Multi-period asset-liability management with cash flows and probability constraints: A mean-field formulation approach☆

Xun Li[a], Xianping Wu[b], Haixiang Yao[c,*]

[a]*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China*
[b]*School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China*
[c]*School of Finance, Guangdong University of Foreign Studies, Guangzhou, 510006, China*

Abstract

Using a multi-period mean-variance model, we investigate an asset-liability portfolio management problem with probability constraints, where an investor intends to control the probability of bankruptcy before the terminal time in the investment. In our model, the wealth process is influenced not only by return on assets and liability but also by uncontrolled cash flows. Applying a mean-field formulation, we obtain closed-form expressions for an efficient investment strategy and its corresponding mean-variance efficient frontier. Sensitivity analysis is also presented to help investors understand the influences of cash flows and probability constraints better.

Keywords: Mean-field formulation; multi-period mean-variance model; asset-liability management; probability constraints; cash flow.

1. Introduction

Since introduced by Markowitz (1952), the mean-variance portfolio selection model has become one of the central themes of modern portfolio theory. In recent years, Li and Ng (2000) and Zhou and Li (2000) respectively extend the Markowitz's mean-variance

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*Corresponding author. Tel.: +8620 37105360.
Email addresses: malixun@polyu.edu.hk (Xun Li), pphappe@sina.com (Xianping Wu), yaohaixiang@gdufs.edu.cn (Haixiang Yao)

model to dynamic discrete-time and continuous-time settings by deriving analytical optimal solutions using embedding techniques. The past decade has witnessed numerous extensions of the mean-variance portfolio analysis in dynamic settings, for example, Li et al. (2002), Li and Zhou (2006), Fu et al. (2010), Li and Xie (2010), Chiu and Wong (2011), Wu and Li (2012), Cui et al (2014a, 2014b).

Along another line, it is well known that asset-liability management (ALM) is extremely important to financial security systems such as pension funds, insurance companies and banks. In ALM, the main concern is the surplus, i.e., the net wealth. Accordingly, ALM is also known as surplus management. Based on a mean-variance criterion, Sharpe and Tint (1990) initiate an ALM problem in a single-period setting. Leippold et al. (2004) discuss a multi-period ALM problem using a geometric approach and Yi et al. (2008) extend their work to the case with uncertain exit time. In addition, Cui et al. (2018) study the same problem by taking advantage of the mean-field formulation. Ferstl and Weissensteiner (2011) consider a multi-stage setting under time-varying investment opportunities and analyze ALM by stochastic programming. Gülpinar and Pachamanova (2013) propose a robust optimization approach to dynamic ALM under time-varying investment opportunities. Consigli et al. (2017) adopt a fairly general framework to study a variety of financial dynamic optimization models including ALM. Chiu and Li (2006) investigate continuous-time ALM problems, where the liability process is described by a geometric Brownian motion. Xie et al. (2008) further consider the case where the liability is described as a Brownian motion with drift. Zeng and Li (2011) propose continuous-time ALM under benchmark in a jump diffusion market, where the risky asset's price is governed by an exponential Levy process and the liability evolves according to a Levy process. Yao et al. (2013a) consider mean-variance portfolio selection problems with endogenous liabilities in a multi-period setting. The ALM problems are also studied associating with regime switching, with asset correlation risks, with constant elasticity of variance processes, with state-dependent risk aversion, and with stochastic interest rates and inflation risks. See Chen and Yang (2011), Chiu and Wong (2014), Zhang and Chen (2016), Zhang et al. (2017), Pan and Xiao (2017), for details.

However, there still exists a gap between academic research and practice. The literature mentioned above ignores cash flows of investors including individual investors and financial institutions. In the real-world, investors might face the situations of capital injections or withdrawals during their investment processes. For example, households may need cash to maintain their daily lives or add their residual income into the investment; insurers can receive insurance premium and need to pay for claim; pension funds may get contributions or issue distributions for their members. In most cases, cash flows are random. Hence, many investors, such as households, insures, pension funds and banks, need to take into account their stochastic cash flows during their investments and ALM processes. Recently, Yao et al. (2013b) study a multi-period mean-variance ALM problem with uncontrolled cash flow in the liability process. Yao et al. (2016) incorporate stochastic cash flow in both the liability and wealth processes, and investigate an ALM problem in the Markov regime-switching setting. In this paper, we incorporate stochastic cash flow into the wealth process, and study an ALM problem with probability constraints. Our model with stochastic cash flow in the wealth process is useful for real-life investors. For example, a company may unexpectedly receive government subsidies for its product development and a household may receive income. This type of cash flow substantially tests an investor's investment and ALM skills.

For another example, the optimal investment policies of Yao et al. (2013b, 2016) do not eliminate the possibility that investors may go bankruptcy on or before the terminal time. In practice, when investors (e.g., pension funds, insurance companies and banks, etc.) go bankruptcy, the benefits of investors cannot be guaranteed. Therefore, it is important and meaningful to choose optimal ALM strategies under the bankruptcy control. Technically, the bankruptcy control (probability constraints) is not directly incorporated, and therefore we characterize this constraint using the Tchebycheff inequality. Under the mean-variance criterion, Zhu et al. (2004) and Bielecki et al. (2005) consider multi-period and continuous-time portfolio selection problems with bankruptcy control, respectively; Wei and Ye (2007) investigate a multi-period portfolio selection problem with bankruptcy control in the Markov regime-switching market; Wu and Zeng (2013) study the case in the regime-switching market with a state of bankruptcy. In particular, Li and Xu study continuous-time mean-variance portfolio within the framework of no-shorting and bankruptcy prohibition. But all these studies on dynamic portfolio selections with bankruptcy control do not consider the ALM problem. For the ALM problem, Li and Li (2012) discuss a multi-period portfolio selection problem with bankruptcy control; Wu et al. (2018) study the case with probability constraints using the mean-filed formulation. However, they do not consider the cash flow. In the above multi-period models, researchers do not successfully obtain analytical solutions for the auxiliary problem induced by the embedding scheme. They use a numerical algorithm to compute both the Lagrangian multiplier vector and the embedding parameter vector. In this work, we introduce mean-field formulation to overcome its fundamental difficulty and derive analytical policies of the Lagrangian problem involved in bankruptcy probability. We only need to use an analytical iterative algorithm to compute the Lagrangian multiplier vector. Therefore, the mean-field formulation powerfully offers a more efficient and more accurate solution scheme in solving the multi-period portfolio selection problem with bankruptcy control.

To our knowledge, all the existing literature on dynamic ALM analysis is either without cash flow or without probability constraints (or bankruptcy control). This is the first study considering a multi-period ALM problem with both cash flows and probability constraints. In this paper, incorporating the uncontrolled cash flow factor into the wealth dynamic process, we further explore the multi-period ALM problem with probability constraints under the mean-variance criterion. From the mathematical point of view, consideration of a stochastic cash flow and the probability constraints makes the problem harder. The probability constraints lead to inequality constraints on the surplus over intermediate periods in our model. Incorporating stochastic cash flow into the wealth dynamic process further increases the computational complexity in obtaining closed-form solutions to the model. Different from most literature of multi-period ALM mean-variance models which adopt the embedding technique, we apply the mean-field formulation initiated by Cui et al. (2014b) to the proposed model in this paper. Compared to the embedding technique, the mean-field approach is relatively simple yet more direct. However, Cui et al. (2014b) study one state process only including the wealth process without the liability process and cash flow. We further develop the mean-field formulation to consider a more general model of two state processes including not only the wealth but also the liability and cash flow within framework of probability constraints. Our work, therefore, is fundamental yet important for dynamic mean-variance portfolio.

This paper proceeds as follows. In Section 2, we describe our multi-period mean-variance ALM problem with both cash flows and probability constraints. In Section 3, using mean-field formulation, we derive the explicit expressions of the optimal strategy and the efficient frontier for the problem. In Section 4, we present results from numerical analysis. Finally, we conclude this paper in Section 5.

2. Formulation

Assume that the capital market consists of one risk-free asset, n risky assets and one liability. An investor joining the market at the beginning of period 0 with an initial wealth x_0 and initial liability l_0, plans to invest his/her wealth within a time horizon T. And there would be a cash flow during the investment process. The investor can reallocate his/her portfolio at the beginning of each following $T-1$ consecutive periods.

At time period t, the given deterministic return of the risk-free asset, the random returns of the n risky assets, the random return of the liability and the uncontrolled cash flow are denoted by s_t (> 1), vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$, q_t and c_t, respectively. The random vector $\mathbf{e}_t$, the random variables q_t and c_t are defined over the probability space $(\Omega, \mathcal{F}, P)$ and are supposed to be statistically independent at different time period. We further assume that the only information known about $\mathbf{e}_t$, q_t and c_t are their first two unconditional moments, $\mathbb{E}[\mathbf{e}_t] = \big(\mathbb{E}[e_t^1], \cdots, \mathbb{E}[e_t^n]\big)'$, $\mathbb{E}[q_t]$, $\mathbb{E}[c_t]$ and $(n+2) \times (n+2)$ positive definite covariance

$$\operatorname{Cov}\left(\begin{pmatrix} \mathbf{e}_t \\ q_t \\ c_t \end{pmatrix}\right) = \mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t \\ q_t \\ c_t \end{pmatrix} \begin{pmatrix} \mathbf{e}_t' & q_t & c_t \end{pmatrix}\right] - \mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t \\ q_t \\ c_t \end{pmatrix}\right] \mathbb{E}\left[\begin{pmatrix} \mathbf{e}_t' & q_t & c_t \end{pmatrix}\right].$$

From the above assumptions, we have

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] & s_t\mathbb{E}[c_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] & \mathbb{E}[\mathbf{e}_t c_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] & \mathbb{E}[q_t c_t] \\ s_t\mathbb{E}[c_t] & \mathbb{E}[c_t\mathbf{e}_t'] & \mathbb{E}[c_t q_t] & \mathbb{E}[c_t^2] \end{pmatrix} \succ 0.$$

We further define the excess return vector of risky assets $\mathbf{P}_t = (P_t^1, \cdots, P_t^n)'$ as $(e_t^1 - s_t, \cdots, e_t^n - s_t)'$. Then the following is held for $t = 0, 1, \cdots, T-1$:

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{P}_t'] & s_t\mathbb{E}[q_t] & s_t\mathbb{E}[c_t] \\ s_t\mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] & \mathbb{E}[\mathbf{P}_t q_t] & \mathbb{E}[\mathbf{P}_t c_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{P}_t'] & \mathbb{E}[q_t^2] & \mathbb{E}[q_t c_t] \\ s_t\mathbb{E}[c_t] & \mathbb{E}[c_t\mathbf{P}_t'] & \mathbb{E}[c_t q_t] & \mathbb{E}[c_t^2] \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{0}' & 0 & 0 \\ -\mathbf{1} & I & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0}' & 1 & 0 \\ 0 & \mathbf{0}' & 0 & 1 \end{pmatrix} \begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] & s_t\mathbb{E}[c_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] & \mathbb{E}[\mathbf{e}_t c_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] & \mathbb{E}[q_t c_t] \\ s_t\mathbb{E}[c_t] & \mathbb{E}[c_t\mathbf{e}_t'] & \mathbb{E}[c_t q_t] & \mathbb{E}[c_t^2] \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{1}' & 0 & 0 \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0}' & 1 & 0 \\ 0 & \mathbf{0}' & 0 & 1 \end{pmatrix} \succ 0,$$

where $\mathbf{1}$ and $\mathbf{0}$ are the n-dimensional all-one and all-zero vectors, respectively, and I is the $n \times n$ identity matrix. The above positive definite matrix further implies

$$\begin{aligned} \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] &\succ 0, \\ s_t^2(1 - \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]) &> 0, \\ \mathbb{E}[q_t^2] - \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t] &> 0, \\ \mathbb{E}[c_t^2] - \mathbb{E}[c_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t c_t] &> 0, \end{aligned}$$

for $t = 0, 1, \cdots, T-1$.

