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An Optimal Investment Problem with Nonsmooth and Nonconcave Utility over a Finite Time Horizon*

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Abstract. In this paper, we study a class of optimal investment problems with a nonsmooth and nonconcave utility function, where the value function is the expected utility determined by the state process and time. We adopt partial differential equation methods to prove that the value function belongs to $C^{2,1}$ under some proper conditions of the utility function. Moreover, we analyze the continuity of the optimal strategy and obtain some of its properties around the boundary and the terminal time. Also, an example sheds light on the theoretical results established.

Key words. optimal investment, parabolic quasi-linear equation, nonsmooth, nonconcave, dual transformation

AMS subject classifications. 35K10, 93E20, 91B70, 91G80

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1. Introduction.

Continuous-time stochastic optimal control problems have been widely studied and developed in finance. Two of the main methods to solve these problems are partial differential equation (PDE) and martingale theory. The PDE approach requires that the underlying state process should be Markovian so that the corresponding Hamilton–Jacobi–Bellman (HJB) equation can be derived using the principle of dynamic programming. For example, Zariphopoulou [34] studied a general investment and consumption problem with borrowing constraints for an agent. Vila and Zariphopoulou [29] extended the research to an infinite horizon problem. Xu and Yi [31] considered a continuous-time model with a constraint on the consumption rate. Guan et al. [13] investigated a compensation problem with nonsmooth and nonconcave utility over a finite time horizon. Usually, in addition to proving the existence and uniqueness of the solution to the PDE, a prior estimation and smoothness of the solution should be discussed to guarantee that the solution is the value function using a verification theorem. To employ the martingale method, one adopts the basic idea that the state process and feasible trading strategies are generated by measurable random payoffs by means of a linear representation under an expectation formula weighted by a variable. Hence, the dynamic optimization problem can be reduced to a static optimization one. For example, Karatzas, Lehoczky, and Shreve [16] derived the explicit solution of a general consumption and investment decision problem. Cox and Huang [6] applied the martingale technique to achieve

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the explicit construction of optimal portfolios. Carpenter [5] introduced the utility problem of a risk-averse manager with compensation schemes and derived its optimal portfolios.

The convex duality approach is an effective method to further deal with such problems. From the perspective of PDE, we can apply the dual (Fenchel–Legendre) transformation to reduce a fully nonlinear and degenerate problem into a linear nondegenerate problem, which can be studied by classical PDE theory, and then the solution of the original problem can be constructed by inverse duality transformation. For example, Jian, Li, and Yi [15] investigated optimal stopping investment problems over a finite time horizon. Xu and Yi [31] considered a continuous-time model with an upper bound constraint on the consumption rate. Guan et al. [13] initially discussed a free boundary problem with a nonsmooth and nonconcave utility function. Ma, Yi, and Guan [23] analyzed the consumption problem whose rate involves lower and upper constraints. To employ the martingale method, we can transfer a dynamic optimization problem into a static optimization one, and apply convex analysis to show the existence of the optimal solutions to the original problem and establish their dual relationship, and then adopt the martingale representation theorem or more general optional decomposition theorems to super-replicate the optimal terminal wealth/consumption. Details can be found in books written by Fleming and Soner [10], Karatzas and Shreve [18], and Pham [25] and the paper by Kramkov and Schachermayer [20].

In the classical case where the utility function is concave and smooth, the model and its solution are well known. Kramkov and Schachermayer [20] considered the problem of expected utility maximization in an incomplete market, where asset prices are semimartingales. Karatzas and Zitkovic [19] established a general existence and uniqueness result using techniques of convex duality. Hugonnier and Kramkov [14] studied the problem where an agent receives random endowments at maturity. Biagini and Frittelli [1] discussed the case in incomplete markets where price processes are described by unbounded semimartingales. Bian and Zheng [3] analyzed the turnpike property in financial economics. The nonstandard case, where the utility function is nonconcave or nonsmooth, has also been widely investigated. Bouchard, Touzi, and Zeghal [4] studied the dual formulation of the utility maximization problem where the utility function is nonsmooth. The problem is solved through approximating the utility function by a sequence of functions with a bounded negative domain. Westray and Zheng [30] considered a nonsmooth utility maximization problem on the positive real line. Bian, Miao, and Zheng [2] constructed a smooth and strictly convex solution to the dual HJB equation. Its conjugate function is proved to be a smooth and strictly concave solution to the primal HJB equation satisfying the terminal and boundary conditions. From the perspective of PDE, the smoothness of the value function is a fundamental property, which plays an important role in proving a verification theorem. For general nonsmooth and/or nonstrictly concave utility functions, Bian, Miao, and Zheng [2] showed that there exists a smooth classical solution to the HJB equation for a large class of constrained problems.

In this paper, we consider an optimal investment problem with nonsmooth and nonconcave utility, which is significantly different from Bian, Miao, and Zheng [2], and the corresponding explicit solution cannot be derived. Under some restrictions on the utility functions, we prove that the value function is a classical solution, which belongs to $C^{2,1}$ and is strictly increasing yet strictly concave in the wealth variable. We also prove that the optimal portfolios on risky assets are continuous. While the time variable approaches the terminal date and the corresponding wealth process belongs to the nonconcave region, the investor will seek enough risks to make the expectation of return reach the concave envelope. While the wealth process approaches zero, the corresponding optimal portfolio on the risky assets will approach zero, but the value function will increase rapidly.

The main contributions and difficulties of this paper are listed below. First, we prove that the value function is a classical solution and strictly increasing yet strictly concave for a large class of constrained utility problems. Second, we adopt the PDE technique with a convex duality method to study the optimal expected utility problem involving general nonsmooth and nonconcave utility. Third, the fundamental difficulty is that the proof requires a large number of a priori estimates of dual problem when we prove that the inverse transformation of the solution to the dual problem is the solution to the original problem. Fourth, in addition to properties of the value function, we also derive the behavior of the optimal investment strategy, such as its continuity in the domain, and its convergence around the terminal date and left boundary.

The remainder of the paper is organized as follows. In section 2, the mathematical formulation of the model is presented, and the terminal condition of the value function with a nonconcave utility function is discussed. In section 3, we derive the HJB equation and formulate the fully nonlinear problem. In section 4, we show the main results of this paper, including the existence and some properties of the solution to the fully nonlinear problem, and study the behavior of the optimal strategy around the boundary and terminal time. In section 5, we introduce an example of this model when there is an uncertain exit time forcing the investor to leave the market. In section 6, we conclude the main results of this paper. In Appendix A, we present the proofs of Theorems 2.1, 4.3, and 4.7 and Lemma 4.8, respectively. In Appendix B, we discuss the growth condition for using the maximum principle on proving $v_y \leq 0$ and $v_{yy} \geq 0$ in the proof of Theorem 2.1.

2. Model formulation.

Consider a complete, arbitrage-free, and continuous-time financial market consisting of one riskless asset and n risky assets. The riskless asset price $S_{0,t}$ is governed by the ordinary differential equation

$$\frac{\mathrm{d}S_{0,t}}{S_{0,t}} = r\mathrm{d}t,$$

and the risky asset prices $S_{i,t}$ are governed by the stochastic differential equations

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = (r+\mu_i)\mathrm{d}t + \sum_{j=1}^{n} \sigma_{i,j}\mathrm{d}W_t^j \quad \text{for } i = 1, 2, \dots, n, \tag{2.1}$$

where the interest rate r of riskless assets, the excess appreciation rates μ_i, and the volatility $\sigma_{i,j}$ of risky assets are constants, and W_t $(t \geq 0)$ is a standard n-dimensional Brownian motion. In addition, the covariance matrix $\sigma\sigma'$ is strongly nondegenerate.

Over a finite time horizon $[0, T]$, a trading strategy for the manager, for given time $t \in [0, T]$, is an n-dimensional process π_s $(s \in [t, T])$, whose ith component $\pi_{i,s}$ is the amount invested in the ith risky asset in the portfolio at time s. An admissible trading strategy π_s must be progressively measurable with respect to the σ-fields $\{\mathcal{F}_s\}$ generated by the Brownian motion and guarantee that $X_s \geq 0$. Note that $X_s = \pi_{0,s} + \sum_{i=1}^{n} \pi_{i,s}$, where $\pi_{0,s}$ is the amount invested in the riskless asset. Hence, the wealth process, $X_s (s \in [t, T])$, follows:

$$\begin{cases} \mathrm{d}X_s = (rX_s + \mu'\pi_s)\mathrm{d}s + \pi_s'\sigma\mathrm{d}W_s, \quad t \leq s \leq T, \\ X_t = x. \end{cases} \tag{2.2}$$

Within the general framework, the dynamic problem is to choose an admissible trading strategy π_s $(s \in [t, T])$ to maximize

$$V(x, t) = \sup_{\pi} \mathbb{E}\left[\int_t^T f(X_s, s)\mathrm{d}s + g(X_T)\right], \tag{2.3}$$

where $f(x, t)$ and $g(x)$, which represent the investor's degree of "happiness" or "satiation" for a given wealth level, are nonnegative continuous functions defined in $\Omega_T = \{(x, t) | x > 0,\ 0 < t < T\}$. Also, suppose that $f(x, t)$ and $g(x)$ are increasing in x; then the convexity means that the investor is risk-seeking, and concavity means risk-averseness.

