LIFETIME RUIN UNDER HIGH-WATERMARK FEES AND DRIFT UNCERTAINTY*

JUNBEOM LEE†, XIANG YU‡, AND CHAO ZHOU§

Abstract. This paper aims to study lifetime ruin minimization problem by considering investment in two hedge funds with high-watermark fees and drift uncertainty. Due to multi-dimensional performance fees that are charged whenever each fund profit exceeds its historical maximum, the value function is expected to be multi-dimensional. New mathematical challenges arise as the standard dimension reduction cannot be applied, and the convexity of the value function and Isaacs condition may not hold in our probability minimization problem with drift uncertainty. We propose to employ the stochastic Perron's method to characterize the value function as the unique viscosity solution to the associated *Hamilton–Jacobi–Bellman* (HJB) equation without resorting to the proof of dynamic programming principle. The required comparison principle is also established in our setting to close the loop of stochastic Perron's method.

Key words. Lifetime ruin, multiple hedge funds, high-watermark fees, drift uncertainty, stochastic Perron's method, comparison principle

AMS subject classifications. Primary, 49L20, 49L25, 60G46; Secondary, 91G10, 93E20

1. Introduction.

Hedge funds have existed for many decades in financial markets and have become increasingly popular in recent times. As opposed to the individual investment, hedge funds pool capital and invest in a variety of assets and it is administered by professionals. Hedge fund managers charge performance fees for their service to individual investors as some regular fees proportional to fund's component assets plus a fraction of the fund's profits. The most common scheme entails annual fees of 2% of assets and 20% of fund profit whenever the profit exceeds its historical maximum—the so-called *high-watermark*. In the present paper, we are interested in investment opportunities among several hedge funds and we intend to study a stochastic control problem given the path-dependent trading frictions as multi-dimensional high-watermark fees.

The existing research on high-watermark fees mainly has focused on the asset management problem from the point of view of the fund manager, see some examples by [21], [29], [1], [23] and [24]. Meanwhile, the high-watermark process is also mathematically related to wealth drawdown constraints studied in [22], [17], [19] and also discussed in [15] after the transformation into expectation constraint. Recently, the high-watermark fees have been incorporated also into Merton problem for individual investor together with consumption choice in [26] and [27]. In the presence with consumption control, analytical solutions can no longer be promised as in some of the previous work for fund managers. After identifying the state processes, the path-dependent feature from high-watermark fees can be hidden so that the dynamic programming argument can be recalled to derive the HJB equation heuristically. The homogeneity of power utility function in [26] and [27] enables the key dimension reduction of the value function and the associated HJB equations can be reduced into ODE problems. Although the regularity can hardly be expected, classical Perron's method can be applied and the nice upgrade of regularity of the viscosity solution can be exercised

*The first and third authors acknowledge the support from the Singapore MOE AcRF grants R-146-000-271-112 and R-146-000-255-114 as well as the French Ministry of Foreign Affairs and the Merlion programme. The second author is partially supported by the Hong Kong Early Career Scheme under grant no. 25302116 and the Hong Kong Polytechnic University central research grant under no.15304317. In addition, the first author received the financial support from the Singapore MOE AcRF grant R-146-000-243-114 and the third author received the financial support from the NSFC Grant 11871364.

†Department of Sales and Trading, Yuanta Securities Korea, 04538 Seoul, Korea. Email: junbeoml22@gmail.com

‡Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. Email: xiang.yu@polyu.edu.hk

§Department of Mathematics, National University of Singapore, Singapore 119076, Singapore. Email: matzc@nus.edu.sg

afterwards using the convexity property of the transformed one-dimensional value function. As the last step, the verification theorem can be concluded with the aid of the smoothness of value function and standard Itô calculus.

In the present paper, we focus on the standpoint of the individual investor who confronts multiple hedge fund accounts in the market. However, we aim to minimize the probability that the investor outlives her wealth, also known as the probability of lifetime ruin, instead of the Merton problem on portfolio or consumption. We determine the optimal investment strategy of an individual among some hedge funds who targets a given rate of consumption by minimizing the probability that the ruin occurs before the death time. For the studies of lifetime ruin probability problem, readers can refer to [35, 10, 12, 11, 36]. In contrast to Merton problem, the dimension reduction of the value function will fail for our probability minimization problem. The auxiliary controlled state process, the so-called process of distance to pay performance fees defined in (2.7), can no longer be absorbed to simplify the PDE problem. Furthermore, comparing with [26] and [27] or the lifetime ruin problem with ambiguity aversion in [11], we need to handle a genuine multi-dimensional control problem with reflections as there exist multiple hedge funds in the market. In other words, the distance process itself is already multi-dimensional, which spurs many new mathematical challenges. To wit, one can still exploit the classical Perron's method as in [26], [27] and [11], and obtain the existence of viscosity solution to the associated HJB equation. Nevertheless, the upgrade of regularity of the viscosity solution can hardly be attained for our multi-dimensional problem. Consequently, the proof of verification theorem, which requires certain regularity of the solution, cannot be completed. To relate the value function to the viscosity solution in our setting using classical Perron's method, we have to provide the technical proof of dynamic programming principle at the beginning.

