

Within the no-toll area (i.e., $x \in (0, x_c)$), the rail cost curve may lie below or intersect with the auto cost curve. For the former case, all commuters residing between the CBD and the modal boundary $\bar{x}$ prefer to use rail mode. Thus, the total number of rail passengers N^{r} can be calculated by

$$N^{\mathrm{r}} = \int_0^{\bar{x}} n(x) dx . \tag{C.12}$$

Taking the derivative of Equation (C.12) with respect to τ yields

$$\frac{dN^{\mathrm{r}}}{d\tau} = \int_0^{\bar{x}} \frac{\partial n}{\partial u} \frac{du}{d\tau} dx + \int_0^{\bar{x}} \frac{\partial n}{\partial \tau} dx + n(\bar{x}) \frac{d\bar{x}}{d\tau} . \tag{C.13}$$

The second term on the right-hand side of Equation (C.13) is equal to zero according to Equation (C.1), since there is no auto user within the region $(0, \bar{x})$. Accordingly, applying $\partial n / \partial u < 0$, $du / d\tau \le 0$ and $d\bar{x} / d\tau > 0$ to Equation (C.13), one immediately derives $dN^{\mathrm{r}} / d\tau \ge 0$. Inversely, we have $dN^{\mathrm{a}} / d\tau \le 0$ since $N^{\mathrm{a}} = N - N^{\mathrm{r}}$, which implies that an increase in the toll level would make partial auto users switch to rail mode.

For the latter case, we denote $\bar{x}_0 \in (0, x_c)$ as the intersection between auto and rail cost curves at no-toll area. The commuters residing within the regions $(0, \bar{x}_0)$ and $(x_c, \bar{x})$ would choose to use railway. Thus, the total number of rail passengers N^{r} can be calculated by

$$N^{\mathrm{r}} = \int_0^{\bar{x}_0} n(x) dx + \int_{x_c}^{\bar{x}} n(x) dx . \tag{C.14}$$

Taking the derivative of Equation (C.14) with respect to τ produces

$$\frac{dN^{\mathrm{r}}}{d\tau} = \int_0^{\bar{x}_0} \frac{\partial n}{\partial u} \frac{du}{d\tau} dx + \int_{x_c}^{\bar{x}} \frac{\partial n}{\partial u} \frac{du}{d\tau} dx + n(\bar{x}) \frac{d\bar{x}}{d\tau} \ge 0 . \tag{C.15}$$

Similarly, we have $dN^{\mathrm{a}} / d\tau \le 0$. Based on the above analysis, $dN^{\mathrm{r}} / d\tau \ge 0$ and $dN^{\mathrm{a}} / d\tau \le 0$ hold for both cases. This completes the proof of proposition 3.

Table 1. Change of the urban system variables with level of auto travel time uncertainty ρ and punctual arrival requirements λ_1 and λ_2.

	u	B	$\bar{x}$	$r(0)$	$n(0)$	N^{r}	N^{a}
ρ	-	-	+	+	+	+	-
λ_1	-	-	+	+	+	+	-
λ_2	-	-	+	+	+	+	-

Note: $r(0)$ and $n(0)$ respectively represent the land rent and residential density at the CBD; "+" and "-" respectively indicate positive and negative correlations.

Table 2. Change of the urban system variables with toll level τ.

	u	B	$\bar{x}$	$r(0)$	$n(0)$	N^{r}	N^{a}
τ	-	-	+	+	+	+	-

Table 3. Input parameters for the numerical illustration.

Parameters	Baseline value
N (total number of households in the city)	400,000
Y (annual household income, RMB/household/year)	100,000
α (housing parameter in the quasi-linear utility function)	20,000
r_a (agricultural rent at the city boundary, RMB/km^2)	40,000,000
K (capacity of the linear corridor, veh/h)	15,000
v_0^{a} (fixed component of monetary auto travel cost, RMB)	15
v^{a} (variable component of monetary auto travel cost, RMB/veh-km)	0.1
t_0^{a} (free-flow travel time per unit of distance on highway, h/km)	0.01
μ_1, μ_2 (parameters in the BPR function)	0.15 and 4
δ (value of travel time, RMB /h)	40
ϖ (average annual number of trips to the city center per household)	365
η (the average daily number of trips to the CBD per household)	1.0
υ (peak-hour factor)	10%
a_0^{r} (average access/egress time to the railway station, h)	0.2
t_0^{r} (average travel time per unit distance by railway, h/km)	0.04
f_0^{r} (fixed component of the rail fare, RMB)	3
f^{r} (variable component of the rail fare, RMB/km)	0.2
λ_1 and λ_2 (punctual arrival requirement in the morning and evening peak)	1.29 and 0
b and θ (parameters in housing production function)	0.5 and 0.015
k (annual interest rate)	5%