For later use, denote

$$\begin{aligned}
B_t &\triangleq \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],\\
\widehat{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],\\
\widetilde{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t],\\
\widehat{BC}_t &\triangleq \mathbb{E}[c_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],\\
\widetilde{BC}_t &\triangleq \mathbb{E}[c_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t c_t],\\
\overline{BC}_t &\triangleq \mathbb{E}[c_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t],
\end{aligned}$$

where $\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']$ is the inverse of matrix $\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'](\succ 0)$. Similar to Cui et al (2014b), we have $0 < B_t < 1$, $\mathbb{E}[q_t^2] - \widetilde{B}_t > 0$ and $\mathbb{E}[c_t^2] - \widetilde{BC}_t > 0$ for $t = 0, 1, \cdots, T-1$.

Let x_t and l_t be the wealth and liability of the investor at the beginning of period t respectively, then $x_t - l_t$ is the surplus. At period t, if π_t^i $(i = 1, 2, \cdots, n)$ is the amount invested in the i-th risky asset, then, $x_t - \sum_{i=1}^n \pi_t^i$ is the amount invested in the risk-free asset. We assume in this paper that the liability is exogenous, which means it is uncontrollable and cannot be affected by the investor's strategies. Denote the information set at the beginning of period t, $t = 1, 2, \cdots, T-1$, as

$$\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1}, c_0, c_1, \cdots, c_{t-1}, q_0, q_1, \cdots, q_{t-1})$$

and the trivial σ-algebra over Ω as $\mathcal{F}_0$, where $\mathbf{P}_t = (P_t^1, \cdots, P_t^n)' = (e_t^1 - s_t, \cdots, e_t^n - s_t)'$ is the excess return vector of risky assets during period t. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is just the unconditional expectation $\mathbb{E}[\cdot]$. We confine all admissible investment strategies to be $\mathcal{F}_t$-adapted Markov controls, i.e., $\pi_t = (\pi_t^1,\ \pi_t^2,\ \cdots,\ \pi_t^n)' \in \mathcal{F}_t$. Then, $\mathbf{P}_t$ and π_t are independent, $\{x_t, l_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t, l_t)$.

The mean-variance model for multi-period assets and liability portfolio selection with cash flows and probability constraints is to seek the best strategy, $\pi_t^* = [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the optimizer of the following optimal stochastic control problem,

$$\left\{\begin{array}{ll}
\min & \mathrm{Var}(x_T - l_T) - w\mathbb{E}[x_T - l_T],\\
\text{s.t.} & x_{t+1} = \displaystyle\sum_{i=1}^n e_t^i\pi_t^i + \left(x_t - \sum_{i=1}^n \pi_t^i\right)s_t + c_t\\
& \qquad = s_t x_t + \mathbf{P}_t'\pi_t + c_t, \qquad t = 0, 1, \cdots, T-1,\\
& l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1,\\
& \Pr(x_t \le l_t) \le a_t, \qquad t = 1, 2, \cdots, T-1,
\end{array}\right. \tag{1}$$

where $w > 0$ is the trade-off parameter between the mean and the variance, and a_t is the probability of bankruptcy, that is, the probability of the wealth less than the liability at period t.

Remark 1. *We consider stochastic cash flows in the wealth process. This setting allows us cover different types of ALM, which can be applied by many kinds of investors, including household, firm, investment company (e.g. mutual fund and pension fund) and bank. Now we give some examples for some cases.*

Case 1 (general case): $c_t \neq 0$ for $t = 0, 1, \cdots, T-1$. There is a stochastic cash flow in the wealth process. For example, a firm can receive subsidies from various government agencies; a firm is required to make contingent payments; a pension fund can receive contributions from pensioners and/or make pension payments to beneficiaries.

Case 2: $c_t = 0$ for $t = 0, 1, \cdots, T-1$. There is no stochastic cash flow in the wealth process. For example, when a company makes purchases/sales on credit.

Case 3: $l_0 = 0$ for $t = 0, 1, \cdots, T-1$. There is no liability. For example, an equity financed firm.

Case 4: $l_0 = 0$ and $c_t = 0$ for $t = 0, 1, \cdots, T-1$. There is no liability and no cash flow. In this case our model degenerates to a conventional portfolio selection model.

Since the probability constraint $\Pr(x_t \le l_t)$ is not easy to conquer in dynamic portfolio selection, we turn it to its upper bound $\mathrm{Var}(x_t - l_t)/(\mathbb{E}[x_t - l_t])^2$ by Tchebycheff inequality. Then the mean-variance model (1) can be equivalently re-written to the following problem,

$$\begin{cases} \min & \mathrm{Var}(x_T - l_T) - w\mathbb{E}[x_T - l_T], \\ \text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t + c_t, \qquad t = 0, 1, \cdots, T-1, \\ & l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1, \\ & \mathrm{Var}(x_t - l_t) \le a_t (\mathbb{E}[x_t - l_t])^2, \qquad t = 1, \cdots, T-1. \end{cases} \tag{2}$$

The optimal solution to problem (2) is feasible in problem (1), thus it serves as an approximated solution to problem (1). To solve problem (2), we consider the following Lagrangian minimization problem,

$$\begin{cases} \min & \mathrm{Var}(x_T - l_T) - w\mathbb{E}[x_T - l_T] + \sum_{t=1}^{T-1} \lambda_t \Big(\mathrm{Var}(x_t - l_t) - a_t (\mathbb{E}[x_t - l_t])^2 \Big), \\ \text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t + c_t, \qquad t = 0, 1, \cdots, T-1, \\ & l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1, \end{cases} \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{T-1})' \in \mathbb{R}_+^{T-1}$ is the vector of Lagrangian multiplier.

Due to the fact that variance operation does not satisfy the smoothing property, problem (3) is nonseparable in the sense of dynamic programming, i.e., it can not be decomposed by a stage-wise backward recursion and then is difficult to be solved directly. Recently, Elliott et al. (2013), Cui et al. (2014) and Ni et al. (2015) introduce mean-field formulations to deal with a class of discrete-time/multi-period nonseparable problems. We further develop their mean-field formulations to tackle our model in this paper. For $t = 0, 1, \cdots, T-1$, taking the expectation operator of the dynamic system specified in (3), we can derive

$$\begin{cases} \mathbb{E}[x_{t+1}] = s_t \mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t'] \mathbb{E}[\pi_t] + \mathbb{E}[c_t], \\ \mathbb{E}[l_{t+1}] = \mathbb{E}[q_t] \mathbb{E}[l_t], \\ \quad \mathbb{E}[x_0] = x_0, \\ \quad \mathbb{E}[l_0] = l_0, \end{cases} \tag{4}$$

since $\mathbf{P}_t$ is independent of x_t and π_t, and q_t is independent of l_t. Combining the dynamic system of (3) and (4) yields the following for $t = 0, 1, \cdots, T-1$,

$$\begin{cases} x_{t+1} - \mathbb{E}[x_{t+1}] = s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\pi_t - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + c_t - \mathbb{E}[c_t] \\ \qquad\qquad\qquad = s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t - \mathbb{E}[\pi_t]\big) + \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\ \qquad\qquad\qquad\quad + c_t - \mathbb{E}[c_t], \\ l_{t+1} - \mathbb{E}[l_{t+1}] = q_t l_t - \mathbb{E}[q_t]\mathbb{E}[l_t] \\ \qquad\qquad\qquad = q_t\big(l_t - \mathbb{E}[l_t]\big) + \big(q_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t], \\ x_0 - \mathbb{E}[x_0] = 0, \\ l_0 - \mathbb{E}[l_0] = 0. \end{cases} \tag{5}$$

Then the state space (x_t, l_t) and the control space (π_t) are enlarged into $(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t])$ and $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$, respectively. Although we can select the control vector $\mathbb{E}[\pi_t]$ and $\pi_t - \mathbb{E}[\pi_t]$ independently at time t, they should be chosen such that

$$\mathbb{E}(\pi_t - \mathbb{E}[\pi_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1,$$

and then

$$\mathbb{E}(x_t - \mathbb{E}[x_t]) = 0, \quad t = 0, 1, \cdots, T-1,$$

is satisfied. We also confine admissible investment strategies $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$ to be $\mathcal{F}_t$-measurable Markov controls.

Problem (3) can be now reformulated as the following mean-field type of linear quadratic optimal stochastic control problem

$$\begin{cases} \min \ \ \mathbb{E}\big[\big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2\big] - w\mathbb{E}[x_T - l_T] \\ \qquad\quad + \sum_{t=1}^{T-1} \Big\{\lambda_t \mathbb{E}\big[\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2\big] - \lambda_t a_t (\mathbb{E}[x_t - l_t])^2\Big\}, \\ \text{s.t.} \ \ \{\mathbb{E}[x_t], \mathbb{E}[l_t], \mathbb{E}[\pi_t]\} \text{ satisfies dynamic equation (4)}, \\ \qquad \{x_t - \mathbb{E}[x_t], l_t - \mathbb{E}[l_t], \pi_t - \mathbb{E}[\pi_t]\} \text{ satisfies dynamic equation (5)}, \\ \qquad \mathbb{E}(\pi_t - \mathbb{E}[\pi_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{6}$$

It is indeed a separable linear quadratic optimal stochastic control problem which can be solved by classic dynamic programming approach.