If $X_t = 0$, in order to keep $X_s \geq 0$, we get that $\pi_s = 0$ and $X_s \equiv 0$, $t \leq s \leq T$. Thus, we obtain a left boundary condition

$$V(0, t) = \int_t^T f(0, s)\mathrm{d}s + g(0). \tag{2.4}$$

In order to make (2.3) well defined, some constraints should be imposed on $f(x, t)$ and $g(x)$. We suppose the following.

Condition I. *There exist a $\gamma \in (0, 1)$ and an $M > 0$ such that for all x, $y \geq 0$, we have*

$$\begin{cases} |g(x) - g(y)| \leq \frac{M}{\gamma}|x - y|^{\gamma}, \\ |f(x, t) - f(y, t)| \leq \frac{M}{\gamma}|x - y|^{\gamma}, \end{cases} \tag{2.5}$$

which also implies the growth condition

$$\begin{cases} g(x) \leq g(0) + \frac{M}{\gamma}x^{\gamma}, \\ f(x, t) \leq f(0, t) + \frac{M}{\gamma}x^{\gamma}. \end{cases} \tag{2.6}$$

Condition II. *Suppose*

$$\lim_{x \to +\infty} g(x) = +\infty. \tag{2.7}$$

2.1. The case that $g(x)$ is nonconcave. If $g(x)$ is nonconcave, we denote $\varphi(x)$ as its concave envelope; i.e., $\varphi(x)$ is the minimal concave function not being less than $g(x)$ (see Figure 2.1).

Since $g(x)$ is increasing and continuous and $\varphi(x)$ is increasing and continuous in x, $\{x > 0 | \varphi(x) > g(x)\}$ is an open set which can be expressed as

$$\Big\{x > 0 \Big| \varphi(x) > g(x)\Big\} = \bigcup_{m=1}^{\infty} (\underline{x}_m, \overline{x}_m), \tag{2.8}$$

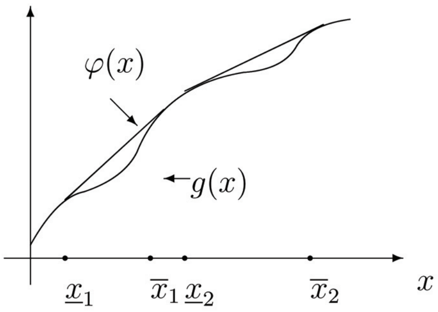

Figure 2.1. $\varphi(x)$.

where $\{(\underline{x}_m, \overline{x}_m)\}_{m=1}^{\infty}$ are countable disjoint open intervals. In the interior of these intervals, $\varphi(x)$ is a linear function.

Note that the portfolio π_t is unconstrained: we point out that the terminal condition of V should be $\varphi(x)$ but not $g(x)$. In fact, within a short time, the behavior of the wealth process is like a martingale. While time t approaches the terminal date and the current wealth process x is located in $(\underline{x}_m, \overline{x}_m)(m \in \mathbb{Z})$, the investor could adopt a strategy such that he/she will hold sufficiently risky assets, and then X_s will rapidly touch $\underline{x}_m$ or $\overline{x}_m$ (with the probability approximately equal to $\frac{x-\underline{x}_m}{\overline{x}_m-\underline{x}_m}$ and $\frac{\overline{x}_m-x}{\overline{x}_m-\underline{x}_m}$, respectively) so that $\mathbb{E}[g(X_T)]$ in the right-hand side of (2.3) approximates to

$$\frac{x-\underline{x}_m}{\overline{x}_m-\underline{x}_m}g(\underline{x}_m) + \frac{\overline{x}_m-x}{\overline{x}_m-\underline{x}_m}g(\overline{x}_m) = \varphi(x).$$

Therefore, the value function is not less than $\varphi(x)$ around the terminal date. Adopting this idea, we could prove the following theorem.

Theorem 2.1. *The value function satisfies*

$$\lim_{t\to T-} V(x,t) = \varphi(x). \tag{2.9}$$

Proof. We prove it in Appendix A.1. ■

3. The HJB equation.

First, according to (2.3), we obtain the following HJB equation:

$$-V_t - \sup_{\pi}\left[\frac{1}{2}(\pi'\sigma\sigma'\pi)V_{xx} + \mu'\pi V_x\right] - rxV_x = f(x,t) \quad \text{in} \quad \Omega_T, \tag{3.1}$$

where

$$\Omega_T := (0,+\infty)\times(0,T).$$

Then we prove that the solution of (3.1) under boundary condition (2.4) and terminal condition (2.9) belongs to $C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$.

Since $f(x,t)$ may not be smooth, concave, or convex, we suppose the following.

Condition III. *$f(x,t)$ is differentiable in x almost everywhere, and there exists the following decomposition in Ω_T:*

$$f_x(x,t) = P(x,t) - Q(x,t) \quad a.e., \tag{3.2}$$

where $P(x,t)$ and $Q(x,t)$ are locally bounded and increasing in x; namely, $f_x(\cdot,t)$ is a bounded variation function for all $t \in (0,T)$ (see [28]). In other words, $f(\cdot,t)$ can be decomposed into a convex function and a concave function.

From the definition in (2.3), we easily observe that V is increasing in x. From (3.1), we know that its solution must be concave. Otherwise, the maximum in the equation will be infinite. These analyses lead us to seek the solution of (2.4) satisfying

$$V_x > 0, \quad V_{xx} < 0, \quad x > 0, \quad 0 < t < T. \tag{3.3}$$

Note that the gradient of $\pi'\sigma\sigma'\pi$ with respect to π is

$$\bigtriangledown_\pi(\pi'\sigma\sigma'\pi) = 2\sigma\sigma'\pi.$$

Hence,

$$\pi^* = -(\sigma\sigma')^{-1}\mu\frac{V_x}{V_{xx}}.$$

Define $a^2 = \mu'(\sigma\sigma')^{-1}\mu$; then we obtain the following terminal-boundary problem:

$$\begin{cases} -V_t + \dfrac{a^2}{2}\dfrac{V_x^2}{V_{xx}} - rxV_x = f(x,t) \quad \text{in} \quad \Omega_T, \\ V(0,t) = \displaystyle\int_t^T f(0,s)\mathrm{d}s + g(0), \quad 0 < t < T, \\ V(x,T) = \varphi(x), \quad x > 0. \end{cases} \tag{3.4}$$

We will show that this problem has a (unique) solution $\widehat{V} \in C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$ which satisfies (3.3) under Conditions I–III (see Theorem 4.6).

Remark 3.1. The concavity of the value function is attributed to the fact that there is no restriction on borrowing. If the agent chooses the proportion on risky assets without limit, then the terminal condition does not follow the utility function but satisfies its concave envelope, and the concavity will be transferred to the interior of the region through the equation. However, if there exists a borrowing constraint and the utility function is not concave, such as given in [8], the value function may not be concave.

4. Main results.

In this section, we prove the solvability of problem (3.4), and study the behavior of the optimal strategy around the boundary and terminal time.

4.1. The solvability of (3.4).

Note that (3.4) is a fully nonlinear and singular equation. Since its dual equation is a standard quasi-linear equation, we start with the dual problem. For the convenience of readers, we will first introduce some knowledge of dual transformation and then derive the dual equation. The process is heuristic because the derivation depends on some a priori assumptions on the primal problem. However, we are able to construct the solution of problem (3.4) by inverse transformation and prove those a priori properties using the conclusions of the dual problem.

4.1.1. The dual transformation of $\varphi(x)$. First, we introduce the concept of dual transformation.

Definition 4.1. *If $u : (0, +\infty) \to \mathbb{R}$ is increasing and concave on $(0, +\infty)$, then the dual transformation is the function $\widetilde{u} : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\widetilde{u}(y) = \sup_{x>0}(u(x) - xy), \quad y > 0.$$

The next proposition collects some useful results utilized in this section.

Proposition 4.2. *If $\widetilde{u}$ is a decreasing function and is convex on $(0, +\infty)$, then the conjugate relation is*

$$u(x) = \inf_{y>0}(\widetilde{u}(y) + xy), \quad x > 0.$$

Denote $\text{dom}(\widetilde{u}) = \{y > 0 | \widetilde{u}(y) < +\infty\}$. *Suppose that one of the two following equivalent conditions is satisfied:*

(i) *u is differentiable on $(0, +\infty)$;*

(ii) *$\widetilde{u}$ is strictly convex on* $\text{int}(\text{dom}(\widetilde{u}))$.

Then the derivative u' is a mapping from $(0, +\infty)$ into $\text{int}(\text{dom}(\widetilde{u})) \neq \emptyset$ *and*

$$u'(x) = \underset{y\geq 0}{\text{argmin}}(\widetilde{u}(y) + xy) \quad \forall x > 0.$$

Moreover, define $\widetilde{u}'(y\pm) = \lim_{z\to y\pm} \frac{\widetilde{u}(z)-\widetilde{u}(y)}{z-y}$; then

$$\widetilde{u}'(y-) \leq \widetilde{u}'(y+) \leq 0 \quad \forall y \in dom(\widetilde{u})$$

and

$$\underset{x\geq 0}{\text{argmax}}(u(x) - xy) = \{x \geq 0 | u'(x) = y\} = [-\widetilde{u}'(y+), -\widetilde{u}'(y-)] \quad \forall y \in \text{dom}(\widetilde{u}). \tag{4.1}$$

If u is strictly concave, then $\widetilde{u}$ is differentiable with $\widetilde{u}'(y) = -(u')^{-1}(y)$.