Note: The data is mainly based on Li et al. (2013) and Liu et al. (2009). ‘RMB’ stands for Chinese currency and US$1.0 approximates RMB 6.70 on 1 October 2016.

Table 4. Performances of urban system before and after the optimal cordon toll pricing with different levels of travel time uncertainty.

Performance index	$\rho = 0$		$\rho = 0.6$		$\rho = 1.2$	
	Before	After	Before	After	Before	After
Toll level (RMB)	**-**	**13.7**	**-**	**7.1**	**-**	**1.8**
Cordon location (km)	**-**	**3.4**	**-**	**6.6**	**-**	**11.2**
Urban length (km)	38.8	36.8	33.2	31.6	27.5	26.9
Modal boundary between the travel modes (km)	5.7	9.4	8.2	10.4	11.4	12.3
Percentage of auto users (%)	66.5	47.2	48.3	36.9	27.2	23.5
Percentage of rail users (%)	33.5	52.8	51.7	63.1	72.8	76.5
Average residential density (households/km^2)	10,323	10,884	12,063	12,666	14,546	14,864
Average housing space per household (m^2/household)	63.4	61.0	57.8	56.0	51.9	51.2
Average housing price (RMB/m^2)	315.6	328.0	346.0	357.4	385.6	390.7
Average land value (million RMB/km^2)	103.2	108.8	120.6	126.7	145.5	148.6
Average capital investment intensity (billion RMB/km^2)	2.06	2.18	2.41	2.53	2.91	2.97
Annual total buffer time cost (10^6 h)	**0**	**0**	**25.6**	**18.9**	**26.4**	**23.0**
Household utility level (10^3 RMB)	**144.0**	**142.5**	**141.9**	**141.1**	**140.2**	**140.0**
Annual social welfare (billion RMB)	**60.03**	**60.46**	**59.43**	**59.57**	**58.96**	**58.97**

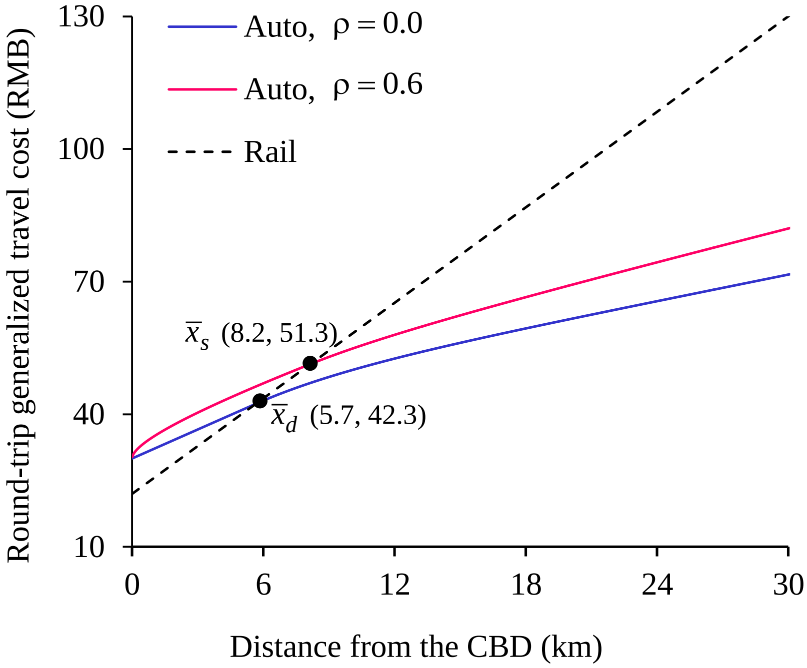


Figure 1. Generalized travel costs by rail and auto under deterministic case (i.e., $\rho=0.0$) and stochastic case (i.e., $\rho=0.6$).