In addition, the individual investor usually cannot keep a real-time track of the performance of hedge funds from fund managers. Moreover, a reliable estimation of the return from hedge fund that consists of a bunch of various assets is almost impossible in practice. Even in the hedge fund performance report, the predicted future return in short term from fund manager is provided as a certain range instead of a fixed number. It is more realistic to assume that the investor allows drift misspecification and starts with a family of plausible probability measures of the underlying model. This leads to a robust investment strategy with Knightian model uncertainty. In particular, we assume that the investor would like to use the available data as a reference model and work on a robust control problem with the penalty on other plausible models based on the deviation from the reference one. One new mathematical challenge from this formulation is that the value function may lose convexity for some parameters and the Issacs condition may fail. Adding our previous difficulties coming from multi-dimensional performance fees, the feedback optimal investment strategy and the saddle point choice of probability measure cannot be obtained. The combination of market imperfections such as trading frictions together with model ambiguity renders many problems mathematically intractable. Some workable examples in this direction can only be found in robust Merton problem with proportional transaction costs, see [28], [14] and [18]. The methodology introduced in these paper may not work for our purpose with path-dependent high-watermark fees.

To tackle our stochastic control problem, we choose to employ the stochastic Perron's method (SPM) and characterize the value function as the unique viscosity solution to the associated HJB equation. This stochastic version of Perron's method, introduced by [7], can avoid the technical and lengthy proof of dynamic programming principle (DPP) and can obtain it as a by-product. We choose SPM over the weak DPP introduced in [13] because SPM can better handle the path-dependent structure of our control problem with additional model uncertainty. Let us note that the comparison principle is needed anyway in both methods. SPM requires the comparison principle to complete the squeeze argument and establish the equivalence between value function and the viscosity solution, while weak DPP needs the comparison principle to guarantee the uniqueness of the viscosity solution to the associated HJB equation. We actually find that the proof of comparison principle for SPM is relatively easier as the applicable class of state processes can be larger than that of weak DPP. We refer a short list of previous work on stochastic control using SPM such as [7], [9], [5], [6], [8], [30], [31], [32], [4] and [34].

To establish the viscosity semisolution property of stochastic envelopes, it is usually crucial to check the boundary viscosity semisolution property. In our framework, we can take advantage of the problem structure from lifetime ruin probability minimization and explicitly construct a *stochastic super-solution* and a *stochastic sub-solution* which satisfy the desired boundary conditions. We note that our arguments using stochastic Perron's method differ from [12] that solves the lifetime ruin problem with transaction costs and [4] that examines the robust optimal switching problem. Some nontrivial issues need to be carefully addressed, which are caused by the uncertainty of drift term and the structure of the auxiliary state process defined as the distance to pay fees. The path-dependent running maximum part coming from high-watermark fees do not appear in [12] nor [4], which deserves some novel and tailor-made treatment in the present paper.

It is the scope of this paper to investigate a multi-dimensional stochastic control problem on the strength of stochastic Perron's method, which integrates the drift ambiguity and high-watermark fees from multiple hedge funds. The generality of the mathematical problem comes at the cost that the associated HJB equation becomes numerically challenging. First, our HJB equation naturally has three spatial variables and a dimension reduction technique cannot be applied to our objective function. In addition, due to the nature of ruin probability mimization and the high-water mark fees, both Dirichlet and Neumann boundary conditions are imposed for our HJB equation. It is well known that the stability and efficiency of numerical schemes may become big issues for the high dimensional nonlinear PDE with mixed type boundary conditions. The numerical analysis and the study of quantitative impacts by high-water mark fees and parameter uncertainty will be pursued in our future research. It will be interesting to apply the deep learning method in the future work to tackle our multi-dimensional nonlinear PDE with mixed boundary conditions as in [33].

The rest of the paper is organized as follows. Section 2 introduces the market model with multiple hedge funds and related high-watermark fees, the default time as well as the set up with drift uncertainty. The robust lifetime ruin problem is defined afterwards. In Section 3, we derive the associated HJB equation for the control problem heuristically and define the viscosity solution accordingly. The main theorem to characterize the value function as the unique viscosity solution is presented. Section 4 provides the proof of all main results using stochastic Perron's method. The proof of the comparison principle of the HJB equation is also reported therein.