Finally, under the additional conditions

$$u'(0) = +\infty, \quad u'(+\infty) = 0,$$

we have $\text{int}(\text{dom}(\widetilde{u})) = \text{dom}(\widetilde{u}) = (0, +\infty)$.

Proof. See Appendix B of [25]. ■

Now define the dual transformation of $\varphi(x)$ as

$$\widetilde{\varphi}(y) = \sup_{x>0}(\varphi(x) - xy), \quad y > 0$$

(see Figure 4.1).

Then, by Proposition 4.2, $\widetilde{\varphi}(y)$ is a decreasing and convex function and

$$\varphi(x) = \inf_{y>0}(\widetilde{\varphi}(y) + xy).$$

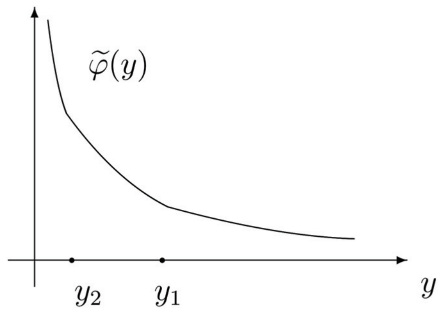

Figure 4.1. $\widetilde{\varphi}(y)$.

Due to the fact that $\varphi(x)$ is not strictly concave, we know that $\widetilde{\varphi}(y)$ is not continuously differentiable. In addition, since $\widetilde{\varphi}(y)$ is convex, we can define

$$\widetilde{\varphi}'(y\pm) = \lim_{z\to y\pm} \frac{\widetilde{\varphi}(z) - \widetilde{\varphi}(y)}{z - y}.$$

Corresponding to the description of $\varphi(x)$ in (2.8), we define

$$y_m = \varphi'(x), \quad x \in (\underline{x}_m, \overline{x}_m), \quad m = 1, 2, \ldots,$$

and we have

$$\widetilde{\varphi}'(y_m+) = -\underline{x}_m, \quad \widetilde{\varphi}'(y_m-) = -\overline{x}_m, \quad m = 1, 2, \ldots$$

(see Figure 4.2).

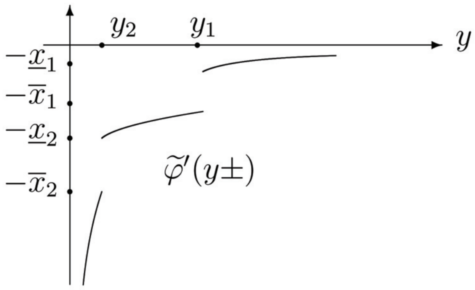

Figure 4.2. $\widetilde{\varphi}'(y\pm)$.

On the other hand,

$$\widetilde{\varphi}(y) = \sup_{x>0}(\varphi(x) - xy) \geq \varphi(0) = g(0), \tag{4.2}$$

and by (2.6),

$$\widetilde{\varphi}(y) = \sup_{x>0}(\varphi(x) - xy) \leq \sup_{x>0}\left(g(0) + M\frac{1}{\gamma}x^\gamma - xy\right) = g(0) + M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}}. \tag{4.3}$$

Due to (2.7), we obtain

$$\widetilde{\varphi}(y) = \sup_{x>0}(\varphi(x) - xy) \geq \varphi\left(\frac{1}{y}\right) - 1 \to +\infty, \quad y \to 0+. \tag{4.4}$$

We will use these results later.

4.1.2. The dual problem of (3.4). Now we define a dual transformation of $V(x,t)$. For any $t \in (0,T)$, define

$$v(y,t) = \sup_{x>0}(V(x,t) - xy), \quad y > 0. \tag{4.5}$$

Based on the assumption $V \in C^{2,1}(\Omega_T)$ and (3.3), we further take the a priori assumption that

$$V_x(0+,t) = +\infty, \quad V_x(+\infty,t) = 0. \tag{4.6}$$

Then the optimal x, corresponding to a fixed y (> 0), satisfies

$$\partial_x(V(x,t) - xy) = V_x(x,t) - y = 0,$$

i.e.,

$$x = I(y,t) := (V_x(\cdot,t))^{-1}(y), \tag{4.7}$$

and thus

$$v(y,t) = V(I(y,t),t) - I(y,t)y. \tag{4.8}$$

It follows from (4.8) that

$$v_y(y,t) = V_x(I(y,t),t)I_y(y,t) - yI_y(y,t) - I(y,t) = -I(y,t), \tag{4.9}$$

$$v_{yy}(y,t) = -I_y(y,t) = \frac{-1}{V_{xx}(I(y,t),t)}, \tag{4.10}$$

$$v_t(y,t) = V_t(I(y,t),t) + V_x(I(y,t),t)I_t(y,t) - yI_t(y,t) = V_t(I(y,t),t).$$

Hence, from (3.4) we derive

$$\begin{cases} -v_t - \dfrac{a^2}{2}y^2 v_{yy} + ryv_y = f(-v_y,t) \quad \text{in} \quad \Omega_T, \\ v(y,T) = \widetilde{\varphi}(y), \quad y > 0. \end{cases} \tag{4.11}$$

4.1.3. The solvability of problem (4.11).

Theorem 4.3. *If Conditions I–III hold, then problem* (4.11) *has a solution*

$$v \in C^{2,1}(\Omega_T)\bigcap C\Big(\Omega_T\bigcup\{t=T\}\Big)$$

satisfying

$$\int_t^T f(0,s)\mathrm{d}s + \widetilde{\varphi}(y) \le v(y,t) \le \int_t^T f(0,s)\mathrm{d}s + g(0) + M^{\frac{1}{1-\gamma}} e^{A(T-t)}\frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}}, \tag{4.12}$$

$$v_y(y,t) < 0, \tag{4.13}$$

$$v_{yy}(y,t) > 0 \tag{4.14}$$

in Ω_T*, where* $A = \frac{a^2}{2}\frac{\gamma}{(1-\gamma)^2} + \frac{\gamma r+1}{1-\gamma}$ *is a constant.*

Proof. We prove it in Appendix A.2. ∎

Lemma 4.4. v_y *satisfies*

$$\lim_{y\to 0+} v_y(y,t) = -\infty, \quad 0<t<T, \tag{4.15}$$

$$\lim_{y\to +\infty} v_y(y,t) = 0, \quad 0<t<T, \tag{4.16}$$

$$\lim_{y\to +\infty} y v_y(y,t) = 0, \quad 0<t<T. \tag{4.17}$$

Proof. For any $t\in(0,T)$, the first inequality in (4.12) and (4.4) implies $\lim_{y\to 0+} v(y,t) \ge \lim_{y\to 0+}\widetilde{\varphi}(y) + \int_t^T f(0,s)\mathrm{d}s = +\infty$. Since $v_{yy}>0$, for a fixed $y_0>0$, we derive

$$v_y(y,t) \le \frac{v(y_0,t)-v(y,t)}{y_0-y} \to -\infty, \quad y\to 0+\,.$$

Hence, we prove (4.15).

Owing to $v_{yy}>0$, for any $y>0$, we have

$$v_y(y,t) \ge \frac{v(y,t)-v(\frac{y}{2},t)}{\frac{y}{2}}.$$

Using (4.12), we get

$$\begin{aligned} v_y(y,t) &\ge \frac{\widetilde{\varphi}(y) - g(0) - M^{\frac{1}{1-\gamma}} e^{A(T-t)}\frac{1-\gamma}{\gamma}\left(\frac{y}{2}\right)^{\frac{\gamma}{\gamma-1}}}{\frac{y}{2}} \\ &\ge -Cy^{\frac{1}{\gamma-1}} \to 0, \quad y\to+\infty, \end{aligned}$$

where the last inequality above is derived due to (4.2). Furthermore,

$$y v_y(y,t) \ge -Cy^{\frac{\gamma}{\gamma-1}} \to 0, \quad y\to+\infty.$$

Combining the result with $v_y<0$, we obtain (4.16) and (4.17). ∎

4.1.4. The solution of problem (3.4). Now, set

$$\widehat{V}(x,t) = \inf_{y>0}(v(y,t)+xy) \quad \forall x>0,\ 0<t<T. \tag{4.18}$$

We now prove that $\widehat{V}(x,t)$ defined in (4.18) is the solution to problem (3.4).

According to (4.14), (4.15), and (4.16), the minimum solution to (4.18) is

$$y^* = \arg\min_{y>0}(v(y,t)+xy) = J(x,t) := (v_y(\cdot,t))^{-1}(-x) \quad \forall x>0,\ 0<t<T, \tag{4.19}$$

and thus

$$\widehat{V}(x,t) = v(J(x,t),t) + xJ(x,t), \tag{4.20}$$

where $J(x,t) \in C((0,+\infty)\times(0,T))$ is decreasing in x.

