Linear Quadratic Optimal Control Problems for Mean-Field Backward Stochastic Differential Equations

Xun Li*, Jingrui Sun†, and Jie Xiong‡

Abstract: This paper is concerned with linear quadratic optimal control problems for mean-field backward stochastic differential equations (MF-BSDEs, for short) with deterministic coefficients. The optimality system, which is a linear mean-field forward-backward stochastic differential equation with constraint, is obtained by a variational method. By decoupling the optimality system, two coupled Riccati equations and an MF-BSDE are derived. It turns out that the coupled two Riccati equations are uniquely solvable. Then a complete and explicit representation is obtained for the optimal control.

Key words: linear quadratic optimal control, mean-field backward stochastic differential equation, Riccati equation, optimality system, decoupling

AMS subject classifications. 49N10, 49N35, 93E20

1 Introduction

The mean-field type stochastic control problem is important in various fields such as science, engineering, economics, management, and particularly in financial investment. The theory of mean-field forward stochastic differential equations (MF-FSDEs, for short) can be traced back to Kac [21] who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. Since then, research on related topics and their applications has become a notable and serious endeavor among researchers in applied probability and optimal stochastic control, particularly in financial engineering. Typical representatives include, but not limited to, McKean [27], Dawson [15], Chan [12], Buckdahn–Djehiche–Li–Peng [7], Buckdahn–Li–Peng [8], Borkar–Kumar [5], Crisan–Xiong [14], Andersson–Djehiche [2], Buckdahn–Djehiche–Li [6], Meyer-Brandis–Oksendal–Zhou [28], Yong [35, 36], Huang–Li–Yong [19], Li–Sun–Yong [23], Pham [29], and Sun [30]. In particular, Lasry–Lions [22] introduced mean-field games, derived their important strategies and mentioned many

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China (malixun@polyu.edu.hk). This author was partially supported by Hong Kong RGC under grants 15209614, 15224215 and 15255416.

†Corresponding author, Department of Mathematics, National University of Singapore, 119076, Republic of Singapore (sjr@mail.ustc.edu.cn). This author was partially supported by the National Natural Science Foundation of China (11401556) and the Fundamental Research Funds for the Central Universities (WK 2040000012).

‡Department of Mathematics, University of Macau, Macau, China (jiexiong@umac.mo); and Department of Mathematics, South University of Science and Technology of China, Shenzhen, China. This author acknowledges the financial support from FDCT 025/2016/A1.

open yet interesting problems. Huang–Caines–Malhamé [17, 18] and Huang–Malhamé–Caines [20] studied large population stochastic dynamic mean-field games. In addition, Carmona–Delarue [9, 10] and Carmona–Delarue–Lachapelle [11] developed probabilistic theory of mean-field including *mean-field forward-backward stochastic differential equations* (MF-FBSDEs, for short), control and games. Bensoussan–Sung–Yam [3], Bensoussan–Sung–Yam–Yung [4], and Graber [16] analyzed mean-field linear-quadratic control problems and games with their strategies. The MF-FSDEs can be treated in a forward-looking way by starting with the initial state. In financial investment, however, one frequently encounters financial investment problems with future conditions (as random variables) specified. This naturally results in a *mean-field backward stochastic differential equation* (MF-BSDE, for short) with a given terminal condition (see Buckdahn–Djehiche–Li–Peng [7] and Buckdahn–Li–Peng [8]). This is an important and challenging research topic. Recently there has been increasing interest in studying this type of stochastic control problems as well as their applications. The optimal stochastic control problems under MF-BSDEs are underdeveloped in the literature, and therefore many fundamental questions remain open and methodologies need to be significantly improved.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following controlled linear MF-BSDE:

$$\begin{cases} dY(s) = \Big\{A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] \\ \qquad\qquad + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)]\Big\}ds + Z(s)dW(s), \quad s \in [t, T], \\ Y(T) = \xi, \end{cases} \tag{1.1}$$

where $A(\cdot)$, $\bar{A}(\cdot)$, $B(\cdot)$, $\bar{B}(\cdot)$, $C(\cdot)$, $\bar{C}(\cdot)$, $D(\cdot)$, $\bar{D}(\cdot)$ are given deterministic matrix-valued functions; ξ is an $\mathcal{F}_T$-measurable random vector; and $u(\cdot)$ is the *control process*. The class of *admissible controls* for (1.1) is

$$\mathcal{U}[t, T] = \Big\{u : [t, T] \times \Omega \to \mathbb{R}^m \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E}\int_t^T |u(s)|^2 ds < \infty\Big\}.$$

Under some mild conditions on the coefficients of equation (1.1), for any terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ (the set of all $\mathcal{F}_T$-measurable, square-integrable $\mathbb{R}^n$-valued processes) and any admissible control $u(\cdot) \in \mathcal{U}[t, T]$, equation (1.1) admits a unique square-integrable adapted solution $(Y(\cdot), Z(\cdot)) \equiv (Y(\cdot\,; \xi, u(\cdot)), Z(\cdot\,; \xi, u(\cdot)))$, which is called the *state process* corresponding to ξ and $u(\cdot)$. Now we introduce the following cost functional:

$$\begin{aligned} J(t, \xi; u(\cdot)) \triangleq \mathbb{E}\Big\{ & \langle GY(t), Y(t)\rangle + \big\langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle \\ & + \int_t^T \Big[\langle Q(s)Y(s), Y(s)\rangle + \big\langle \bar{Q}(s)\mathbb{E}[Y(s)], \mathbb{E}[Y(s)]\big\rangle \\ & + \langle R(s)Z(s), Z(s)\rangle + \big\langle \bar{R}(s)\mathbb{E}[Z(s)], \mathbb{E}[Z(s)]\big\rangle \\ & + \langle N(s)u(s), u(s)\rangle + \big\langle \bar{N}(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)]\big\rangle\Big] ds\Big\}, \end{aligned} \tag{1.2}$$

where G, $\bar{G}$ are symmetric matrices and $Q(\cdot)$, $\bar{Q}(\cdot)$, $R(\cdot)$, $\bar{R}(\cdot)$, $N(\cdot)$, and $\bar{N}(\cdot)$ are deterministic, symmetric matrix-valued functions. Our mean-field backward stochastic linear quadratic (LQ, for short) optimal control problem can be stated as follows.

Problem (MF-BSLQ). For any given $t \in [0,T)$ and terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, find a $u^*(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,\xi;u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,\xi;u(\cdot)) \triangleq V(t,\xi). \tag{1.3}$$

Any $u^*(\cdot) \in \mathcal{U}[t,T]$ satisfying (1.3) is called an *optimal control* of Problem (MF-BSLQ) for the terminal state ξ, the corresponding $(Y^*(\cdot),Z^*(\cdot)) \equiv (Y(\cdot\,;\xi,u^*(\cdot)),Z(\cdot\,;\xi,u^*(\cdot)))$ is called an *optimal state process*, and the three-tuple $(Y^*(\cdot),Z^*(\cdot),u^*(\cdot))$ is called an *optimal triple*. The function $V(\cdot\,,\cdot)$ is called the *value function* of Problem (MF-BSLQ). Note that when the mean-field part is absent, Problem (MF-BSLQ) is reduced to a stochastic LQ optimal control of backward stochastic differential equations (see Lim–Zhou [24] for some relevant results). For LQ optimal control problems of forward stochastic differential equations, the interested reader is referred to, for example, [34, 13, 1, 33, 31] and the book of Yong–Zhou [37].

In this paper, we shall construct the optimal control and provide a representation of the value function for Problem (MF-BSLQ). The main idea can be described as follows. We first show that under certain conditions the cost functional is strictly convex and coercive with respect to the control variable. So from the basic theorem in convex analysis we conclude the uniqueness and existence of an optimal control. By a variational method, the optimal control is then characterized in terms of the optimality system, which is a coupled mean-field type forward-backward stochastic differential equation, together with a stationarity condition. In order to obtain an explicit representation of the optimal control, we use a decoupling technique inspired by the four-step scheme introduced in [25, 26] for general FBSDEs to solve the optimality system. This leads to a pair of coupled Riccati equations. By considering their connection with forward mean-field LQ problems, we further establish the unique solvability of the Riccati equations. The optimal control is thus constructed and an explicit formula for the value function can be developed by the method of completing the squares.

The rest of the paper is organized as follows. Section 2 gives some preliminaries. Among other things, we show Problem (MF-BSLQ) is uniquely solvable by using the basic theory of convex analysis. In Section 3, we derive the optimality system by a variational method and the coupled two Riccati equations by a decoupling technique. Section 4 is devoted to the uniqueness and existence of solutions to the Riccati equations. In Section 5, we present explicit formulas of the optimal control and the value function.

2 Preliminaries

Throughout this paper, $\mathbb{R}^{n\times m}$ is the Euclidean space of all $n\times m$ real matrices, $\mathbb{S}^n$ is the space of all symmetric $n\times n$ real matrices, $\mathbb{S}^n_+$ is the subset of $\mathbb{S}^n$ consisting of positive definite matrices, and $\overline{\mathbb{S}^n_+}$ is the closure of $\mathbb{S}^n_+$ in $\mathbb{R}^{n\times n}$. When $m=1$, we simply write $\mathbb{R}^{n\times m}$ as $\mathbb{R}^n$, and when $n=m=1$, we drop the superscript. Recall that the inner product $\langle\cdot\,,\cdot\rangle$ on $\mathbb{R}^{n\times m}$ is given by $\langle M,N\rangle \mapsto \mathrm{tr}\,(M^\top N)$, where the superscript $\top$ denotes the transpose of matrices and $\mathrm{tr}\,(K)$ denotes the trace of a matrix K, and that the induced norm on $\mathbb{R}^{n\times m}$ is given by $|M| = \sqrt{\mathrm{tr}\,(M^\top M)}$. If no confusion is likely, we shall use $\langle\cdot\,,\cdot\rangle$ for inner products in possibly different Hilbert spaces, and denote by $|\cdot|$ the norm induced by $\langle\cdot\,,\cdot\rangle$. Let $t\in[0,T)$ and $\mathbb{H}$ be a given Euclidean space. The space of $\mathbb{H}$-valued continuous functions on $[t,T]$ is denoted by $C([t,T];\mathbb{H})$, and the space of $\mathbb{H}$-valued, pth $(1\leqslant p\leqslant\infty)$ power Lebesgue integrable functions on $[t,T]$ is denoted by $L^p(t,T;\mathbb{H})$. Further, we introduce the following spaces of random variables and stochastic processes:

$$
\begin{aligned}
L^2_{\mathcal{F}_T}(\Omega;\mathbb{H}) &= \Big\{\xi:\Omega\to\mathbb{H}\;\big|\;\xi \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}|\xi|^2<\infty\Big\},\\
L^2_{\mathbb{F}}(t,T;\mathbb{H}) &= \Big\{\varphi:[t,T]\times\Omega\to\mathbb{H}\;\big|\;\varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,}\\
&\qquad\qquad \mathbb{E}\int_t^T|\varphi(s)|^2ds<\infty\Big\},\\
L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H})) &= \Big\{\varphi:[t,T]\times\Omega\to\mathbb{H}\;\big|\;\varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous,}\\
&\qquad\qquad \mathbb{E}\Big[\sup_{t\leqslant s\leqslant T}|\varphi(s)|^2\Big]<\infty\Big\}.
\end{aligned}
$$

Next we introduce the following assumptions that will be in force throughout this paper.

(H1) The coefficients of the state equation satisfy the following:

$$A(\cdot),\bar{A}(\cdot)\in L^1(0,T;\mathbb{R}^{n\times n}),\quad B(\cdot),\bar{B}(\cdot)\in L^2(0,T;\mathbb{R}^{n\times m}),\quad C(\cdot),\bar{C}(\cdot)\in L^2(0,T;\mathbb{R}^{n\times n}).$$

(H2) The weighting coefficients in the cost functional satisfy

$$\begin{cases} G,\bar{G}\in\mathbb{S}^n, & Q(\cdot),\bar{Q}(\cdot)\in L^1(0,T;\mathbb{S}^n),\\ R(\cdot),\bar{R}(\cdot)\in L^\infty(0,T;\mathbb{S}^n), & N(\cdot),\bar{N}(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\end{cases}$$

and there exists a constant $\delta>0$ such that for a.e. $s\in[0,T]$,

$$\begin{cases} G,\;G+\bar{G}\geqslant 0, & Q(s),\;Q(s)+\bar{Q}(s)\geqslant 0,\\ R(s),\;R(s)+\bar{R}(s)\geqslant 0, & N(s),\;N(s)+\bar{N}(s)\geqslant \delta I.\end{cases}$$

Now we present a result concerning the well-posedness of the state equation (1.1).

Theorem 2.1. *Let* (H1) *hold. Then for any* $(\xi,u(\cdot))\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)\times\mathcal{U}[t,T]$, *MF-BSDE* (1.1) *admits a unique adapted solution* $(Y(\cdot),Z(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$. *Moreover, there exists a constant* $K>0$, *independent of* ξ *and* $u(\cdot)$, *such that*

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T}|Y(s)|^2+\int_t^T|Z(s)|^2ds\right]\leqslant K\mathbb{E}\left[|\xi|^2+\int_t^T|u(s)|^2ds\right]. \tag{2.1}$$

Note that (H1) allows the coefficients $A(\cdot)$ and $C(\cdot)$ to be unbounded, which is a little different from the standard case [7, 8]. However, the proof of Theorem 2.1 is similar to that of the case without mean-field. We only present a short proof here and refer the interested reader to Sun–Yong [32, Proposition 2.1] for more details.