3. The Optimal Strategy

In order to obtain the explicit expressions of the the cost-to-go functional and the optimal strategy for Problem (6), we define nine deterministic sequences of parameters $\{\xi_t\}$, $\{\eta_t\}$, $\{\epsilon_t\}$, $\{\beta_t\}$, $\{\zeta_t\}$, $\{\phi_t\}$, $\{\theta_t\}$, $\{\delta_t\}$ and $\{\psi_t\}$ by the following backward recursions as follows

$$\begin{aligned}
\xi_t &= \xi_{t+1} s_t^2 (1 - B_t) + \lambda_t, \\
\eta_t &= \eta_{t+1} s_t (\mathbb{E}[q_t] - \widehat{B}_t) - \lambda_t, \\
\epsilon_t &= \epsilon_{t+1} \mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}} \widetilde{B}_t + \lambda_t, \\
\beta_t &= \beta_{t+1} s_t^2 - \frac{\beta_{t+1}^2 s_t^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} - \lambda_t a_t, \\
\zeta_t &= \big(2\beta_{t+1}\mathbb{E}[c_t] + \zeta_{t+1}\big) s_t - \frac{2B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \beta_{t+1} s_t \big((\beta_{t+1} - \xi_{t+1})\mathbb{E}[c_t] + \frac{1}{2}\zeta_{t+1}\big) \\
&\quad - \frac{2\widehat{BC}_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \beta_{t+1} s_t \xi_{t+1}, \\
\phi_t &= \phi_{t+1} s_t \mathbb{E}[q_t] - \frac{B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \beta_{t+1} s_t \big(\phi_{t+1} - \eta_{t+1}\big)\mathbb{E}[q_t] \\
&\quad - \frac{\widehat{B}_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \beta_{t+1} s_t \eta_{t+1} + \lambda_t a_t,
\end{aligned}$$

$$
\begin{aligned}
\theta_t &= \theta_{t+1}\mathbb{E}[q_t] + 2\eta_{t+1}(\mathbb{E}[q_t c_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]) + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[c_t] \\
&\quad - \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big(\phi_{t+1}-\eta_{t+1}\big)\mathbb{E}[q_t]\big((\beta_{t+1}-\xi_{t+1})\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\big) \\
&\quad - \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big(\phi_{t+1}-\eta_{t+1}\big)\mathbb{E}[q_t]\xi_{t+1} \\
&\quad - \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\eta_{t+1}\big((\beta_{t+1}-\xi_{t+1})\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\big) \\
&\quad - 2\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\overline{BC}_t\Big)\eta_{t+1}\xi_{t+1}, \\
\delta_t &= \delta_{t+1}\big(\mathbb{E}[q_t]\big)^2 + \epsilon_{t+1}\big(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2\big) - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big(\phi_{t+1}-\eta_{t+1}\big)^2\big(\mathbb{E}[q_t]\big)^2 \\
&\quad - \Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{B}_t\Big)\eta_{t+1}^2 \\
&\quad - \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big(\phi_{t+1}-\eta_{t+1}\big)\mathbb{E}[q_t]\eta_{t+1} - \lambda_t a_t, \\
\psi_t &= \psi_{t+1} + \beta_{t+1}\big(\mathbb{E}[c_t]\big)^2 + \zeta_{t+1}\mathbb{E}[c_t] + \xi_{t+1}\mathbb{E}[(c_t-\mathbb{E}[c_t])^2] \\
&\quad - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big((\beta_{t+1}-\xi_{t+1})\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\big)^2 \\
&\quad - \Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{BC}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{BC}_t\Big)\xi_{t+1}^2 \\
&\quad - \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\big((\beta_{t+1}-\xi_{t+1})\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\big)\xi_{t+1},
\end{aligned}
$$

for $t = T-1, T-2, \cdots, 0$, with terminal conditions

$$
\xi_T = 1,\ \eta_T = -1,\ \epsilon_T = 1,\ \beta_T = 0,\ \zeta_T = -w,\ \phi_T = 0,\ \theta_T = w,\ \delta_T = 0,\ \psi_T = 0,
$$

where $\lambda_0 = 0$.

Now, we present the solution to the cost-to-go functional and the optimal portfolio strategy of Problem (3) by the following theorem.

Theorem 1. *Assume that the assets and liability are correlated at every period. Then, the cost-to-go functional is presented by*

$$
\begin{aligned}
&J_t\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t], \mathbb{E}[l_t], l_t-\mathbb{E}[l_t]\big) \\
&= \xi_t\big(x_t-\mathbb{E}[x_t]\big)^2 + 2\eta_t\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \epsilon_t\big(l_t-\mathbb{E}[l_t]\big)^2 \\
&\quad + \beta_t(\mathbb{E}[x_t])^2 + \zeta_t\mathbb{E}[x_t] + 2\phi_t\mathbb{E}[l_t]\mathbb{E}[x_t] + \theta_t\mathbb{E}[l_t] + \delta_t(\mathbb{E}[l_t])^2 + \psi_t, \qquad (7)
\end{aligned}
$$

and the optimal strategy of Problem (3) is given by

$$\begin{aligned}\pi_t^* = &-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\Bigg(s_t x_t - \frac{(\xi_{t+1}-\beta_{t+1})(1-B_t)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} s_t \mathbb{E}[x_t] \\ &+ \frac{\phi_{t+1}\mathbb{E}[q_t] + \eta_{t+1}\big((1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t - \mathbb{E}[q_t]\big)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[l_t] \\ &+ \frac{\big(\beta_{t+1}-\xi_{t+1}\big)\mathbb{E}[c_t] + \frac{1}{2}\zeta_{t+1} + (\xi_{t+1}-\beta_{t+1})\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\Bigg) \\ &- \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t] l_t - \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[c_t\mathbf{P}_t],\end{aligned} \tag{8}$$

where

$$\begin{aligned}\mathbb{E}[x_t] = &\, x_0 \prod_{j=0}^{t-1} \frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j} - \sum_{k=0}^{t-1}\Bigg(\prod_{j=k+1}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\Bigg) \\ &\times \Bigg(\frac{\phi_{k+1}\mathbb{E}[q_k]B_k + \eta_{k+1}\big(\widehat{B}_k - \mathbb{E}[q_k]B_k\big)}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}\Big(\prod_{j=0}^{k-1}\mathbb{E}[q_j]\Big) l_0 \\ &+ \frac{\xi_{k+1}\mathbb{E}[c_k] - \frac{1}{2}\zeta_{k+1}B_k}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k} - \frac{\xi_{k+1}\widehat{BC}_k}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}\Bigg).\end{aligned} \tag{9}$$

for $t = 0, 1, \cdots, T-1$.

The detailed proof of Theorem 1 can be found in Appendix A. By dynamic programming method, we post the cost-to-go functional at terminal time T according to our model first. Then we present and prove the expression of the cost-to-go functional at time t. The optimal strategy is obtained in the procedure of this proof.

Remark 2. *We observe from Theorem 1 that the cost-to-go functional is a quadratic polynomial function of* $\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t]$ *and* $l_t - \mathbb{E}[l_t]$*, and the optimal strategy is a linear function of the current wealth* x_k *and the current liability* l_k*. To express the structure of the optimal strategy more clearly, we simply reformulate the optimal strategy (8) to the following form:*

$$\pi_t^* = H_{t,1}x_t + H_{t,2}l_t + H_{t,3}, \tag{10}$$

where

$$
\begin{aligned}
H_{t,1} &= -s_t \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t], \\
H_{t,2} &= -\frac{\eta_{t+1}}{\xi_{t+1}} \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t q_t], \\
H_{t,3} &= -\frac{\frac{1}{2}\zeta_{t+1} + (\xi_{t+1} - \beta_{t+1})(\widehat{BC}_t - \mathbb{E}[c_t])}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] - \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[c_t \mathbf{P}_t] \\
&\quad + \frac{-\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \\
&\quad \cdot \Bigg(-(\xi_{t+1} - \beta_{t+1})(1 - B_t) s_t \mathbb{E}[x_t] + \big[\phi_{t+1} \mathbb{E}[q_t] + \eta_{t+1}\big((1 - \tfrac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t - \mathbb{E}[q_t]\big)\big] \mathbb{E}[l_t] \Bigg).
\end{aligned}
$$

It is obvious that the derived analytical optimal portfolio policy consists of three terms: the investor's current wealth, current liability, and current cash flow with his risk attitude specified by w and a_t. It is also a function of the initial wealth x_0 and the initial liability l_0. In other words, it is of a feedback form, but not Markovian. At each period t, the optimal control policy depends on two pieces of information from the given information set, the current state (x_t, l_t) and the initial state (x_0, l_0).

Remark 3. *Let $\zeta_k = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]$, $\xi_k = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t q_t]$ and $\chi_k = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[c_t \mathbf{P}_t]$. Also, let*

$$
\begin{aligned}
\mu_k^1 &= -\Bigg(s_t x_t - \tfrac{(\xi_{t+1} - \beta_{t+1})(1 - B_t)}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} s_t \mathbb{E}[x_t] + \frac{\phi_{t+1} \mathbb{E}[q_t] + \eta_{t+1}\big((1 - \frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t - \mathbb{E}[q_t]\big)}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \mathbb{E}[l_t], \\
&\quad + \frac{(\beta_{t+1} - \xi_{t+1}) \mathbb{E}[c_t] + \frac{1}{2}\zeta_{t+1} + (\xi_{t+1} - \beta_{t+1}) \widehat{BC}_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1} B_t} \Bigg) \\
\mu_k^2 &= -\tfrac{\eta_{t+1}}{\xi_{t+1}} l_t, \\
\mu_k^3 &= -1,
\end{aligned}
$$

which depend only on the current market parameters. Then, the optimal strategy can be expressed as another simple structure which is a linear function of ζ_k, ξ_k and χ_k, i.e.,

$$
\pi_t^* = \mu_k^1 \zeta_k + \mu_k^2 \xi_k + \mu_k^3 \chi_k. \tag{11}
$$

If one takes ζ_k, ξ_k and χ_k as three mutual funds, then the corresponding investment amount invested in these three mutual funds are μ_k^1, μ_k^2 and μ_k^3, respectively. This leads to a three-fund separation theorem, which means that though there are one risk-free asset and n risky assets in the market, the investor only need to allocate his/her wealth among the risk-free asset and three artificial mutual funds of ζ_k, ξ_k and χ_k.

Remark 4. *If the assets, the liability and the cash flow are uncorrelated with each other, the optimal strategy can be simplified as*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\bigg(s_t x_t - \frac{(\xi_{t+1}-\beta_{t+1})(1-B_t)s_t\mathbb{E}[x_t] + \phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}$$
$$-\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big) + \frac{\frac{1}{2}\zeta_{t+1}+\beta_{t+1}}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[c_t]\bigg),$$

where

$$\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j} - \sum_{k=0}^{t-1}\bigg(\prod_{j=k+1}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\bigg)$$
$$\times\bigg(\frac{\phi_{k+1}\mathbb{E}[q_k]B_k}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}\Big(\prod_{j=0}^{k-1}\mathbb{E}[q_j]\Big)l_0 + \frac{\xi_{k+1}(1-B_k)\mathbb{E}[c_k]-\frac{1}{2}\zeta_{k+1}B_k}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}\bigg).$$

In this case, the optimal strategy is linear in $\zeta_k = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]$. Then our three-fund separation theorem aforementioned degenerates to the well-known one-fund theorem in the conventional dynamic portfolio selection problem with neither cash flows nor probability constraints (see Li and Ng (2000) for more details), which means that the investor only need to allocate his/her wealth among the risk-free asset and the artificial mutual fund ζ_k, and can obtain the exact investment effect as that invested in the risk-free asset and the n risky asset. This finding implies that the correlation among the assets, the liability and the cash flow is the hidden reason behind the transition from the three-fund property to the one-fund property in the optimal strategy.