Lemma 4.5. *The function J defined in (4.19) satisfies*

$$\lim_{x\to 0+} J(x,t) = +\infty, \quad 0<t<T, \tag{4.21}$$

$$\lim_{x\to 0+} xJ(x,t) = 0, \quad 0<t<T, \tag{4.22}$$

$$\lim_{x\to +\infty} J(x,t) = 0, \quad 0<t<T. \tag{4.23}$$

Proof. The above results (4.21), (4.22), and (4.23) can be directly derived from (4.16), (4.17), and (4.15), respectively. ∎

Theorem 4.6. *The function $\widehat{V}$ defined in (4.18) belongs to $C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$ and is the solution to problem (3.4). Moreover, it satisfies*

$$\widehat{V}_x > 0, \quad \widehat{V}_{xx} < 0 \quad \forall x>0,\ 0<t<T \tag{4.24}$$

and

$$\lim_{x\to 0+}\widehat{V}_x(x,t) = +\infty, \quad \lim_{x\to +\infty}\widehat{V}_x(x,t) = 0, \quad 0<t<T. \tag{4.25}$$

Proof. It follows from (4.20) that

$$\widehat{V}_x(x,t) = v_y(J(x,t),t)J_x(x,t) + xJ_x(x,t) + J(x,t) = J(x,t) > 0, \tag{4.26}$$

$$\widehat{V}_{xx}(x,t) = J_x(x,t) = \partial_x[(v_y(\cdot,t))^{-1}(x)] = \frac{-1}{v_{yy}(J(x,t),t)} < 0, \tag{4.27}$$

$$\widehat{V}_t(x,t) = v_y(J(x,t),t)J_t(x,t) + v_t(J(x,t),t) + xJ_t(x,t) = v_t(J(x,t),t). \tag{4.28}$$

Then we derive $\widehat{V}(x,t) \in C^{2,1}(\Omega_T)$ and

$$\left(-\widehat{V}_t + \frac{a^2}{2}\frac{\widehat{V}_x^2}{\widehat{V}_{xx}} - rx\widehat{V}_x + \beta\widehat{V}\right)(x,t) = \left(-v_t - \frac{a^2}{2}y^2v_{yy} - (\beta-r)yv_y + \beta v\right)(J(x,t),t) = 0.$$

Next, we verify the boundary and terminal conditions. According to (4.21), (4.22), and (4.12), we have

$$\lim_{x\to 0+}\widehat{V}(x,t)=\lim_{x\to 0+}[v(J(x,t),t)+xJ(x,t)]=\lim_{y\to+\infty}v(y,t)=\int_t^T f(0,t)+g(0),\quad t\in(0,T).$$

Then $\widehat{V}$ satisfies the boundary condition in (3.4).

According to (4.12), we have $v(y,t)\ge\widetilde{\varphi}(y)$. Also, we have

$$\widehat{V}(x,t)=\inf_{y>0}(v(y,t)+xy)\ge\inf_{y>0}(\widetilde{\varphi}(y)+xy)=\varphi(x).$$

On the other hand, we obtain

$$\widehat{V}(x,t)=\inf_{y>0}(v(y,t)+xy)\le v(\varphi'(x),t)+x\varphi'(x).$$

Let $t\to T-$. Then we have

$$\limsup_{t\to T-}\widehat{V}(x,t)\le\lim_{t\to T-}v(\varphi'(x),t)+x\varphi'(x)=\widetilde{\varphi}(\varphi'(x))+x\varphi'(x)=\varphi(x).\tag{4.29}$$

Hence, $\widehat{V}$ satisfies the terminal condition in (3.4).

Actually, since $\varphi'(x)$ is continuous and $\lim_{t\to T-}v(y)=\widetilde{\varphi}(y)$ is locally uniform to y, the limit in (4.29) will be locally uniform to x. ∎

4.1.5. Verification theorem. Now we present the following verification theorem.

Theorem 4.7. *The solution $\widehat{V}$ to problem (3.4) constructed in section 4.1.4 is the value function V defined in (2.3). Moreover, the optimal portfolio in risky assets π_t^* is a continuous vector function of the wealth x and the current time t, which can be expressed as*

$$\widehat{\pi}(x,t)=-(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}.$$

Thus, $\pi_s^*=\widehat{\pi}(X_s,s)$, $t\le s\le T$.

Proof. We prove it in Appendix A.3. ∎

4.2. Behavior of strategy around boundary and terminal line. In this part, we further study the behavior of π^* around $x=0$ and $t=T$.

4.2.1. Behavior of strategy around $x=0$.

Lemma 4.8. *The solution to problem (4.11) has the following limits:*

$$\lim_{y\to+\infty}y^2v_{yy}(y,t)=0,\quad 0<t<T,\tag{4.30}$$

$$\lim_{y\to+\infty}v_t(y,t)=-f(0,t),\quad 0<t<T.\tag{4.31}$$

Proof. We prove it in Appendix A.4. ∎

Following Lemma 4.8, we have the following.

Theorem 4.9. $\widehat{\pi}$ *satisfies*

$$\lim_{x\to 0+}\widehat{\pi}(x,t)=\lim_{x\to 0+}\left[-(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}\right]=0,\quad 0<t<T. \tag{4.32}$$

Proof. By (4.26), (4.27), and (4.30), we derive

$$\lim_{x\to 0+}\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}=\lim_{y\to+\infty}(-yv_{yy}(y,t))=\lim_{y\to+\infty}\frac{1}{y}\lim_{y\to+\infty}(-y^2v_{yy}(y,t))=0. \qquad \blacksquare$$

Generally speaking, in order to prevent bankruptcy, when wealth is scarce, we should avoid investing in risky securities to reduce risk.

4.2.2. Behavior of strategy around $t = T$.

Lemma 4.10. v_y *satisfies*

$$\widetilde{\varphi}'(y-)\le\liminf_{t\to T-}v_y(y,t)\le\limsup_{t\to T-}v_y(y,t)\le\widetilde{\varphi}'(y+),\quad y>0. \tag{4.33}$$

Proof. For fixed $y > 0$, if (4.33) is not true, there exists a sequence $\{t_n\}_{n\in\mathbb{Z}^+}$ such that $\lim_{n\to\infty} t_n = T$, and

$$\theta:=\lim_{n\to\infty}v_y(y,t_n)>\widetilde{\varphi}'(y+)\quad(\text{or }<\widetilde{\varphi}'(y-)).$$

Note that

$$v(y,t_n)=\widehat{V}(-v_y(y,t_n),t_n)+v_y(y,t_n)y.$$

Letting $n\to\infty$, we have

$$\lim_{n\to\infty}v(y,t_n)=\varphi(-\theta)+\theta y<\max_{x\ge 0}(\varphi(x)-xy)=\widetilde{\varphi}(y),$$

where the inequality above is due to (4.1). This contradicts $\lim_{y\to T-} v(y,t) = \widetilde{\varphi}(y)$. Thus, we prove (4.33). $\blacksquare$

Lemma 4.11. V_x *satisfies*

$$\lim_{t\to T-}V_x(x,t)=\varphi'(x)=y_m=\frac{\varphi(\overline{x}_m)-\varphi(\underline{x}_m)}{\overline{x}_m-\underline{x}_m},\quad x\in(\underline{x}_m,\overline{x}_m),\ m=1,2,\dots. \tag{4.34}$$

Proof. For $x\in(\underline{x}_m,\overline{x}_m)$, according to Lemma 4.10, for any small $\varepsilon>0$, we have

$$\begin{aligned}\liminf_{t\to T-}v_y(y_m+\varepsilon,t)&\ge\widetilde{\varphi}'((y_m+\varepsilon)-)\ge\widetilde{\varphi}'(y_m+)\\&=-\underline{x}_m>-x>-\overline{x}_m\\&=\widetilde{\varphi}'(y_m-)\ge\widetilde{\varphi}'((y_m-\varepsilon)+)\ge\limsup_{t\to T-}v_y(y_m-\varepsilon,t).\end{aligned}$$

If t is sufficiently close to T, then

$$v_y(y_m+\varepsilon,t) > -x > v_y(y_m-\varepsilon,t).$$

Due to the fact that $v_{yy}>0$, we get

$$y_m+\varepsilon > (v_y(\cdot,t))^{-1}(-x) > y_m-\varepsilon,$$

i.e.,

$$y_m+\varepsilon > V_x(x,t) > y_m-\varepsilon.$$

Thus, we get

$$y_m+\varepsilon \ge \limsup_{t\to T-} V_x(x,t) \ge \liminf_{t\to T-} V_x(x,t) \ge y_m-\varepsilon.$$

Since ε is arbitrary, we obtain the desired result of (4.34). ■

Theorem 4.12. *When $t \to T-$, we have $\frac{1}{\widehat{\pi}(x,t)} \to 0$ in $L^1([b,c])$ for any fixed $[b,c] \subset (\underline{x}_m, \overline{x}_m)$, i.e.,*

$$\int_b^c \frac{1}{\widehat{\pi}(x,t)}dx \to 0, \quad t\to T-.$$

Proof. Using Lemma 4.11, we get

$$\begin{aligned}\int_b^c \frac{1}{\widehat{\pi}(x,t)}dx &= \frac{-1}{(\sigma\sigma')^{-1}\mu}\int_b^c \frac{\widehat{V}_{xx}(x,t)}{\widehat{V}_x(x,t)}dx\\ &= \frac{-1}{(\sigma\sigma')^{-1}\mu}\Big[\ln(\widehat{V}_x(c,t)) - \ln(\widehat{V}_x(b,t))\Big]\\ &\to \frac{-1}{(\sigma\sigma')^{-1}\mu}\Big[\ln(y_m) - \ln(y_m)\Big] = 0, \quad t\to T-.\end{aligned}$$ ■

This conclusion is consistent with Theorem 2.1. In the sense of finance, if your utility of the wealth level is locally convex, when moving towards the end of the trading activities, one makes a wild bet. This will improve one's happiness in the long run.