Proof. To show the uniqueness, let $(Y_1(\cdot),Z_1(\cdot)),(Y_2(\cdot),Z_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ be two adapted solutions to (1.1). Then $(Y_0(\cdot),Z_0(\cdot))\triangleq(Y_1(\cdot)-Y_2(\cdot),Z_1(\cdot)-Z_2(\cdot))$ satisfies

$$\begin{cases} dY_0=\big\{AY_0+\bar{A}\mathbb{E}[Y_0]+CZ_0+\bar{C}\mathbb{E}[Z_0]\big\}ds+Z_0dW, & s\in[t,T],\\ Y_0(T)=0.\end{cases}$$

In the above we have suppressed the time variable s and will do so frequently in the sequel to simplify notation. By Itô's formula, we have for any $r\in[t,T]$,

$$|Y_0(r)|^2=-\int_r^T\Big(2\big\langle Y_0,AY_0+\bar{A}\mathbb{E}[Y_0]+CZ_0+\bar{C}\mathbb{E}[Z_0]\big\rangle+|Z_0|^2\Big)ds-2\int_r^T\langle Y_0,Z_0\rangle dW(s).$$

Taking expectation and making use of the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}|Y_0(r)|^2+\mathbb{E}\int_r^T|Z_0(s)|^2ds \\
&\leqslant \mathbb{E}\int_r^T\Big[\big(2|A(s)|+2|C(s)|^2\big)|Y_0(s)|^2+\frac{1}{2}|Z_0(s)|^2\Big]ds \\
&\quad+\mathbb{E}\int_r^T\Big\{2|\bar{A}(s)||Y_0(s)||\mathbb{E}[Y_0(s)]|+4|\bar{C}(s)|^2|Y_0(s)|^2+\frac{1}{4}|\mathbb{E}[Z_0(s)]|^2\Big\}ds \\
&\leqslant \int_r^T\Big\{\Big[2|A(s)|+2|\bar{A}(s)|+2|C(s)|^2+4|\bar{C}(s)|^2\Big]\mathbb{E}|Y_0(s)|^2+\frac{3}{4}\mathbb{E}|Z_0(s)|^2\Big\}ds,
\end{aligned}
$$

from which follows

$$
\mathbb{E}|Y_0(r)|^2+\frac{1}{4}\mathbb{E}\int_r^T|Z_0(s)|^2ds\leqslant\int_r^T\Big[2|A(s)|+2|\bar{A}(s)|+2|C(s)|^2+4|\bar{C}(s)|^2\Big]\mathbb{E}|Y_0(s)|^2ds.
$$

We now conclude by Gronwall's inequality that $\mathbb{E}|Y_0(r)|^2=0$ and hence $\mathbb{E}\int_r^T|Z_0(s)|^2ds=0$ for all $r\in[t,T]$. This proves the uniqueness.

Next we will use the contraction mapping theorem to prove the existence. For notational simplicity, we denote $f(s)=B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]$. For any $\beta\in\mathbb{R}$, we define $\mathcal{M}_\beta[t,T]$ to be the Banach space

$$
\mathcal{M}_\beta[t,T]=L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)
$$

equipped with the norm

$$
\|(Y,Z)\|_{\mathcal{M}_\beta[t,T]}\triangleq\Big\{\mathbb{E}\Big[\sup_{t\leqslant s\leqslant T}|Y(s)|^2e^{\beta h(s)}\Big]+\mathbb{E}\int_t^T|Z(s)|^2e^{\beta h(s)}ds\Big\}^{\frac{1}{2}},
$$

where

$$
h(s)=\int_t^s\Big[|A(r)|+|\bar{A}(r)|+|C(r)|^2+|\bar{C}(r)|^2\Big]dr,\quad s\in[t,T].
$$

Since all the norms $\|\cdot\|_{\mathcal{M}_\beta[t,T]}$ with different β are equivalent, we denote $\mathcal{M}_\beta[t,T]$ simply by $\mathcal{M}[t,T]$. Now according to the theory of classical BSDE (see, for example, Sun-Yong [32, Proposition 2.1] and Yong-Zhou [37, Chapter 7]), for any $(y(\cdot),z(\cdot))\in\mathcal{M}[t,T]$, the BSDE

$$
\begin{cases}dY(s)=\{Ay+\bar{A}\mathbb{E}[y]+Cz+\bar{C}\mathbb{E}[z]+f\}ds+ZdW(s),\quad s\in[t,T],\\ Y(T)=\xi\end{cases}\tag{2.2}
$$

admits a unique adapted solution $(Y(\cdot),Z(\cdot))\in\mathcal{M}[t,T]$, and the following estimate holds:

$$
\mathbb{E}\Big[\sup_{t\leqslant s\leqslant T}|Y(s)|^2+\int_t^T|Z(s)|^2ds\Big]\leqslant K'\mathbb{E}\Big[|\xi|^2+\Big(\int_t^T|g(s)|ds\Big)^2\Big],\tag{2.3}
$$

where $g=Ay+\bar{A}\mathbb{E}[y]+Cz+\bar{C}\mathbb{E}[z]+f$ and $K'>0$ is constant independent of ξ and g. So we can define an operator $\mathcal{T}$ from $\mathcal{M}[t,T]$ to itself by $\mathcal{T}(y,z)=(Y,Z)$ via the BSDE (2.2). Once we find a $\beta>0$ such that

$$
\|\mathcal{T}(y,z)-\mathcal{T}(\bar{y},\bar{z})\|_{\mathcal{M}_\beta[t,T]}\leqslant\frac{1}{2}\|(y,z)-(\bar{y},\bar{z})\|_{\mathcal{M}_\beta[t,T]},\quad\forall(y,z),(\bar{y},\bar{z})\in\mathcal{M}[t,T],\tag{2.4}
$$

the existence will follow immediately from the contraction mapping theorem. To this end, take any $(y_1(\cdot), z_1(\cdot)), (y_2(\cdot), z_2(\cdot)) \in \mathcal{M}[t,T]$, and let

$$(y(\cdot), z(\cdot)) = (y_1(\cdot) - y_2(\cdot), z_1(\cdot) - z_2(\cdot)), \quad (Y(\cdot), Z(\cdot)) = \mathcal{T}(y_1, z_1) - \mathcal{T}(y_2, z_2).$$

By applying Itô's formula to $s \mapsto |Y(s)|^2 e^{\beta h(s)}$ and using the Cauchy-Schwarz inequality, we obtain for any $r \in [t,T]$,

$$\begin{aligned}
&|Y(r)|^2 e^{\beta h(r)} + \int_r^T |Z|^2 e^{\beta h} ds + 2\int_r^T e^{\beta h} \langle Y, Z\rangle dW(s) \\
&= -\int_r^T e^{\beta h}\Big\{\beta h'|Y|^2 + 2\big\langle Y, Ay + \bar{A}\mathbb{E}[y] + Cz + \bar{C}\mathbb{E}[z]\big\rangle\Big\} ds \\
&\leqslant \int_r^T e^{\beta h}\Big\{\Big[-\beta h' + \beta\Big(|A| + |\bar{A}| + |C|^2 + |\bar{C}|^2\Big)\Big]|Y|^2 \\
&\quad + \beta^{-1}\Big(|A||y|^2 + |\bar{A}|\mathbb{E}|y|^2 + |z|^2 + \mathbb{E}|z|^2\Big)\Big\} ds \\
&= \beta^{-1}\int_r^T e^{\beta h}\Big(|A||y|^2 + |\bar{A}|\mathbb{E}|y|^2 + |z|^2 + \mathbb{E}|z|^2\Big) ds \\
&\leqslant \alpha\Big\{\sup_{t\leqslant s\leqslant T}\Big[|y(s)|^2 e^{\beta h(s)}\Big] + \sup_{t\leqslant s\leqslant T}\mathbb{E}\Big[|y(s)|^2 e^{\beta h(s)}\Big] + \int_r^T |z|^2 e^{\beta h} ds + \mathbb{E}\int_r^T |z|^2 e^{\beta h} ds\Big\},
\end{aligned} \tag{2.5}$$

where

$$\alpha = \beta^{-1}\left[\int_r^T \Big(|A(s)| + |\bar{A}(s)|\Big) ds + 1\right].$$

Taking expectation, one obtains

$$\mathbb{E}\left\{|Y(r)|^2 e^{\beta h(r)} + \int_r^T |Z|^2 e^{\beta h} ds\right\} \leqslant 2\alpha\|(y,z)\|^2_{\mathcal{M}_\beta[t,T]}. \tag{2.6}$$

On the other hand, by the Burkholder-Davis-Gundy inequalities and the Cauchy-Schwarz inequality, there exists a constant $K > 0$ such that

$$\begin{aligned}
\mathbb{E}\left\{\sup_{t\leqslant r\leqslant T}\left|\int_r^T e^{\beta h}\langle Y, Z\rangle dW(s)\right|\right\} &\leqslant K\mathbb{E}\left\{\int_t^T e^{2\beta h}|Y|^2|Z|^2 ds\right\}^{1/2} \\
&\leqslant K\mathbb{E}\left\{\left[\sup_{t\leqslant s\leqslant T}|Y(s)|^2 e^{\beta h(s)}\right]^{1/2}\left[\int_t^T |Z|^2 e^{\beta h} ds\right]^{1/2}\right\} \\
&\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{t\leqslant s\leqslant T}|Y(s)|^2 e^{\beta h(s)}\right] + K^2\mathbb{E}\left[\int_t^T |Z|^2 e^{\beta h} ds\right].
\end{aligned} \tag{2.7}$$

Hereafter, we shall use $K > 0$ to represent a generic constant which can be different from line to line. Now combining (2.5), (2.7), and (2.6) we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leqslant r\leqslant T}|Y(r)|^2 e^{\beta h(r)}\right] &\leqslant 2\alpha\|(y,z)\|^2_{\mathcal{M}_\beta[t,T]} + 2\mathbb{E}\left\{\sup_{t\leqslant r\leqslant T}\left|\int_r^T e^{\beta h}\langle Y, Z\rangle dW(s)\right|\right\} \\
&\leqslant \alpha K\|(y,z)\|^2_{\mathcal{M}_\beta[t,T]} + \frac{1}{2}\mathbb{E}\left[\sup_{t\leqslant r\leqslant T}|Y(r)|^2 e^{\beta h(r)}\right].
\end{aligned}$$

This, together with (2.6), in turn yields

$$\|(Y,Z)\|^2_{\mathcal{M}_\beta[t,T]} \leqslant \beta^{-1}K\left[\int_t^T \Big(|A| + |\bar{A}|\Big) ds + 1\right]\|(y,z)\|^2_{\mathcal{M}_\beta[t,T]}.$$

Thus we can choose β sufficiently large so that (2.4) holds, and the existence therefore follows.

Finally, to prove the estimate (2.1), we let $(Y(\cdot), Z(\cdot))$ be the adapted solution to the MF-BSDE (1.1) so that $(Y, Z) = \mathcal{T}(Y, Z)$ and let $(Y_0(\cdot), Z_0(\cdot))$ be the adapted solution to (2.2) with respect to $(y(\cdot), z(\cdot)) = (0, 0)$ so that $(Y_0, Z_0) = \mathcal{T}(0, 0)$. By (2.4) and (2.3),

$$\begin{aligned}
\|(Y,Z)\|_{\mathcal{M}_\beta[t,T]} &\leqslant \|(Y,Z)-(Y_0,Z_0)\|_{\mathcal{M}_\beta[t,T]} + \|(Y_0,Z_0)\|_{\mathcal{M}_\beta[t,T]} \\
&\leqslant \frac{1}{2}\|(Y,Z)\|_{\mathcal{M}_\beta[t,T]} + K\left[\mathbb{E}|\xi|^2 + \mathbb{E}\left(\int_t^T |f(s)|ds\right)^2\right]^{1/2} \\
&\leqslant \frac{1}{2}\|(Y,Z)\|_{\mathcal{M}_\beta[t,T]} + K\left[\mathbb{E}|\xi|^2 + \mathbb{E}\int_t^T |u(s)|^2 ds\right]^{1/2},
\end{aligned}$$

and the estimate (2.1) follows readily. □

From Theorem 2.1, one can easily see that under (H1)–(H2), the quadratic cost functional $J(t, \xi; u(\cdot))$ is well-defined (i.e., finite) for all $(t, \xi) \in [0, T] \times L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t, T]$. Thus, Problem (MF-BSLQ) makes sense. The following result tells us that under (H1)–(H2), Problem (MF-BSLQ) is actually uniquely solvable for any terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

Theorem 2.2. *Let* (H1)–(H2) *hold. Then for any terminal state* $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$*, Problem* (MF-BSLQ) *admits a unique optimal control.*