Based on the proof of Theorem 1, the optimal objective of problem (3) is as follows:

$$J_0\big(\mathbb{E}[x_0], 0, \mathbb{E}[l_0], 0\big) = \beta_0 x_0^2 + \zeta_0 x_0 + 2\phi_0 l_0 x_0 + \theta_0 l_0 + \delta_0 l_0^2 + \psi_0. \tag{12}$$

By sequences $\{\beta_t\}$, $\{\zeta_t\}$, $\{\phi_t\}$, $\{\theta_t\}$, $\{\delta_t\}$ and $\{\psi_t\}$ (see the beginning of this section for more details), we know that $\{\beta_0\}$, $\{\zeta_0\}$, $\{\phi_0\}$, $\{\theta_0\}$, $\{\delta_0\}$ and $\{\psi_0\}$ depend on the vector of Lagrangian multiplier $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{T-1})'$. It follows from the Lagrange dual theory (see Luenberger (1968)) that the optimal value for the original mean-variance Problem (3) can be obtained by maximizing $J_0\big(\mathbb{E}[x_0], 0, \mathbb{E}[l_0], 0; \lambda\big)$ over $\lambda \in \mathbb{R}_+^{T-1}$. According to (12), we can also derive the minimum variance term as below.

Theorem 2. *Assume that the assets and liability are correlated at every period. Then, the efficient frontier of problem (3) is given by*

$$\mathrm{Var}(x_T - l_T) = \max_{\lambda\in\mathbb{R}_+^{T-1}} J_0\big(x_0, 0, l_0, 0\big) + w\mathbb{E}[x_T - l_T] \tag{13}$$

for $\mathbb{E}[x_T - l_T] \geq \zeta_0 x_0 - \theta_0 l_0$.

Remark 5. *We calculate* $\mathbb{E}[x_t - l_t]$ *and* $\mathrm{Var}(x_t - l_t)$, $t = 1, 2, \cdots, T-1$ *in Appendix B. In addition, See Table 2 and 3 in the following section. In fact,* $J_0(\cdot)$ *is convex in* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{T-1})' \in \mathbb{R}_+^{T-1}$. *From its explicit form, we can find an optimal Lagrangian multiplier vector* λ^* *using the steepest descent algorithm or interior point algorithm directly in the code via MATLAB.*

4. Numerical examples

4.1. An empirical example

In this subsection, using the data from Yao et al. (2016), we consider an empirical example for the ALM of a defined contribution pension fund based on the real market data in USA. In this example, we calculate the relevant market parameters according to real data. Moreover, we further obtain the dynamic optimal investment strategy, and the mean and variance of the terminal surplus using our theoretical results established. As a result, we illustrate how the pension fund can keep a tradeoff between its return and risk measured by variance.

With an initial wealth $x_0 = 3$, an initial liability $l_0 = 1$ and a trade-off parameter $w = 1$, an pension fund is scheduled to enter the market at time 0, where the unit is one thousand dollars. We adopt monthly data and take one month as time unit in this scenario. An investment plan for $T = 5$ periods (months) is made by the pension fund manager. At the beginning of each period, the wealth of the pension fund can be dynamically allocated in the risk-free asset and three risky assets. We choose Cisco Systems, Forest City Enterprises and Tandy Brands Accessories as three risky assets from NYSE, AMEX and NASDAQ. The sample size is 156 months between January 2000 and December 2012. The monthly American BAA corporate bonds is used as a proxy of the uncontrollable liability return and the pension contributions for each period are modeled as stochastic cash flows. We adopt American average monthly salary level to calibrate these stochastic cash flows and assume that the pension fund charges 12.4 percent of monthly wages as contributions. This fixed proportion is consistent with the retirement 401(k) plan in USA. The BAA bond and three stocks returns data come from CRSP (Center for Research in Security Prices) database, while average hourly wage data are from Bureau of labor statistics. For simplicity, we assume that there are four members in the pension fund and parameters of this model are independent of the time period t. Using the data stated above, we calculate the related parameters of the model for $t = 0, 1, \cdots, T-1$, as follows:

$$\mathbb{E}[\mathbf{P}_t c_t] = (-0.0065, -0.0055, 0.0001)',$$

$$\mathbb{E}[\mathbf{P}_t q_t] = (\ 0.0140, -0.0054, -0.0055)',$$

$$\mathbb{E}[\mathbf{P}_t] = (0.0139, -0.0054, -0.0054)',$$

$$\mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] = \begin{pmatrix} 0.0257 & 0.0046 & 0.006 \\ 0.0046 & 0.0135 & 0.0046 \\ 0.006 & 0.0046 & 0.0194 \end{pmatrix},$$

$$\mathbb{E}[c_t] = 0.4284, \quad \mathbb{E}[c_t^2] = 0.6355,$$
$$\mathbb{E}[q_t] = 1.0056, \quad \mathbb{E}[q_t^2] = 1.0113, \quad \mathbb{E}[c_t q_t] = 0.4308.$$

Next, we choose average interest rates on American total marketable treasury securities as the risk-free interest[1]. Since the annual rate (average interest rates on American total marketable treasury securities) of the last month (December 2012) in our sample is 2.103%. Hence, the monthly risk-free interest is $s_t = 1 + 2.103\%/12$. Using the data above, we have

$$\mathbf{K}_1 = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = (0.7219, -0.5168, -0.3791)',$$
$$\mathbf{K}_2 = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t] = (0.7273, -0.5163, -0.3860)',$$
$$\mathbf{K}_3 = \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t']\mathbb{E}[\mathbf{P}_t c_t] = (-0.2222, -0.3882, 0.1659)'.$$

According to Theorem 1, we can derive the optimal strategy of problem (3) as follows

$$\pi_0^* = -s_0(x_0 - 2.9479)\mathbf{K}_1 + 1.0218\mathbf{K}_2 l_0 - \mathbf{K}_3,$$
$$\pi_1^* = -s_1(x_1 - 3.3851)\mathbf{K}_1 + 1.0163\mathbf{K}_2 l_1 - \mathbf{K}_3,$$
$$\pi_2^* = -s_2(x_2 - 3.8224)\mathbf{K}_1 + 1.0108\mathbf{K}_2 l_2 - \mathbf{K}_3,$$
$$\pi_3^* = -s_3(x_3 - 4.2598)\mathbf{K}_1 + 1.0054\mathbf{K}_2 l_3 - \mathbf{K}_3,$$
$$\pi_4^* = -s_4(x_4 - 4.6972)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4 - \mathbf{K}_3.$$

The final optimal expected expectation and variance of the surplus are $\mathbb{E}[x_5 - l_5] = 4.1981$ and $\mathrm{Var}(x_5 - l_5) = 2.1757$, respectively. In Table 1, we choose $a_t = 0.2$ and w from 0.5 to 1.25 with a step size 0.05. We can see from Table 1 that both the final expectation and variance of the surplus increase. This is obvious and coincident with the real financial market. As we all know, the investor who can bear a higher risk may achieve more wealth. In addition, the higher the risk, the easier to go bankruptcy.

[1]The average interest rates are calculated on the total un-matured interest-bearing debt. See http://www.treasurydirect.gov/govt/rates/pd/avg/avg.htm

Table 1: The impact of w ($a_t = 0.2$)

w	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85
$\mathbb{E}[x_T - l_T]$	4.1786	4.1806	4.1825	4.1845	4.1864	4.1884	4.1903	4.1922
$\mathrm{Var}(x_T - l_T)$	2.1611	2.1621	2.1632	2.1644	2.1657	2.1671	2.1687	2.1703
w	0.9	0.95	1	1.05	1.1	1.15	1.2	1.25
$\mathbb{E}[x_T - l_T]$	4.1942	4.1961	4.1981	4.2000	4.2020	4.2039	4.2059	4.2078
$\mathrm{Var}(x_T - l_T)$	2.1720	2.1738	2.1757	2.1776	2.1797	2.1819	2.1842	2.1866

4.2. Another example

Since the above empirical example is the correlated case, in this subsection we provide another example to let readers/practitioners know how to calculate numerical example for the correlated and the uncorrelated cases using our theoretical results, and present the difference between these two cases.

We consider another example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of US market and a bank account. Based on the data provided in Elton et al. (2007), Table 2 presents the expected values, standard deviations and correlation coefficients of the annual return rates of these three indices, the liability and the cash flow per thousand dollars (see Yao et al. (2016)).

Table 2: Data

	SP	EM	MS	liability	cash flow (thousand dollars)
Expected return	14%	16%	17%	10%	0.438
Standard deviation	18.5%	30%	24%	20%	0.672
Correlation coefficient					
SP	1	0.64	0.79	ρ_1	ρ_{1c}
EM	0.64	1	0.75	ρ_2	ρ_{2c}
MS	0.79	0.75	1	ρ_3	ρ_{3c}
liability	ρ_1	ρ_2	ρ_3	1	ρ_{lc}
cash flow (thousand dollars)	ρ_{1c}	ρ_{2c}	ρ_{3c}	ρ_{lc}	1

Thus, for any time t, we have

$$\mathbb{E}[\mathbf{P}_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \quad \mathrm{Cov}(\mathbf{P}_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

We consider 5 time periods and the annual risk-free rate is 5% ($s_t = 1.05$). Assume that the investor has an initial wealth $x_0 = 3$ (thousand dollars), an initial liability $l_0 = 1$ (thousand dollars), a trade-off parameter $w = 1$. Furthermore, for $t = 0, 1, 2, 3, 4$, assume that the probability $a_t = 0.1$, the correlation of assets and liability is $\rho = (\rho_1, \rho_2, \rho_3)$, where

$$\rho_i = \frac{\mathrm{Cov}(q_t, P_t^i)}{\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}}$$

is the correlation coefficient of the i-th asset and liability. This means

$$\mathbb{E}[P_t^i q_t] = \mathbb{E}[q_t]\mathbb{E}[P_t^i] + \rho_i\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}.$$

Also, the correlation of assets and cash flow is defined as $\rho_c = (\rho_{1c}, \rho_{2c}, \rho_{3c})$, where

$$\rho_{ic} = \frac{\mathrm{Cov}(c_t, P_t^i)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(P_t^i)}}$$

is the correlation coefficient of the i-th asset and cash flow. Hence,

$$\mathbb{E}[P_t^i c_t] = \mathbb{E}[c_t]\mathbb{E}[P_t^i] + \rho_{ic}\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(P_t^i)}.$$

The correlation of the liability and cash flow is defined as

$$\rho_{lc} = \frac{\mathrm{Cov}(q_t, c_t)}{\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(c_t)}},$$

and thus

$$\mathbb{E}[q_t c_t] = \mathbb{E}[q_t]\mathbb{E}[c_t] + \rho_{lc}\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(c_t)}.$$

