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Dividend optimization for jump-diffusion model with solvency constraints

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Abstract

Belhaj (2010) established that a barrier strategy is optimal for the dividend payment problem under the jump-diffusion model. However, if the optimal dividend barrier level is set too low, then the bankruptcy probability (after a dividend payment) in the near future may be too high to be acceptable. This paper aims to address this issue by taking the solvency constrain into consideration. Precisely, we consider a dividend payment problem with solvency constraint under a jump-diffusion surplus process model. Using involved stochastic control and partial integro-differential equation theories, we derive the optimal dividend payment strategy and the value function of the problem. The optimal dividend barrier level must be put higher if one can not bear the high bankruptcy probability after a dividend payment in the near future.

Keywords: Dividend payment, jump-diffusion, solvency constraints, barrier strategy, partial integro-differential equation.

1 Introduction

Dividend optimization is a classical theme in the research fields of finance and insurance. One aim of this type of problem is to seek the optimal dividend payment strategy (DPS for

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short), which maximizes the expected discounted dividends received by the shareholders until the time of the company's bankruptcy. In the real world, the policymakers have the desire to allocate the dividend as much as possible to shareholders, and they also have the need to maintain liquidity in order to reduce the company's default risk (see Cadenillas et al. (2007) for an analysis). Therefore, in order to determine the optimal dividend payment strategy (ODPS for short), the policymakers have to make a trade-off between these two factors.

Albrecher and Thonhauser (2009) and Avanzi (2009) survey the stochastic control method for dividend optimization problem. Under the assumption that the surplus of an insurance company is described by a Brownian motion with drift, Asmussen and Taksar (1997) study an optimal dividend problem for an insurance company by applying the singular stochastic control theory. For other related works we refer to Asmussen et al. (2000), Taksar (2000), Paulsen (2007, 2008), Schmidli (2008), Wei et al. (2010), Bai et al. (2012), Zhang (2012), Jiang and Yang (2013), and references therein. For the case of surplus process being spectrally negative Lévy process and the case of involving transaction costs, we refer to Loeffen (2008, 2009) and Hunting and Paulsen (2013) and references therein.

Belhaj (2010) studies a problem of optimal dividend payment for a company whose surplus follows a jump-diffusion process. The company chooses its DPS to maximize the expected value of discounted future dividend payments to its shareholders. The result shows that barrier strategy is optimal, that is, the company pays out all the reserve above a critical threshold (namely, the barrier) to shareholders as dividends, and distributes nothing whenever the surplus reserve is less than the threshold. However, if the optimal barrier level is very low, the company is more likely to go bankrupt in the near future; in another words, the bankruptcy probability in the near future is extremely high. In practice, on the other hand, policyholders pay their premiums to the insurance company in advance; they certainly expect to have their claims covered when claims occur. In order to protect the policyholders' benefit, the ODPS should not be allowed to carry out by the company. Whereas it is reasonable that the company have to find the optimal "admissive" DPS and proper provisions have been made with regard to the policyholders in the admissive DPS. So it would be realistic that the company itself imposes a solvency requirement. In fact, a provision can be set by a proper governmental insurance regulatory agency or by the company itself. Therefore, it is very necessary to consider solvency constraint in the optimal dividend payment problem.

In recent years, much attention has been paid to the stochastic control and the optimization problems with constraints, see, for example, Lions (1985), Zhu et al. (2004), Hu and Zhou (2005), Ji and Zhou (2006), Tiesler et al. (2012), van den Broek et al. (2011), Li and Xu (2016), Xu and Yi (2016). Particularly, the optimization problems involving probability constraint have been widely applied in engineering, economics and finance. Such problems are difficult to handle since they are generally non-convex. The optimal dividend payment problems with constraints have also been considered in the literature. For instance, Choulli et al. (2001, 2003, 2004) study the optimal dividend policies with risk exposure constraints. Paulsen (2003) considers an insurance company's optimal dividend problem with solvency constraints and shows that the optimal strategy is a barrier type. He et al. (2008) and He and Liang (2008) investigate the dividend optimization problems involving proportion reinsurance and solvency constraints under a simple diffusion model. Liang and Huang (2011) and Liang and Sun (2011) further study the proportional reinsurance, investment and dividend problems for insurance company under solvency constraints.

In the aforementioned works, the dividend optimization problems with constraints are all considered in the framework of diffusion model. In this paper, we assume that the insurance company's surplus process follows a jump-diffusion process, so that the insurance company faces two types of risks: unstable income (reflected by a Brownian motion) and potential losses (reflected by a Poisson process). In order to make this problem to be tractable, we further assume that the size of jump is an exponential random variable. Using the stochastic control and the partial differential-integral equation theories, we first derive the properties of the bankruptcy probability and the value function, and then the optimal dividend payment policy and the corresponding value function of the problem with solvency constraints.

This paper considers the dividend payment problem, following Dufresne and Gerber (1991), under a particular constraint, that is the probability of bankruptcy occurring within a period T after any dividend payment must be small. As de Finetti (1957) pointed out in his seminal paper that it makes no sense if $T = +\infty$, we consider, in this paper, the bankruptcy probability occurring within a finite time case, which is defined on the modified surplus after distribution of dividends. The finite time component is mandatory, otherwise the probability would be 100%, irrespective of the payment (and thus uninformative). And the fact of using the modified surplus addresses de Finetti's criticism. The main contribution of this paper is twofold. On one hand, this paper extends the investigation on the dividend optimization problems with bankruptcy probability constraint to the classical insurance surplus model with a diffusion perturbation, and it also investigates the effect of bankruptcy probability constraint on the DPS. On the other hand, this paper analyzes in detail the various properties of the finite-time bankruptcy probability and the value function under the jump-diffusion risk model.