5. An example: Carpenter's model with uncontrollable exit time. In this section, we give a specific application example on our general model.

The compensation of the manager is the sum of the payoff of a call option on the remaining assets and a constant compensation $K>0$ when he/she leaves the market. Suppose that the strike price $b>0$ is postulated as a constant; then the compensation is denoted as

$$Y_\tau = (X_\tau - b)^+ + K,$$

where τ is the exit time.

The manager chooses an investment policy to maximize his/her expected utility of wealth at any possible future exit time. The utility function U, which shows the behavior of the risk-averse manager, is strictly increasing and strictly concave. It can be expressed as

$$U(y) = \frac{1}{\gamma}y^\gamma$$

with $0 < \gamma < 1$.

We suppose that there is an exit time and the investor may leave the financial market for some uncontrollable reasons (see, e.g., [26, 27, 33, 32]). At time t, the exit time denoted by τ is usually considered to be a random variable under exponential distribution with mean value $\frac{1}{\lambda}$, and it is assumed to be independent of $\{\mathcal{F}_t\}$. Therefore,

$$\mathbb{P}(\tau \le s | \tau > t) = 1 - e^{-\lambda(s-t)}.$$

Hence, we can adopt the value function as the expectation of discounted utility on compensation

$$\begin{aligned} V_d(x,t) &= \sup_\pi \mathbb{E}\Big[e^{-\beta(T-t)} g(X_{\tau\wedge T})\Big| \tau > t\Big] \\ &= \sup_\pi \mathbb{E}\bigg[\int_t^T \lambda e^{-(\beta+\lambda)(s-t)} g(X_s)\mathrm{d}s + e^{-(\beta+\lambda)(T-t)} g(X_T)\bigg], \end{aligned}$$

where the constant $\beta > 0$ is the discounted rate and

$$g(x) := U[(x-b)^+ + K] = \frac{1}{\gamma}((x-b)^+ + K)^\gamma.$$

Denote $\varphi(x)$ as its concave envelope; see Figure 5.1.

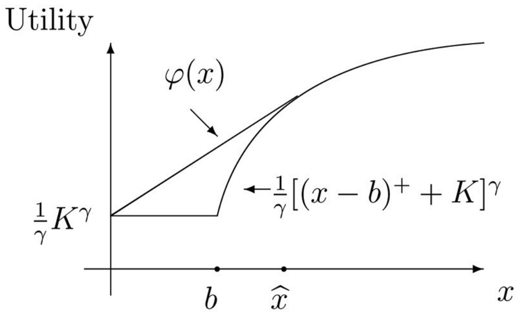

Figure 5.1. $\varphi(x)$.

Let $V(x,t) = e^{(\beta+\lambda)(T-t)} V_d(x,t)$; then

$$V(x,t) = \sup_\pi \mathbb{E}\bigg[\int_t^T \lambda e^{-(\beta+\lambda)(s-T)} g(X_s)\mathrm{d}s + g(X_T)\bigg].$$

Note that

$$g(x) = \bigg[\frac{1}{\gamma}((x-b)^+ + K)^\gamma - K^{\gamma-1}(x-b)^+\bigg] - \Big[-K^{\gamma-1}(x-b)^+\Big],$$

where $\frac{1}{\gamma}((x-b)^+ + K)^\gamma - K^{\gamma-1}(x-b)^+$ and $-K^{\gamma-1}(x-b)^+$ are concave functions in x. Therefore, $g(x)$ and $f(x,t) = e^{-(\beta+\lambda)(T-t)} g(x)$ satisfy Conditions I–III. Thus, we can use the conclusions of the general case to obtain $V \in C^{2,1}(\Omega_T) \bigcap C(\overline{\Omega_T})$ and the behavior of optimal strategy π_t^* when $x \to 0$ and $t \to T$.

6. Conclusion. In Theorem 4.3, we prove that the value function $V(x,t)$ defined in (2.3) is a $C^{2,1}$ smooth function and is strictly increasing and strictly concave in x under Conditions I–III. Moreover, it follows from (4.25) that the value function increases rapidly when x is small. In Theorem 4.7, we demonstrate that the optimal strategy π^* is a continuous function of the current wealth and current time. In Theorem 4.9, we prove that when x approaches zero, π^* also approaches zero. This means that we recommend not to take much more risk when the wealth approaches the bankruptcy. In Theorem 4.12, we prove that if x belongs to the set $\{x>0|\varphi(x)>g(x)\}$, which is the nonconcave region of $g(x)$, then π^* approaches infinity when t approaches to T. This means that it is worth it to seek risk in this situation.

Appendix A.

A.1. Proof of Theorem 2.1.

Proof. The proof of Theorem 2.1 can be accomplished by proving $\limsup_{t\to T-} V(x,t) \le \varphi(x)$ and $\liminf_{t\to T-} V(x,t) \ge \varphi(x)$. We begin with proving the two inequalities, respectively.

Proof of the first inequality. Define

$$\zeta_s = e^{-(r+\frac{1}{2}\mu'(\sigma'\sigma)^{-1}\mu)s-\mu'\sigma^{-1}W_s};$$

then we get

$$\mathrm{d}\zeta_s = \zeta_s\big(-r\mathrm{d}s - \mu'\sigma^{-1}\mathrm{d}W_s\big),$$

and

$$\begin{aligned}\mathrm{d}(\zeta_s X_s) &= \zeta_s \mathrm{d}X_s + X_s\mathrm{d}\zeta_s + \mathrm{d}\zeta_s\mathrm{d}X_s\\ &= \zeta_s[(rX_s+\mu'\pi_s)\mathrm{d}s + \pi_s'\sigma\mathrm{d}W_s - rX_s\mathrm{d}s - \mu'\sigma^{-1}X_s\mathrm{d}W_s - (\mu'\sigma^{-1})(\pi_s'\sigma)'\mathrm{d}s]\\ &= \zeta_s[\pi_s'\sigma - \mu'\sigma^{-1}X_s]\mathrm{d}W_s. \end{aligned} \tag{A.1}$$

Thus, $\zeta_s X_s$ is a positive local martingale and thus a supermartingale. For any admissible π, by Jensen's inequality, we have

$$\mathbb{E}\left[\varphi\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right] \le \varphi\left(\mathbb{E}\left[\frac{\zeta_T}{\zeta_t}X_T\right]\right) \le \varphi(x).$$

Then

$$\limsup_{t\to T-}\sup_{\pi}\mathbb{E}\left[\varphi\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right] \le \varphi(x). \tag{A.2}$$

We now prove

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}\left[\left|\varphi(X_T) - \varphi\left(\frac{\zeta_T}{\zeta_t}X_\tau\right)\right|\right] = 0. \tag{A.3}$$

It is not hard to check from (2.5) that for all $0<y<x$,

$$|\varphi(x)-\varphi(y)| \le C|x-y|^\gamma.$$

Indeed, by (2.5), we derive

$$g(x) \le g(y) + C|x-y|^\gamma \le \varphi(y) + C|x-y|^\gamma.$$

Since $\varphi(y) + C|x-y|^\gamma$ is concave on x for any fixed y, we obtain

$$\varphi(x) \le \varphi(y) + C|x-y|^\gamma.$$

Thus, for any admissible π, we get

$$\mathbb{E}\left[\left|\varphi(X_T) - \varphi\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right|\right] \le C\mathbb{E}\left[\left(\frac{\zeta_T}{\zeta_t}X_T\right)^\gamma \left|\frac{\zeta_t}{\zeta_T} - 1\right|^\gamma\right].$$

Using the Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\varphi(X_T) - \varphi\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right|\right] &\le C\left[\mathbb{E}\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right]^\gamma \left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T} - 1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma} \\ &\le Cx^\gamma \left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T} - 1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma}.\end{aligned}$$

Hence,

$$\lim_{t\to T-} \sup_\pi \mathbb{E}\left[\left|\varphi(X_T) - \varphi\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right|\right] \le Cx^\gamma \lim_{t\to T-} \left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T} - 1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma} = 0.$$

Meanwhile, we turn to proving

$$\lim_{t\to T-} \sup_\pi \mathbb{E}\left[\int_t^T f(X_s, s)\mathrm{d}s\right] = 0. \tag{A.4}$$

Using (2.6), we have

$$\begin{aligned}\mathbb{E}\left[\int_t^T f(X_s, s)\mathrm{d}s\right] &\le \mathbb{E}\left[\int_t^T CX_s^\gamma \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T f(0, s)\mathrm{d}s\right] \\ &\le C\int_t^T \left(\mathbb{E}\left[\left|\frac{\zeta_s}{\zeta_t}X_s\right|\right]\right)^\gamma \left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_s}\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma} \mathrm{d}s + \mathbb{E}\left[\int_t^T f(0, s)\mathrm{d}s\right] \\ &= C\int_t^T x^\gamma \left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_s}\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma} \mathrm{d}s + \int_t^T f(0, s)\mathrm{d}s.\end{aligned}$$

Note that the right-hand side of the last equality above is independent of π. Let $t \to T-$, and we have (A.4).