Proof. Let $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ be given. For a control $u(\cdot) \in \mathcal{U}[t, T]$, we denote by $(Y^u(\cdot), Z^u(\cdot))$ the unique adapted solution to the state equation (1.1). In terms of $Y^u(\cdot)$ and $Z^u(\cdot)$, the cost functional (1.2) can be written as

$$\begin{aligned}
J(t,\xi;u(\cdot)) &= \mathbb{E}\langle G\{Y^u(t)-\mathbb{E}[Y^u(t)]\}, Y^u(t)-\mathbb{E}[Y^u(t)]\rangle + \big\langle (G+\bar{G})\mathbb{E}[Y^u(t)], \mathbb{E}[Y^u(t)]\big\rangle \\
&\quad + \mathbb{E}\int_t^T \Big[\langle Q\{Y^u-\mathbb{E}[Y^u]\}, Y^u-\mathbb{E}[Y^u]\rangle + \big\langle (Q+\bar{Q})\mathbb{E}[Y^u], \mathbb{E}[Y^u]\big\rangle \\
&\quad + \langle R\{Z^u-\mathbb{E}[Z^u]\}, Z^u-\mathbb{E}[Z^u]\rangle + \big\langle (R+\bar{R})\mathbb{E}[Z^u], \mathbb{E}[Z^u]\big\rangle \\
&\quad + \langle N\{u-\mathbb{E}[u]\}, u-\mathbb{E}[u]\rangle + \big\langle (N+\bar{N})\mathbb{E}[u], \mathbb{E}[u]\big\rangle\Big] ds\Big\}.
\end{aligned} \tag{2.8}$$

By the linearity of the differential equation in (1.1), we have for any $u(\cdot), v(\cdot) \in \mathcal{U}[t, T]$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$,

$$Y^{\alpha u+\beta v}(\cdot) = \alpha Y^u(\cdot) + \beta Y^v(\cdot), \quad Z^{\alpha u+\beta v}(\cdot) = \alpha Z^u(\cdot) + \beta Z^v(\cdot).$$

Recall that for any positive semidefinite matrix $M \in \mathbb{S}^k$, $x, y \in \mathbb{R}^k$, and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$,

$$\langle M(\alpha x + \beta y), \alpha x + \beta y\rangle \leqslant \alpha\langle Mx, x\rangle + \beta\langle My, y\rangle,$$

and the inequality is strict when M is positive semidefinite and $x \neq y$. Thus, by the assumption (H2) and (2.8), we have for any two different controls $u(\cdot), v(\cdot) \in \mathcal{U}[t, T]$ and any $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$,

$$J(t, \xi; \alpha u(\cdot) + \beta v(\cdot)) < \alpha J(t, \xi; u(\cdot)) + \beta J(t, \xi; v(\cdot)).$$

This shows the map $u(\cdot) \mapsto J(t, \xi; u(\cdot))$ is strictly convex. Further, we see from (2.8) that under the assumption (H2),

$$J(t,\xi;u(\cdot)) \geqslant \delta\mathbb{E}\int_t^T \Big[|u(s)-\mathbb{E}[u(s)]|^2 + |\mathbb{E}[u(s)]|^2\Big] ds = \delta\mathbb{E}\int_t^T |u(s)|^2 ds,$$

which implies the coercivity of $u(\cdot) \mapsto J(t,\xi;u(\cdot))$. Therefore, by the basic theorem in convex analysis, for any given $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, Problem (MF-BSLQ) has a unique optimal control. $\square$

3 Optimality system, decoupling, and Riccati equations

Let us first derive the optimality system for the optimal control of Problem (MF-BSLQ). For simplicity of notation, in what follows we shall often suppress the time variable s if no confusion can arise.

Theorem 3.1. *Let* (H1)–(H2) *hold. Let* $(Y^*(\cdot),Z^*(\cdot),u^*(\cdot))$ *be the optimal triple for the terminal state* $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. *Then the solution* $X^*(\cdot)$ *to the mean-field forward stochastic differential equation (MF-FSDE, for short)*

$$\begin{cases} dX^* = \Big\{-A^\top X^* - \bar{A}^\top\mathbb{E}[X^*] + QY^* + \bar{Q}\mathbb{E}[Y^*]\Big\}\,ds \\ \qquad\qquad + \Big\{-C^\top X^* - \bar{C}^\top\mathbb{E}[X^*] + RZ^* + \bar{R}\mathbb{E}[Z^*]\Big\}\,dW, \quad s\in[t,T], \\ X^*(t) = GY^*(t) + \bar{G}\mathbb{E}[Y^*(t)], \end{cases} \tag{3.1}$$

satisfies

$$Nu^* + \bar{N}\mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top\mathbb{E}[X^*] = 0, \quad \text{a.e. } s\in[t,T], \text{ a.s.} \tag{3.2}$$

Proof. For any $u(\cdot)\in\mathcal{U}[t,T]$ and any $\varepsilon\in\mathbb{R}$, let $(Y(\cdot),Z(\cdot))$ be the solution of

$$\begin{cases} dY = \big\{AY + \bar{A}\mathbb{E}[Y] + Bu + \bar{B}\mathbb{E}[u] + CZ + \bar{C}\mathbb{E}[Z]\big\}\,ds + ZdW, \quad s\in[t,T], \\ Y(T) = 0, \end{cases}$$

and let $(Y^\varepsilon(\cdot),Z^\varepsilon(\cdot))$ be the solution to the perturbed state equation

$$\begin{cases} dY^\varepsilon = \big\{AY^\varepsilon + \bar{A}\mathbb{E}[Y^\varepsilon] + B(u^*+\varepsilon u) + \bar{B}\mathbb{E}[u^*+\varepsilon u] + CZ^\varepsilon + \bar{C}\mathbb{E}[Z^\varepsilon]\big\}\,ds + Z^\varepsilon dW, \ s\in[t,T], \\ Y^\varepsilon(T) = \xi. \end{cases}$$

It is clear that $(Y^\varepsilon(\cdot),Z^\varepsilon(\cdot)) = (Y^*(\cdot)+\varepsilon Y(\cdot), Z^*(\cdot)+\varepsilon Z(\cdot))$, and hence

$$\begin{aligned} &J(t,\xi;u^*(\cdot)+\varepsilon u(\cdot)) - J(t,\xi;u^*(\cdot)) \\ &\quad = 2\varepsilon\mathbb{E}\bigg\{\langle GY^*(t),Y(t)\rangle + \big\langle\bar{G}\mathbb{E}[Y^*(t)],\mathbb{E}[Y(t)]\big\rangle \\ &\qquad + \int_t^T\Big[\langle QY^*,Y\rangle + \langle RZ^*,Z\rangle + \langle Nu^*,u\rangle\Big]ds \\ &\qquad + \int_t^T\Big[\big\langle\bar{Q}\mathbb{E}[Y^*],\mathbb{E}[Y]\big\rangle + \big\langle\bar{R}\mathbb{E}[Z^*],\mathbb{E}[Z]\big\rangle + \big\langle\bar{N}\mathbb{E}[u^*],\mathbb{E}[u]\big\rangle\Big]ds\bigg\} \\ &\qquad + \varepsilon^2\mathbb{E}\bigg\{\langle GY(t),Y(t)\rangle + \big\langle\bar{G}\mathbb{E}[Y(t)],\mathbb{E}[Y(t)]\big\rangle \\ &\qquad + \int_t^T\Big[\langle QY,Y\rangle + \langle RZ,Z\rangle + \langle Nu,u\rangle\Big]ds \\ &\qquad + \int_t^T\Big[\big\langle\bar{Q}\mathbb{E}[Y],\mathbb{E}[Y]\big\rangle + \big\langle\bar{R}\mathbb{E}[Z],\mathbb{E}[Z]\big\rangle + \big\langle\bar{N}\mathbb{E}[u],\mathbb{E}[u]\big\rangle\Big]ds\bigg\}. \end{aligned}$$

Applying Itô's formula to $s\mapsto\langle X^*(s),Y(s)\rangle$, we have

$$-\mathbb{E}\big\{\langle GY^*(t),Y(t)\rangle + \big\langle\bar{G}\mathbb{E}[Y^*(t)],\mathbb{E}[Y(t)]\big\rangle\big\}$$

$$
\begin{aligned}
&= -\mathbb{E}\big\langle GY^*(t) + \bar{G}\mathbb{E}[Y^*(t)], Y(t)\big\rangle \\
&= \mathbb{E}\int_t^T \Big\{\big\langle QY^* + \bar{Q}\mathbb{E}[Y^*], Y\big\rangle + \big\langle RZ^* + \bar{R}\mathbb{E}[Z^*], Z\big\rangle + \big\langle B^\top X^* + \bar{B}^\top\mathbb{E}[X^*], u\big\rangle\Big\}ds.
\end{aligned}
$$

It follows that for any $u(\cdot) \in \mathcal{U}[t,T]$ and any $\varepsilon \in \mathbb{R}$,

$$
\begin{aligned}
&J(t,\xi;u^*(\cdot)+\varepsilon u(\cdot)) - J(t,\xi;u^*(\cdot)) \\
&= 2\varepsilon\mathbb{E}\int_t^T \langle Nu^* + \bar{N}\mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top\mathbb{E}[X^*], u\rangle ds \\
&\quad + \varepsilon^2\mathbb{E}\bigg\{\langle GY(t), Y(t)\rangle + \big\langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle + \int_t^T \Big[\langle QY,Y\rangle + \langle RZ,Z\rangle + \langle Nu,u\rangle\Big]ds \\
&\quad + \int_t^T \Big[\big\langle \bar{Q}\mathbb{E}[Y], \mathbb{E}[Y]\big\rangle + \big\langle \bar{R}\mathbb{E}[Z], \mathbb{E}[Z]\big\rangle + \big\langle \bar{N}\mathbb{E}[u], \mathbb{E}[u]\big\rangle\Big]ds\bigg\}.
\end{aligned} \tag{3.3}
$$

Set $\phi(\varepsilon) = J(t,\xi;u^*(\cdot)+\varepsilon u(\cdot)) - J(t,\xi;u^*(\cdot))$. It follows form (3.3) that $\phi(\cdot)$ is continuously differentiable with derivative at $\varepsilon = 0$ given by

$$
\phi'(0) = 2\mathbb{E}\int_t^T \langle Nu^* + \bar{N}\mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top\mathbb{E}[X^*], u\rangle ds.
$$

Since $u^*(\cdot)$ is the optimal control of Problem (MF-BSLQ) for the terminal state ξ, we have

$$
\phi(\varepsilon) = J(t,\xi;u^*(\cdot)+\varepsilon u(\cdot)) - J(t,\xi;u^*(\cdot)) \geqslant 0, \quad \forall \varepsilon \in \mathbb{R}; \quad \phi(0) = 0.
$$

Thus, $\varepsilon = 0$ is a global extremum of $\phi(\cdot)$ and hence $\phi'(0) = 0$. Now (3.2) follows easily since $u(\cdot)$ is arbitrary. □

From the above result, we see that if $u(\cdot)$ happens to be an optimal control of Problem (MF-BSLQ) for terminal state ξ, then the following MF-FBSDE admits an adapted solution $(X(\cdot), Y(\cdot), Z(\cdot))$:

$$
\begin{cases}
dX = \Big\{-A^\top X - \bar{A}^\top\mathbb{E}[X] + QY + \bar{Q}\mathbb{E}[Y]\Big\}ds \\
\qquad\quad + \Big\{-C^\top X - \bar{C}^\top\mathbb{E}[X] + RZ + \bar{R}\mathbb{E}[Z]\Big\}dW, \quad s \in [t,T], \\
dY = \Big\{AY + \bar{A}\mathbb{E}[Y] + Bu + \bar{B}\mathbb{E}[u] + CZ + \bar{C}\mathbb{E}[Z]\Big\}ds + ZdW, \quad s \in [t,T], \\
X(t) = GY(t) + \bar{G}\mathbb{E}[Y(t)], \qquad Y(T) = \xi,
\end{cases} \tag{3.4}
$$

and the following *stationarity condition* holds:

$$
Nu + \bar{N}\mathbb{E}[u] - B^\top X - \bar{B}^\top\mathbb{E}[X] = 0, \quad \text{a.e. } s \in [t,T], \text{ a.s.} \tag{3.5}
$$

We call (3.4), together with the stationarity condition (3.5), the *optimality system* for the optimal control of Problem (MF-BSLQ). Note that (3.5) brings a coupling into the MF-FBSDE (3.4) and does not provide a representation for $u(\cdot)$ because the equation for $X(\cdot)$ involves $Y(\cdot)$ and $Z(\cdot)$.