The remainder of this paper is organized as follows. We formulate the model in Section 2. In Section 3 we give a feasibility analysis of the problem as it may fail to provide a feasible solution due to the constraint involved. Properties of the bankruptcy probability and the value function, and the main result are given in Section 4. Section 5 concludes this paper.

2 Model formulation

We first give a mathematical formulation of our problem. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions, i.e., $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous and $\mathbb{P}$-complete. We suppose that all stochastic processes and random variables below are defined on this filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We note that $\mathcal{F}_t$ represents the information available up to time t, based on which any decision is made at time t.

Following Dufresne and Gerber (1991), we consider an insurance company whose surplus, without dividend control, evolves according to

$$R_t = x + \mu t + \sigma W_t - \sum_{k=1}^{N(t)} Y_k, \tag{2.1}$$

where $x > 0$ is the initial (capital) reserve level, μ is the expected premium rate, σ is the volatility of the surplus, W_t is a standard Brownian motion adapted to the filtration $\mathbb{F}$, $N(t)$ is the number of claims occurred up to time t, and Y_k is the amount of the k-th claim.

In this paper, we assume that $N(t)$, adapted to the filtration $\mathbb{F}$, follows a homogeneous Poisson process with intensity $\lambda > 0$, i.e., $N(t) \sim \mathrm{Poi}(\lambda t)$. Moreover, the Poisson process $N(t)$ is independent of the Brownian motion W_t, and $\{Y_k\}_{k\in N}$ is a sequence of positive independent and identically distributed (i.i.d.) random variables with a common exponential distribution density

$$p(y) = \delta e^{-\delta y}, \; y \geqslant 0. \tag{2.2}$$

So $S_t = \sum_{k=1}^{N(t)} Y_k$, the total claim amount up to time t, composes a compound Poisson process $S = (S_t)_{t\geqslant 0}$. Furthermore, we assume that $N(t)$ is independent of $\{Y_k\}_{k\in N}$. In particular, when $\sigma = 0$, our risk model (2.1) reduces to the classical Cramér-Lundberg model. The insurance company cannot obtain external financing so that it has to declare bankruptcy once its reserve falls below zero. In fact, either after a run of negative realizations of the Brownian motion (so that $R_t = 0$) or suddenly when the size of a jump (claim) is larger than the surplus reserve (so that $R_t \leqslant 0$), the company will declare bankruptcy. In addition, suppose that the net profit condition is valid, that is, $\mu > \mathbb{E}(S_1) = \frac{\lambda}{\delta}$, where $\frac{1}{\delta} = \mathbb{E}(Y_1)$. It means that the expected premium per unit of time is larger than the expected loss.

Let L_t be the cumulative amount of dividend payment up to time t. Then a process $L = \{L_t\}_{t\geqslant 0}$ is called a DPS if it is nonnegative, nondecreasing, càdlàg and $\mathbb{F}$-adapted. Under a dividend payment strategy (DPS) L, the corresponding reserve (the controlled surplus process) evolves according to

$$R_t^L = x + \mu t + \sigma W_t - \sum_{k=1}^{N(t)} Y_k - L_t, \quad \text{for } t \geqslant 0, \tag{2.3}$$

and $R^L_{0-} = x > 0$. A DPS L is said to be admissible if: (i) $L_{0-} = 0$, (ii) $0 \leqslant \Delta L_t = L_t - L_{t-} \leqslant R^L_{t-}$ for all $t \geqslant 0$, a.s., where R^L_t is the controlled surplus process under L. The condition (ii) means that the insurance company cannot pay an amount of dividends larger than surplus reserves. We denote by Π the set of all admissible DPSs. The objective of the insurance company is to choose an admissible DPS to maximize the expected value of discounted future dividend payments until the time of bankruptcy

$$V(L; x) = \mathbb{E}\left(\int_{[0,\tau^L]} e^{-\rho t}\, \mathrm{d}L_t\right),$$

where

$$\tau^L = \inf\left\{t \geqslant 0 \,\middle|\, R^L_t \leqslant 0\right\}$$

is the bankruptcy time under the DPS L, that is, the first passage time of R^L_t below zero, and ρ is a constant discounting factor. The associated optimization problem then is to find the value function

$$v(x) = \sup_{L \in \Pi} V(L; x), \tag{2.4}$$

and an admissible DPS $L^* \in \Pi$, which is called the optimal dividend payment strategy (ODPS), such that $v(x) = V(L^*; x)$.