4.2.1. Correlation example

In this subsection, assume that the returns of the assets and liability are correlated with $\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25)$, the returns of the assets and cash flow are correlated with $\rho_c = (\rho_{1c}, \rho_{2c}, \rho_{3c}) = (0.25, 0.25, 0.25)$ and the correlation of the liability and cash flow is $\rho_{qc} = 0.25$. Hence,

$$\mathrm{Cov}\left(\begin{pmatrix}\mathbf{P}_t\\ q_t\\ c_t\end{pmatrix}\right) = \begin{pmatrix}\mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t,\mathbf{P}_t) & \mathrm{Cov}(c_t,\mathbf{P}_t)\\ \mathrm{Cov}(q_t,\mathbf{P}_t') & \mathrm{Var}(q_t) & \mathrm{Cov}(q_t,c_t)\\ \mathrm{Cov}(c_t,\mathbf{P}_t') & \mathrm{Cov}(q_t,c_t) & \mathrm{Var}(c_t)\end{pmatrix}$$
$$= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 & 0.0311\\ 0.0355 & 0.0900 & 0.0540 & 0.0300 & 0.0504\\ 0.0351 & 0.0540 & 0.0576 & 0.0120 & 0.0403\\ -0.0092 & 0.0300 & 0.0120 & 0.0400 & 0.0336\\ 0.0311 & 0.0504 & 0.0403 & 0.0336 & 0.4516\end{pmatrix} \succ 0.$$

Using the above formula of $\mathbb{E}[P_t^i q_t]$, we have $\mathbb{E}[\mathbf{P}_t q_t] = (0.0898, 0.1510, 0.1440)'$. Similarly, we obtain $\mathbb{E}[\mathbf{P}_t c_t] = (0.0705, 0.0986, 0.0929)'$. Moreover,

$$\mathbf{K}_1 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = (1.0580, -0.1207, 1.1052)',$$
$$\mathbf{K}_2 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t] = (-0.2398, 0.4374, 1.7446)',$$
$$\mathbf{K}_3 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t c_t] = (0.8152, 0.2481, 0.5390)'.$$

By interior point algorithm of "fmincon" with the initial point $\lambda = (0.5, 0.5, 0.5, 0.5)'$, we can obtain $\lambda^* = (0, 0.082, 0, 0)'$. According to Theorem 1, we can derive the optimal strategy of problem (3) as follows

$$\pi_0^* = -1.05(x_0 - 3.3047)\mathbf{K}_1 + 1.1877\mathbf{K}_2 l_0 - \mathbf{K}_3,$$
$$\pi_1^* = -1.05(x_1 - 3.8005)\mathbf{K}_1 + 1.1335\mathbf{K}_2 l_1 - \mathbf{K}_3,$$
$$\pi_2^* = -1.05(x_2 - 4.3634)\mathbf{K}_1 + 1.0979\mathbf{K}_2 l_2 - \mathbf{K}_3,$$
$$\pi_3^* = -1.05(x_3 - 4.9122)\mathbf{K}_1 + 1.0478\mathbf{K}_2 l_3 - \mathbf{K}_3,$$
$$\pi_4^* = -1.05(x_4 - 5.4884)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4 - \mathbf{K}_3.$$

The terminal optimal expected expectation and variance of the surplus are $\mathbb{E}[x_5 - l_5] = 5.2628$ and $\mathrm{Var}(x_5 - l_5) = 1.8843$, respectively. In addition, the expectations and variances of the surplus at other periods are also given in Table 3.

Table 3: The expectations and variances of the surplus

t	1	2	3	4
λ_t	0	0.082	0	0
$\mathbb{E}[x_t - l_t]$	2.6714	3.3233	3.9767	4.6215
$\mathrm{Var}(x_t - l_t)$	0.6431	1.1044	1.4567	1.7069
$\mathrm{Var}(x_t - l_t) - a_t\big(\mathbb{E}[x_t - l_t]\big)^2$	-0.0706	0	-0.1247	-0.4289

Table 4 and Table 5 present the impact of the risk aversion factor w and the probability of bankruptcy a_t on the the expectations and variances of the surplus, respectively. In Table 4, we choose $a_t = 0.1$ and w from 0.5 to 8 with a step size 0.5. In Table 5, w is fixed to 5 and a_t changes from 0.1 to 0.25 with a step size 0.01. We can see from the two tables that when w and a_t increase respectively, both the final expectation and variance of the surplus increase. Since both of the parmeters w and a_t represent the risk, these results are coincident with the real financial market as the empirical example in subsection 4.1.

Table 4: The impact of w ($a_t = 0.1$)

w	0.5	1	1.5	2	2.5	3	3.5	4
$\mathbb{E}[x_T - l_T]$	4.7215	5.2628	5.5301	5.7018	5.8573	5.9589	6.0273	6.0956
$\mathrm{Var}(x_T - l_T)$	1.4880	1.8843	2.2183	2.5154	2.8652	3.1395	3.3616	3.6178
w	4.5	5	5.5	6	6.5	7	7.5	8
$\mathbb{E}[x_T - l_T]$	6.1639	6.2322	6.3005	6.3688	6.4371	6.5054	6.5737	6.6420
$\mathrm{Var}(x_T - l_T)$	3.9082	4.2326	4.5912	4.9839	5.4108	5.8718	6.3670	6.8963

Table 5: The impact of a_t ($w = 5$)

a_t	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17
$\mathbb{E}[x_T - l_T]$	6.2322	6.4977	6.7388	6.9663	7.1854	7.3984	7.5791	7.7151
$\mathrm{Var}(x_T - l_T)$	4.2326	4.6992	5.1977	5.7342	6.3125	6.9352	7.4606	7.8021
a_t	0.18	0.19	0.20	0.21	0.22	0.23	0.24	0.25
$\mathbb{E}[x_T - l_T]$	7.8491	7.9815	8.1129	8.2436	8.3740	8.5024	8.5750	8.6472
$\mathrm{Var}(x_T - l_T)$	8.1636	8.5458	8.9498	9.3764	9.8266	10.2922	10.4957	10.7069

4.2.2. Uncorrelation example

In this subsection, assume that the returns of the assets, liability and cash flow are uncorrelated. Hence,

$$
\mathrm{Cov}\left(\begin{pmatrix}\mathbf{P}_t\\ q_t\\ c_t\end{pmatrix}\right) = \begin{pmatrix}\mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t,\mathbf{P}_t) & \mathrm{Cov}(c_t,\mathbf{P}_t)\\ \mathrm{Cov}(q_t,\mathbf{P}_t') & \mathrm{Var}(q_t) & \mathrm{Cov}(q_t,c_t)\\ \mathrm{Cov}(c_t,\mathbf{P}_t') & \mathrm{Cov}(q_t,c_t) & \mathrm{Var}(c_t)\end{pmatrix}
$$

$$
= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & 0 & 0\\ 0.0355 & 0.0900 & 0.0540 & 0 & 0\\ 0.0351 & 0.0540 & 0.0576 & 0 & 0\\ 0 & 0 & 0 & 0.04 & 0\\ 0 & 0 & 0 & 0 & 0.4516 \end{pmatrix} \succ 0.
$$

By interior point algorithm of "fmincon" with the initial point $\lambda = (0.5, 0.5, 0.5, 0.5)'$, we can obtain $\lambda^* = (0, 1.1829, 0, 0)'$. According to Theorem 1, we can derive the optimal strategy of problem (3) as follows

$$
\begin{aligned}
\pi_0^* &= -1.05(x_0 - 3.9938 + 1.1558 l_0)\mathbf{K}_1,\\
\pi_1^* &= -1.05(x_1 - 4.5631 + 1.1032 l_1)\mathbf{K}_1,\\
\pi_2^* &= -1.05(x_2 - 5.4046 + 1.1498 l_2)\mathbf{K}_1,\\
\pi_3^* &= -1.05(x_3 - 6.0443 + 1.0975 l_3)\mathbf{K}_1,\\
\pi_4^* &= -1.05(x_4 - 6.7161 + 1.0476 l_4)\mathbf{K}_1.
\end{aligned}
$$

Remark 6. *From the above numerical results, we find that the optimal investment strategy depends on $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ for the correlated case but the optimal investment strategy only depends on $\mathbf{K}_1$ for the uncorrelated case. In fact, $\mathbf{K}_1 = \zeta_k = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]$, $\mathbf{K}_2 = \xi_k = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]$ and $\mathbf{K}_3 = \chi_k = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[c_t\mathbf{P}_t]$ are the three mutual funds. As stated in Remark 3 and Remark 4 in Section 3, the three-fund separation theorem holds and the optimal strategy can be expressed as a line function of three mutual funds for the correlated case; however, for the uncorrelated case, the one-fund separation theorem holds and the optimal strategy can be expressed as a line function of only one mutual fund.*

The expectation and variance of the terminal optimal surplus are $\mathbb{E}[x_5 - l_5] = 5.5519$ and $\mathrm{Var}(x_5 - l_5) = 2.4118$, respectively. In addition, the expectations and variances of the surplus at other periods are given in Table 6.

Table 6: The expectations and variances of the surplus

t	1	2	3	4
λ_t	0	1.1829	0	0
$\mathbb{E}[x_t - l_t]$	2.6637	3.3249	4.0694	4.8100
$\text{Var}(x_t - l_t)$	0.6046	1.1055	1.6267	2.0510
$\text{Var}(x_t - l_t) - a_t\big(\mathbb{E}[x_t - l_t]\big)^2$	-0.1049	0	-0.0294	-0.2626

In the following, we give two tables to show the impact of w and a_t when the returns of assets and liability and cash flow are all uncorrelated. That is, $\rho = (\rho_1, \rho_2, \rho_3) = \rho_c = (\rho_{1c}, \rho_{2c}, \rho_{3c}) = (0, 0, 0)$ and $\rho_{qc} = 0$. It is easy to see that the results are similar with that of correlated case in the last subsection.

Table 7: The impact of w ($a_t = 0.1$)

w	0.5	1	1.5	2	2.5	3	3.5	4
$\mathbb{E}[x_T - l_T]$	5.2249	5.5519	5.7484	5.9037	6.0421	6.1104	6.1787	6.2470
$\text{Var}(x_T - l_T)$	2.1789	2.4118	2.6510	2.9229	3.2307	3.4185	3.6405	3.8967
w	4.5	5	5.5	6	6.5	7	7.5	8
$\mathbb{E}[x_T - l_T]$	6.3153	6.3836	6.4519	6.5202	6.5885	6.6568	6.7251	6.7934
$\text{Var}(x_T - l_T)$	4.1869	4.5114	4.8699	5.2626	5.6895	6.1505	6.6456	7.1749

Table 8: The impact of a_t ($w = 5$)

a_t	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17
$\mathbb{E}[x_T - l_T]$	6.3836	6.8327	7.1548	7.4368	7.6981	7.9406	8.1000	8.2539
$\text{Var}(x_T - l_T)$	4.5114	5.1664	5.7902	6.4430	7.1382	7.8548	8.2302	8.6268
a_t	0.18	0.19	0.20	0.21	0.22	0.23	0.24	0.25
$\mathbb{E}[x_T - l_T]$	8.4041	8.5515	8.6969	8.8409	8.9687	9.0482	9.1269	9.2050
$\text{Var}(x_T - l_T)$	9.0457	9.4882	9.9553	10.4483	10.8911	11.1108	11.3389	11.5758

Remark 7. *From the numerical results between Tables 3-5 and Tables 6-8, we find that both the* $\mathbb{E}[x_T - l_T]$ *and* $\text{Var}(x_T - l_T)$ *in the uncorrelated case are greater than those in the correlated case. This mean that, in the uncorrelated market, the investor will become more radical, he is willing to take on greater risks to get a higher return.*

5. Conclusion

In this paper, we consider a multi-period asset-liability mean-variance portfolio selection problem with probability constraints and cash flows. Using mean-field formulation, the analytical optimal strategy and efficient frontier are strictly derived. The numerical examples show how to apply the theoretical results. We also present the impact of the risk aversion fact and bankruptcy probability on the expectation and variance of the terminal surplus via numerical examples, which coincides with the real life.