2. Market Model and Problem Formulation.

2.1. Multiple Hedge Funds with High-watermark Fees. Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space such that $\mathbb{G}$ satisfies the *usual conditions* and $\mathbb{E}$ denote the expectation operator under $\mathbb{P}$. Let $(W_t)_{t\geq 0}$ denote an independent 2-dimensional Brownian motion and $\mathbb{F} \coloneqq (\mathcal{F}_t)_{t\geq 0}$ be the *natural filtration* generated by $(W_t)_{t\geq 0}$ and it is assumed that $\mathcal{F}_t \subset \mathcal{G}_t$. Later, we will characterize $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ more precisely.

We consider the financial market consisting of one risk-less bond with interest rate $r > 0$ and two hedge fund accounts $(F^i_t)_{t\geq 0}$, $i \in \{1, 2\}$, described by

$$\mathrm{d}F^i_t = \mu^i F^i_t \,\mathrm{d}t + \sigma^i F^i_t \,\mathrm{d}W_t,$$

for some constant $\mu^i \geq 0$ and constant vector $\sigma^i \in \mathbb{R}^2$. To simplify the presentation, we only focus on two hedge funds henceforth. The mathematical arguments and main results can be easily extended to the multi-dimensional case of $N \geq 2$ hedge funds without any technical difficulty. We shall denote

$$F \coloneqq \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}, \quad \mu \coloneqq \begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix}, \quad \sigma \coloneqq \begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix},$$

and assume that σ is invertible.

Contrary to some standard investment problems in liquid risky assets such as stocks, we are considering the model when the investor is facing the wealth allocation among some hedge fund accounts that charge proportional fees on the profit as trading frictions. In particular, the investor needs to pay some high-watermark fees to the fund manager whenever the accumulative profit reaches the highest value. The 2/20-rule is common for hedge funds in the sense that 2% per year of the total investment and 20% of the additional profits are paid to the fund manager whenever the high-watermark exceeds the previously attained profit maximum. To explain this in a more explicit manner, let $\pi = (\pi^1, \pi^2) \in \mathbb{R}^2$ denote the investment strategy in two hedge funds F. The accumulative profit $\overline{P}^\pi = [\overline{P}^{1,\pi}, \overline{P}^{2,\pi}]^\top$ from the hedge fund before the deduction of the high-watermark fee, is characterized by the stochastic integral

$$\overline{P}_t^{i,\pi} := \int_0^t \pi_s^i \frac{\mathrm{d}F_s^i}{F_s^i}. \tag{2.1}$$

In practice, the investor and fund manager may agree to choose a *benchmark* to measure the manager's performance, see [27]. High-watermark fees are only deducted when the profit process of the fund exceeds the benchmark level. For example, the fund manager may only receive incentives when the fund account outperforms the S&P index.

The initial high-watermark fee is denoted by some non-negative constant vector $y = [y^1, y^2]^\top$. Let $F^B \in \mathbb{R}^2$ be the *benchmark* process given by

$$\mathrm{d}F_t^B = \mathrm{diag}(F_t^B)[\mu^B \,\mathrm{d}t + \sigma^B \,\mathrm{d}W_t],$$

for some $\mu^B \in \mathbb{R}^2$, $\sigma^B \in \mathbb{R}^{2\times 2}$. We denote by $\overline{B}^\pi = [\overline{B}^{1,\pi}, \overline{B}^{2,\pi}]^\top$ the accumulated *benchmark* profit process if the same strategy π is adopted, i.e.,

$$\overline{B}_t^{i,\pi} := \int_0^t \pi_s^i \frac{\mathrm{d}F_s^B}{F_s^B}.$$

Let $q = [q^1, q^2]^\top$ represent the proportional rates of high-watermark fee of each hedge fund and $P^{y,\pi} = [P^{1,y,\pi}, P^{2,y,\pi}]$ be the realized profit after charging the high-watermark fee. Moreover, we define $M^{i,y,\pi}$ as the historical high-watermark of the i-th hedge fund. The realized profit process $P^{i,y,\pi}$, $i \in \{1, 2\}$, is given by

$$\begin{cases} \mathrm{d}P_t^{i,y,\pi} := \mathrm{d}\overline{P}_t^{i,\pi} - q^i \,\mathrm{d}M_t^{i,y,\pi}, & P_0^{i,y,\pi} = 0, \\ M_t^{i,y,\pi} := \sup_{0\le s\le t} \left\{(P_s^{i,y,\pi} - \overline{B}_s^{i,\pi}) \vee y^i\right\}, & y^i \ge 0. \end{cases} \tag{2.2}$$