Belhaj (2010) proved that there exists an optimal barrier DPS L^b (with dividend barrier $b > 0$) for the problem (2.4). This strategy pays out immediately everything in excess of this level b as dividend at any time, and distributes nothing whenever the surplus reserves level is equal to or less than b. Mathematically, it is defined by

$$L^b_t = \sup_{s \leqslant t} (R_s - b)^+, \quad \text{for } t \geqslant 0, \tag{2.5}$$

and $L^b_{0-} = 0$, where R_s is the surplus process (2.1). The corresponding controlled surplus process (2.3) (with $L \equiv L^b$) is therefore reflected at b at any time. In this paper we focus on such barrier dividend strategies only. However, as mentioned earlier, if the dividend barrier level is too low, the bankruptcy probability in the near future after a payment will be unacceptably high. In order to avoid the company goes bankrupt soon, the company's decision-makers will consider the solvency requirement when they pay the dividend to its shareholders. One natural requirement is that, after the last dividend payment, the probability of bankruptcy occurring within a fixed time T is not allowed to exceed a certain ruin tolerance level $\varepsilon > 0$, namely,

$$\psi(b, T) \leqslant \varepsilon, \tag{2.6}$$

where

$$\begin{aligned}
\psi(b, T) &= \mathbb{P}\left(\tau_b \leqslant T \,\middle|\, \text{there exists at least one dividend payment before or at the time } \tau^{L^b}\right) \\
&= \mathbb{P}\left(\tau_b \leqslant T \,\middle|\, \text{there exists } 0 \leqslant t \leqslant \tau^{L^b} \text{ such that } L^b_t > 0\right) \\
&= \mathbb{P}\left(\tau_b \leqslant T \,\middle|\, \text{there exists } 0 \leqslant t \leqslant \tau^{L^b} \text{ such that } R_t > b\right) \\
&= \mathbb{P}\left(\tau_b \leqslant T \,\middle|\, \eta_b \geqslant 0\right),
\end{aligned}$$

and τ_b is the time from the last payment time to the bankruptcy time, namely

$$\tau_b = \tau^{L^b} - \eta_b,$$

and η_b is the last dividend payment time,

$$\eta_b = \sup\left\{0 \leqslant t \leqslant \tau^{L^b} \,\middle|\, \Delta L_t^b > 0\right\},$$

with $\sup \emptyset = -\infty$ by convention. In our model, the last dividend payment and bankruptcy may happen at the same time.

In this paper, we consider those admissible DPSs which pay dividends only when the reserve is above certain threshold. Mathematically, we denote, for any $b \geqslant 0$,

$$\Pi_b = \left\{L^a \,\middle|\, a \geqslant b\right\}.$$

Obviously, Π_0 is the set of all barrier dividend strategies, and $\Pi_{b_1} \subset \Pi_{b_2}$ if $b_1 > b_2$.

Define the set of barriers that satisfy the constraint (2.6)

$$\mathfrak{B} := \left\{b \,\middle|\, \text{ The constraint (2.6) is satisfied } \right\} = \left\{b \,\middle|\, \psi(b,T) \leqslant \varepsilon\right\},$$

and define the corresponding value function $V_c(x)$ under this constraint

$$V_c(x) := \sup_{b \in \mathfrak{B}} V(x,b), \tag{2.7}$$

where

$$V(x,b) := \sup_{L \in \Pi_b} V(L;x) = \sup_{L \in \Pi_b} \mathbb{E}\left(\int_{[0,\tau^L]} e^{-\rho s}\, \mathrm{d}L_s\right) \tag{2.8}$$

is the optimal value for the level b.

Our main purpose is to find the value function $V_c(x)$, the optimal barrier $b^* \in \mathfrak{B}$ solving the problem (2.7), and the ODPS $L^* \in \Pi_{b^*}$ solving the problem (2.8).

To solve the problem (2.7), because $V(x,b)$ is decreasing in b, we will seek the minimum barrier b which ensures that the constraint (2.6) is satisfied.

3 Feasibility analysis

Since the problem (2.7) involves probability constraint, the first issue is its feasibility (that is, whether the constraint set $\mathfrak{B}$ is empty or not), which is the subject of this section.

Firstly, we know that compound Poisson process $S_t = \sum_{k=1}^{N(t)} Y_k$ can be represented in the following form:

$$S_t = \sum_{k=1}^{N(t)} Y_k = \int_{[0,t]\times\mathbb{R}^+} xJ(\mathrm{d}s \times \mathrm{d}x),$$

where J is a Poisson random measure with intensity measure $\lambda p(dx)\,\mathrm{d}t$ (p is given by (2.2)). For every measurable set $A \subset \mathbb{R}^+$, $J([t_1, t_2] \times A)$ counts the number of jump times of S in the period $[t_1, t_2]$ such that their jump sizes are in A (see Cont and Tankov (2004) for details).

In terms of Poisson random measure, the controlled surplus process (under the barrier DPS L^b) can be written equivalently as

$$\mathrm{d}R_t^{L^b} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \int_{\mathbb{R}^+} xJ(\mathrm{d}s \times \mathrm{d}x) - \mathrm{d}L_t^b, \quad R_{0-}^{L^b} = x.$$

Define the first dividend payment time

$$\theta_b = \inf\left\{0 \leqslant t \leqslant \tau^{L^b} \,\middle|\, \Delta L_t^b > 0\right\},$$

with $\inf \emptyset = +\infty$ by convention. We notice that $0 \leqslant \theta_b \leqslant \eta_b$ whenever at least one payment paid (so that $\eta_b \geqslant 0$). Moreover θ_b is a stopping time (by contrast, η_b is not).