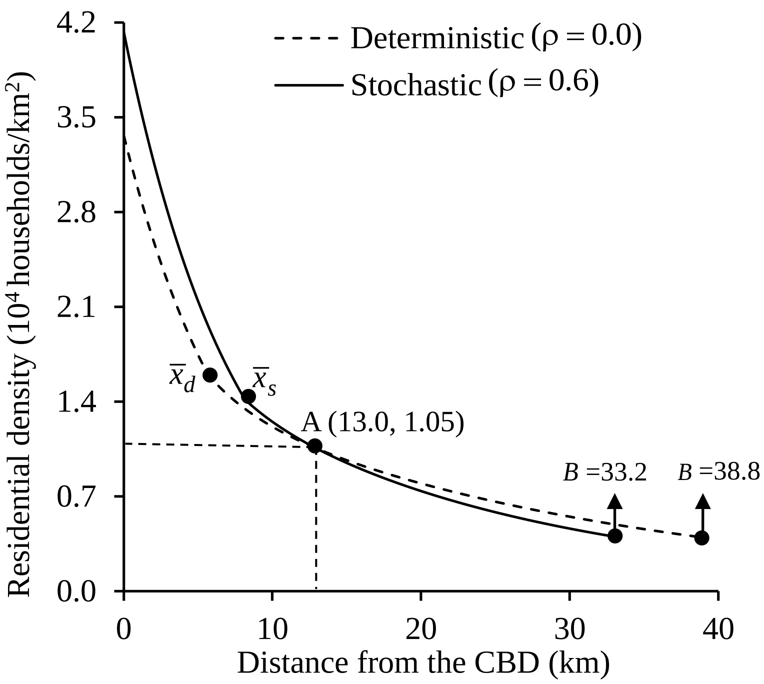
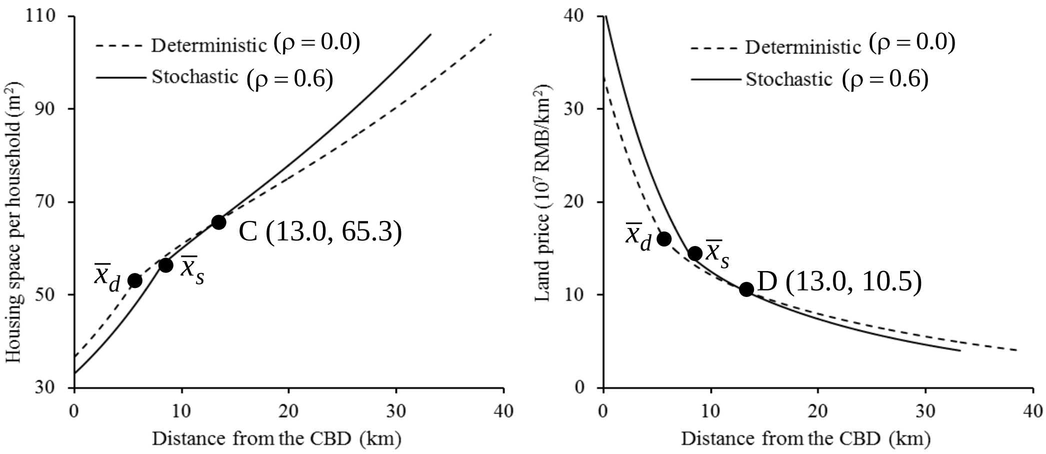


Figure 2. Comparison of residential distributions under deterministic case (i.e., $\rho=0.0$) and stochastic case (i.e., $\rho=0.6$).



(a) (b)

Figure 3. (a) Housing space per household and (b) land price under deterministic case (i.e., ρ=0.0) and stochastic case (i.e., ρ=0.6).