To represent (2.2) in a more convenient form, let us define

$$\overline{M}_t^{i,y,\pi} := \sup_{0\le s\le t} \left\{(\overline{P}_s^{i,\pi} - \overline{B}_s^{i,\pi}) \vee y^i\right\}, \quad i \in \{1, 2\}. \tag{2.3}$$

Then by (2.2),

$$\begin{aligned} \overline{M}_t^{i,y,\pi} - y^i &= \sup_{0\le s\le t} \left\{[\overline{P}_s^{i,\pi} - \overline{B}_s^{i,\pi}] - y^i\right\}^+ \\ &= \sup_{0\le s< t} \left\{[P_s^{i,y,\pi} - \overline{B}_s^{i,\pi}] - y^i + q^i[M_s^{i,y,\pi} - y^i]\right\}^+ \\ &=(1 + q^i)(M_t^{i,y,\pi} - y^i). \end{aligned} \tag{2.4}$$

Therefore, in view of (2.2) and (2.4), for $i \in \{1,2\}$, we have

$$P_t^{i,y,\pi} = \overline{P}_t^{i,\pi} - \frac{q^i}{1+q^i}[\overline{M}_t^{i,y,\pi} - y^i], \tag{2.5}$$

Equivalently, $P^{i,y,\pi}$ can be rewritten as

$$\mathrm{d}P_t^{i,y,\pi} = \mu^i \pi_t^i \,\mathrm{d}t + \sigma^i \pi_t^i \,\mathrm{d}W_t - q^i(1+q^i)^{-1}\,\mathrm{d}\overline{M}_t^{i,y,\pi}. \tag{2.6}$$

As the high-watermark fee is only deducted whenever $\overline{M}^i - [\overline{P}^i - \overline{B}^i] = 0$, the distance between $\overline{M}^i$ and $\overline{P}^i - \overline{B}^i$ will be considered in the investment decision. Therefore, let us introduce the distance process $Y^{y,\pi} = [Y^{1,y,\pi}, Y^{2,y,\pi}]^\top$ as the difference

$$Y^{i,y,\pi} := M^{i,y,\pi} - [P^{i,y,\pi} - \overline{B}^{i,\pi}]. \tag{2.7}$$

In view of (2.4), (2.5), and (2.7), it clearly follows that $Y^{y,\pi} = \overline{M}^{y,\pi} - [\overline{P}^\pi - \overline{B}^\pi]$. To facilitate the future analysis using dynamic programming argument, we expect to deal with a multi-dimensional value function of the control problem depending on the two dimensional initial distance $Y_0 = (y^1, y^2)$ and the investor's initial wealth x. The precise formulation will be introduced later.

We continue to characterize the investor's wealth more explicitly. The amount of the risky position (hedge funds) is $\mathbb{1}^\top \pi$ and the rest of the investor's wealth is put into the risk-less bond. Furthermore, it is assumed that the investor consumes at a constant rate $c \geq 0$ all the time. Let $X^{x,y,\pi}$ denote the process of investor's wealth with initial value x. Then the controlled state processes are given by

$$\begin{cases} \mathrm{d}X_t^{x,y,\pi} = [rX_t^{x,y,\pi} - c + \pi_t^\top \mu_\Delta^r]\,\mathrm{d}t + \pi_t^\top \sigma \,\mathrm{d}W_t - q^\top \,\mathrm{d}M_t^{y,\pi}, & X_0 = x, \\ \mathrm{d}Y_t^{y,\pi} = -\operatorname{diag}(\pi_t)[\mu_\Delta^B \,\mathrm{d}t + \sigma_\Delta^B \,\mathrm{d}W_t] + \operatorname{diag}(\mathbb{1}+q)\,\mathrm{d}M_t^{y,\pi}, & Y_0 = y. \end{cases} \tag{2.8}$$

where we denote $\mu_\Delta^r := [\mu^1 - r,\ \mu^2 - r]^\top$, $\mu_\Delta^B := \mu - \mu^B$, $\sigma_\Delta^B := \sigma - \sigma^B$, and

$$\operatorname{diag}(\mathbb{1}+q) := \begin{bmatrix} 1+q^1 & 0 \\ 0 & 1+q^2 \end{bmatrix}.$$