To solve the optimality system (3.4)–(3.5), we use the decoupling technique inspired by the four-step scheme introduced in [25, 26] for general FBSDEs. This will lead to a derivation of two Riccati equations. To be precise, we conjecture that $X(\cdot)$ and $Y(\cdot)$ are related by the following:

$$Y(s)=-\Sigma(s)\big\{X(s)-\mathbb{E}[X(s)]\big\}-\Gamma(s)\mathbb{E}[X(s)]-\varphi(s),\quad s\in[t,T], \tag{3.6}$$

where $\Sigma(\cdot),\Gamma(\cdot):[0,T]\to\mathbb{S}^n$ are absolutely continuous and $\varphi(\cdot)$ satisfies

$$d\varphi(s)=\alpha(s)ds+\beta(s)dW(s),\qquad \varphi(T)=-\xi, \tag{3.7}$$

for some $\mathbb{F}$-progressively measurable processes $\alpha(\cdot)$ and $\beta(\cdot)$. Note that

$$\begin{cases} d\mathbb{E}[X]=\Big\{-\big(A+\bar{A}\big)^{\top}\mathbb{E}[X]+\big(Q+\bar{Q}\big)\mathbb{E}[Y]\Big\}\,ds \\ d\mathbb{E}[Y]=\Big\{\big(A+\bar{A}\big)\mathbb{E}[Y]+\big(B+\bar{B}\big)\mathbb{E}[u]+\big(C+\bar{C}\big)\mathbb{E}[Z]\Big\}ds, \\ \mathbb{E}[X(t)]=\big(G+\bar{G}\big)\mathbb{E}[Y(t)],\qquad \mathbb{E}[Y(T)]=\mathbb{E}[\xi], \\ \big(N+\bar{N}\big)\mathbb{E}[u]-\big(B+\bar{B}\big)^{\top}\mathbb{E}[X]=0. \end{cases} \tag{3.8}$$

Thus,

$$\begin{cases} d\big(X-\mathbb{E}[X]\big)=\Big\{-A^{\top}\big(X-\mathbb{E}[X]\big)+Q\big(Y-\mathbb{E}[Y]\big)\Big\}\,ds \\ \qquad\qquad\qquad\quad +\Big\{-C^{\top}X-\bar{C}^{\top}\mathbb{E}[X]+RZ+\bar{R}\mathbb{E}[Z]\Big\}dW, \\ d\big(Y-\mathbb{E}[Y]\big)=\Big\{A\big(Y-\mathbb{E}[Y]\big)+B\big(u-\mathbb{E}[u]\big)+C\big(Z-\mathbb{E}[Z]\big)\Big\}ds+ZdW, \\ X(t)-\mathbb{E}[X(t)]=G\big(Y(t)-\mathbb{E}[Y(t)]\big),\qquad Y(T)-\mathbb{E}[Y(T)]=\xi-\mathbb{E}[\xi], \\ N\big(u-\mathbb{E}[u]\big)-B^{\top}\big(X-\mathbb{E}[X]\big)=0. \end{cases} \tag{3.9}$$

From (3.6) we have

$$Y-\mathbb{E}[Y]=-\Sigma\big(X-\mathbb{E}[X]\big)-\big(\varphi-\mathbb{E}[\varphi]\big),\qquad \mathbb{E}[Y]=-\Gamma\mathbb{E}[X]-\mathbb{E}[\varphi]. \tag{3.10}$$

Denoting $\eta(\cdot)=\varphi(\cdot)-\mathbb{E}[\varphi(\cdot)]$ and $\gamma(\cdot)=\alpha(\cdot)-\mathbb{E}[\alpha(\cdot)]$, we have from (3.7) that

$$d\eta(s)=\gamma(s)ds+\beta(s)dW(s),\qquad \eta(T)=\mathbb{E}[\xi]-\xi. \tag{3.11}$$

Then (3.9)–(3.11) yield

$$\begin{aligned} 0 &= d(Y-\mathbb{E}[Y])+\dot{\Sigma}(X-\mathbb{E}[X])ds+\Sigma d(X-\mathbb{E}[X])+d\eta \\ &= \big\{A(Y-\mathbb{E}[Y])+B(u-\mathbb{E}[u])+C(Z-\mathbb{E}[Z])\big\}ds+ZdW \\ &\quad +\dot{\Sigma}(X-\mathbb{E}[X])ds+\big\{-\Sigma A^{\top}(X-\mathbb{E}[X])+\Sigma Q(Y-\mathbb{E}[Y])\big\}ds \\ &\quad +\big\{-\Sigma C^{\top}X-\Sigma\bar{C}^{\top}\mathbb{E}[X]+\Sigma RZ+\Sigma\bar{R}\mathbb{E}[Z]\big\}dW+\gamma ds+\beta dW \\ &= \big\{A(Y-\mathbb{E}[Y])+B(u-\mathbb{E}[u])+C(Z-\mathbb{E}[Z])+\dot{\Sigma}(X-\mathbb{E}[X]) \\ &\quad -\Sigma A^{\top}(X-\mathbb{E}[X])+\Sigma Q(Y-\mathbb{E}[Y])+\gamma\big\}ds \\ &\quad +\big\{Z-\Sigma C^{\top}X-\Sigma\bar{C}^{\top}\mathbb{E}[X]+\Sigma RZ+\Sigma\bar{R}\mathbb{E}[Z]+\beta\big\}dW \\ &= \big\{-A\Sigma(X-\mathbb{E}[X])-A\eta+BN^{-1}B^{\top}(X-\mathbb{E}[X])+C(Z-\mathbb{E}[Z]) \\ &\quad +\dot{\Sigma}(X-\mathbb{E}[X])-\Sigma A^{\top}(X-\mathbb{E}[X])-\Sigma Q\Sigma(X-\mathbb{E}[X])-\Sigma Q\eta+\gamma\big\}ds \\ &\quad +\big\{Z-\Sigma C^{\top}X-\Sigma\bar{C}^{\top}\mathbb{E}[X]+\Sigma RZ+\Sigma\bar{R}\mathbb{E}[Z]+\beta\big\}dW \end{aligned}$$

$$
\begin{aligned}
&= \big\{\big(\dot\Sigma - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BN^{-1}B^\top\big)(X-\mathbb{E}[X]) \\
&\quad + C(Z-\mathbb{E}[Z]) - (A+\Sigma Q)\eta + \gamma\big\}ds \\
&\quad + \big\{Z - \Sigma C^\top X - \Sigma \bar C^\top \mathbb{E}[X] + \Sigma R Z + \Sigma \bar R \mathbb{E}[Z] + \beta\big\}dW.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\big(\dot\Sigma - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BN^{-1}B^\top\big)(X-\mathbb{E}[X]) + C(Z-\mathbb{E}[Z]) - (A+\Sigma Q)\eta + \gamma &= 0, \quad (3.12)\\
Z - \Sigma C^\top X - \Sigma \bar C^\top \mathbb{E}[X] + \Sigma R Z + \Sigma \bar R \mathbb{E}[Z] + \beta &= 0. \quad (3.13)
\end{aligned}
$$

Now from (3.13) we have

$$
(I + \Sigma R + \Sigma \bar R)\mathbb{E}[Z] - \Sigma (C+\bar C)^\top \mathbb{E}[X] + \mathbb{E}[\beta] = 0. \tag{3.14}
$$

Subtracting (3.14) from (3.13), we obtain

$$
(I+\Sigma R)(Z-\mathbb{E}[Z]) - \Sigma C^\top (X - \mathbb{E}[X]) + (\beta - \mathbb{E}[\beta]) = 0. \tag{3.15}
$$

Assuming that $I+\Sigma R$ and $I + \Sigma R + \Sigma \bar R$ are invertible, we obtain from (3.14) and (3.15):

$$
\begin{aligned}
\mathbb{E}[Z] &= (I+\Sigma R + \Sigma \bar R)^{-1}\big\{\Sigma(C+\bar C)^\top \mathbb{E}[X] - \mathbb{E}[\beta]\big\}, && (3.16)\\
Z - \mathbb{E}[Z] &= (I+\Sigma R)^{-1}\big\{\Sigma C^\top (X-\mathbb{E}[X]) - (\beta - \mathbb{E}[\beta])\big\}. && (3.17)
\end{aligned}
$$

Substitution of (3.17) into (3.12) now gives

$$
\begin{aligned}
&\Big[\dot\Sigma - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BN^{-1}B^\top + C(I+\Sigma R)^{-1}\Sigma C^\top\Big](X-\mathbb{E}[X]) \\
&- C(I+\Sigma R)^{-1}(\beta - \mathbb{E}[\beta]) - (A+\Sigma Q)\eta + \gamma = 0,
\end{aligned}
$$

from which one should let

$$
\begin{cases}
\dot\Sigma - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BN^{-1}B^\top + C(I+\Sigma R)^{-1}\Sigma C^\top = 0,\\
\gamma - C(I+\Sigma R)^{-1}(\beta - \mathbb{E}[\beta]) - (A+\Sigma Q)\eta = 0.
\end{cases} \tag{3.18}
$$

Also, we have from (3.8), (3.10), and (3.16):

$$
\begin{aligned}
0 &= \frac{d}{ds}\big(\mathbb{E}[Y] + \Gamma \mathbb{E}[X] + \mathbb{E}[\varphi]\big)\\
&= (A+\bar A)\mathbb{E}[Y] + (B+\bar B)\mathbb{E}[u] + (C+\bar C)\mathbb{E}[Z]\\
&\quad + \dot\Gamma \mathbb{E}[X] - \Gamma(A+\bar A)^\top \mathbb{E}[X] + \Gamma(Q+\bar Q)\mathbb{E}[Y] + \mathbb{E}[\alpha]\\
&= -(A+\bar A)\Gamma\mathbb{E}[X] - (A+\bar A)\mathbb{E}[\varphi] + (B+\bar B)(N+\bar N)^{-1}(B+\bar B)^\top \mathbb{E}[X]\\
&\quad + (C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\big\{\Sigma(C+\bar C)^\top \mathbb{E}[X] - \mathbb{E}[\beta]\big\}\\
&\quad + \dot\Gamma\mathbb{E}[X] - \Gamma(A+\bar A)^\top \mathbb{E}[X] - \Gamma(Q+\bar Q)\Gamma\mathbb{E}[X] - \Gamma(Q+\bar Q)\mathbb{E}[\varphi] + \mathbb{E}[\alpha]\\
&= \big\{\dot\Gamma - (A+\bar A)\Gamma - \Gamma(A+\bar A)^\top - \Gamma(Q+\bar Q)\Gamma + (B+\bar B)(N+\bar N)^{-1}(B+\bar B)^\top\\
&\quad + (C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\Sigma(C+\bar C)^\top\big\}\mathbb{E}[X]\\
&\quad - \big[(A+\bar A) + \Gamma(Q+\bar Q)\big]\mathbb{E}[\varphi] - (C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\mathbb{E}[\beta] + \mathbb{E}[\alpha].
\end{aligned}
$$

Hence, one should let

$$
\begin{cases}
\dot\Gamma - (A+\bar A)\Gamma - \Gamma(A+\bar A)^\top - \Gamma(Q+\bar Q)\Gamma + (B+\bar B)(N+\bar N)^{-1}(B+\bar B)^\top\\
\quad + (C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\Sigma(C+\bar C)^\top = 0,\\
\mathbb{E}[\alpha] - \big[(A+\bar A) + \Gamma(Q+\bar Q)\big]\mathbb{E}[\varphi] - (C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\mathbb{E}[\beta] = 0.
\end{cases} \tag{3.19}
$$

Moreover, comparing the terminal values on both sides of the two equations in (3.10), one has

$$\Sigma(T) = 0, \qquad \Gamma(T) = 0.$$

Therefore, by (3.18)–(3.19), we see that $\Sigma(\cdot)$ and $\Gamma(\cdot)$ should satisfy the following Riccati-type equations, respectively:

$$\begin{cases} \dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q \Sigma + BN^{-1}B^\top + C(I+\Sigma R)^{-1}\Sigma C^\top = 0, \quad s \in [0,T], \\ \Sigma(T) = 0, \end{cases} \tag{3.20}$$

$$\begin{cases} \dot{\Gamma} - (A+\bar{A})\Gamma - \Gamma(A+\bar{A})^\top - \Gamma(Q+\bar{Q})\Gamma + (B+\bar{B})(N+\bar{N})^{-1}(B+\bar{B})^\top \\ \quad + (C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1}\Sigma(C+\bar{C})^\top = 0, \quad s \in [0,T], \\ \Gamma(T) = 0, \end{cases} \tag{3.21}$$

and $\varphi(\cdot)$ should satisfy the following MF-BSDE on $[0,T]$:

$$\begin{cases} d\varphi = \Big\{ (A+\Sigma Q)\varphi + \big[\bar{A} + \Gamma(Q+\bar{Q}) - \Sigma Q\big]\mathbb{E}[\varphi] + C(I+\Sigma R)^{-1}\beta \\ \qquad + \big[(C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1} - C(I+\Sigma R)^{-1}\big]\mathbb{E}[\beta] \Big\} ds + \beta dW, \\ \varphi(T) = -\xi. \end{cases} \tag{3.22}$$

4 Unique solvability of Riccati equations

In this section we shall establish the unique solvability of the Riccati equations (3.20) and (3.21). Once $\Sigma(\cdot)$ and $\Pi(\cdot)$ are known, the existence of a solution to MF-BSDE (3.22) will immediately follows from Theorem 2.1.