We define the new controlled surplus process starting from the first dividend payment time

$$\widetilde{R}_t^{L^b} = R_{t+\theta_b}^{L^b}, \quad \text{for } t \geqslant 0,$$

then

$$\mathrm{d}\widetilde{R}_t^{L^b} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \int_{\mathbb{R}^+} xJ(\mathrm{d}s \times \mathrm{d}x) - \mathrm{d}L_t^b.$$

We notice that $\widetilde{R}_{0-}^{L} = R_{\theta_b -}^{L^b} = b$. To study the probability of bankruptcy $\psi(b, T)$, we allow the initial value of R_{0-}^{L} to change and define

$$\xi_x^{L^b} = \inf\left\{t \geqslant 0 \,\middle|\, \widetilde{R}_t^{L^b} \leqslant 0,\ \widetilde{R}_{0-}^{L^b} = x\right\}, \quad \text{for } x \leqslant b,$$

and let

$$\varphi_b(x, T) = \mathbb{P}\left(\xi_x^{L^b} \leqslant T\right).$$

Obviously,

$$\varphi_b\,(b, T) = \psi(b, T).$$

Proposition 3.1. *We have*

$$\lim_{b \to +\infty} \psi(b, T) = \lim_{b \to +\infty} \varphi_b(b, T) = 0.$$

Proof. It is easily seen that

$$\xi_b^{L^b} = \inf\left\{t \geqslant 0 \,\middle|\, \widetilde{R}_t^{L^b} \leqslant 0,\ \widetilde{R}_{0-}^{L^b} = b\right\} \leqslant \inf\left\{t \geqslant 0 \,\middle|\, b + \mu t + \sigma W_t \leqslant 0\right\}.$$

It follows that

$$\left\{\ \xi_b^{L^b} \leqslant T\right\} \subseteq \left\{\inf_{0\leqslant t\leqslant T}(b+\mu t+\sigma W_t)\leqslant 0\right\}.$$

So, we have

$$\begin{aligned}\varphi_b(b,T) &= \mathbb{P}\left(\xi_b^{L^b} \leqslant T\right)\\ &\leqslant \mathbb{P}\left(\inf_{0\leqslant t\leqslant T}(b+\mu t+\sigma W_t)\leqslant 0\right)\\ &\leqslant \mathbb{P}\left(\inf_{0\leqslant t\leqslant T} W_t \leqslant -\tfrac{b}{\sigma}\right) \to 0, \qquad \text{as } b\to+\infty.\end{aligned}$$

The desired result is thus proved. □

From the above proposition, we know that the risk constrained set $\mathfrak{B}$ is non-empty for any $\varepsilon > 0$. This guarantees the problem (2.7) is well defined for any ruin tolerance level.

4 Main result

In order to derive our main result, we firstly give several lemmas.

For $x \geqslant 0$ we define

$$v_b(x) = \begin{cases} \frac{f(x)}{f'(b)}, & 0\leqslant x\leqslant b,\\ \frac{f(b)}{f'(b)}+x-b, & x>b,\end{cases}$$

where

$$f(x) = (\tfrac{1}{2}\sigma^2\theta_1^2+\mu\theta_1)(e^{\theta_3 x}-e^{\theta_2 x}) + (\tfrac{1}{2}\sigma^2\theta_2^2+\mu\theta_2)(e^{\theta_1 x}-e^{\theta_3 x}) + (\tfrac{1}{2}\sigma^2\theta_3^2+\mu\theta_3)(e^{\theta_2 x}-e^{\theta_1 x}),$$

and $\theta_1,\ \theta_2,\ \theta_3$ are the roots of the equation

$$\tfrac{1}{2}\sigma^2\theta^3 + (\mu+\tfrac{1}{2}\delta\sigma^2)\theta^2 + (\mu\delta-(\rho+\lambda))\theta - \delta\rho = 0,$$

which satisfy $\theta_1 < -\delta < \theta_2 < 0 < \theta_3$.

From Proposition 3.1 of Belhaj (2010), we know that $f''(x)$ is strictly increasing mapping from $(0,\infty)$ onto $(-\infty,+\infty)$, so that it has a unique positive root. We denote the root by $\beta > 0$ in the rest part of this paper.

Lemma 4.1 (Belhaj (2010)). *The value function of the dividend payment problem (2.4), which is without the bankruptcy probability constraint, is given by $v(x) = v_\beta(x)$. Moreover, the ODPS is the barrier DPS L^β.*

Proof. See Section 3 in Belhaj (2010). □

Lemma 4.2. *We have* $\frac{\partial}{\partial b}v_b(x) \leqslant 0$ *for* $b \geqslant \beta$.

Proof. From Belhaj (2010), we know that

$$
\begin{aligned}
(\tfrac{1}{2}\sigma^2\theta_2^2 + \mu\theta_2 - \tfrac{1}{2}\sigma^2\theta_3^2 - \mu\theta_3) &= -\tfrac{1}{2}\sigma^2(\theta_1+\delta)(\theta_3-\theta_2) < 0,\\
(\tfrac{1}{2}\sigma^2\theta_3^2 + \mu\theta_3 - \tfrac{1}{2}\sigma^2\theta_1^2 - \mu\theta_1) &= -\tfrac{1}{2}\sigma^2(\theta_2+\delta)(\theta_1-\theta_3) < 0,\\
(\tfrac{1}{2}\sigma^2\theta_1^2 + \mu\theta_1 - \tfrac{1}{2}\sigma^2\theta_2^2 - \mu\theta_2) &= -\tfrac{1}{2}\sigma^2(\theta_3+\delta)(\theta_2-\theta_1) > 0.
\end{aligned}
$$