Sometimes, we omit the superscripts x, y, π for simplicity and we also denote

$$Z := (X, Y^1, Y^2), \quad \mathbf{z} := (x, y^1, y^2).$$

2.2. Default Time and Preliminaries. Another important ingredient of our model is the default time of the individual investor, such as the death time independent with $(W_t)_{t\geq 0}$, which is defined as a random variable

$$\tau_D : (\Omega, \mathcal{G}) \to (\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+))$$

satisfying $\mathbb{P}(\tau_D = 0) = 0$ and $\mathbb{P}(\tau_D > t) > 0$, for any $t \geq 0$. From this point onward, the *full market filtration* $\mathbb{G}$ is precisely defined by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0} := (\mathcal{F}_t \vee \sigma(\{\tau_D \leq u\} : u \leq t))_{t\geq 0}$. It is worth noting that τ_D is a $\mathbb{G}$-stopping time but may fail to be an $\mathbb{F}$-stopping time. In what follows, we assume that there exists a constant $\lambda^D > 0$ such that

$$G_t^D := \mathbb{P}(\tau_D > t | \mathcal{F}_t) = e^{-\lambda^D t}.$$

We call λ^D the intensity of default time τ_D with respect to $\mathbb{F}$. Under this assumption,

$$(\mathcal{M}_t^D)_{t\geq 0} := \left(\mathbb{1}_{\tau_D \leq t} - \lambda^D (t \wedge \tau_D)\right)_{t\geq 0} \tag{2.9}$$

is a ($\mathbb{G}$)-martingale. Moreover, for any $\mathbb{F}$-martingale $(\xi_t)_{t\geq 0}$, $(\xi_{t\wedge \tau^D})_{t\geq 0}$ is a $\mathbb{G}$-martingale. Therefore, $(W_t)_{t\geq 0}$ is a $\mathbb{G}$-Brownian motion; see [20].

Remark 2.1. 1. In view of the existence of the intensity, τ_D is *totally inaccessible*. In other words, the default of the investor comes with total surprise. On the other hand, a *ruin time*, which will be introduced later, is defined as a hitting time that the controlled wealth process crosses a given level and it is therefore *predictable*. In the present paper, we envision an individual investor who chooses her portfolio to minimize the probability involving the *ruin time* before the default time occurs.

2. Although investment strategies are defined as $\mathbb{G}$-adapted processes, the *full filtration* $\mathbb{G}$ is not fully observable for the investor. However, in this filtration setup, for any $\mathbb{G}$-adapted process, we can find an $\mathbb{F}$-reduction, where $\mathbb{F}$ is the observable information. Therefore, the strictly $\mathbb{G}$-adapted strategies only describe an immediate action taken by the investor at the default time. Note that an $\mathbb{F}$-adapted process is not necessarily determined independently of the default time τ_D, because the (constant) default intensity λ^D is trivially $\mathbb{F}$-adapted.

2.3. Life Time Ruin Problem with Drift Uncertainty. Based on previous building blocks, we are ready to introduce the primary stochastic control problem that the investor confronts. In particular, the investor concerns the viability of her investment before the default time and she wishes to maintain the amount of her wealth above a certain level, say $R \geq 0$, before the default time happens. To this end, it is natural to introduce the so-called ruin time

$$\tau_R^{x,y,\pi} := \inf\{t \geq 0 \,:\, X_t^{x,y,\pi} \leq R\ \}.$$

Mathematically speaking, the investor chooses π from an admissible set $\mathscr{A}$ so that τ_R occurs as late as possible. As the investor cannot control the *totally inaccessible time* τ_D, she aims to minimize the probability that the ruin occurs before the default time.

However, we consider a more practical scenario in the present paper that the return of hedge funds may not be revealed by fund manager to the investor very frequently. The investor usually can only get access to the performance of the fund from some reports on regular dates. Moreover, as the hedge fund consists of components from various assets, the estimation of return can hardly be provided on a timely basis. Based on these observations, it is reasonable to assume that the investor may not have a precise knowledge of the dynamics of hedge funds. This naturally leads to the so-called Knightian model uncertainty.

In this paper, we will only focus on the case with drift uncertainty, i.e. the investor conceives a family of plausible return terms from the hedge fund dynamics and proceeds to solve the control problem in a robust sense. Indeed, the precise estimation of the drift term is much more challenging than the estimation of volatility term, which motivates our research. In particular, we aim to minimize the probability of lifetime ruin by choosing wealth allocation among multiple hedge funds with high-watermark fees and drift uncertainty, which is new to the existing literature. To this end, let us first introduce a class of probability measures equivalent to the reference probability $\mathbb{P}$ and denote this class by $\mathscr{L}$.