Theorem 4.1. *Let* (H1)–(H2) *hold. Then the Riccati equations* (3.20) *and* (3.21) *admit unique solutions* $\Sigma(\cdot) \in C([0,T];\overline{\mathbb{S}^n_+})$ *and* $\Gamma(\cdot) \in C([0,T];\overline{\mathbb{S}^n_+})$*, respectively.*

Proof. For $\lambda > 0$ and $\varepsilon \geqslant 0$, let us consider the forward stochastic differential equation (FSDE, for short)

$$\begin{cases} dX(s) = \Big\{ A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] \\ \qquad\qquad + C(s)v(s) + \bar{C}(s)\mathbb{E}[v(s)] \Big\} ds + v(s)dW(s), \quad s \in [t,T], \\ X(t) = \xi, \end{cases} \tag{4.1}$$

and the cost functional

$$\begin{aligned} J_{\lambda,\varepsilon}(t,\xi;u(\cdot),v(\cdot)) \;=\; & \mathbb{E}\Big\{ \int_t^T \Big[\langle Q(s)X(s),X(s)\rangle + \big\langle \bar{Q}(s)\mathbb{E}[X(s)],\mathbb{E}[X(s)]\big\rangle \\ & + \langle [\varepsilon I + R(s)]v(s),v(s)\rangle + \big\langle \bar{R}(s)\mathbb{E}[v(s)],\mathbb{E}[v(s)]\big\rangle \\ & + \langle N(s)u(s),u(s)\rangle + \big\langle \bar{N}(s)\mathbb{E}[u(s)],\mathbb{E}[u(s)]\big\rangle \Big] ds + \lambda |X(T)|^2 \Big\}. \end{aligned}$$

We pose the following forward mean-field LQ problem: For any given initial pair $(t,\xi) \in [0,T] \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, find a pair $(u^*(\cdot),v^*(\cdot)) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ such that

$$J_{\lambda,\varepsilon}(t,\xi;u^*(\cdot),v^*(\cdot)) = \inf_{u(\cdot),v(\cdot)} J_{\lambda,\varepsilon}(t,\xi;u(\cdot),v(\cdot)) \triangleq V_{\lambda,\varepsilon}(t,\xi)$$

as $(u(\cdot),v(\cdot))$ ranges over the space $L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$. By (H2), we have for any $(t,\xi)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and any $(u(\cdot),v(\cdot))\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$,

$$\begin{aligned} &J_{\lambda,\varepsilon}(t,\xi;u(\cdot),v(\cdot)) \\ &\geqslant \int_t^T\Big\{\langle Q(s)\mathbb{E}[X(s)],\mathbb{E}[X(s)]\rangle+\big\langle \bar Q(s)\mathbb{E}[X(s)],\mathbb{E}[X(s)]\big\rangle \\ &\quad+\langle R(s)\mathbb{E}[v(s)],\mathbb{E}[v(s)]\rangle+\big\langle \bar R(s)\mathbb{E}[v(s)],\mathbb{E}[v(s)]\big\rangle \\ &\quad+\langle [N(s)-\delta/2]\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle+\big\langle \bar N(s)\mathbb{E}[u(s)],\mathbb{E}[u(s)]\big\rangle\Big\}ds \\ &\quad+\varepsilon\mathbb{E}\int_t^T|v(s)|^2ds+\frac{\delta}{2}\mathbb{E}\int_t^T|u(s)|^2ds \\ &\geqslant\Big(\varepsilon\wedge\frac{\delta}{2}\Big)\mathbb{E}\int_t^T\Big[|v(s)|^2+|u(s)|^2\Big]ds. \end{aligned} \tag{4.2}$$

Then it follows from [30, Theorem 5.2] (see also [35, Theorem 4.1]) that for any $\lambda,\varepsilon>0$, the following two Riccati equations

$$\begin{cases} \dot P_{\lambda,\varepsilon}+P_{\lambda,\varepsilon}A+A^\top P_{\lambda,\varepsilon}+Q-P_{\lambda,\varepsilon}(B,C)\begin{pmatrix} N & 0\\ 0 & \varepsilon I+R+P_{\lambda,\varepsilon}\end{pmatrix}^{-1}(B,C)^\top P_{\lambda,\varepsilon}=0,\\ P_{\lambda,\varepsilon}(T)=\lambda I, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dot\Pi_{\lambda,\varepsilon}+\Pi_{\lambda,\varepsilon}(A+\bar A)+(A+\bar A)^\top\Pi_{\lambda,\varepsilon}+Q+\bar Q\\ -\Pi_{\lambda,\varepsilon}(B+\bar B,C+\bar C)\begin{pmatrix} N+\bar N & 0\\ 0 & \varepsilon I+R+\bar R+P_{\lambda,\varepsilon}\end{pmatrix}^{-1}(B+\bar B,C+\bar C)^\top\Pi_{\lambda,\varepsilon}=0,\\ \Pi_{\lambda,\varepsilon}(T)=\lambda I, \end{cases} \tag{4.4}$$

admit unique solutions $P_{\lambda,\varepsilon}(\cdot)$ and $\Pi_{\lambda,\varepsilon}(\cdot)$, respectively, such that

$$\begin{aligned} V_{\lambda,\varepsilon}(t,\xi)=\mathbb{E}\langle P_{\lambda,\varepsilon}(t)(\xi-\mathbb{E}[\xi]),\xi-\mathbb{E}[\xi]\rangle+\langle\Pi_{\lambda,\varepsilon}(t)\mathbb{E}[\xi],\mathbb{E}[\xi]\rangle,&\\ \forall(t,\xi)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n).& \end{aligned} \tag{4.5}$$

For fixed $\lambda>0$, we have

$$\begin{aligned} V_{\lambda,\varepsilon}(t,\xi)=\inf_{u(\cdot),v(\cdot)}J_{\lambda,\varepsilon}(t,\xi;u(\cdot),v(\cdot))\leqslant\inf_{u(\cdot),v(\cdot)}J_{\lambda,\varepsilon'}(t,\xi;u(\cdot),v(\cdot))=V_{\lambda,\varepsilon'}(t,\xi),&\\ \forall(t,\xi)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n),& \end{aligned} \tag{4.6}$$

whenever $0\leqslant\varepsilon\leqslant\varepsilon'$. This implies

$$P_{\lambda,\varepsilon}(t)\leqslant P_{\lambda,\varepsilon'}(t),\quad \Pi_{\lambda,\varepsilon}(t)\leqslant\Pi_{\lambda,\varepsilon'}(t),\qquad\forall\, t\in[0,T];\quad\forall\, 0<\varepsilon\leqslant\varepsilon'. \tag{4.7}$$

On the other hand, we claim that

$$V_{\lambda,0}(t,\xi)>0,\qquad\forall(t,\xi)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\text{ with }\xi\neq0. \tag{4.8}$$

Indeed, if (4.8) is false then $V_{\lambda,0}(t,\xi)=0$ for some (t,ξ) with $\xi\neq0$. Let $\{(u_k(\cdot),v_k(\cdot))\}_{k=1}^\infty$ be a minimizing sequence of $(u(\cdot),v(\cdot))\mapsto J_{\lambda,0}(t,\xi;u(\cdot),v(\cdot))$ and let $X_k(\cdot)$ be the corresponding solution of (4.1). By making use of (H2), we have

$$J_{\lambda,0}(t,\xi;u_k(\cdot),v_k(\cdot))$$

$$
\begin{aligned}
&\geqslant \int_t^T \Big\{\langle Q(s)\mathbb{E}[X_k(s)], \mathbb{E}[X_k(s)]\rangle + \big\langle \bar{Q}(s)\mathbb{E}[X_k(s)], \mathbb{E}[X_k(s)]\big\rangle \\
&\quad + \langle R(s)\mathbb{E}[v_k(s)], \mathbb{E}[v_k(s)]\rangle + \big\langle \bar{R}(s)\mathbb{E}[v_k(s)], \mathbb{E}[v_k(s)]\big\rangle \\
&\quad + \langle [N(s)-\delta/2]\mathbb{E}[u_k(s)], \mathbb{E}[u_k(s)]\rangle + \big\langle \bar{N}(s)\mathbb{E}[u_k(s)], \mathbb{E}[u_k(s)]\big\rangle \Big\} ds \\
&\quad + \frac{\delta}{2}\mathbb{E}\int_t^T |u_k(s)|^2 ds + \lambda \mathbb{E}|X_k(T)|^2 \\
&\geqslant \frac{\delta}{2}\mathbb{E}\int_t^T |u_k(s)|^2 ds + \lambda \mathbb{E}|X_k(T)|^2, \quad \forall k \geqslant 1.
\end{aligned}
$$

Since $J_{\lambda,0}(t,\xi;u_k(\cdot),v_k(\cdot)) \to V_{\lambda,0}(t,\xi) = 0$, the above implies

$$
\lim_{k\to\infty} \mathbb{E}\int_t^T |u_k(s)|^2 ds = 0, \qquad \lim_{k\to\infty} \mathbb{E}|X_k(T)|^2 = 0.
$$

Regarding $(X_k(\cdot), v_k(\cdot))$ as the adapted solution to the MF-BSDE

$$
\begin{cases}
dY(s) = \Big\{A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u_k(s) + \bar{B}(s)\mathbb{E}[u_k(s)] \\
\qquad\qquad + C(s)v_k(s) + \bar{C}(s)\mathbb{E}[v_k(s)]\Big\} ds + v_k(s)dW(s), \quad s \in [t,T], \\
Y(T) = X_k(T),
\end{cases}
$$

we apply Theorem 2.1 to obtain

$$
\begin{aligned}
\mathbb{E}|\xi|^2 &= \mathbb{E}|X_k(t)|^2 \leqslant \mathbb{E}\left[\sup_{t\leqslant s\leqslant T} |X_k(s)|^2\right] \\
&\leq K\mathbb{E}\left[|X_k(T)|^2 + \int_t^T |u_k(s)|^2 ds\right] \\
&\to 0 \quad \text{as} \quad k \to \infty,
\end{aligned}
$$

which leads to a contradiction. Now (4.8), together with (4.6) and (4.7), implies that the limits $\lim_{\varepsilon\to 0} P_{\lambda,\varepsilon}(t)$ and $\lim_{\varepsilon\to 0} \Pi_{\lambda,\varepsilon}(t)$ exist, and

$$
P_\lambda(t) \triangleq \lim_{\varepsilon\to 0} P_{\lambda,\varepsilon}(t) > 0, \quad \Pi_\lambda(t) \triangleq \lim_{\varepsilon\to 0} \Pi_{\lambda,\varepsilon}(t) > 0, \qquad \forall t \in [0,T].
$$

By (4.3), we get

$$
\begin{aligned}
P_{\lambda,\varepsilon}(t) = \lambda I + \int_t^T \Bigg[& P_{\lambda,\varepsilon}A + A^\top P_{\lambda,\varepsilon} + Q \\
& - P_{\lambda,\varepsilon}(B,C)\begin{pmatrix} N & 0 \\ 0 & \varepsilon I + R + P_{\lambda,\varepsilon}\end{pmatrix}^{-1}(B,C)^\top P_{\lambda,\varepsilon}\Bigg] ds.
\end{aligned}
$$

Passing to limit as $\varepsilon \to 0$, by the bounded convergence theorem, we have

$$
P_\lambda(t) = \lambda I + \int_t^T \left[P_\lambda A + A^\top P_\lambda + Q - P_\lambda(B,C)\begin{pmatrix} N & 0 \\ 0 & R + P_\lambda\end{pmatrix}^{-1}(B,C)^\top P_\lambda\right] ds.
$$

Therefore,

$$
\begin{cases}
\dot{P}_\lambda + P_\lambda A + A^\top P_\lambda + Q - P_\lambda(B,C)\begin{pmatrix} N & 0 \\ 0 & R + P_\lambda\end{pmatrix}^{-1}(B,C)^\top P_\lambda = 0, \\
P_\lambda(T) = \lambda I.
\end{cases}
$$

Similarly using (4.4), we have

$$\begin{cases} \dot{\Pi}_\lambda + \Pi_\lambda(A+\bar{A}) + (A+\bar{A})^\top\Pi_\lambda + Q + \bar{Q} \\ \qquad - \Pi_\lambda(B+\bar{B}, C+\bar{C}) \begin{pmatrix} N+\bar{N} & 0 \\ 0 & R+\bar{R}+P_\lambda \end{pmatrix}^{-1} (B+\bar{B}, C+\bar{C})^\top\Pi_\lambda = 0, \\ \Pi_\lambda(T) = \lambda I. \end{cases}$$

Next, for fixed $\varepsilon > 0$, we have

$$\begin{aligned} V_{\lambda,\varepsilon}(t,\xi) = \inf_{u(\cdot),v(\cdot)} J_{\lambda,\varepsilon}(t,\xi;u(\cdot),v(\cdot)) \leqslant \inf_{u(\cdot),v(\cdot)} J_{\lambda',\varepsilon}(t,\xi;u(\cdot),v(\cdot)) = V_{\lambda',\varepsilon}(t,\xi), \\ \forall (t,\xi) \in [0,T] \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n), \end{aligned}$$

whenever $0 < \lambda \leqslant \lambda'$. It follows that

$$P_{\lambda,\varepsilon}(t) \leqslant P_{\lambda',\varepsilon}(t), \quad \Pi_{\lambda,\varepsilon}(t) \leqslant \Pi_{\lambda',\varepsilon}(t), \qquad \forall t \in [0,T],$$

and hence

$$0 < P_\lambda(t) \leqslant P_{\lambda'}(t), \quad 0 < \Pi_\lambda(t) \leqslant \Pi_{\lambda'}(t), \quad \forall t \in [0,T]; \qquad 0 < \lambda \leqslant \lambda'.$$

Therefore, the families $\{\Sigma_\lambda(t) \triangleq P_\lambda(t)^{-1} : \lambda > 0\}$ and $\{\Gamma_\lambda(t) \triangleq \Pi_\lambda(t)^{-1} : \lambda > 0\}$ are decreasing in $\mathbb{S}^n_+$ and hence converge. We denote

$$\Sigma(t) = \lim_{\lambda\to\infty} \Sigma_\lambda(t) \geqslant 0, \quad \Gamma(t) = \lim_{\lambda\to\infty} \Gamma_\lambda(t) \geqslant 0, \qquad t \in [0,T].$$