Using these inequalities together with $\theta_1 < \theta_2 < 0 < \theta_3$ we obtain

$$
\begin{aligned}
f(x) &= (\tfrac{1}{2}\sigma^2\theta_1^2 + \mu\theta_1)(e^{\theta_3 x} - e^{\theta_2 x}) + (\tfrac{1}{2}\sigma^2\theta_2^2 + \mu\theta_2)(e^{\theta_1 x} - e^{\theta_3 x})\\
&\quad + (\tfrac{1}{2}\sigma^2\theta_3^2 + \mu\theta_3)(e^{\theta_2 x} - e^{\theta_1 x})\\
&= (\tfrac{1}{2}\sigma^2\theta_2^2 + \mu\theta_2 - \tfrac{1}{2}\sigma^2\theta_3^2 - \mu\theta_3)e^{\theta_1 x} + (\tfrac{1}{2}\sigma^2\theta_3^2 + \mu\theta_3 - \tfrac{1}{2}\sigma^2\theta_1^2 - \mu\theta_1)e^{\theta_2 x}\\
&\quad + (\tfrac{1}{2}\sigma^2\theta_1^2 + \mu\theta_1 - \tfrac{1}{2}\sigma^2\theta_2^2 - \mu\theta_2)e^{\theta_3 x}\\
&> (\tfrac{1}{2}\sigma^2\theta_2^2 + \mu\theta_2 - \tfrac{1}{2}\sigma^2\theta_3^2 - \mu\theta_3) + (\tfrac{1}{2}\sigma^2\theta_3^2 + \mu\theta_3 - \tfrac{1}{2}\sigma^2\theta_1^2 - \mu\theta_1)\\
&\quad + (\tfrac{1}{2}\sigma^2\theta_1^2 + \mu\theta_1 - \tfrac{1}{2}\sigma^2\theta_2^2 - \mu\theta_2)\\
&= 0.
\end{aligned}
$$

From Proposition 3.1 of Belhaj (2010), we also have $f''(x)$ is a monotone increasing function, so $f''(b) \geqslant f''(\beta) = 0$ for $b \geqslant \beta$. Therefore, if $x < b$, then

$$
\frac{\partial}{\partial b}v_b(x) = -\frac{f(x)f''(b)}{[f'(b)]^2} \leqslant 0,
$$

and if $x > b$, then

$$
\frac{\partial}{\partial b}v_b(x) = \frac{[f'(b)]^2 - f(b)f''(b)}{[f'(b)]^2} - 1 = -\frac{f(b)f''(b)}{[f'(b)]^2} \leqslant 0.
$$

The proof is complete as we notice that $\frac{\partial}{\partial b}v_b(x)$ is continuous at $x = b$. □

Lemma 4.3. *The bankruptcy probability* $\psi(b, T)$ *is a decreasing function with respect to* b *for any fixed* T.

Proof. By the comparison theorem of stochastic differential equation, one can show that $\varphi_b(x, T)$ is decreasing in x and b respectively. Therefore, $\psi(b, T) = \varphi_b(b, T)$ is a decreasing function of b. □

The following result is crucial for deriving our main result below.

Theorem 4.4. *For any fixed* $b > 0$, *the following partial integro-differential equation*

$$
\begin{cases}
\frac{\partial \phi}{\partial t}(t,x) = \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial x^2}(t,x) + \mu\frac{\partial \phi}{\partial x}(t,x) + \lambda\delta\Big(\int_0^\infty [\phi(t,x-y) - \phi(t,x)]e^{-\delta y}\,\mathrm{d}y\Big), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \;\; t > 0,\; 0 < x < b,\\
\phi(0,x) = 1, \qquad \textit{for} \;\; 0 < x \leqslant b,\\
\phi(0,x) = 0, \qquad \textit{for} \;\; x \leqslant 0,\\
\phi(t,x) = 0,\; \phi_x(t,b) = 0, \qquad \textit{for} \;\; t > 0,\; x \leqslant 0,
\end{cases}
\tag{4.1}
$$

admits a solution $\phi(t,x) \in C^{1,2}\{([0,+\infty)\times[0,b])\backslash(0,0)\}\cap C\{([0,+\infty)\times(-\infty,b])\backslash(0,0)\}$*. Moreover,* $\phi(T,x)$ *is the probability to survive on* $[0,T]$*, i.e.,* $\phi(T,x) = 1-\varphi_b(x,T)$ *for* $x\in[0,b]$*.*

Proof. We note that

$$\begin{aligned}
\int_0^{\infty}\phi(t,x-y)e^{-\delta y}\,\mathrm{d}y &= \int_x^{-\infty}\phi(t,z)e^{-\delta(x-z)}(-\,\mathrm{d}z)\\
&= e^{-\delta x}\int_{-\infty}^{x}\phi(t,z)e^{\delta z}\,\mathrm{d}z\\
&= e^{-\delta x}\int_0^{x}\phi(t,z)e^{\delta z}\,\mathrm{d}z,
\end{aligned}$$

so (4.1) is equivalent to

$$\begin{cases}
\frac{\partial\phi}{\partial t}(t,x) = \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2}(t,x) + \mu\frac{\partial\phi}{\partial x}(t,x) - \lambda\phi(t,x) + \lambda\delta e^{-\delta x}\int_0^x\phi(t,z)e^{\delta z}\,\mathrm{d}z,\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for}\quad t>0,\ 0<x<b,\\
\phi(0,x)=1, \qquad\qquad \text{for}\quad 0<x\leqslant b,\\
\phi(0,x)=0, \qquad\qquad \text{for}\quad x\leqslant 0,\\
\phi(t,x)=0,\ \ \phi_x(t,b)=0, \quad \text{for}\quad t>0,\ x\leqslant 0.
\end{cases} \tag{4.2}$$

Applying standard fixed point technique, we can easily prove that the above problem has a unique solution $\phi(t,x)\in C^{1,2}\{([0,+\infty)\times[0,b])\backslash(0,0)\}\cap C\{([0,+\infty)\times(-\infty,b])\backslash(0,0)\}$ with $0\leqslant\phi(t,x)\leqslant 1$. We leave the details of proof to the interested readers.