Definition 2.2. *$\mathbb{Q} \in \mathscr{L}$ if for any $0 \leq t$,*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t} = \exp\left(-\frac{1}{2}\int_0^t \|\theta_s\|^2\,\mathrm{d}s + \int_0^t \theta_s^\top\,\mathrm{d}W_s\right), \tag{2.10}$$

for some $\mathbb{G}$-predictable process θ valued in a closed set $\mathcal{L} \subseteq \mathbb{R}^2$ containing $\mathbf{0}$ such that

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s\Big] < \infty,$$

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^t \|\theta_s\|^2\,\mathrm{d}s\Big)\Big] < \infty, \quad \textit{for any } t \geq 0.$$

In what follows, an equivalent measure $\mathbb{Q}$ is generated by θ by the representation in (2.10), and we call $\mathbb{Q}$ the θ-measure. The investor intends to minimize the ruin probability under some $\mathbb{Q} \in \mathscr{L}$, but the deviation of the measure from $\mathbb{P}$ is penalized by a relative entropy process up to the default time τ_D:

$$H_t(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{G}_t}\Big)\Big], \quad \text{for } t \geq 0. \tag{2.11}$$

The investor's robust stochastic control problem is then defined by

$$V(x,y;\varepsilon) := \inf_{\pi\in\mathscr{A}} \sup_{\mathbb{Q}\in\mathscr{L}} \Big\{\mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(\mathbb{Q} \mid \mathbb{P})\Big\}. \tag{2.12}$$

Here $\mathscr{A}$ denotes the set of all admissible controls defined in the following sense.

Definition 2.3. *$\pi \in \mathscr{A}$ if π is $\mathbb{G}$-predictable and valued in a compact set $\mathcal{K} \subseteq \mathbb{R}^2$ such that* $(0,0) \in \mathcal{K}$.

Remark 2.4. The coefficient ε in the penalty term of (2.12) corresponds to the investor's level of model ambiguity about the reference probability $\mathbb{P}$. For instance, the case $\varepsilon \to 0$ implies that

$$\sup_{\mathbb{Q}\in\mathscr{L}} \Big\{\mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(\mathbb{Q} \mid \mathbb{P})\Big\} \to \mathbb{P}(\tau_R^{x,y,\pi} < \tau_D),$$

which indicates that the investor is completely confident about the probability measure $\mathbb{P}$. On the other hand, if the agent is extremely uncertain as $\varepsilon \to \infty$, we get that

$$\sup_{\mathbb{Q}\in\mathscr{L}} \Big\{\mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(\mathbb{Q} \mid \mathbb{P})\Big\} \to \sup_{\mathbb{Q}\in\mathscr{L}} \mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D),$$

which reduces to the worst-case scenario. It is worth noting that the formulation involving the penalty term only works for drift uncertainty. If some plausible probabilities are mutually singular due to volatility uncertainty, i.e. there is no dominating reference probability $\mathbb{P}$, the entropy cannot be defined as in (2.11). Another interesting issue we can consider in the robust framework is to incorporate the investor's ambiguity attitude towards a given set of plausible priors. Similar to [25], one can employ the alpha-maxmin preference and formulate the ruin probability problem under model uncertainty as

$$\inf_{\pi\in\mathscr{A}} \Big[\alpha \sup_{\mathbb{Q}\in\mathscr{L}} \mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D) + (1-\alpha) \inf_{\mathbb{Q}\in\mathscr{L}} \mathbb{Q}(\tau_R^{x,y,\pi} < \tau_D)\Big].$$

This formulation allows for both drift and volatility uncertainty and the constant coefficient $\alpha \in [0,1]$ can represent how much ambiguity averse the investor is. Nevertheless, this problem becomes time inconsistent and we need to look for some equilibrium portfolio strategies instead of the optimal one, which is beyond the scope of this paper and will be left as future research.

Remark 2.5. The compactness of $\mathcal{K}$ in the definition of admissible set $\mathscr{A}$ can be understood that the investor does not take an extreme strategy and the immediate liquidation is also admissible. Moreover, as π is $\mathbb{G}$-predictable, it is also $\mathbb{F}$-predictable before τ_D. Therefore, there is a unique continuous $\overline{P}^{\pi}$ satisfying (2.1). Thanks to (2.3) and (2.5), $P^{y,\pi}$ is well-defined. More importantly, the compactness of $\mathcal{K}$ is necessary for the associated HJB equation to be continuous. Otherwise, it becomes difficult to prove the comparison principle for its viscosity solutions because the typical doubling argument relies on Crandall-Ishii's lemma and the closure of *super/sub-jets*, which require the compactness of $\mathcal{K}$. In other words, if the comparison principle is already guaranteed, we can relax the conditions on $\mathscr{A}$ only with care for $P^{y,\pi}$ to be well-defined.