Therefore, by (A.2), (A.3), and (A.4), we get

$$\begin{aligned}\limsup_{t\to T-} V(x,t) &= \limsup_{t\to T-} \sup_\pi \mathbb{E}\left[\int_t^T f(X_s, s)\mathrm{d}s + g(X_T)\right] \\ &= \limsup_{t\to T-} \sup_\pi \mathbb{E}\left[g(X_T)\right] \\ &\le \limsup_{t\to T-} \sup_\pi \mathbb{E}\left[\varphi(X_\tau)\right] \\ &\le \limsup_{t\to T-} \sup_\pi \mathbb{E}\left[\varphi\left(\frac{\zeta_\tau}{\zeta_t}X_\tau\right)\right] + \lim_{t\to T-} \sup_\pi \mathbb{E}\left[\left|\varphi(X_\tau) - \varphi\left(\frac{\zeta_\tau}{\zeta_t}X_\tau\right)\right|\right] \\ &\le \varphi(x).\end{aligned}$$

Thus, we prove that $\limsup_{t\to T-} V(x,t) \le \varphi(x)$.

Proof of the second inequality. For fixed $t<T$, if $x \in \{x | \varphi(x) = g(x)\}$, set $\pi = 0$ and we can get

$$V(x,t) \ge g(x) = \varphi(x).$$

So $\liminf_{t\to T-} V(x,t) \ge \varphi(x)$.

Otherwise, if $x \in (\underline{x}_m, \overline{x}_m)$ for an $m \in \mathbb{Z}$, we choose strategy $\pi_s = \pi_s^N$ for each $N \in \mathbb{Z}^+$ to make the coefficient of (A.1) with corresponding wealth process $X_s = X_s^N$ satisfying

$$\frac{\zeta_s}{\zeta_t}\left[(\pi_s^N)'\sigma - \mu'\sigma^{-1}X_s^N\right] = (a_s^N)' := N\chi_{\left\{\underline{x}_m < \frac{\zeta_s}{\zeta_t}X_s^N < \overline{x}_m\right\}} I_n' \quad \forall\, N > 0,$$

where I_n is an n-dimensional unit column vector. Let $Y_s^N = \frac{\zeta_s}{\zeta_t}X_s^N$. Then using (A.1) we have

$$\mathrm{d}Y_s^N = (a_s^N)'\mathrm{d}W_s, \quad t \le s \le T.$$

It is not hard to obtain that

$$\underline{x}_m \le Y_s^N \le \overline{x}_m, \quad t \le s \le T.$$

Since

$$\begin{aligned}\{\underline{x}_m < Y_T^N < \overline{x}_m\} &= \{\underline{x}_m < Y_s^N = x + NI_n'(W_s - W_t) < \overline{x}_m,\ t \le s \le T\}\\ &\subset \{\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m\},\end{aligned}$$

we have

$$\mathbb{P}(\underline{x}_m < Y_T^N < \overline{x}_m) \le \mathbb{P}(\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m) \to 0, \quad N \to \infty.$$

Then we get

$$\underline{x}_m\mathbb{P}(Y_T^N = \underline{x}_m) + \overline{x}_m\mathbb{P}(Y_T^N = \overline{x}_m) \to \mathbb{E}Y_T^N = x, \quad N \to \infty.$$

Therefore,

$$\lim_{N\to\infty}\mathbb{P}(Y_T^N = \underline{x}_m) = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m}, \quad \lim_{N\to\infty}\mathbb{P}(Y_T^N = \overline{x}_m) = \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m}.$$

Hence,

$$\lim_{N\to\infty}\mathbb{E}[g(Y_T^N)] = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m}g(\underline{x}_m) + \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m}g(\overline{x}_m) = \varphi(x).$$

Thus, since Y^N is bounded,

$$\sup_\pi \mathbb{E}\left[\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right] \ge \lim_{N\to\infty}\mathbb{E}[g(Y_T^N)] = \varphi(x).$$

Further, similar to (A.3), we have

$$\lim_{t\to T-}\sup_\pi \mathbb{E}\left[\left|g(X_T) - g\left(\frac{\zeta_T}{\zeta_t}X_T\right)\right|\right] = 0.$$

Hence,

$$\begin{aligned}\liminf_{t\to T-} V(x,t) &\geq \liminf_{t\to T-}\sup_{\pi}\mathbb{E}\Big[g(X_T)\Big]\\ &\geq \liminf_{t\to T-}\sup_{\pi}\mathbb{E}\Big[g\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big] - \lim_{t\to T-}\sup_{\pi}\mathbb{E}\Big[\Big|g(X_T) - g\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big|\Big]\\ &\geq \varphi(x).\end{aligned}$$

We prove the desired result. ■

A.2. Proof of Theorem 4.3.

Proof. Denote the lower bound and upper bound in (4.12) by $w(y,t)$ and $W(y,t)$, respectively, i.e.,

$$\begin{aligned} w(y,t) &= \int_t^T f(0,s)\mathrm{d}s + \widetilde{\varphi}(y),\\ W(y,t) &= \int_t^T f(0,s)\mathrm{d}s + g(0) + M^{\frac{1}{1-\gamma}} e^{A(T-t)}\frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}}.\end{aligned}$$

From (4.3) we see that $W(y,t) \geq w(y,t)$.

Consider the approximation problem on bounded domain $\Omega_T^N := (1/N, N)\times(0,T)$ as follows:

$$\text{(A.5)}\qquad \begin{cases} -v_t^N - \frac{a^2}{2}y^2 v_{yy}^N + ryv_y^N = f(|v_y^N|,t) \quad \text{in} \quad \Omega_T^N,\\ v^N(1/N,t) = w(1/N,t), \quad 0<t<T,\\ v^N(N,t) = w(N,t), \quad 0<t<T,\\ v^N(y,T) = \widetilde{\varphi}(y), \quad 1/N < y < N.\end{cases}$$

By the theorem of a quasi-linear equation, since $f(x,t)$ is Hölder continuous, we obtain a solution $v^N \in C^{2,1}(\Omega_T^N)\bigcap C(\overline{\Omega_T^N})$ to (A.5) (see [22, 24]).

In the following, we prove

$$\text{(A.6)}\qquad w \leq v^N \leq W \quad \text{in} \quad \Omega_T^N.$$

Note that

$$w_t(y,t) = -f(0,t), \quad w_y(y,t)\leq 0, \quad w_{yy}(y,t)\geq 0.$$

Thus,

$$-w_t - \frac{a^2}{2}y^2 w_{yy} + ryw_y - f(|w_y|,t) \leq f(0,t) - f(-w_y,t) \leq 0.$$

Together with $w = v^N$ on $\partial_p\Omega_T^N$, using the comparison principle (see [12]), we know that w is a subsolution of (A.5).

On the other hand, note that

$$
\begin{aligned}
&-W_t-\frac{a^2}{2}y^2W_{yy}+ryW_y-f(|W_y|,t)\\
&=f(0,t)+M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r\right)-f\left(M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{1}{\gamma-1}},t\right)\\
&\geq f(0,t)+M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r\right)-f(0,t)-M^{\frac{1}{1-\gamma}}\frac{1}{\gamma}e^{A\gamma(T-t)}y^{\frac{\gamma}{\gamma-1}}\\
&\geq M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r-\frac{1}{\gamma}\right)\\
&\geq 0,
\end{aligned}
$$

where the first inequality is derived due to (2.5). Together with $W\geq w=v^N$ on $\partial_p\Omega_T^N$, using the comparison principle, we obtain that W is a supersolution of (A.5).

Using uniform Schauder interior estimation (see [21]), letting $N\to+\infty$ we obtain a solution $v\in C^{2,1}(\Omega_T)\bigcap C(\Omega_T\bigcup\{t=T\})$ to problem

$$
\text{(A.7)}\qquad \begin{cases} -v_t-\frac{a^2}{2}y^2v_{yy}+ryv_y=f(|v_y|,t) \quad \text{in} \quad \Omega_T,\\ v(y,T)=\widetilde{\varphi}(y), \quad y>0, \end{cases}
$$

satisfying (4.12).

In the following, we prove

$$v_y\leq 0,\quad v_{yy}\geq 0 \quad \text{in} \quad \Omega_T.$$

Since v satisfies the power growth condition for $y\to 0$ and $y\to+\infty$, we can prove v_y and v_{yy}, or at least their approximations, which satisfy the power growth condition, so that we can use the maximum principle in the unbounded domain Ω_T. On the other hand, we may assume that $f\in C(\Omega_T)$ and $\varphi\in C^2((0,+\infty))$ in proving $v_y\leq 0$ and $v_{yy}\geq 0$ since we can adopt the approximation approach. The discussion is tedious, so we put it in Appendix B.