Appendix A

Appendix A.1. Lemmas 1 and 2

In this appendix, we present some lemmas which are useful for us to prove the Theorem 1 in this paper.

Lemma 1 (Sherman-Morrison formula). *Suppose that A is an invertible square matrix and μ and ν are two given vectors. If*

$$1 + \nu' A^{-1} \mu \neq 0,$$

then the following holds

$$(A + \mu\nu')^{-1} = A^{-1} - \frac{A^{-1}\mu\nu' A^{-1}}{1 + \nu' A^{-1}\mu}.$$

Lemma 2. *Suppose that $z_1 \neq 0$, $z_2 \neq 0$ and $z_1(1 - B_t) + z_2 B_t \neq 0$ hold. Then*

$$(i) \quad \Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t] = \frac{1}{z_1(1 - B_t) + z_2 B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t];$$

$$(ii) \quad \Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t]$$
$$= \frac{\big(1 - \frac{z_2}{z_1}\big)\widehat{B}_t}{z_1(1 - B_t) + z_2 B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] + \frac{1}{z_1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t];$$

$$(iii) \quad \Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t c_t]$$
$$= \frac{\big(1 - \frac{z_2}{z_1}\big)\widehat{BC}_t}{z_1(1 - B_t) + z_2 B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] + \frac{1}{z_1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t c_t];$$

$$(iv) \quad \mathbb{E}[\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t] = \frac{\widehat{B}_t}{z_1(1 - B_t) + z_2 B_t};$$

$$(v)\quad \mathbb{E}[\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t c_t] = \frac{\widehat{BC}_t}{z_1(1-B_t)+z_2B_t};$$

$$(vi)\quad \mathbb{E}[q_t\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t] = \frac{1-\frac{z_2}{z_1}}{z_1(1-B_t)+z_2B_t}\widehat{B}_t^2 + \frac{1}{z_1}\widetilde{B}_t;$$

$$(vii)\quad \mathbb{E}[c_t\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t c_t] = \frac{1-\frac{z_2}{z_1}}{z_1(1-B_t)+z_2B_t}\widehat{BC}_t^2 + \frac{1}{z_1}\widetilde{BC}_t.$$

Proof. (i) Applying Sherman-Morrison formula yields

$$\begin{aligned}
&\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t]\\
&= \left(z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'] + \frac{z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'](z_1-z_2)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']}{1-z_1^{-1}(z_1-z_2)\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]}\right)\mathbb{E}[\mathbf{P}_t]\\
&= \frac{1}{z_1(1-B_t)+z_2B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t].
\end{aligned}$$

(ii) Applying Sherman-Morrison formula yields

$$\begin{aligned}
&\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t]\\
&= \left(z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'] + \frac{z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'](z_1-z_2)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']}{1-z_1^{-1}(z_1-z_2)\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]}\right)\mathbb{E}[\mathbf{P}_t q_t]\\
&= \frac{\big(1-\frac{z_2}{z_1}\big)\widehat{B}_t}{z_1(1-B_t)+z_2B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] + \frac{1}{z_1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t].
\end{aligned}$$

(iii) Applying Sherman-Morrison formula yields

$$\begin{aligned}
&\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[c_t\mathbf{P}_t]\\
&= \left(z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'] + \frac{z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'](z_1-z_2)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']z_1^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']}{1-z_1^{-1}(z_1-z_2)\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]}\right)\mathbb{E}[c_t\mathbf{P}_t]\\
&= \frac{\big(1-\frac{z_2}{z_1}\big)\widehat{BC}_t}{z_1(1-B_t)+z_2B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] + \frac{1}{z_1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[c_t\mathbf{P}_t].
\end{aligned}$$

(iv) Applying the above (ii) yields

$$\begin{aligned}
&\mathbb{E}[\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t]\\
&= \frac{\big(1-\frac{z_2}{z_1}\big)\widehat{B}_t}{z_1(1-B_t)+z_2B_t}B_t + \frac{1}{z_1}\widehat{B}_t\\
&= \frac{\widehat{B}_t}{z_1(1-B_t)+z_2B_t}.
\end{aligned}$$

(v) Applying the above (iii) yields

$$\begin{aligned}
&\mathbb{E}[\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t c_t] \\
&= \frac{\big(1 - \frac{z_2}{z_1}\big)\widehat{BC}_t}{z_1(1-B_t) + z_2 B_t} B_t + \frac{1}{z_1}\widehat{BC}_t \\
&= \frac{\widehat{BC}_t}{z_1(1-B_t) + z_2 B_t}.
\end{aligned}$$

(vi) Applying the above (ii) yields

$$\begin{aligned}
&\mathbb{E}[q_t\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t q_t] \\
&= \frac{\big(1 - \frac{z_2}{z_1}\big)\widehat{B}_t}{z_1(1-B_t) + z_2 B_t}\widehat{B}_t + \frac{1}{z_1}\widetilde{B}_t \\
&= \frac{1 - \frac{z_2}{z_1}}{z_1(1-B_t) + z_2 B_t}\widehat{B}_t^2 + \frac{1}{z_1}\widetilde{B}_t.
\end{aligned}$$

(vii) Applying the above (iii) yields

$$\begin{aligned}
&\mathbb{E}[c_t\mathbf{P}_t']\Big(z_1\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(z_1 - z_2\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t c_t] \\
&= \frac{\big(1 - \frac{z_2}{z_1}\big)\widehat{BC}_t}{z_1(1-B_t) + z_2 B_t}\widehat{BC}_t + \frac{1}{z_1}\widetilde{BC}_t \\
&= \frac{1 - \frac{z_2}{z_1}}{z_1(1-B_t) + z_2 B_t}\widehat{BC}_t^2 + \frac{1}{z_1}\widetilde{BC}_t.
\end{aligned}$$

This completes the proof. □

Appendix A.2: The Proof of Theorem 1

Proof. We prove the main results by dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional at period t is calculated by

$$\begin{aligned}
&J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
&= \min_{\{\pi_t - \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}} \mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\
&\quad + \lambda_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t - l_t]\big)^2.
\end{aligned}$$

The cost-to-go functional at terminal time T is

$$\begin{aligned}
&J_T\big(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]\big) \\
&= \big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2 - w\mathbb{E}\big[x_T - l_T\big] \\
&= \xi_T\big(x_T - \mathbb{E}[x_T]\big)^2 + 2\eta_T\big(l_T - \mathbb{E}[l_T]\big)\big(x_T - \mathbb{E}[x_T]\big) + \epsilon_T\big(l_T - \mathbb{E}[l_T]\big)^2 \\
&\quad + \beta_T(\mathbb{E}[x_T])^2 + \zeta_T\mathbb{E}[x_T] + 2\phi_T\mathbb{E}[l_T]\mathbb{E}[x_T] + \theta_T\mathbb{E}[l_T] + \delta_T(\mathbb{E}[l_T])^2 + \psi_T.
\end{aligned}$$

Assume that the cost-to-go functional at time $t+1$ is the following expression

$$
\begin{aligned}
&J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big) \\
&= \xi_{t+1}\big(x_{t+1}-\mathbb{E}[x_{t+1}]\big)^2 + 2\eta_{t+1}\big(l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big(x_{t+1}-\mathbb{E}[x_{t+1}]\big) + \epsilon_{t+1}\big(l_{t+1}-\mathbb{E}[l_{t+1}]\big)^2 \\
&\quad +\beta_{t+1}(\mathbb{E}[x_{t+1}])^2 + \zeta_{t+1}\mathbb{E}[x_{t+1}] + 2\phi_{t+1}\mathbb{E}[l_{t+1}]\mathbb{E}[x_{t+1}] + \theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}(\mathbb{E}[l_{t+1}])^2 + \psi_{t+1}.
\end{aligned}
$$

We prove that the above statement still holds at time t. For given information set $\mathcal{F}_t$, we have

$$
\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\
&= \mathbb{E}\bigg[\xi_{t+1}\Big[s_t\big(x_t-\mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big) + \big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + \big(c_t-\mathbb{E}[c_t]\big)\Big]^2 \\
&\quad + 2\eta_{t+1}\Big[q_t\big(l_t-\mathbb{E}[l_t]\big) + \big(q_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]\Big] \\
&\quad \times\Big[s_t\big(x_t-\mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big) + \big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + \big(c_t-\mathbb{E}[c_t]\big)\Big] \\
&\quad + \epsilon_{t+1}\big[q_t(l_t-\mathbb{E}[l_t]) + (q_t-\mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2 + \beta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big)^2 \\
&\quad + \zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big) + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big) \\
&\quad + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1}\bigg|\mathcal{F}_t\bigg] \\
&= \xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + \big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) + \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
&\quad + 2\big(\pi_t-\mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + 2\big(\mathbb{E}[c_t\mathbf{P}_t'] - \mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\big)\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + 2\big(\mathbb{E}[c_t\mathbf{P}_t'] - \mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + \mathbb{E}[(c_t-\mathbb{E}[c_t])^2]\Big] \\
&\quad + 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t-\mathbb{E}[l_t]\big)\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\Big(\mathbb{E}[l_t]\big(\pi_t-\mathbb{E}[\pi_t]\big) + \big(l_t-\mathbb{E}[l_t]\big)\mathbb{E}[\pi_t] + \mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big) \\
&\quad + \big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\Big(\big(l_t-\mathbb{E}[l_t]\big) + \mathbb{E}[l_t]\Big)\Big] \\
&\quad + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + 2(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(l_t-\mathbb{E}[l_t])\mathbb{E}[l_t] + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
&\quad + \beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2 + 2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[\pi_t']\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] \\
&\quad + 2s_t\mathbb{E}[c_t]\mathbb{E}[x_t] + 2\mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \big(\mathbb{E}[c_t]\big)^2\Big] + \zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big) \\
&\quad + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big) + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1}.
\end{aligned}
$$