For $x\in[0,b]$, let $T_0 = T\wedge\xi_x^{L^b}$.[1] Applying the generalized Itô formula to $\phi(T-t,\widetilde{R}_t^{L^b})$ from $t=0-$ to $t=T_0$, and noticing $\widetilde{R}_{0-}^{L^b}=x$, we obtain

$$\begin{aligned}
&\phi\left(T-T_0,\widetilde{R}_{T_0}^{L^b}\right)\\
&= \phi(T,x) + \int_{[0,T_0]}\left(-\phi_t(T-s,\widetilde{R}_s^{L^b}) + \frac{1}{2}\sigma^2\phi_{xx}(T-s,\widetilde{R}_s^{L^b}) + \mu\phi_x(T-s,\widetilde{R}_s^{L^b})\right)\mathrm{d}s\\
&\quad + \int_{[0,T_0]}\lambda\delta\left(\int_0^{\infty}\left[\phi(T-s,\widetilde{R}_s^{L^b}-y)-\phi(T-s,\widetilde{R}_s^{L^b})\right]e^{-\delta y}\,\mathrm{d}y\right)\mathrm{d}s\\
&\quad + \int_{[0,T_0]}\sigma\phi_x(T-s,\widetilde{R}_s^{L^b})\,\mathrm{d}W_s - \int_{[0,T_0]}\sigma\phi_x(T-s,\widetilde{R}_s^{L^b})\,\mathrm{d}L_s\\
&= \phi(T,x) + \int_0^{T_0}\sigma\phi_x(T-s,\widetilde{R}_s^{L^b})\,\mathrm{d}W_s - \int_{[0,T_0]}\sigma\phi_x(T-s,\widetilde{R}_s^{L^b})\,\mathrm{d}L_s.
\end{aligned}$$

Notice that $\phi_x(T-s,\widetilde{R}_s^{L^b})=0$ when $\widetilde{R}_s^{L^b}=b$, and $\mathrm{d}L_s=0$ when $\widetilde{R}_s^{L^b}<b$, so we conclude the last integral in the above equation is 0 and hence

$$\phi\left(T-T_0,\widetilde{R}_{T_0}^{L^b}\right) = \phi(T,x) + \int_0^{T_0}\sigma\phi_x(T-s,\widetilde{R}_s^{L^b})\,\mathrm{d}W_s.$$

[1] As convention, we write $x\wedge y=\min\{x,y\}$ and $x\vee y=\max\{x,y\}$ for real numbers x and y.

Taking mathematical expectation on both sides of the above equality, we have

$$\begin{aligned}\phi(T,x) &= \mathbb{E}\left[\phi\left(T-(T\wedge\xi_x^{L^b}),\widetilde{R}^{L^b}_{T\wedge\xi_x^{L^b}}\right)\right]\\&=\mathbb{E}\left[\phi\left(0,\widetilde{R}^{L^b}_T\right)1_{T<\xi_x^{L^b}}\right]+\mathbb{E}\left[\phi\left(T-\xi_x^{L^b},\widetilde{R}^{L^b}_{\xi_x^{L^b}}\right)1_{T\geqslant\xi_x^{L^b}}\right]\\&=\mathbb{E}\left[1_{T<\xi_x^{L^b}}\right]\\&=1-\varphi_b(x,T),\end{aligned}$$

where we used the fact that $0<\widetilde{R}^{L^b}_T\leqslant b$ if $T<\xi_x^{L^b}$ and $\widetilde{R}^{L^b}_{\xi_x^{L^b}}\leqslant 0$ if $T\geqslant\xi_x^{L^b}$. This completes the proof. □

Lemma 4.5. *Let $\phi^b(t,x)$ be the solution of (4.1), then $b\mapsto\phi^b(T,x)$ is continuous on $[\beta\vee x,+\infty)$.*

Proof. Fix any $(t,b_1)\in[0,T]\times[\beta,+\infty)$. We will show

$$\lim_{b\to b_1}\phi^b(t,x)=\phi^{b_1}(t,x),\quad\forall\,(t,x)\in[0,T]\times[0,b_1].$$

First we see from Theorem 4.4 that $\phi^b(t,x)$ is increasing in b and

$$0\leqslant\phi^b(t,x)\leqslant 1,\quad\forall\,(t,x)\in[0,T]\times[0,b_1].$$

Hence there is a function $\overline{\phi}(t,x)\in C\{([0,T]\times[0,b_1])\setminus(0,0)\}$ such that

$$\lim_{b\to b_1}\phi^b(t,x)=\overline{\phi}(t,x),\quad\forall\,(t,x)\in[0,T]\times[0,b_1].$$

Now we prove $\overline{\phi}(t,x)$ is just $\phi^{b_1}(t,x)$. Let

$$x=bz,\quad\theta^b(t,z)=\phi^b(t,x),\quad\overline{\theta}(t,z)=\overline{\phi}(t,x).$$

It is clear that

$$0\leqslant\theta^b(t,z)\leqslant 1,\quad\lim_{b\to b_1}\theta^b(t,z)=\overline{\theta}(t,z).$$

We only need to prove that

$$\overline{\theta}(t,z)=\theta^{b_1}(t,z),\quad\forall\,(t,z)\in[0,T]\times[0,1].$$