Now using the fact

$$\begin{cases} \frac{d}{dt}\big[P_\lambda(t)^{-1}P_\lambda(t)\big] = 0, \qquad \frac{d}{dt}\big[\Pi_\lambda(t)^{-1}\Pi_\lambda(t)\big] = 0, \\ [R(t)+P_\lambda(t)]^{-1} = [I + P_\lambda(t)^{-1}R(t)]^{-1}P_\lambda(t)^{-1}, \\ [R(t)+\bar{R}(t)+P_\lambda(t)]^{-1} = \big\{I + P_\lambda(t)^{-1}[R(t)+\bar{R}(t)]\big\}^{-1}P_\lambda(t)^{-1}, \end{cases}$$

one can easily show that $\Sigma_\lambda(\cdot)$ is a solution of

$$\begin{cases} \dot{\Sigma}_\lambda - A\Sigma_\lambda - \Sigma_\lambda A^\top - \Sigma_\lambda Q\Sigma_\lambda + BN^{-1}B^\top + C(I+\Sigma_\lambda R)^{-1}\Sigma_\lambda C^\top = 0, \\ \Sigma_\lambda(T) = \lambda^{-1}I, \end{cases} \tag{4.9}$$

and $\Gamma_\lambda(\cdot)$ is a solution of

$$\begin{cases} \dot{\Gamma}_\lambda - (A+\bar{A})\Gamma_\lambda - \Gamma_\lambda(A+\bar{A})^\top - \Gamma_\lambda(Q+\bar{Q})\Gamma_\lambda + (B+\bar{B})(N+\bar{N})^{-1}(B+\bar{B})^\top \\ \qquad + (C+\bar{C})\big[I + \Sigma_\lambda(R+\bar{R})\big]^{-1}\Sigma_\lambda(C+\bar{C})^\top = 0, \\ \Gamma_\lambda(T) = \lambda^{-1}I. \end{cases} \tag{4.10}$$

Note that (4.9) is equivalent to

$$\Sigma_\lambda(t) = \lambda^{-1}I - \int_t^T \Big[A\Sigma_\lambda + \Sigma_\lambda A^\top + \Sigma_\lambda Q\Sigma_\lambda - BN^{-1}B^\top - C(I+\Sigma_\lambda R)^{-1}\Sigma_\lambda C^\top\Big]ds.$$

Because $\{\Sigma_\lambda(t)\}_{\lambda\geqslant 1}$ and $\{[I+\Sigma_\lambda(t)R(t)]^{-1}\Sigma_\lambda(t)\}_{\lambda\geqslant 1}$ are uniformly bounded on $[0,T]$, by letting $\lambda\to\infty$, we obtain from the dominated convergence theorem:

$$\Sigma(t)=-\int_t^T\Big[A\Sigma+\Sigma A^\top+\Sigma Q\Sigma-BN^{-1}B^\top-C(I+\Sigma R)^{-1}\Sigma C^\top\Big]ds,$$

so $\Sigma(\cdot)$ is a solution of the Riccati equation (3.20). Likewise, $\Gamma(\cdot)$ is a solution of the Riccati equation (3.21).

To prove the uniqueness, let us suppose that $\Sigma_1(\cdot),\Sigma_2(\cdot)\in C([0,T];\overline{\mathbb{S}^n_+})$ are two solutions of (3.20). Then it is easy to show that $\Delta(\cdot)\triangleq\Sigma_1(\cdot)-\Sigma_2(\cdot)$ is a solution to the equation

$$\begin{cases}\dot\Delta-(A+\Sigma_1Q)\Delta-\Delta(A+\Sigma_2Q)^\top+C(I+\Sigma_1R)^{-1}\Delta\big[I-R(I+\Sigma_2R)^{-1}\Sigma_2\big]C^\top=0,\\ \Delta(T)=0.\end{cases}$$

Note that the functions Σ_i and $(I+\Sigma_iR)^{-1},\ i=1,2$ are bounded on $[0,T]$. Then a standard argument using the Gronwall inequality will show that $\Delta(\cdot)=0$. The uniqueness of the solution to equation (3.21) is proved similarly. $\square$

5 Representations of optimal controls and value function

This section is going to give explicit formulas of the optimal controls and the value function, via the solutions to the Riccati equations (3.20), (3.21), and the MF-BSDE (3.22). Our first result can be stated as follows.

Theorem 5.1. *Let* (H1)–(H2) *hold and let* $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ *be given. Let* $\Sigma(\cdot)$ *and* $\Gamma(\cdot)$ *be the unique solutions to the Riccati equations* (3.20) *and* (3.21)*, respectively, and let* $(\varphi(\cdot),\beta(\cdot))$ *be the unique adapted solution to the MF-BSDE* (3.22)*. Then the following MF-FSDE admits a unique solution* $X(\cdot)$*:*

$$\begin{cases}dX=\Big\{-(A+\Sigma Q)^\top X-\big[\bar A-\Sigma Q+\Gamma(Q+\bar Q)\big]^\top\mathbb{E}[X]-Q\varphi-\bar Q\mathbb{E}[\varphi]\Big\}ds\\ \qquad\quad+\Big\{\big[R(I+\Sigma R)^{-1}\Sigma-I\big]C^\top X+\Big(-\bar C^\top-R(I+\Sigma R)^{-1}\Sigma C^\top\\ \qquad\quad+(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\Sigma(C+\bar C)^\top\Big)\mathbb{E}[X]\\ \qquad\quad-R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])-(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\mathbb{E}[\beta]\Big\}dW,\\ X(t)=-[I+G\Sigma(t)]^{-1}G\{\varphi(t)-\mathbb{E}[\varphi(t)]\}-[I+(G+\bar G)\Gamma(t)]^{-1}(G+\bar G)\mathbb{E}[\varphi(t)],\end{cases}\tag{5.1}$$

and the unique optimal control of Problem (MF-BSLQ) *for the terminal state* ξ *is given by*

$$u=N^{-1}B^\top(X-\mathbb{E}[X])+(N+\bar N)^{-1}(B+\bar B)^\top\mathbb{E}[X].\tag{5.2}$$

Proof. It is clear that (5.1) has a unique solution $X(\cdot)$. So we only need to prove that $u(\cdot)$ defined by (5.2) is the unique optimal control of Problem (MF-BSLQ) for the terminal state ξ. To this end, we define

$$Y=-\Sigma(X-\mathbb{E}[X])-\Gamma\mathbb{E}[X]-\varphi,\tag{5.3}$$

$$\begin{aligned}Z=&(I+\Sigma R)^{-1}\big\{\Sigma C^\top(X-\mathbb{E}[X])-(\beta-\mathbb{E}[\beta])\big\}\\ &+(I+\Sigma R+\Sigma\bar R)^{-1}\big\{\Sigma(C+\bar C)^\top\mathbb{E}[X]-\mathbb{E}[\beta]\big\}.\end{aligned}\tag{5.4}$$

Then we have $Y(T)=\xi$ and

$$\mathbb{E}[Y]=-\Gamma\mathbb{E}[X]-\mathbb{E}[\varphi], \tag{5.5}$$

$$\mathbb{E}[Z]=(I+\Sigma R+\Sigma\bar{R})^{-1}\big\{\Sigma(C+\bar{C})^\top\mathbb{E}[X]-\mathbb{E}[\beta]\big\}. \tag{5.6}$$

Also, from (5.4) and (5.6) we have

$$\begin{aligned}
&Z+\Sigma RZ+\Sigma\bar{R}\mathbb{E}[Z]-\Sigma C^\top X-\Sigma\bar{C}^\top\mathbb{E}[X]+\beta \\
&\quad=(I+\Sigma R)Z+\Sigma\bar{R}\mathbb{E}[Z]-\Sigma C^\top X-\Sigma\bar{C}^\top\mathbb{E}[X]+\beta \\
&\quad=\Sigma C^\top(X-\mathbb{E}[X])-(\beta-\mathbb{E}[\beta])+(I+\Sigma R)\mathbb{E}[Z] \\
&\qquad+\Sigma\bar{R}\mathbb{E}[Z]-\Sigma C^\top X-\Sigma\bar{C}^\top\mathbb{E}[X]+\beta \\
&\quad=-\Sigma(C+\bar{C})^\top\mathbb{E}[X]+\mathbb{E}[\beta]+(I+\Sigma R+\Sigma\bar{R})\mathbb{E}[Z] \\
&\quad=0.
\end{aligned} \tag{5.7}$$

Thus, making use of (3.20), (3.21), and (5.7), we have

$$\begin{aligned}
dY&=-\dot{\Sigma}(X-\mathbb{E}[X])ds-\Sigma d(X-\mathbb{E}[X])-\dot{\Gamma}\mathbb{E}[X]ds-\Gamma d\mathbb{E}[X]-d\varphi \\
&=-\dot{\Sigma}(X-\mathbb{E}[X])ds+\Sigma\Big\{(A+\Sigma Q)^\top(X-\mathbb{E}[X])+Q(\varphi-\mathbb{E}[\varphi])\Big\}ds \\
&\quad-\Sigma\Big\{\big[R(I+\Sigma R)^{-1}\Sigma-I\big]C^\top X+\Big(-\bar{C}^\top-R(I+\Sigma R)^{-1}\Sigma C^\top \\
&\quad+(R+\bar{R})(I+\Sigma R+\Sigma\bar{R})^{-1}\Sigma(C+\bar{C})^\top\Big)\mathbb{E}[X] \\
&\quad-R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])-(R+\bar{R})(I+\Sigma R+\Sigma\bar{R})^{-1}\mathbb{E}[\beta]\Big\}dW \\
&\quad-\dot{\Gamma}\mathbb{E}[X]ds+\Gamma\Big\{\big[A+\bar{A}+\Gamma(Q+\bar{Q})\big]^\top\mathbb{E}[X]+(Q+\bar{Q})\mathbb{E}[\varphi]\Big\}ds \\
&\quad-\Big\{(A+\Sigma Q)\varphi+\big[\bar{A}+\Gamma(Q+\bar{Q})-\Sigma Q\big]\mathbb{E}[\varphi]+C(I+\Sigma R)^{-1}\beta \\
&\quad+\big[(C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1}-C(I+\Sigma R)^{-1}\big]\mathbb{E}[\beta]\Big\}ds-\beta dW \\
&=\Big\{\Big(-\dot{\Sigma}+\Sigma(A+\Sigma Q)^\top\Big)(X-\mathbb{E}[X])+\Big(-\dot{\Gamma}+\Gamma\big[A+\bar{A}+\Gamma(Q+\bar{Q})\big]^\top\Big)\mathbb{E}[X] \\
&\quad-A\varphi-\bar{A}\mathbb{E}[\varphi]-C(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])-(C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1}\mathbb{E}[\beta]\Big\}ds \\
&\quad-\Big\{\Sigma R(I+\Sigma R)^{-1}\Sigma C^\top(X-\mathbb{E}[X])-\Sigma C^\top X-\Sigma\bar{C}^\top\mathbb{E}[X] \\
&\quad+\Sigma(R+\bar{R})(I+\Sigma R+\Sigma\bar{R})^{-1}\Sigma(C+\bar{C})^\top\mathbb{E}[X]-\Sigma R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta]) \\
&\quad-\Sigma(R+\bar{R})(I+\Sigma R+\Sigma\bar{R})^{-1}\mathbb{E}[\beta]+\beta\Big\}dW \\
&=\Big\{\Big(-A\Sigma+BN^{-1}B^\top+C(I+\Sigma R)^{-1}\Sigma C^\top\Big)(X-\mathbb{E}[X])+\Big(-(A+\bar{A})\Gamma \\
&\quad+(B+\bar{B})(N+\bar{N})^{-1}(B+\bar{B})^\top+(C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1}\Sigma(C+\bar{C})^\top\Big)\mathbb{E}[X] \\
&\quad-A\varphi-\bar{A}\mathbb{E}[\varphi]-C(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])-(C+\bar{C})(I+\Sigma R+\Sigma\bar{R})^{-1}\mathbb{E}[\beta]\Big\}ds \\
&\quad-\Big\{\Sigma R(I+\Sigma R)^{-1}\big\{\Sigma C^\top(X-\mathbb{E}[X])-(\beta-\mathbb{E}[\beta])\big\}-\Sigma C^\top X-\Sigma\bar{C}^\top\mathbb{E}[X]+\beta \\
&\quad+\Sigma(R+\bar{R})(I+\Sigma R+\Sigma\bar{R})^{-1}\big\{\Sigma(C+\bar{C})^\top\mathbb{E}[X]-\mathbb{E}[\beta]\big\}\Big\}dW \\
&=\Big\{-A\Big(\Sigma(X-\mathbb{E}[X])+\Gamma\mathbb{E}[X]+\varphi\Big)-\bar{A}\Big(\Gamma\mathbb{E}[X]+\mathbb{E}[\varphi]\Big)+BN^{-1}B^\top(X-\mathbb{E}[X]) \\
&\quad+(B+\bar{B})(N+\bar{N})^{-1}(B+\bar{B})^\top\mathbb{E}[X]+C(I+\Sigma R)^{-1}\big\{\Sigma C^\top(X-\mathbb{E}[X])-(\beta-\mathbb{E}[\beta])\big\}
\end{aligned}$$

$$
\begin{aligned}
&+(C+\bar C)(I+\Sigma R+\Sigma\bar R)^{-1}\big\{\Sigma(C+\bar C)^\top\mathbb{E}[X]-\mathbb{E}[\beta]\big\}\Big\}ds\\
&-\Big\{\Sigma R(Z-\mathbb{E}[Z])-\Sigma C^\top X-\Sigma\bar C^\top\mathbb{E}[X]+\beta+\Sigma(R+\bar R)\mathbb{E}[Z]\Big\}dW\\
=&\Big\{AY+\bar A\mathbb{E}[Y]+B(u-\mathbb{E}[u])+(B+\bar B)\mathbb{E}[u]+C(Z-\mathbb{E}[Z])+(C+\bar C)\mathbb{E}[Z]\Big\}ds\\
&-\Big\{\Sigma RZ+\Sigma\bar R\mathbb{E}[Z]-\Sigma C^\top X-\Sigma\bar C^\top\mathbb{E}[X]+\beta\Big\}dW\\
=&\Big\{AY+\bar A\mathbb{E}[Y]+Bu+\bar B\mathbb{E}[u]+CZ+\bar C\mathbb{E}[Z]\Big\}ds+ZdW.
\end{aligned}
$$