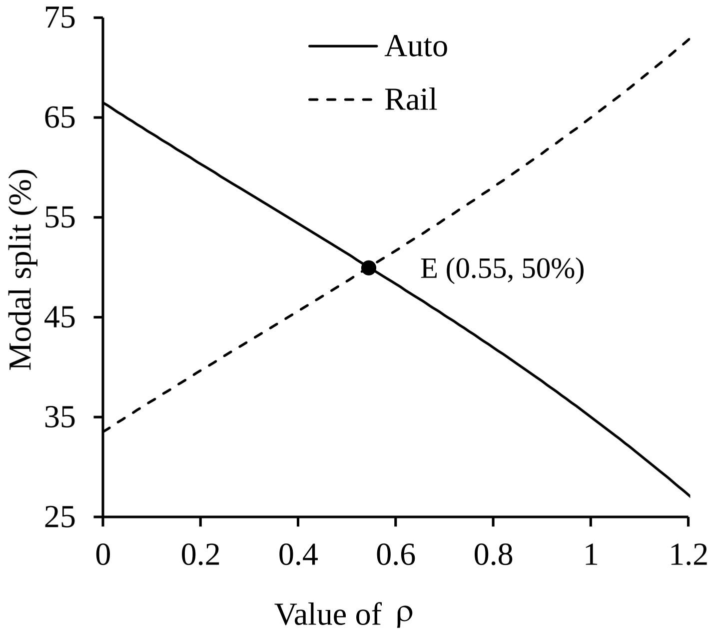


Figure 4. Change of modal split with the level of auto travel time uncertainty.

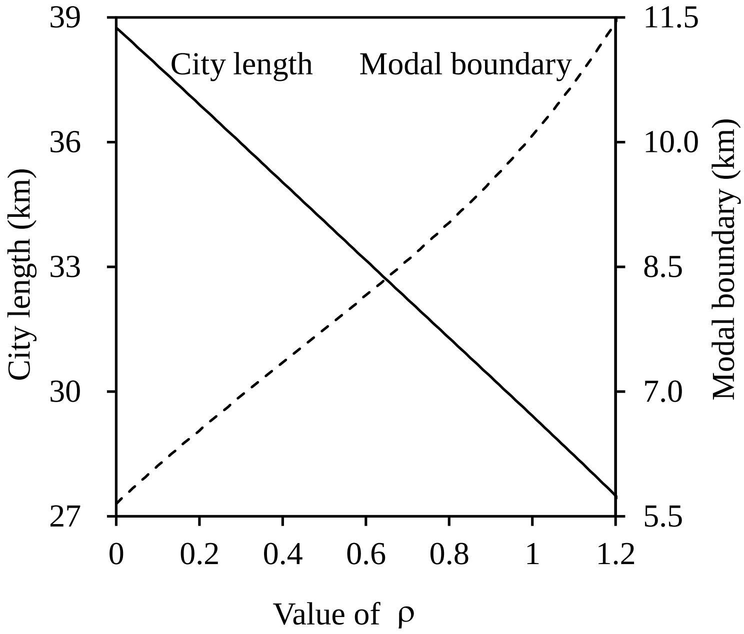


Figure 5. Change of city length and modal boundary with the level of auto travel time uncertainty.