We see from (4.2) that $\theta^b(t,z)$ satisfies

$$\begin{cases}\frac{\partial\theta^b}{\partial t}(t,z)=\frac{\sigma^2}{2b^2}\frac{\partial^2\theta^b}{\partial x^2}(t,z)+\frac{\mu}{b}\frac{\partial\theta^b}{\partial x}(t,z)-\lambda\theta^b(t,z)+\lambda\delta be^{-\delta bz}\int_0^z\theta^b(t,y)e^{\delta by}\,\mathrm{d}y,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad t>0,\ 0<z<1,\\ \theta^b(0,z)=1,\qquad\text{for}\quad 0<z\leqslant 1,\\ \theta^b(0,z)=0,\qquad\text{for}\quad z\leqslant 0,\\ \theta^b(t,z)=0,\ \theta^b_z(t,1)=0,\quad\text{for}\quad t>0,\ z\leqslant 0.\end{cases}\tag{4.3}$$

Multiplying the first equation in (4.3) by any test function $g(t,z)\in C^{1,2}([0,T]\times[0,1])$ with $g(0,z)=g(T,z)=0$ and $g(t,0)=g_z(t,1)=0$, and then integrating on $[0,T]\times[0,1]$, after integration by parts, we have that

$$\begin{aligned}&\int_0^T\int_0^1\Big(-g_t(t,z)-\frac{\sigma^2}{2b^2}g_{zz}(t,z)+\frac{\mu}{b}g_z(t,z)+\lambda g(t,z)\Big)\theta^b(t,z)\,\mathrm{d}t\,\mathrm{d}z\\ =&\int_0^T\int_0^1\Big(\lambda\delta b e^{-\delta bz}\int_0^z\theta^b(t,y)e^{\delta by}\,\mathrm{d}y\Big)g(t,z)\,\mathrm{d}t\,\mathrm{d}z.\end{aligned}$$

Letting $b\to b_1$ and applying the dominated convergence theorem, we get

$$\begin{aligned}&\int_0^T\int_0^1\Big(-g_t(t,z)-\frac{\sigma^2}{2b_1^2}g_{zz}(t,z)+\frac{\mu}{b_1}g_z(t,z)+\lambda g(t,z)\Big)\overline{\theta}(t,z)\,\mathrm{d}t\,\mathrm{d}z\\ =&\int_0^T\int_0^1\Big(\lambda\delta b_1 e^{-\delta b_1z}\int_0^z\overline{\theta}(t,y)e^{\delta b_1y}\,\mathrm{d}y\Big)g(t,z)\,\mathrm{d}t\,\mathrm{d}z.\end{aligned}$$

It means that $\overline{\theta}(t,z)$ is a weak solution of (4.3) with $b=b_1$. Thus $\overline{\theta}(t,z)\equiv\theta^{b_1}(t,z)$ by the uniqueness of solution for the problem (4.2). □

We define the following infinitesimal generator of the jump-diffusion process (for details see Cont and Tankov (2004) or Øksendal and Sulem (2009)), which is applied to a function $g\in C^2[0,\infty)$,

$$\mathcal{L}g(x):=\frac{1}{2}\sigma^2g''(x)+\mu g'(x)+\lambda\delta\left(\int_0^\infty[g(x-y)-g(x)]e^{-\delta y}\,\mathrm{d}y\right).$$

Lemma 4.6. *We have*

- *If $b\leqslant\beta$, then $V(x,b)=V(x,\beta)=V_c(x)=v_\beta(x)$;*
- *If $b>\beta$, then $V(x,b)=v_b(x)$.*

Proof. The first conclusion is obvious because L^β, the ODPS for the case without the bankruptcy probability constraint, is a feasible strategy.

Now soppose $b>\beta$. Let $L\in\Pi_b$ be an arbitrary DPS. Applying the generalized Itô formula to $e^{-\rho t}v_b(\widetilde{R}_t^L)$, we have

$$\begin{aligned}e^{-\rho(t\wedge\xi_x^L)}v_b(\widetilde{R}^L_{t\wedge\xi_x^L})=v_b(x)&+\int_0^{t\wedge\xi_x^L}e^{-\rho s}(\mathcal{L}-\rho)v_b(\widetilde{R}_s^L)\,\mathrm{d}s\\ &+\int_0^{t\wedge\xi_x^L}\sigma e^{-\rho s}v_b'(\widetilde{R}_s^L)\,\mathrm{d}W_s-\int_{[0,t\wedge\xi_x^L]}e^{-\rho s}v_b'(\widetilde{R}_s^L)\,\mathrm{d}L_s.\end{aligned}\tag{4.4}$$

From Belhaj (2010), we know that v_b is a concave function, so $1=v_b'(+\infty)\leqslant v_b'(x)\leqslant v_b'(0)$. Taking expectations on both sides of (4.4), and noting that $(\mathcal{L}-\rho)v_b\leqslant 0$, we obtain

$$\begin{aligned}\mathbb{E}\Big[e^{-\rho(t\wedge\xi_x^L)}v_b(\widetilde{R}^L_{t\wedge\xi_x^L})\Big]&\leqslant v_b(x)-\mathbb{E}\Big[\int_{[0,t\wedge\xi_x^L]}e^{-\rho s}v_b'(\widetilde{R}_s^L)\,\mathrm{d}L_s\Big]\\ &\leqslant v_b(x)-\mathbb{E}\Big[\int_{[0,t\wedge\xi_x^L]}e^{-\rho s}\,\mathrm{d}L_s\Big],\end{aligned}$$

i.e.,

$$\mathbb{E}\Big[e^{-\rho(t\wedge\xi_x^L)}v_b(\widetilde{R}^L_{t\wedge\xi_x^L})\Big]+\mathbb{E}\Big[\int_{[0,t\wedge\xi_x^L]}e^{-\rho s}\,\mathrm{d}L_s\Big]\leqslant v_b(x).$$