Since any admissible strategy of $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$ satisfies $\mathbb{E}[\pi_t - \mathbb{E}[\pi_t]] = \mathbf{0}$ and $\mathbb{E}[l_t - \mathbb{E}[l_t]] = 0$ holds, we have

$$\begin{aligned}
&\mathbb{E}\Big[\big(\pi_t - \mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\big(\pi_t - \mathbb{E}[\pi_t]\big)\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\big[\big(\mathbb{E}[c_t\mathbf{P}_t'] - \mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\big)\big(\pi_t - \mathbb{E}[\pi_t]\big)\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(l_t - \mathbb{E}[l_t])\mathbb{E}[l_t]\Big|\mathcal{F}_0\Big] = 0.
\end{aligned}$$

We first identify the optimal solution $(\mathbb{E}[\pi_t^*], \pi_t^* - \mathbb{E}[\pi_t^*])$ by minimizing the following equivalent cost functional,

$$\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]\\
&= \xi_{t+1}\Big[s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 + \big(\pi_t - \mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big)\\
&\quad + 2s_t\big(x_t - \mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big) + \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\\
&\quad + 2\big(\mathbb{E}[c_t\mathbf{P}_t'] - \mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + \mathbb{E}[(c_t - \mathbb{E}[c_t])^2]\Big]\\
&\quad + 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\big(\pi_t - \mathbb{E}[\pi_t]\big)\\
&\quad + \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\pi_t] + \big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\Big]\\
&\quad + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big]\\
&\quad + \beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2 + 2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[\pi_t']\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\\
&\quad + 2s_t\mathbb{E}[c_t]\mathbb{E}[x_t] + 2\mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \big(\mathbb{E}[c_t]\big)^2\Big] + \zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big)\\
&\quad + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[c_t]\big) + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1}\\
&= \xi_{t+1}\Big[s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 + \big(\pi_t - \mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big) + 2s_t\big(x_t - \mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big)\Big]\\
&\quad + \mathbb{E}[\pi_t']\big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\xi_{t+1} - \beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + 2\xi_{t+1}\big(\mathbb{E}[c_t\mathbf{P}_t'] - \mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\\
&\quad + 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\big(\pi_t - \mathbb{E}[\pi_t]\big)\\
&\quad + \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\pi_t] + \big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\Big]
\end{aligned}$$

$$+\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2+(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big]+\xi_{t+1}\mathbb{E}[(c_t-\mathbb{E}[c_t])^2]$$
$$+\beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2+2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]+2s_t\mathbb{E}[c_t]\mathbb{E}[x_t]+2\mathbb{E}[c_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]+\big(\mathbb{E}[c_t]\big)^2\Big]$$
$$+\zeta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]+\mathbb{E}[c_t]\big)+2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]+\mathbb{E}[c_t]\big)$$
$$+\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]+\delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2+\psi_{t+1}.$$

It is easy to see that $\pi_t^*-\mathbb{E}[\pi_t^*]$ can be expressed by the linear form of states and their expected states, and $\mathbb{E}[\pi_t^*]$ can be constructed by the linear form of the expected states, i.e.,

$$\pi_t^*-\mathbb{E}[\pi_t^*]=-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t-\mathbb{E}[x_t]\big)-\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\big(l_t-\mathbb{E}[l_t]\big),\quad(14)$$

$$\begin{aligned}\mathbb{E}[\pi_t^*]=&-\big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-(\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\Bigg(\beta_{t+1}s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t]+\xi_{t+1}\mathbb{E}[c_t\mathbf{P}_t]\\&+\Big((\beta_{t+1}-\xi_{t+1})\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\Big)\mathbb{E}[\mathbf{P}_t]+\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\mathbb{E}[l_t]\\&+\eta_{t+1}\big(\mathbb{E}[\mathbf{P}_tq_t]-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\Bigg)\\=&-\big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-(\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\Bigg(K_{t+1,1}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[x_t]\\&+K_{t+1,2}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[l_t]+K_{t+1,3}\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t]+K_{t+1,4}\mathbb{E}[\mathbf{P}_t]+K_{t+1,5}\mathbb{E}[c_t\mathbf{P}_t]\Bigg)\\=&-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\frac{1}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\Bigg(\beta_{t+1}s_t\mathbb{E}[x_t]+\big(\beta_{t+1}-\xi_{t+1}\big)\mathbb{E}[c_t]\\&+\frac{1}{2}\zeta_{t+1}+(\xi_{t+1}-\beta_{t+1})\widehat{BC}_t+\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]+\eta_{t+1}\big((1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]\Bigg)\\&-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[c_t\mathbf{P}_t]-\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t],\end{aligned}\quad(15)$$

where

$$\begin{cases}K_{t+1,1}=\beta_{t+1}s_t,\\K_{t+1,2}=\big(\phi_{t+1}-\eta_{t+1}\big)\mathbb{E}[q_t],\\K_{t+1,3}=\eta_{t+1},\\K_{t+1,4}=\big(\beta_{t+1}-\xi_{t+1}\big)\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1},\\K_{t+1,5}=\xi_{t+1}.\end{cases}$$

Hence, combining with (14) and (15), we derive the desired result (8).

In order to get the explicit expression of the cost-to-go functional at time t, we substitute $\pi_t^*-\mathbb{E}[\pi_t^*]$ and $\mathbb{E}[\pi_t^*]$ back to $\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}],x_{t+1}-\mathbb{E}[x_{t+1}],\mathbb{E}[l_{t+1}],l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]$, and further derive

$$
\begin{aligned}
&J_t\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t], \mathbb{E}[l_t], l_t-\mathbb{E}[l_t]\big)\\
&= \min_{\{\pi_t-\mathbb{E}[\pi_t],\mathbb{E}[\pi_t]\}} \mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big]\\
&\quad + \lambda_t\big(x_t-l_t-\mathbb{E}[x_t-l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t-l_t]\big)^2\\
&= \xi_{t+1}s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + 2\eta_{t+1}s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \beta_{t+1}s_t^2\big(\mathbb{E}[x_t]\big)^2\\
&\quad + \big(2\beta_{t+1}\mathbb{E}[c_t]+\zeta_{t+1}\big)s_t\mathbb{E}[x_t] + \big(\theta_{t+1}\mathbb{E}[q_t] + 2\eta_{t+1}(\mathbb{E}[q_tc_t]-\mathbb{E}[q_t]\mathbb{E}[c_t]) + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\\
&\quad + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] + 2\phi_{t+1}s_t\mathbb{E}[q_t]\mathbb{E}[l_t]\mathbb{E}[x_t]\\
&\quad + \beta_{t+1}\big(\mathbb{E}[c_t]\big)^2 + \zeta_{t+1}\mathbb{E}[c_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \xi_{t+1}\mathbb{E}[(c_t-\mathbb{E}[c_t])^2] + \psi_{t+1}\\
&\quad - \xi_{t+1}\Big[-\mathbb{E}[\mathbf{P}_t]s_t\big(x_t-\mathbb{E}[x_t]\big) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}[\mathbf{P}_tq_t]\big(l_t-\mathbb{E}[l_t]\big)\Big]'\\
&\quad \times \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\Big[-\mathbb{E}[\mathbf{P}_t]s_t\big(x_t-\mathbb{E}[x_t]\big) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}[\mathbf{P}_tq_t]\big(l_t-\mathbb{E}[l_t]\big)\Big]\\
&\quad - \Bigg[K_{t+1,1}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[x_t] + K_{t+1,2}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[l_t] + K_{t+1,3}\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t] + K_{t+1,4}\mathbb{E}[\mathbf{P}_t] + K_{t+1,5}\mathbb{E}[c_t\mathbf{P}_t]\Bigg]'\\
&\quad \times \big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\\
&\quad \times \Bigg[K_{t+1,1}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[x_t] + K_{t+1,2}\mathbb{E}[\mathbf{P}_t]\mathbb{E}[l_t] + K_{t+1,3}\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t] + K_{t+1,4}\mathbb{E}[\mathbf{P}_t] + K_{t+1,5}\mathbb{E}[c_t\mathbf{P}_t]\Bigg]\\
&\quad + \lambda_t\big(x_t-l_t-\mathbb{E}[x_t-l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t-l_t]\big)^2\\
&= \xi_{t+1}s_t^2(1-B_t)\big(x_t-\mathbb{E}[x_t]\big)^2 + 2\eta_{t+1}s_t(\mathbb{E}[q_t]-\widehat{B}_t)\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big)\\
&\quad + \Big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}}\widetilde{B}_t\Big)(l_t-\mathbb{E}[l_t])^2 + \beta_{t+1}s_t^2\big(\mathbb{E}[x_t]\big)^2\\
&\quad + \big(2\beta_{t+1}\mathbb{E}[c_t]+\zeta_{t+1}\big)s_t\mathbb{E}[x_t] + \big(\theta_{t+1}\mathbb{E}[q_t] + 2\eta_{t+1}(\mathbb{E}[q_tc_t]-\mathbb{E}[q_t]\mathbb{E}[c_t]) + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\\
&\quad + \epsilon_{t+1}(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2 + 2\phi_{t+1}s_t\mathbb{E}[q_t]\mathbb{E}[l_t]\mathbb{E}[x_t]\\
&\quad + \beta_{t+1}\big(\mathbb{E}[c_t]\big)^2 + \zeta_{t+1}\mathbb{E}[c_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \xi_{t+1}\mathbb{E}[(c_t-\mathbb{E}[c_t])^2] + \psi_{t+1}\\
&\quad - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}^2\big(\mathbb{E}[x_t]\big)^2 - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,2}^2\big(\mathbb{E}[l_t]\big)^2\\
&\quad - \Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{B}_t\Big)K_{t+1,3}^2\big(\mathbb{E}[l_t]\big)^2 - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,4}^2\\
&\quad - \Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{BC}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{BC}_t\Big)K_{t+1,5}^2 - \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,2}\mathbb{E}[l_t]\mathbb{E}[x_t]\\
&\quad - \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,3}\mathbb{E}[l_t]\mathbb{E}[x_t] - \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,4}\mathbb{E}[x_t]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,1}K_{t+1,5}\mathbb{E}[x_t] - \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,2}K_{t+1,3}\big(\mathbb{E}[l_t]\big)^2 \\
&- \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,2}K_{t+1,4}\mathbb{E}[l_t] - \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,2}K_{t+1,5}\mathbb{E}[l_t] \\
&- \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,3}K_{t+1,4}\mathbb{E}[l_t] - 2\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\overline{BC}_t\Big) K_{t+1,3}K_{t+1,5}\mathbb{E}[l_t] \\
&- \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} K_{t+1,4}K_{t+1,5} + \lambda_t\big(x_t - l_t - \mathbb{E}[x_t-l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t-l_t]\big)^2.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&J_t\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t], \mathbb{E}[l_t], l_t-\mathbb{E}[l_t]\big) \\
&= \Big(\xi_{t+1}s_t^2(1-B_t)+\lambda_t\Big)\big(x_t-\mathbb{E}[x_t]\big)^2 \\
&\quad + 2\Big(\eta_{t+1}s_t(\mathbb{E}[q_t]-\widehat{B}_t)-\lambda_t\Big)\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) \\
&\quad + \Big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}}\widetilde{B}_t + \lambda_t\Big)(l_t-\mathbb{E}[l_t])^2 \\
&\quad + \Big(\beta_{t+1}s_t^2 - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}^2 - \lambda_t a_t\Big)\big(\mathbb{E}[x_t]\big)^2 \\
&\quad + \Big[\big(2\beta_{t+1}\mathbb{E}[c_t]+\zeta_{t+1}\big)s_t - \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,4} \\
&\qquad - \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,5}\Big]\mathbb{E}[x_t] \\
&\quad + 2\Big[\phi_{t+1}s_t\mathbb{E}[q_t] - \frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,2} \\
&\qquad - \frac{\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,1}K_{t+1,3} + \lambda_t a_t\Big]\mathbb{E}[l_t]\mathbb{E}[x_t] \\
&\quad + \Big[\theta_{t+1}\mathbb{E}[q_t] + 2\eta_{t+1}(\mathbb{E}[q_tc_t]-\mathbb{E}[q_t]\mathbb{E}[c_t]) + 2\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[c_t] \\
&\qquad - \frac{2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,2}K_{t+1,4} - \frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,2}K_{t+1,5} \\
&\qquad - \frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,3}K_{t+1,4} - 2\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\overline{BC}_t\Big)K_{t+1,3}K_{t+1,5}\Big]\mathbb{E}[l_t]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\delta_{t+1}\big(\mathbb{E}[q_t]\big)^2+\epsilon_{t+1}\big(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2\big)-\frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,2}^2\right.\\
&\quad\left.-\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}+\frac{1}{\xi_{t+1}}\widetilde{B}_t\Big)K_{t+1,3}^2-\frac{2\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,2}K_{t+1,3}-\lambda_t a_t\right]\big(\mathbb{E}[l_t]\big)^2\\
&+\beta_{t+1}\big(\mathbb{E}[c_t]\big)^2+\zeta_{t+1}\mathbb{E}[c_t]+\xi_{t+1}\mathbb{E}[(c_t-\mathbb{E}[c_t])^2]+\psi_{t+1}\\
&\quad-\frac{B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,4}^2-\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{BC}_t^2}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}+\frac{1}{\xi_{t+1}}\widetilde{BC}_t\Big)K_{t+1,5}^2\\
&\quad-\frac{2\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}K_{t+1,4}K_{t+1,5}\\
=\;&\xi_t\big(x_t-\mathbb{E}[x_t]\big)^2+2\eta_t\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big)+\epsilon_t\big(l_t-\mathbb{E}[l_t]\big)^2\\
&+\beta_t(\mathbb{E}[x_t])^2+\zeta_t\mathbb{E}[x_t]+2\phi_t\mathbb{E}[l_t]\mathbb{E}[x_t]+\theta_t\mathbb{E}[l_t]+\delta_t(\mathbb{E}[l_t])^2+\psi_t.
\end{aligned}
$$