Remark 2.6.
1. One can naturally generalize our model to include ambiguity on the hazard rate as well. Nevertheless, the additional ambiguity on default time does not complicate our analysis on the associated HJB equation and our methodology still holds valid. For a related work on life time ruin problem with uncertain hazard rate (but without high-watermark fees), we refer to [36], in which the one-dimensional HJB equation can be solved by a verification argument.
2. The main mathematical challenge comes from the multi-dimensional high-water mark fees. Even without drift uncertainty, our stochastic control problem is still three dimensional together with mixed boundary conditions, which does not admit any closed form solution. The examination of the impact by the uncertainty parameter ε in our model would be appealing, which nevertheless relies on some stable and efficient numerical schemes. As some conventional numerical methods may not work well for our multi-dimensional nonlinear PDE with mixed boundary conditions, we will not explore this direction further in the left of the paper and leave the numerical treatment and sensitivity analysis as future work.

3. Dynamic Programming Equation and Main Results. In this section, we first heuristically derive the HJB equation associated with the value function using dynamic programming argument or martingale optimality principle. For technical reason, when default occurs, we assign a coffin state $\mathbf{\Delta}$ to the underlying process Z. Moreover, for any domain in what follows, we consider its one point compactification and any function u is extended by assigning $u(\mathbf{\Delta}) = 0$. Denote the $(\mathbb{Q}, \mathbb{G})$-Brownian motion by $W^{\mathbb{Q}}$, where $\mathbb{Q}$ is generated by θ. For $t < \tau_D$, (2.8) can be written as

$$\begin{cases} \mathrm{d}X_t^{x,y,\pi} = [rX_t^{x,y,\pi} - c + \pi_t^\top(\mu_\Delta^r + \sigma\theta)]\,\mathrm{d}t + \pi_t^\top\sigma\,\mathrm{d}W_t^{\mathbb{Q}} - q^\top\,\mathrm{d}M_t^{y,\pi}, & X_0 = x, \\ \mathrm{d}Y_t^{y,\pi} = -\operatorname{diag}(\pi_t)[(\mu_\Delta^B + \sigma_\Delta^B\theta)\,\mathrm{d}t + \sigma_\Delta^B\,\mathrm{d}W_t^{\mathbb{Q}}] + \operatorname{diag}(\mathbb{1}+q)\,\mathrm{d}M_t^{y,\pi}, & Y_0 = y. \end{cases} \tag{3.1}$$

To obtain the associated HJB equation, we apply Itô's formula to a smooth function φ that

$$\begin{aligned} \mathrm{d}\varphi(Z_t) - \frac{1}{2\varepsilon}\|\theta_t\|^2\,\mathrm{d}t = & \big[-\lambda^D[\varphi(Z_t) - \varphi(\mathbf{\Delta})] + (rX_t - c)\varphi_x + \mathcal{A}^{\pi_t,\theta_t}[\varphi](Z_t)\big]\,\mathrm{d}t \\ & - \sum_{i=1,2}\big[q^i\varphi_x(Z_t) - (1+q^i)\varphi_{y^i}(Z_t)\big]\mathbb{1}_{Y_t^i=0}\,\mathrm{d}M_t^i \\ & + \big[\varphi_x(Z_t)\pi_t^\top\sigma - \nabla_y\varphi(Z_t)^\top\operatorname{diag}(\pi_t)\sigma_\Delta^B\big]\,\mathrm{d}W_t^{\mathbb{Q}} + [\varphi(\mathbf{\Delta}) - \varphi(Z_{t-})]\,\mathrm{d}\mathcal{M}_t^D, \end{aligned} \tag{3.2}$$

where $\nabla_y\varphi \coloneqq [\partial_{y^1}\varphi,\ \partial_{y^2}\varphi]^\top$ and

$$\begin{aligned} \mathcal{A}^{\pi,\theta}[\varphi](x,y^1,y^2) &\coloneqq -\frac{1}{2\varepsilon}\|\theta\|^2 + b[\pi,\theta]^\top\nabla\varphi + \frac{1}{2}\operatorname{Tr}(\Sigma[\pi]\nabla^2\varphi), \\ b[\pi,\theta] &\coloneqq \begin{bmatrix} \pi^\top(\mu_\Delta^r + \sigma\theta) \\ -\operatorname{diag}(\pi)(\mu_\Delta^B + \sigma_\Delta^B\theta) \end{bmatrix}, \\ \Sigma[\pi] &\coloneqq \begin{bmatrix} \pi^\top\sigma \\ -\operatorname{diag}(\pi)\sigma_\Delta^B \end{bmatrix}\begin{bmatrix} \pi^\top\sigma \\ -\operatorname{diag}(\pi)\sigma_\Delta^B \end{bmatrix}^\top. \end{aligned}$$