Moreover, the first equation in (5.1) can be written as

$$
\begin{aligned}
dX=&\Big\{-A^\top X-\bar A^\top\mathbb{E}[X]-Q\Big(\Sigma(X-\mathbb{E}[X])+\Gamma\mathbb{E}[X]+\varphi\Big)-\bar Q\Big(\Gamma\mathbb{E}[X]+\mathbb{E}[\varphi]\Big)\Big\}ds\\
&+\Big\{-C^\top X-\bar C^\top\mathbb{E}[X]+R(I+\Sigma R)^{-1}\Big(\Sigma C^\top(X-\mathbb{E}[X])-(\beta-\mathbb{E}[\beta])\Big)\\
&+(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\Big(\Sigma(C+\bar C)^\top\mathbb{E}[X]-\mathbb{E}[\beta]\Big)\Big\}dW\\
=&\Big\{-A^\top X-\bar A^\top\mathbb{E}[X]+QY+\bar Q\mathbb{E}[Y]\Big\}ds+\Big\{-C^\top X-\bar C^\top\mathbb{E}[X]+RZ+\bar R\mathbb{E}[Z]\Big\}dW.
\end{aligned}
$$

From the second equation in (5.1), we see

$$
\mathbb{E}[X(t)] = -[I+(G+\bar G)\Gamma(t)]^{-1}(G+\bar G)\mathbb{E}[\varphi(t)], \tag{5.8}
$$

$$
X(t)-\mathbb{E}[X(t)] = -[I+G\Sigma(t)]^{-1}G\{\varphi(t)-\mathbb{E}[\varphi(t)]\}. \tag{5.9}
$$

(5.5) and (5.8) yield

$$
[I+(G+\bar G)\Gamma(t)]\mathbb{E}[X(t)]=-(G+\bar G)\mathbb{E}[\varphi(t)]=(G+\bar G)\big\{\Gamma\mathbb{E}[X(t)]+\mathbb{E}[Y(t)]\big\},
$$

from which follows

$$
\mathbb{E}[X(t)]=(G+\bar G)\mathbb{E}[Y(t)]. \tag{5.10}
$$

Note that by (5.3) and (5.5),

$$
Y(t)-\mathbb{E}[Y(t)]=-\Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\}-\big\{\varphi(t)-\mathbb{E}[\varphi(t)]\big\},
$$

which, together with (5.9), yields

$$
\begin{aligned}
[I+G\Sigma(t)]\big\{X(t)-\mathbb{E}[X(t)]\big\} &= -G\big\{\varphi(t)-\mathbb{E}[\varphi(t)]\big\}\\
&= G\Big(\Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\}+Y(t)-\mathbb{E}[Y(t)]\Big),
\end{aligned}
$$

from which follows

$$
X(t)-\mathbb{E}[X(t)]=G\big\{Y(t)-\mathbb{E}[Y(t)]\big\}. \tag{5.11}
$$

Combining (5.10)–(5.11) we have

$$
X(t)=GY(t)+\bar G\mathbb{E}[Y(t)].
$$

Finally, observing that $u(\cdot)$ defined by (5.2) satisfies

$$
Nu+\bar N\mathbb{E}[u]-B^\top X-\bar B^\top\mathbb{E}[X]=0,
$$

we see that $(X(\cdot),Y(\cdot),Z(\cdot),u(\cdot))$ solves the optimality system (3.4)–(3.5). The result then follows immediately from Theorem 3.1. $\square$

We next present a formula for the value function of Problem (MF-BSLQ).

Theorem 5.2. *Let* (H1)–(H2) *hold. Then the value function of Problem* (MF-BSLQ) *is given by*

$$
\begin{aligned}
V(t,\xi) = \mathbb{E}\Big\{ & \big\langle G[I+\Sigma(t)G]^{-1}(\varphi(t)-\mathbb{E}[\varphi(t)]), \varphi(t)-\mathbb{E}[\varphi(t)]\big\rangle \\
& + \big\langle (G+\bar{G})[I+\Gamma(t)(G+\bar{G})]^{-1}\mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)]\big\rangle \\
& + \int_t^T \Big[\langle Q(\varphi-\mathbb{E}[\varphi]), \varphi-\mathbb{E}[\varphi]\rangle + \big\langle (Q+\bar{Q})\mathbb{E}[\varphi], \mathbb{E}[\varphi]\big\rangle \\
& + \big\langle (I+R\Sigma)^{-1}R(\beta-\mathbb{E}[\beta]), \beta-\mathbb{E}[\beta]\big\rangle \\
& + \big\langle [I+(R+\bar{R})\Sigma]^{-1}(R+\bar{R})\mathbb{E}[\beta], \mathbb{E}[\beta]\big\rangle \Big] ds \Big\}.
\end{aligned}
$$

where $\Sigma(\cdot)$ *and* $\Gamma(\cdot)$ *are the unique solutions to the Riccati equations* (3.20) *and* (3.21)*, respectively, and* $(\varphi(\cdot),\beta(\cdot))$ *is the unique adapted solution to the MF-BSDE* (3.22)*.*

Proof. Let $(Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ be the optimal triple corresponding to the terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, and let $X^*(\cdot)$ be the solution to MF-FSDE (3.1). According to Theorem 3.1, $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ satisfies the optimality system (3.4)–(3.5). On the other hand, let $X(\cdot)$ be the solution to (5.1), and let $u(\cdot)$, $Y(\cdot)$, and $Z(\cdot)$ be defined by (5.2), (5.3), and (5.4), respectively. We recall from the proof of Theorem 5.1 that $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot))$ also satisfies the optimality system (3.4)–(3.5). By the uniqueness of optimal controls, we must have

$$
(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot)) = (X(\cdot), Y(\cdot), Z(\cdot), u(\cdot)).
$$

Thus, the value $V(t,\xi)$ is equal to

$$
\begin{aligned}
J(t,\xi;u(\cdot)) &= \mathbb{E}\Big\{ \int_t^T \Big[\langle QY, Y\rangle + \big\langle \bar{Q}\mathbb{E}[Y], \mathbb{E}[Y]\big\rangle + \langle RZ, Z\rangle + \big\langle \bar{R}\mathbb{E}[Z], \mathbb{E}[Z]\big\rangle \\
&\quad + \langle Nu, u\rangle + \big\langle \bar{N}\mathbb{E}[u], \mathbb{E}[u]\big\rangle \Big] ds + \langle GY(t), Y(t)\rangle + \big\langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle \Big\} \\
&= \mathbb{E}\Big\{ \int_t^T \Big[\langle Q(Y-\mathbb{E}[Y]), Y-\mathbb{E}[Y]\rangle + \big\langle (Q+\bar{Q})\mathbb{E}[Y], \mathbb{E}[Y]\big\rangle \\
&\quad + \langle R(Z-\mathbb{E}[Z]), Z-\mathbb{E}[Z]\rangle + \big\langle (R+\bar{R})\mathbb{E}[Z], \mathbb{E}[Z]\big\rangle \\
&\quad + \langle N(u-\mathbb{E}[u]), u-\mathbb{E}[u]\rangle + \big\langle (N+\bar{N})\mathbb{E}[u], \mathbb{E}[u]\big\rangle \Big] ds \\
&\quad + \langle G(Y(t)-\mathbb{E}[Y(t)]), Y(t)-\mathbb{E}[Y(t)]\rangle + \big\langle (G+\bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle \Big\}.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\mathbb{E}[Y] &= -\Gamma\mathbb{E}[X] - \mathbb{E}[\varphi], \\
\mathbb{E}[Z] &= \big[I+\Sigma(R+\bar{R})\big]^{-1}\big\{\Sigma(C+\bar{C})^\top\mathbb{E}[X] - \mathbb{E}[\beta]\big\}, \\
\mathbb{E}[u] &= (N+\bar{N})^{-1}(B+\bar{B})^\top\mathbb{E}[X], \\
Y-\mathbb{E}[Y] &= -\Sigma(X-\mathbb{E}[X]) - (\varphi-\mathbb{E}[\varphi]), \\
Z-\mathbb{E}[Z] &= (I+\Sigma R)^{-1}\big\{\Sigma C^\top(X-\mathbb{E}[X]) - (\beta-\mathbb{E}[\beta])\big\}, \\
u-\mathbb{E}[u] &= N^{-1}B^\top(X-\mathbb{E}[X]),
\end{aligned}
$$

and using the fact that

$$(I+MN)^{-1}M = M(I+NM)^{-1}, \qquad \forall M,N\in\overline{\mathbb{S}^n_+},$$

it can be shown by a straightforward computation that

$$\begin{aligned}
&\mathbb{E}\int_t^T \Big[\langle Q(Y-\mathbb{E}[Y]),Y-\mathbb{E}[Y]\rangle+\langle R(Z-\mathbb{E}[Z]),Z-\mathbb{E}[Z]\rangle+\langle N(u-\mathbb{E}[u]),u-\mathbb{E}[u]\rangle\Big]ds \qquad (5.12)\\
&=\mathbb{E}\int_t^T\Big\{\Big\langle\Big[\Sigma Q\Sigma+C(I+\Sigma R)^{-1}\Sigma R\Sigma(I+R\Sigma)^{-1}C^\top+BN^{-1}B^\top\Big](X-\mathbb{E}[X]),X-\mathbb{E}[X]\Big\rangle\\
&\quad+2\langle X-\mathbb{E}[X],\Sigma Q(\varphi-\mathbb{E}[\varphi])\rangle-2\big\langle X-\mathbb{E}[X],C(I+\Sigma R)^{-1}\Sigma R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])\big\rangle\\
&\quad+\langle Q(\varphi-\mathbb{E}[\varphi]),\varphi-\mathbb{E}[\varphi]\rangle+\big\langle(I+R\Sigma)^{-1}R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta]),\beta-\mathbb{E}[\beta]\big\rangle\Big\}ds,
\end{aligned}$$

and that

$$\begin{aligned}
&\int_t^T\Big[\big\langle(Q+\bar Q)\mathbb{E}[Y],\mathbb{E}[Y]\big\rangle+\big\langle(R+\bar R)\mathbb{E}[Z],\mathbb{E}[Z]\big\rangle+\big\langle(N+\bar N)\mathbb{E}[u],\mathbb{E}[u]\big\rangle\Big]ds \qquad (5.13)\\
&=\int_t^T\Big\{\Big\langle\Big[\Gamma(Q+\bar Q)\Gamma+(C+\bar C)\big[I+\Sigma(R+\bar R)\big]^{-1}\Sigma(R+\bar R)\Sigma\big[I+(R+\bar R)\Sigma\big]^{-1}(C+\bar C)^\top\\
&\quad+(B+\bar B)(N+\bar N)^{-1}(B+\bar B)^\top\Big]\mathbb{E}[X],\mathbb{E}[X]\Big\rangle+2\big\langle\mathbb{E}[X],\Gamma(Q+\bar Q)\mathbb{E}[\varphi]\big\rangle\\
&\quad-2\big\langle\mathbb{E}[X],(C+\bar C)\big[I+\Sigma(R+\bar R)\big]^{-1}\Sigma(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\mathbb{E}[\beta]\big\rangle\\
&\quad+\big\langle(Q+\bar Q)\mathbb{E}[\varphi],\mathbb{E}[\varphi]\big\rangle+\big\langle\big[I+(R+\bar R)\Sigma\big]^{-1}(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\mathbb{E}[\beta],\mathbb{E}[\beta]\big\rangle\Big\}ds.
\end{aligned}$$