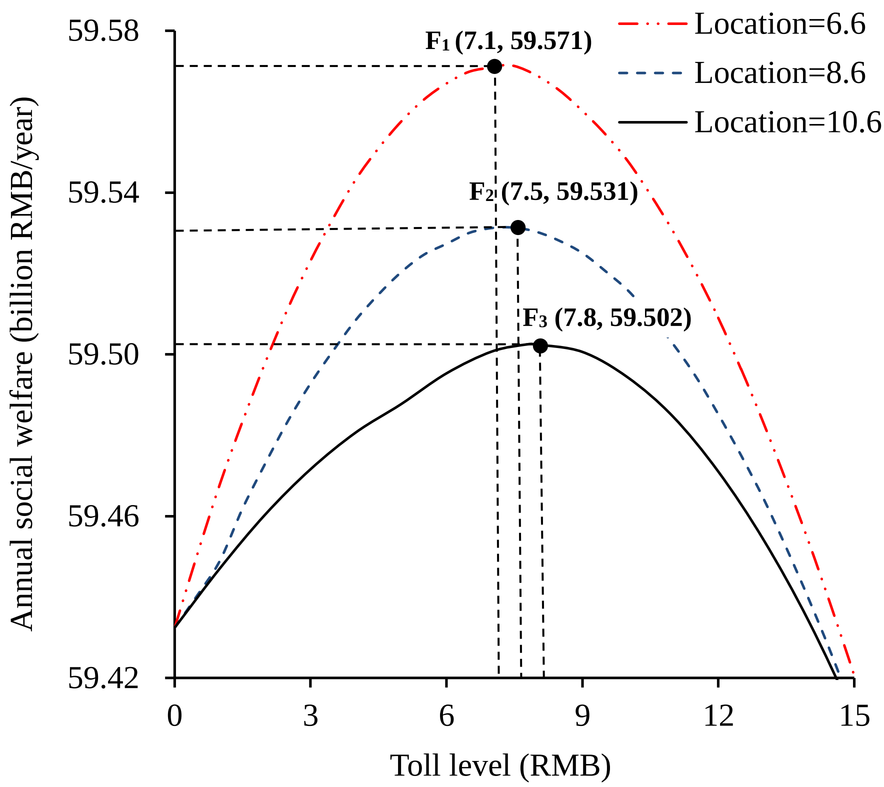


Figure 6. Variation in the total social welfare with change in the toll level and location.

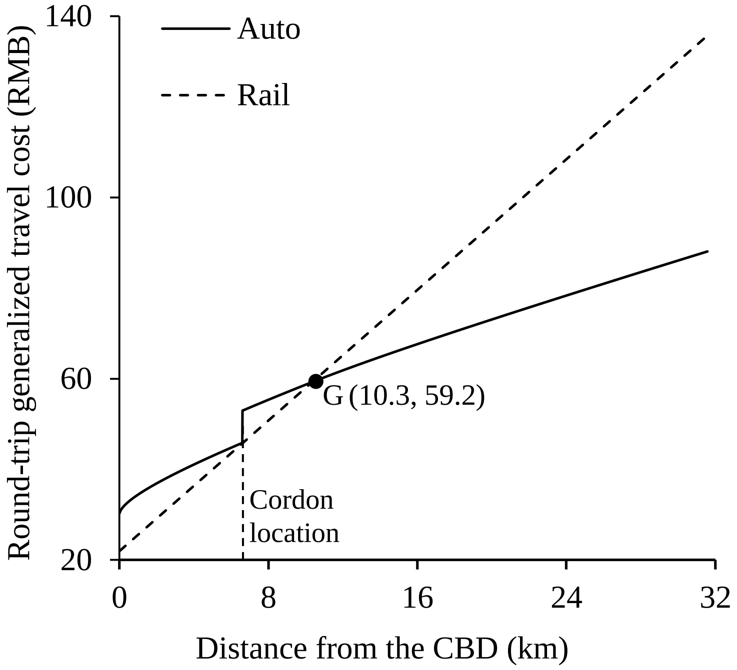


Figure 7. Generalized travel costs by rail and auto under the optimal cordon toll pricing scheme.