Differentiating the equation in (A.7) w.r.t. y, we have

$$
\text{(A.8)}\qquad -\partial_t v_y-\frac{a^2}{2}y^2\partial_{yy}v_y+(r-a^2)y\partial_y v_y+rv_y-f_x(|v_y|,t)\operatorname{sgn}(v_y)v_{yy}=0.
$$

Differentiate w.r.t. y again, we get

$$
\text{(A.9)}\qquad -\partial_t v_{yy}-\partial_y\left[\frac{a^2}{2}y^2\partial_y v_{yy}\right]+(r-a^2)y\partial_y v_{yy}+(2r-a^2)v_{yy}+\partial_y\Big[f_x(-v_y,t)v_{yy}\Big]=0.
$$

Owing to $v(y,T)=\widetilde{\varphi}(y)$ being decreasing and convex, we obtain $v_y\leq 0$ and $v_{yy}\geq 0$ by using the maximum principle.

Now we further prove

$$v_y<0,\quad v_{yy}>0 \quad \text{in} \quad \Omega_T.$$

Using (A.8) and the strong maximum principle, we have $v_y<0$ in Ω_T.

Applying (3.2), (A.9) can be rewritten as

$$-\partial_t v_{yy} - \partial_y\Big[\frac{a^2}{2}y^2\partial_y v_{yy} + Q(-v_y,t)v_{yy}\Big] + (r-a^2)y\partial_y v_{yy} + (2r-a^2)v_{yy} = -\partial_y\Big[P(-v_y,t)v_{yy}\Big].$$

It can be further changed into

$$-\partial_t v_{yy} - \partial_y\Big[\frac{a^2}{2}y^2\partial_y v_{yy} + Q(-v_y,t)v_{yy}\Big] + [(r-a^2)y + P(-v_y,t)]\partial_y v_{yy} + (2r-a^2)v_{yy} = -\partial_y P(-v_y,t)v_{yy}.$$

We regard it as a linear PDE in v_{yy} with divergence form. Define a parabolic operator $\mathcal{L}$ as

$$\mathcal{L}u := -\partial_t u - \partial_y\Big[\frac{a^2}{2}y^2\partial_y u + Q(-v_y,t)u\Big] + [(r-a^2)y + P(-v_y,t)]\partial_y u + (2r-a^2)u;$$

then we have

$$\mathcal{L}v_{yy} = -\partial_y P(-v_y,t)v_{yy}.$$

Noting that $P(x,t)$ and $Q(x,t)$ are increasing in x and $-v_y(y,t)$ is decreasing in y, we have that $P(-v_y(y,t),t)$ and $Q(-v_y(y,t),t)$ are decreasing in y. So

$$-\partial_y P(-v_y,t) \geq 0, \quad -\partial_y Q(-v_y,t) \geq 0$$

in the weak sense. Therefore,

$$\mathcal{L}v_{yy} \geq 0.$$

Using the strong maximum principle, we have $v_{yy} > 0$. ∎

A.3. Proof of Theorem 4.7.

Proof. Here we present the proof of the verification theorem. Before that, we introduce the dynamic programming principle (see [25]) to (2.3). For any stopping time θ,

$$V(x,t) = \sup_\pi \mathbb{E}\Big[\int_t^{T\wedge\theta} f(x,s)\mathrm{d}s + V(X_{T\wedge\theta}, T\wedge\theta)\Big].$$

Set $\theta = T-\varepsilon$; then we have

$$V(x,t) = \sup_\pi \mathbb{E}\Big[\int_t^{T-\varepsilon} f(x,s)\mathrm{d}s + V(X_{T-\varepsilon}, T-\varepsilon)\Big].$$

Taking $\varepsilon \to 0$ in the above, using (2.9), we get

$$V(x,t) = \sup_\pi \mathbb{E}\Big[\int_t^{T} f(X_s,s)\mathrm{d}s + \varphi(X_T)\Big], \quad t<T. \tag{A.10}$$

This means that definition (2.3) is equivalent to (A.10), where $g(x)$ is replaced by $\varphi(x)$.

We will prove that $\widehat{V}(x,t)$ constructed by (4.18) satisfies (A.10). Fix $x>0$ and $t<T$, for any admissible π_s, if X_s satisfies (2.2); since $\widehat{V}(x,t)\in C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$, we can use Itô's formula and get

$$\begin{aligned}\mathbb{E}[\widehat{V}(X_{T-\varepsilon},T-\varepsilon)-\widehat{V}(x,t)] &= \mathbb{E}\bigg[\int_t^{T-\varepsilon}\bigg[\widehat{V}_t+\frac{1}{2}(\pi'\sigma\sigma'\pi)\widehat{V}_{xx}+\mu'\widehat{\pi}\widehat{V}_x-rx\widehat{V}_x\bigg](X_s,s)\mathrm{d}s\bigg]\\ &\le -\mathbb{E}\bigg[\int_t^{T-\varepsilon}\bigg[-\widehat{V}_t-\max_{\pi}\bigg[\frac{1}{2}(\pi'\sigma\sigma'\pi)\widehat{V}_{xx}+\mu'\widehat{\pi}\widehat{V}_x\bigg]-rx\widehat{V}_x\bigg](X_s,s)\mathrm{d}s\bigg]\\ &= -\mathbb{E}\bigg[\int_t^{T-\varepsilon}f(X_s,s)\mathrm{d}s\bigg].\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\widehat{V}(x,t)\ge\mathbb{E}\bigg[\int_t^T f(X_s,s)\mathrm{d}s+\varphi(X_T)\bigg].$$

Therefore,

$$\widehat{V}(x,t)\ge\sup_{\pi}\mathbb{E}\bigg[\int_t^T f(X_s,s)\mathrm{d}s+\varphi(X_T)\bigg].$$

On the other hand, define

$$\widehat{\pi}(x,t)=-(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}.$$

Let $\widehat{X}_s$ be the solution of the following SDE:

$$\begin{cases}\mathrm{d}X_s=(rX_s+\mu'\widehat{\pi}(X_s,s))\mathrm{d}s+\widehat{\pi}'(X_s,s)\sigma\mathrm{d}W_s, & s\ge t,\\ X_t=x.\end{cases}$$

By Itô's formula, we get

$$\begin{aligned}\mathbb{E}[\widehat{V}(x,T-\varepsilon)-\widehat{V}(x,t)] &= \mathbb{E}\bigg[\int_t\bigg[\widehat{V}_t+\frac{1}{2}(\widehat{\pi}'\sigma\sigma'\widehat{\pi})\widehat{V}_{xx}+\mu'\widehat{\pi}\widehat{V}_x-rx\widehat{V}_x\bigg](\widehat{X}_s,s)\mathrm{d}s\bigg]\\ &= \mathbb{E}\bigg[\int_t^{T-\varepsilon}\bigg[\widehat{V}_t-\frac{a^2}{2}\frac{\widehat{V}_x}{\widehat{V}_{xx}}+rx\widehat{V}_x\bigg](\widehat{X}_s,s)\mathrm{d}s\bigg]\\ &= -\mathbb{E}\bigg[\int_t^{T-\varepsilon}f(\widehat{X}_s,s)\mathrm{d}s\bigg].\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\widehat{V}(x,t)=\mathbb{E}\bigg[\int_t^T f(\widehat{X}_s,s)\mathrm{d}s+\varphi(\widehat{X}_T)\bigg].$$

Thus,

$$\widehat{V}(x,t)\le\sup_{\pi}\mathbb{E}\bigg[\int_t^T f(X_s,s)\mathrm{d}s+\varphi(X_T)\bigg].$$ ■

A.4. Proof of Lemma 4.8.

Proof. Let $z = \ln y$, $u(z,t) = v(y,t) - \varphi(0) - \int_t^T f(0,s)\mathrm{d}s$. Thus,

$$v_t = u_t - f(0,t), \quad yv_y = u_z, \quad y^2 v_{yy} = u_{zz} - u_z.$$

Then, by (4.11), we get

(A.11)
$$\begin{cases} -u_t - \dfrac{a^2}{2}u_{zz} + \Big(r + \dfrac{a^2}{2}\Big)u_z = f(-e^{-z}u_z, t) - f(0,t), & -\infty < z < +\infty,\ 0 < t < T, \\ u(z, T-) = \widetilde{\varphi}(e^z) - \varphi(0), & -\infty < z < +\infty. \end{cases}$$

According to (4.12) and (4.17), we know that

$$\lim_{z\to+\infty} u(z,t) = 0, \quad 0 < t < T, \tag{A.12}$$

$$\lim_{z\to+\infty} u_z(z,t) = 0, \quad 0 < t < T. \tag{A.13}$$

In order to prove (4.30) and (4.31), we only need to prove

$$\lim_{z\to+\infty} u_{zz}(z,t) = 0, \quad 0 < t < T, \tag{A.14}$$

$$\lim_{z\to+\infty} u_t(z,t) = 0, \quad 0 < t < T. \tag{A.15}$$

Set $Q_\delta^{b,c} = \{(z,t) | b < z < c,\ \delta < t < T - \delta\}$. For any $c > b > 0$, we apply the $W_p^{2,1}$ interior estimate (see [22]). For $p > 3$, we obtain

$$|u|_{W_p^{2,1}(Q_{\delta/2}^{n-1,n+2})} \le C\Big(|u|_{L_p(Q_{\delta/3}^{n-2,n+3})} + |f(-e^{-z}u_z, t) - f(0,t)|_{L_p(Q_{\delta/3}^{n-2,n+3})}\Big),$$

where C is independent of n. Using (A.12), (A.13), and the Hölder continuity of $f(x,t)$, we have

$$\lim_{n\to+\infty} |u|_{W_p^{2,1}(Q_{\delta/2}^{n-1,n+2})} = 0.$$

By the Sobolev embedding theorem, we get $\lim_{n\to+\infty} |u_z|_{\alpha,\, Q_{\delta/2}^{n-1,n+2}} = 0$ $(0 < \alpha < 1 - \frac{3}{p})$. Applying the Schauder interior estimate, we obtain

$$|u|_{2+\alpha,\, Q_\delta^{n,n+1}} \le C\Big(|u|_{0,\, Q_{\delta/2}^{n-1,n+2}} + |f(-e^{-z}u_z, t) - f(0,t)|_{\alpha,\, Q_{\delta/2}^{n-1,n+2}}\Big).$$