Observing that

$$\begin{cases}
d\mathbb{E}[X]=-\Big\{\big[A+\bar A+\Gamma(Q+\bar Q)\big]^\top\mathbb{E}[X]+(Q+\bar Q)\mathbb{E}[\varphi]\Big\}ds,\\
d(X-\mathbb{E}[X])=-\Big\{(A+\Sigma Q)^\top(X-\mathbb{E}[X])+Q(\varphi-\mathbb{E}[\varphi])\Big\}ds\\
\qquad-\Big\{(I+R\Sigma)^{-1}C^\top(X-\mathbb{E}[X])+\big[I+(R+\bar R)\Sigma\big]^{-1}(C+\bar C)^\top\mathbb{E}[X]\\
\qquad+(I+R\Sigma)^{-1}R(\beta-\mathbb{E}[\beta])+\big[I+(R+\bar R)\Sigma\big]^{-1}(R+\bar R)\mathbb{E}[\beta]\Big\}dW,
\end{cases}$$

we have by applying Itô's formula to $s\mapsto\langle\Sigma(s)(X(s)-\mathbb{E}[X(s)]),X(s)-\mathbb{E}[X(s)]\rangle$,

$$\begin{aligned}
&-\mathbb{E}\big\langle\Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\},X(t)-\mathbb{E}[X(t)]\big\rangle \qquad (5.14)\\
&=\mathbb{E}\int_t^T\Big\{\Big\langle\Big[\dot\Sigma-(A+\Sigma Q)\Sigma-\Sigma(A+\Sigma Q)^\top+C(I+\Sigma R)^{-1}\Sigma(I+R\Sigma)^{-1}C^\top\Big]\\
&\quad\cdot(X-\mathbb{E}[X]),X-\mathbb{E}[X]\Big\rangle-2\langle X-\mathbb{E}[X],\Sigma Q(\varphi-\mathbb{E}[\varphi])\rangle\\
&\quad+2\big\langle X-\mathbb{E}[X],C(I+\Sigma R)^{-1}\Sigma R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta])\big\rangle\\
&\quad+\langle(I+R\Sigma)^{-1}R\Sigma R(I+\Sigma R)^{-1}(\beta-\mathbb{E}[\beta]),\beta-\mathbb{E}[\beta]\rangle\Big\}ds\\
&\quad+\int_t^T\Big\{\big\langle(C+\bar C)\big[I+\Sigma(R+\bar R)\big]^{-1}\Sigma\big[I+(R+\bar R)\Sigma\big]^{-1}(C+\bar C)^\top\mathbb{E}[X],\mathbb{E}[X]\big\rangle\\
&\quad+2\big\langle\mathbb{E}[X],(C+\bar C)\big[I+\Sigma(R+\bar R)\big]^{-1}\Sigma(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\mathbb{E}[\beta]\big\rangle\\
&\quad+\big\langle\big[I+(R+\bar R)\Sigma\big]^{-1}(R+\bar R)\Sigma(R+\bar R)\big[I+\Sigma(R+\bar R)\big]^{-1}\mathbb{E}[\beta],\mathbb{E}[\beta]\big\rangle\Big\}ds,
\end{aligned}$$

and by applying the integration by parts formula to $s \mapsto \langle \Gamma(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)]\rangle$, we have

$$
\begin{aligned}
&-\langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)]\rangle \\
&\quad= \int_t^T \Big\{\Big\langle\Big(\dot{\Gamma} - \big[A+\bar{A}+\Gamma(Q+\bar{Q})\big]\Gamma - \Gamma\big[A+\bar{A}+\Gamma(Q+\bar{Q})\big]^\top\Big)\mathbb{E}[X], \mathbb{E}[X]\Big\rangle \\
&\qquad - 2\big\langle \mathbb{E}[X], \Gamma(Q+\bar{Q})\mathbb{E}[\varphi]\big\rangle\Big\}ds.
\end{aligned} \tag{5.15}
$$

Now adding equations (5.12), (5.13), (5.14) and (5.15) yields

$$
\begin{aligned}
V(t,\xi) = \mathbb{E}\Big\{&\langle G(Y(t)-\mathbb{E}[Y(t)]), Y(t)-\mathbb{E}[Y(t)]\rangle + \big\langle (G+\bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle \\
&+ \big\langle \Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\}, X(t)-\mathbb{E}[X(t)]\big\rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)]\rangle\Big\} \\
&+ \mathbb{E}\int_t^T \Big\{\langle Q(\varphi-\mathbb{E}[\varphi]), \varphi-\mathbb{E}[\varphi]\rangle + \big\langle (Q+\bar{Q})\mathbb{E}[\varphi], \mathbb{E}[\varphi]\big\rangle \\
&+ \big\langle (I+R\Sigma)^{-1}R(\beta-\mathbb{E}[\beta]), \beta-\mathbb{E}[\beta]\big\rangle + \big\langle \big[I+(R+\bar{R})\Sigma\big]^{-1}(R+\bar{R})\mathbb{E}[\beta], \mathbb{E}[\beta]\big\rangle\Big\}ds.
\end{aligned} \tag{5.16}
$$

Recalling that

$$
\mathbb{E}[Y] = -\Gamma\mathbb{E}[X] - \mathbb{E}[\varphi], \qquad Y - \mathbb{E}[Y] = -\Sigma(X-\mathbb{E}[X]) - (\varphi - \mathbb{E}[\varphi]),
$$

and noting that

$$
\begin{aligned}
\mathbb{E}[X(t)] &= -[I+(G+\bar{G})\Gamma(t)]^{-1}(G+\bar{G})\mathbb{E}[\varphi(t)], \\
X(t)-\mathbb{E}[X(t)] &= -[I+G\Sigma(t)]^{-1}G\{\varphi(t)-\mathbb{E}[\varphi(t)]\},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\mathbb{E}\Big\{\langle G(Y(t)-\mathbb{E}[Y(t)]), Y(t)-\mathbb{E}[Y(t)]\rangle + \big\langle (G+\bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)]\big\rangle \\
&\quad + \big\langle \Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\}, X(t)-\mathbb{E}[X(t)]\big\rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)]\rangle\Big\} \\
&= \mathbb{E}\Big\{\big\langle G\big\{\Sigma(t)(X(t)-\mathbb{E}[X(t)])+(\varphi(t)-\mathbb{E}[\varphi(t)])\big\}, \Sigma(t)(X(t)-\mathbb{E}[X(t)])+(\varphi(t)-\mathbb{E}[\varphi(t)])\big\rangle \\
&\qquad + \big\langle (G+\bar{G})\big\{\Gamma(t)\mathbb{E}[X(t)]+\mathbb{E}[\varphi(t)]\big\}, \Gamma(t)\mathbb{E}[X(t)]+\mathbb{E}[\varphi(t)]\big\rangle \\
&\qquad + \big\langle \Sigma(t)\big\{X(t)-\mathbb{E}[X(t)]\big\}, X(t)-\mathbb{E}[X(t)]\big\rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)]\rangle\Big\} \\
&= \mathbb{E}\Big\{\langle \Sigma(t)[I+G\Sigma(t)](X(t)-\mathbb{E}[X(t)]), X(t)-\mathbb{E}[X(t)]\rangle \\
&\qquad + 2\langle G\Sigma(t)(X(t)-\mathbb{E}[X(t)]), \varphi(t)-\mathbb{E}[\varphi(t)]\rangle + \langle G(\varphi(t)-\mathbb{E}[\varphi(t)]), \varphi(t)-\mathbb{E}[\varphi(t)]\rangle \\
&\qquad + \big\langle \Gamma(t)[I+(G+\bar{G})\Gamma(t)]\mathbb{E}[X(t)], \mathbb{E}[X(t)]\big\rangle + 2\big\langle (G+\bar{G})\Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[\varphi(t)]\big\rangle \\
&\qquad + \big\langle (G+\bar{G})\mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)]\big\rangle\Big\} \\
&= \mathbb{E}\Big\{\big\langle G[I+\Sigma(t)G]^{-1}(\varphi(t)-\mathbb{E}[\varphi(t)]), \varphi(t)-\mathbb{E}[\varphi(t)]\big\rangle \\
&\qquad + \big\langle (G+\bar{G})[I+\Gamma(t)(G+\bar{G})]^{-1}\mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)]\big\rangle\Big\}.
\end{aligned}
$$

Substitution of the above into (5.16) completes the proof. $\square$

Acknowledgement. The authors would like to thank the anonymous referees for their suggestive comments, which lead to an improvement of the paper.

References

[1] Ait Rami, M., Moore, J.B., Zhou, X.Y.: Indefinite stochastic linear quadratic control and generalized differential Riccati equation. SIAM J. Control Optim. **40**, 1296–1311 (2001)

[2] Andersson, D., Djehiche, B.: A maximum principle for SDEs of mean-field type. Appl. Math. Optim. **63**, 341–356 (2011)

[3] Bensoussan, A., Sung, K.C.J., Yam, S.C.P.: Linear-quadratic time-inconsistent mean field games. Dyn. Games Appl. **3**, 537–552 (2013)

[4] Bensoussan, A., Sung, K.C.J., Yam, S.C.P., Yung, S.P.: Linear-quadratic mean field games. J. Optim. Theory Appl. **169**, 496–529 (2016)

[5] Borkar, V.S., Kumar, K.S.: McKean-Vlasov limit in portfolio optimization. Stochastic Anal. Appl. **28**, 884–906 (2010)

[6] Buckdahn, R., Djehiche, B., Li, J.: A general stochastic maximum principle for SDEs of mean-field type. Appl. Math. Optim. **64**, 197–216 (2011)

[7] Buckdahn, R., Djehiche, B., Li, J., Peng, S.: Mean-field backward stochastic differential equations: A limit approach. Ann. Probab. **37**, 1524–1565 (2009)

[8] Buckdahn, R., Li, J., Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. Stochastic Process. Appl. **119**, 3133–3154 (2009)

[9] Carmona, R., Delarue, F.: Forward-backward stochastic differential equations and controlled McKean-Vlasov dynamics. Ann. Probab. **43**, 2647–2700 (2015)

[10] Carmona, R., Delarue, F.: Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games. Springer-Verlag, to appear (2017)

[11] Carmona, R., Delarue, F., Lachapelle, A.: Control of McKean-Vlasov dynamics versus mean field games. Math. Financ. Econ. **7**, 131–166 (2013)

[12] Chan, T.: Dynamics of the McKean-Vlasov equation. Ann. Probab. **22**, 431–441 (1994)

[13] Chen, S., Li, X., Zhou, X.Y.: Stochastic linear quadratic regulators with indefinite control weight costs. SIAM J. Control Optim. **36**, 1685–1702 (1998)

[14] Crisan, D., Xiong, J.: Approximate McKean-Vlasov representations for a class of SPDEs. Stochastics **82**, 53–68 (2010)

[15] Dawson, D.A.: Critical dynamics and fluctuations for a mean-field model of cooperative behavior. J. Statist. Phys. **31**, 29–85 (1983)

[16] Graber, P.J.: Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource. Appl. Math. Optim. **74**, 459–486 (2016)

[17] Huang, M., Caines, P.E., Malhamé, R.P.: Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria. IEEE Trans. Automat. Control **52**, 1560–1571 (2007)

[18] Huang, M., Caines, P.E., Malhamé, R.P.: Social optima in mean field LQG control: centralized and decentralized strategies. IEEE Trans. Automat. Control **57**, 1736–1751 (2012)

[19] Huang, J., Li, X., Yong, J.: A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon. Math. Control Relat. Fields **5**, 97–139 (2015)

[20] Huang, M., Malhamé, R.P., Caines, P.E.: Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst. **6**, 221–252 (2006)

[21] Kac, M.: Foundations of kinetic theory. In: Proc. 3rd Berkeley Symp. Math. Stat. Prob. **3**, 171–197 (1956)

[22] Lasry, J.-M., Lions, P.-L.: Mean field games. Jpn. J. Math. **2**, 229–260 (2007)

[23] Li, X., Sun, J., Yong, J.: Mean-field stochastic linear quadratic optimal control problems: closed-loop solvability. Probab. Uncertain. Quant. Risk **1**, 2 (2016). doi:10.1186/s41546-016-0002-3.

[24] Lim, A.E.B., Zhou, X.Y.: Linear-quadratic control of backward stochastic differential equations. SIAM J. Control Optim. **40**, 450–474 (2001)

[25] Ma, J., Protter, P., Yong, J.: Solving forward-backward stochastic differential equations explicitly: A four-step scheme. Probab. Theory Related Fields **98**, 339–359 (1994)

[26] Ma, J., Yong, J.: Forward-Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Math. 1702, Springer-Verlag, New York (1999)

[27] McKean, H.P.: A class of Markov processes associated with nonlinear parabolic equations. Proc. Natl. Acad. Sci. USA **56**, 1907–1911 (1966)

[28] Meyer-Brandis, T., Oksendal, B., Zhou, X.Y.: A mean-field stochastic maximum principle via Malliavin calculus. Stochastics **84**, 643–666 (2012)

[29] Pham, H.: Linear quadratic optimal control of conditional McKean-Vlasov equation with random coefficients and applications. Probab. Uncertain. Quant. Risk **1**, 7 (2016)

[30] Sun, J.: Mean-field stochastic linear quadratic optimal control problems: Open-loop solvabilities. ESAIM: COCV. **23**, 1099–1127 (2017)

[31] Sun, J., Li, X., Yong, J.: Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. SIAM J. Control Optim. **54**, 2274–2308 (2016)

[32] Sun, J., Yong, J.: Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points. SIAM J. Control Optim. **52**, 4082–4121 (2014)

[33] Tang, S.: General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations. SIAM J. Control Optim. **42**, 53–75 (2003)

[34] Wonham, W.M.: On a matrix Riccati equation of stochastic control. SIAM J. Control **6**, 681–697 (1968)

[35] Yong, J.: Linear-quadratic optimal control problems for mean-field stochastic differential equations. SIAM J. Control Optim. **51**, 2809–2838 (2013)

[36] Yong, J.: Linear-quadratic optimal control problems for mean-field stochastic differential equations – Time-consistent solutions. Transactions of the AMS (2015)

[37] Yong, J., Zhou, X.Y.: Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, New York (1999)