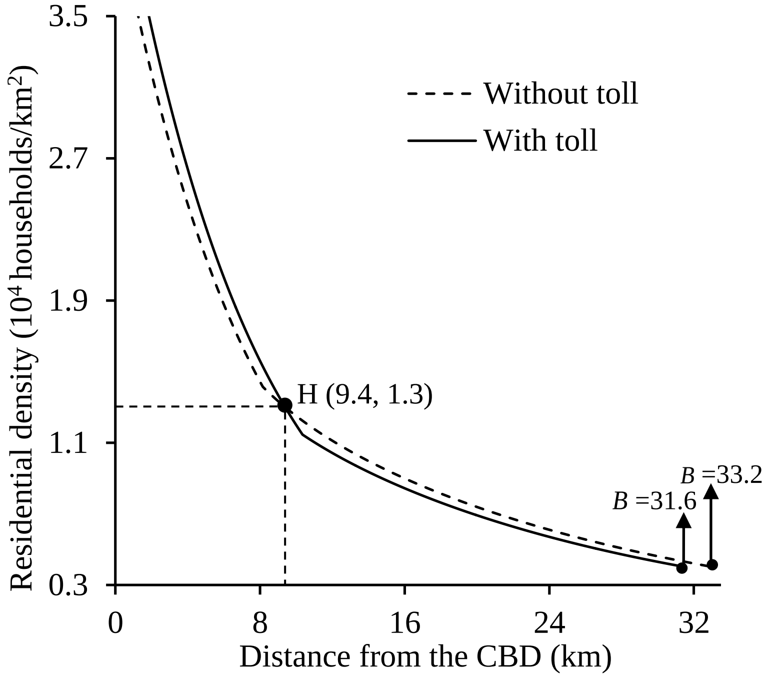


Figure 8. Comparison of residential distributions with and without the optimal cordon toll pricing scheme.

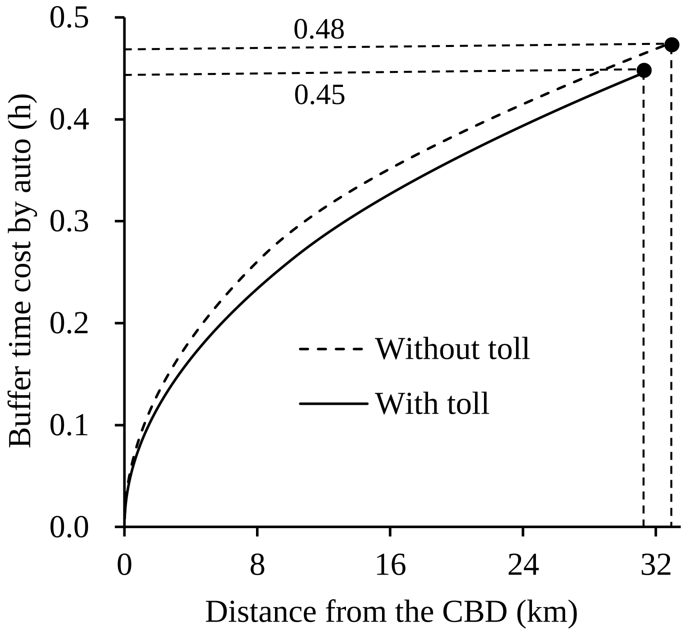


Figure 9. Total buffer time costs for the two-way commuting trip by auto with and without the optimal cordon toll pricing scheme.

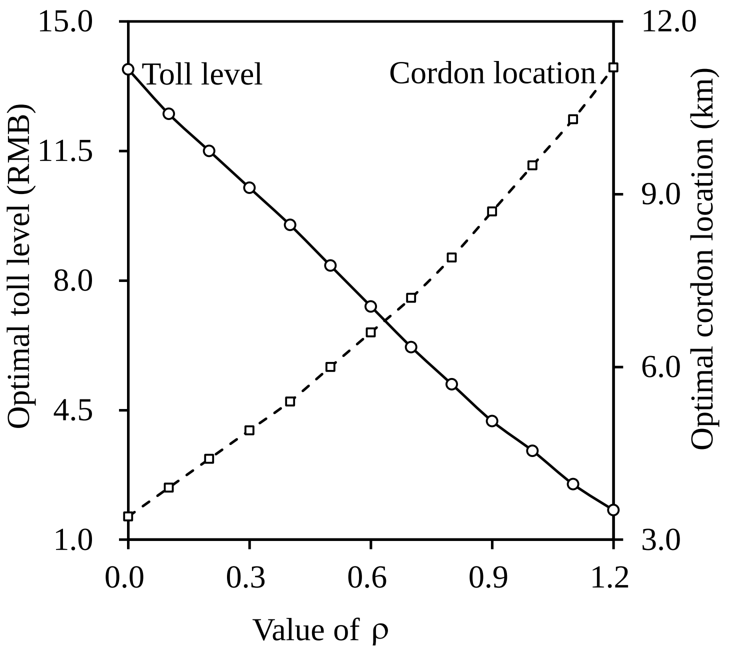


Figure 10. Change of the optimal cordon location and toll level with the level of auto travel time uncertainty.