Thus, we have $\lim_{n\to+\infty} |u|_{2+\alpha,\, Q_\delta^{n,n+1}} = 0$, which implies (A.14) and (A.15) (see [22]). ■

Appendix B. A note on the proof of $v_y \le 0$ and $v_{yy} \ge 0$. In this section, we discuss the growth condition on proving $v_y \le 0$ and $v_{yy} \ge 0$ in the proof of Theorem 2.1, using the maximum principle.

Let us suppose that $f^\varepsilon \in C(\Omega_T)$ and $\varphi_\varepsilon \in C^2((0,+\infty))$, which are the approximates of f and φ, keep all the properties given in Conditions I–III, and $\widetilde{\varphi}_\varepsilon$ is the dual transformation of φ_ε, satisfying

$$\sup_{y>0} |\widetilde{\varphi}'_\varepsilon(y)| + \sup_{y>0} |\widetilde{\varphi}''_\varepsilon(y)| + \sup_{(x,t)\in\Omega_T} |f^\varepsilon(x,t)| < \frac{1}{\varepsilon}, \quad y>0,\ 0<t<T, \tag{B.1}$$

$$\begin{aligned} &\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi} \quad \text{in} \quad C([b,c]) \quad \forall\, [b,c] \subset (-\infty,+\infty),\\ &f^\varepsilon \to f \quad \text{in} \quad C(Q) \quad \forall\, Q \subset\subset \Omega_T. \end{aligned}$$

Denote v^ε as the solution of the corresponding problem

$$\begin{cases} -v^\varepsilon_t - \frac{a^2}{2} y^2 v^\varepsilon_{yy} + ryv^\varepsilon_y = f^\varepsilon(|v^\varepsilon_y|,t) \quad \text{in} \quad \Omega_T,\\ v^\varepsilon(y,T) = \widetilde{\varphi}_\varepsilon(y), \quad 0<y<+\infty. \end{cases} \tag{B.2}$$

Now we prove that v^ε_y and v^ε_{yy} satisfy the power growth condition for $y \to 0$ and $y \to +\infty$, so that we can use the maximum principle in unbounded domain Ω_T.

Let $z = \ln y$, $u^\varepsilon(z,t) = v^\varepsilon(y,t) - \varphi^\varepsilon(0) - \int_t^T f^\varepsilon(0,s)\mathrm{d}s$. Thus,

$$\begin{cases} -u^\varepsilon_t - \dfrac{a^2}{2} u^\varepsilon_{zz} + \Big(r + \dfrac{a^2}{2}\Big) u^\varepsilon_z = f^\varepsilon(|e^{-z}u^\varepsilon_z|,t) - f^\varepsilon(0,t), \quad -\infty<z<+\infty,\ 0<t<T,\\ u^\varepsilon(z,T-) = \widetilde{\varphi}^\varepsilon(e^z) - \varphi^\varepsilon(0), \quad -\infty<z<+\infty. \end{cases} \tag{B.3}$$

It only needs to prove that u^ε_z and u^ε_{zz} satisfy the exponential growth condition for $z \to -\infty$ and $z \to +\infty$.

Set $Q^{b,c} = \{(z,t) | b<z<c,\ 0<t<T\}$. Applying the $W^{2,1}_p$ interior estimate to (B.3), we get

$$|u^\varepsilon|_{W^{2,1}_p(Q^{n-1,n+2})} \le C\Big(|u^\varepsilon|_{L_p(Q^{n-2,n+3})} + |\widetilde{\varphi}_\varepsilon|_{W^2_p((n-2,n+3))} + |f^\varepsilon(e^{-z}|u^\varepsilon_z|,t) - f^\varepsilon(0,t)|_{L_p(Q^{n-2,n+3})}\Big),$$

where C is independent of n. Using (B.1) and the Sobolev embedding theorem, we have

$$|u^\varepsilon_z|_{\alpha,Q^{n-1,n+2}} \le |u^\varepsilon|_{W^{2,1}_p(Q^{n-1,n+2})} \le C\Big(|u^\varepsilon|_{0,Q^{n-2,n+3}} + \frac{1}{\varepsilon}\Big).$$

Since u^ε satisfies exponential growth condition (due to v^ε satisfying the power growth condition), we derive that u^ε_z also satisfies the exponential growth condition.

Furthermore, applying the Schauder interior estimate to problem (B.3), we obtain

$$\begin{aligned} |u^\varepsilon|_{2+\alpha,\, Q^{n,n+1}} &\le C\Big(|u^\varepsilon|_{0,\, Q^{n-1,n+2}} + |\widetilde{\varphi}_\varepsilon|_{2+\alpha,\, Q^{n-1,n+2}} + |f^\varepsilon(e^{-z}|u^\varepsilon_z|,t) - f^\varepsilon(0,t)|_{\alpha,\, Q^{n-1,n+2}}\Big)\\ &\le C\Big(|u^\varepsilon|_{0,\, Q^{n-1,n+2}} + |\widetilde{\varphi}_\varepsilon|_{2+\alpha,\, Q^{n-1,n+2}} + |f^\varepsilon|_{\alpha,\, Q^{n-1,n+2}} |e^{-z}u^\varepsilon_z|_{\alpha,\, Q^{n-1,n+2}}\Big). \end{aligned}$$

Since $|\widetilde{\varphi}_\varepsilon|_{2+\alpha,\, Q^{n-1,n+2}}$, $|f^\varepsilon|_{\alpha,\, Q^{n-1,n+2}}$ are bounded, and u^ε_z satisfies the exponential growth condition, we derive that u^ε_{zz} satisfies the exponential growth condition. Therefore, we obtain that v_y^ε and v_{yy}^ε satisfy the power growth condition. Thus, we can use the maximum principle as in Appendix A.2 to prove that $v_y^\varepsilon \le 0$ and $v_{yy}^\varepsilon \ge 0$ in unbounded domain Ω_T.

In the following, we prove that v, the solution of (4.11), is the limit of v^ε. Rewrite the equation in (B.2) as

$$-v_t^\varepsilon - \frac{a^2}{2} y^2 v_{yy}^\varepsilon + ryv_y^\varepsilon - F^\varepsilon v_y^\varepsilon = f^\varepsilon(0,t) + G^\varepsilon,$$

where

$$F^\varepsilon(y,t) = \begin{cases} \frac{f^\varepsilon(|v_y^\varepsilon|,t) - f^\varepsilon(0,t)}{v_y^\varepsilon} & \text{if} \quad |v_y^\varepsilon| > 1, \\ 0 & \text{if} \quad |v_y^\varepsilon| \le 1, \end{cases}$$

$$G^\varepsilon(y,t) = \begin{cases} 0 & \text{if} \quad |v_y^\varepsilon| > 1, \\ f^\varepsilon(|v_y^\varepsilon|,t) - f^\varepsilon(0,t) & \text{if} \quad |v_y^\varepsilon| \le 1. \end{cases}$$

Since f^ε satisfies (2.5), we have

$$|F^\varepsilon(y,t)| \le \frac{M}{\gamma} \quad \forall\, (y,t) \in \Omega_T.$$

Also, since $f^\varepsilon(x,t)$ is increasing in x,

$$|G^\varepsilon(y,t)| \le f^\varepsilon(1,t).$$

Now, for any fixed $Q \subset\subset \Omega_T$, $Q' \subset\subset \Omega_T \cup \{t = T\}$, we can use the $W_p^{2,1}$ interior estimate and the C^α interior estimate (see [22]) to get

$$|v^\varepsilon|_{W_p^{2,1}(Q)} \le C_Q, \quad |v^\varepsilon|_{\alpha,Q'} \le C_{Q'},$$

where C_Q and $C_{Q'}$ are independent of ε. Then there exists a subsequence of v^ε converging to $\widetilde{v}$ weakly in $W^{2,1}_{p,\,\mathrm{loc}}(\Omega_T)$ and uniformly in $C_{\mathrm{loc}}(\Omega_T \cup \{t = T\})$. This means that $\widetilde{v}$ satisfies the equation, the terminal condition (4.11), and the growth condition (4.12), so $\widetilde{v} = v$.

Hence, $v_y \le 0$ and $v_{yy} \ge 0$ can be deduced from $v_y^\varepsilon \le 0$ and $v_{yy}^\varepsilon \ge 0$.

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