Since $\widetilde{R}^L_{\xi_x^L}\leqslant 0$ and $v_b(0)=0$, by letting $t\to+\infty$ in above and applying the dominated convergence theorem, we deduce

$$\lim_{t\to\infty}\mathbb{E}\Big[e^{-\rho(t\wedge\xi_x^L)}v_b(\widetilde{R}^L_{t\wedge\xi_x^L})\Big]=0,$$

which gives

$$V(L;x)=\mathbb{E}\Big[\int_{[0,\xi_x^L]}e^{-\rho s}\,\mathrm{d}L_s\Big]\leqslant v_b(x).$$

Therefore, by (2.8),

$$V(x,b)\leqslant v_b(x).$$

On the other hand, if we choose the DPS L^b, all of the above inequalities turn out to be equalities. In fact, for $0\leqslant x\leqslant b$, it is easy to see that

$$(\mathcal{L}-\rho)v_b(x)=(\mathcal{L}-\rho)\frac{f(x)}{f'(b)}=0,$$

so we obtain $(\mathcal{L}-\rho)v_b(\widetilde{R}^{L^b}_s)=0$ as $\widetilde{R}^{L^b}_s\leqslant b$. Thus, after taking expectations on both sides of (4.4) with $L\equiv L^b$,

$$\mathbb{E}\Big[e^{-\rho(t\wedge\xi_x^{L^b})}v_b(\widetilde{R}^{L^b}_{t\wedge\xi_x^{L^b}})\Big]=v_b(x)-\mathbb{E}\Big[\int_{[0,t\wedge\xi_x^{L^b}]}e^{-\rho s}v_b'(\widetilde{R}^{L^b}_s)\,\mathrm{d}L^b_s\Big].$$

Since if $\Delta L_s>0$ then $\widetilde{R}^{L^b}_s=b$, using $v_b'(b)=1$, we see that

$$\mathbb{E}\Big[e^{-\rho(t\wedge\xi_x^{L^b})}v_b(\widetilde{R}^{L^b}_{t\wedge\xi_x^{L^b}})\Big]=v_b(x)-\mathbb{E}\Big[\int_{[0,t\wedge\xi_x^{L^b}]}e^{-\rho s}\,\mathrm{d}L^b_s\Big],$$

Letting $t\to\infty$, we obtain

$$V(L^b;x)=\mathbb{E}\Big[\int_{[0,\xi_x^{L^b}]}e^{-\rho s}\,\mathrm{d}L^b_s\Big]=v_b(x).$$

Therefore,

$$V(x,b)\geqslant V(L^b;x)=v_b(x).$$

The proof is complete. □

Now, we present the main result of this paper.

Theorem 4.7. *Let $\varepsilon\in(0,1)$ be a tolerance level of the bankruptcy probability. Then, for the dividend optimization problem (2.7),*

- *If* $\mathbb{P}\Big(\xi_\beta^{L^\beta} \leqslant T\Big) \leqslant \varepsilon$, *then the ODPS is the barrier DPS* L^β, *and the optimal value function is*

$$V_c(x) = v_\beta(x).$$

- *If* $\mathbb{P}\Big(\xi_\beta^{L^\beta} \leqslant T\Big) > \varepsilon$, *then the ODPS is the barrier DPS* L^{b^*} *and the optimal value function is*

$$V_c(x) = v_{b_*}(x), \tag{4.5}$$

where the optimal barrier $b^* > \beta$ *is uniquely determined by*

$$b^* = \min\{b : \mathbb{P}(\xi_b^{L^b} \leqslant T) = \varepsilon\}.$$

Proof. The first claim is obvious since the optimal strategy L^β is feasible under the bankruptcy probability constraint.

Now suppose $\mathbb{P}\Big(\xi_b^{L^b} \leqslant T\Big) > \varepsilon$. From Proposition 3.1 and Lemmas 4.3–4.5, we see that there exists a unique $b^* > \beta$ such that

$$b^* = \min\Big\{b : \mathbb{P}(\xi_b^{L^b} \leqslant T) = \varepsilon\Big\} = \min\{b : b \in \mathfrak{B}\} \in \mathfrak{B}.$$

According to Lemma 4.2, $V(x,b) = v_b(x)$ is a decreasing function with respect to b, and from Lemma 4.6, b^* satisfies (4.5), i.e.,

$$V_c(x) = v_{b^*}(x) = \sup_{b\in\mathfrak{B}} V(x,b) = V(L^{b^*}; x).$$

The proof is completed. □

5 Conclusion remarks

This paper has considered an insurance company's optimal dividend payment problem with the solvency constraint, assuming the surplus of the insurance company follows a jump-diffusion model. Investigating such problems with constraint is a significant and challenging subject both in theoretical study and in practical applications.

The important feasibility issue of the model has been fully studied and a closed-form optimal solution has been given. Both involved stochastic control theory and partial differential-integral equation theory have been employed so as to derive the solution for the problem.

In this paper, we have only considered barrier dividend payment strategies; more general strategies will be studied in future works. This is an interesting and very important problem and new methodologies are called for to solve it.

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