This is the desired result of cost-to-fuctional (7). Substituting $\mathbb{E}[\pi_t^*]$ to dynamics of $\mathbb{E}[x_t]$ in (4) yields

$$
\begin{aligned}
\mathbb{E}[x_{t+1}]&=s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]+\mathbb{E}[c_t]\\
&=s_t\mathbb{E}[x_t]-\frac{1}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\bigg(\beta_{t+1}s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t]+\xi_{t+1}\mathbb{E}[c_t\mathbf{P}_t]\\
&\quad+\Big(\big(\beta_{t+1}-\xi_{t+1}\big)\mathbb{E}[c_t]+\frac{1}{2}\zeta_{t+1}\Big)\mathbb{E}[\mathbf{P}_t]+\phi_{t+1}\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\mathbb{E}[l_t]\\
&\quad+\eta_{t+1}\big(\mathbb{E}[\mathbf{P}_tq_t]-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\bigg)+\mathbb{E}[c_t]\\
&=\frac{\xi_{t+1}(1-B_t)s_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[x_t]-\frac{\phi_{t+1}\mathbb{E}[q_t]B_t+\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[l_t]\\
&\quad-\frac{\big(\beta_{t+1}-\xi_{t+1}\big)\mathbb{E}[c_t]B_t+\frac{1}{2}\zeta_{t+1}B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}-\frac{\xi_{t+1}\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}+\mathbb{E}[c_t]\\
&=\frac{\xi_{t+1}(1-B_t)s_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[x_t]-\frac{\phi_{t+1}\mathbb{E}[q_t]B_t+\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[l_t]\\
&\quad+\frac{\xi_{t+1}\mathbb{E}[c_t]-\frac{1}{2}\zeta_{t+1}B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}-\frac{\xi_{t+1}\widehat{BC}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t},
\end{aligned}
\tag{16}
$$

which implies the expression of $\mathbb{E}[x_t]$ in (9).

Finally, we show that this optimal strategy satisfies the linear constraints. At time 0, $\mathbb{E}[\pi_0^*-\mathbb{E}[\pi_0^*]]=\mathbf{0}$ is obvious due to $x_0=\mathbb{E}[x_0]$ and $l_0=\mathbb{E}[l_0]$. Then, according to the dynamic system of (5), we have $\mathbb{E}[x_1-\mathbb{E}[x_1]]=0$ and $\mathbb{E}[l_1-\mathbb{E}[l_1]]=0$, which further implies $\mathbb{E}[\pi_1^*-\mathbb{E}[\pi_1^*]]=\mathbf{0}$. Repeating this argument, we have $\mathbb{E}[\pi_t^*-\mathbb{E}[\pi_t^*]]=\mathbf{0}$ holds for all t. □

Appendix B: Calculation for $\mathbb{E}[x_t - l_t]$ and $\mathrm{Var}(x_t - l_t)$

It follows from (10) that we have

$$\begin{aligned} x_{t+1}^2 &= s_t^2 x_t^2 + \pi_t' \mathbf{P}_t \mathbf{P}_t' \pi_t + c_t^2 + 2 s_t x_t \mathbf{P}_t' \pi_t + 2 s_t x_t c_t + 2\mathbf{P}_t' \pi_t c_t \\ &= s_t^2 x_t^2 + (H_{t,1} x_t + H_{t,2} l_t + H_{t,3})' \mathbf{P}_t \mathbf{P}_t' (H_{t,1} x_t + H_{t,2} l_t + H_{t,3}) + c_t^2 \\ &\quad + 2 s_t x_t \mathbf{P}_t' (H_{t,1} x_t + H_{t,2} l_t + H_{t,3}) + 2 s_t x_t c_t + 2\mathbf{P}_t' c_t (H_{t,1} x_t + H_{t,2} l_t + H_{t,3}) \end{aligned}$$

and

$$\begin{aligned} x_{t+1} l_{t+1} &= (s_t x_t + \mathbf{P}_t' \pi_t + c_t) l_t q_t \\ &= s_t q_t x_t l_t + \mathbf{P}_t' q_t \pi_t l_t + q_t c_t l_t \\ &= s_t q_t x_t l_t + \mathbf{P}_t' q_t (H_{t,1} x_t l_t + H_{t,2} l_t^2 + H_{t,3} l_t) + q_t c_t l_t. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}[x_{t+1}^2] &= s_t^2 (1 - B_t) \mathbb{E}[x_t^2] + H_{t,2}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,2} \mathbb{E}[l_t^2] + H_{t,3}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \\ &\quad + 2 H_{t,1}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,2} \mathbb{E}[x_t l_t] + 2 H_{t,1}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \mathbb{E}[x_t] + 2 H_{t,2}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \mathbb{E}[l_t] \\ &\quad + \mathbb{E}[c_t^2] + 2 s_t \mathbb{E}[\mathbf{P}_t'] H_{t,2} \mathbb{E}[x_t l_t] + 2 s_t \mathbb{E}[\mathbf{P}_t'] H_{t,3} \mathbb{E}[x_t] + 2 s_t \mathbb{E}[x_t] \mathbb{E}[c_t] \\ &\quad + 2 \mathbb{E}[c_t \mathbf{P}_t'] H_{t,1} \mathbb{E}[x_t] + 2 \mathbb{E}[c_t \mathbf{P}_t'] H_{t,2} \mathbb{E}[l_t] + 2 \mathbb{E}[c_t \mathbf{P}_t'] H_{t,3} \\ &= s_t^2 (1 - B_t) \mathbb{E}[x_t^2] + H_{t,2}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,2} \mathbb{E}[l_t^2] + H_{t,3}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \\ &\quad + 2 H_{t,1}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,2} \mathbb{E}[x_t l_t] + 2 H_{t,1}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \mathbb{E}[x_t] + 2 H_{t,2}' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] H_{t,3} \mathbb{E}[l_t] \\ &\quad + \mathbb{E}[c_t^2] + 2 s_t \mathbb{E}[\mathbf{P}_t'] H_{t,2} \mathbb{E}[x_t l_t] + 2 s_t \mathbb{E}[\mathbf{P}_t'] H_{t,3} \mathbb{E}[x_t] + 2 s_t \mathbb{E}[x_t] \mathbb{E}[c_t] + 2 \mathbb{E}[c_t \mathbf{P}_t'] \mathbb{E}[\pi_t] \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[x_{t+1} l_{t+1}] &= s_t \mathbb{E}[q_t] \mathbb{E}[x_t l_t] + \mathbb{E}[\mathbf{P}_t' q_t] (H_{t,1} \mathbb{E}[x_t l_t] + H_{t,2} \mathbb{E}[l_t^2] + H_{t,3} \mathbb{E}[l_t]) + \mathbb{E}[q_t c_t] \mathbb{E}[l_t], \\ &= \big(s_t \mathbb{E}[q_t] + \mathbb{E}[\mathbf{P}_t' q_t] H_{t,1}\big) \mathbb{E}[x_t l_t] + \mathbb{E}[\mathbf{P}_t' q_t] H_{t,2} \mathbb{E}[l_t^2] + \big(\mathbb{E}[\mathbf{P}_t' q_t] H_{t,3} + \mathbb{E}[q_t c_t]\big) \mathbb{E}[l_t]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\mathrm{Var}(x_{t+1} - l_{t+1}) \\ &= \mathrm{Var}(x_{t+1}) - 2\mathrm{Cov}(x_{t+1}, l_{t+1}) + \mathrm{Var}(l_{t+1}) \\ &= \mathbb{E}[x_{t+1}^2] - \big(\mathbb{E}[x_{t+1}]\big)^2 - 2\big(\mathbb{E}[x_{t+1} l_{t+1}] - \mathbb{E}[x_{t+1}] \mathbb{E}[l_{t+1}]\big) + \mathbb{E}[l_{t+1}^2] - \big(\mathbb{E}[l_{t+1}]\big)^2. \end{aligned}$$

Thus, we can calculate $\mathbb{E}[x_t - l_t]$ and $\mathrm{Var}(x_t - l_t)$ for $t = 1, 2, \cdots, T-1$ by (16) and the above formula.

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