Recall that $\varphi(\mathbf{\Delta}) = 0$ in (3.2). Now, let us deduce related boundary conditions. Recalling (2.12), we can set $V(R, y^1, y^2) = 1$ for any $y^i \geq 0$. In addition, if $X_t = c/r$ at $t \geq 0$, the optimal strategy is liquidating the risky position so that $X_s = c/r$ for any $s \geq t$. Therefore, $V(c/r, y^1, y^2) = 0$ for any $y^i \geq 0$. Thus, motivated by these boundary conditions, we need to consider the following regions and boundaries

$$\begin{aligned}
\mathcal{O} &:= \{(x, y^1, y^2) \colon R < x < c/r,\ y^1 \geq 0,\ y^2 \geq 0\}, \\
\mathcal{O}^+ &:= \{(x, y^1, y^2) \colon R < x < c/r,\ y^1 > 0,\ y^2 > 0\}, \\
\partial\mathcal{O}_i^0 &:= \{(x, y^1, y^2) \in \mathcal{O} \colon R < x < c/r,\ y^i = 0\}, \\
\partial\mathcal{O}^0 &:= \{(x, y^1, y^2) \in \mathcal{O} \colon R < x < c/r,\ y^1 = 0 \text{ or } y^2 = 0\}, \\
\partial\mathcal{O}_R &:= \{(R, y^1, y^2) \colon\ y^1 > 0,\ y^2 > 0\}, \\
\partial\mathcal{O}_{c/r} &:= \{(c/r, y^1, y^2) \colon\ y^1 > 0,\ y^2 > 0\}.
\end{aligned}$$

Note that $\mathcal{O} = \mathcal{O}^+ \cup \partial\mathcal{O}^0$, $\partial\mathcal{O} = \partial\mathcal{O}_R \cup \partial\mathcal{O}_{c/r} \cup \partial\mathcal{O}_1^0 \cup \partial\mathcal{O}_2^0$, and $\partial\mathcal{O}^0 = \partial\mathcal{O}_1^0 \cup \partial\mathcal{O}_2^0$. Moreover, for any set A, we let $\mathrm{cl}\,(A)$ denote the closure of A in what follows. We then consider the following operators

$$\begin{cases} \mathcal{F}[\varphi](\mathbf{z}) := \lambda^D \varphi(\mathbf{z}) - (rx - c)\varphi_x(\mathbf{z}) - \inf_{\pi\in\mathcal{K}} \sup_{\theta\in\mathcal{L}} \mathcal{A}^{\pi,\theta}[\varphi](\mathbf{z}), \\ \mathcal{B}^i[\varphi](\mathbf{z}) := q^i \varphi_x(\mathbf{z}) - (1 + q^i)\varphi_{y^i}(\mathbf{z}), \quad i \in \{1, 2\}, \end{cases} \tag{3.3}$$

and the associated HJB equation can be (formally) written as

$$\begin{cases} \mathcal{F}[\varphi](\mathbf{z}) = 0, & \text{on } \ \mathbf{z} \in \mathcal{O}^+, \\ \mathcal{B}^1[\varphi](\mathbf{z}) = 0, & \text{on } \ \mathbf{z} \in \partial\mathcal{O}_1^0, \\ \mathcal{B}^2[\varphi](\mathbf{z}) = 0, & \text{on } \ \mathbf{z} \in \partial\mathcal{O}_2^0, \\ \varphi(\mathbf{z}) = 1, & \text{on } \ \mathbf{z} \in \partial\mathcal{O}_R, \\ \varphi(\mathbf{z}) = 0, & \text{on } \ \mathbf{z} \in \partial\mathcal{O}_{c/r}. \end{cases} \tag{3.4}$$

Remark 3.1. One may want to solve a benchmark case without uncertainty, namely $\mathcal{L} = \{\mathbf{0}\}$. In this case, while $\mathcal{F}[\varphi]$ becomes simpler as

$$\mathcal{F}^0[\varphi](\mathbf{z}) := \lambda^D \varphi(\mathbf{z}) - (rx - c)\varphi_x(\mathbf{z}) - \inf_{\pi\in\mathcal{K}} \mathcal{A}^{\pi,\mathbf{0}}[\varphi](\mathbf{z}), \tag{3.5}$$

the boundary condition $\mathcal{B}^i$, $i = 1, 2$, still remain unchanged. Note that the major difficulties of our problem are the high dimensionality and the Neumann-type boundary conditions. Thus, considering the benchmark case does not provide an easier problem, and classical solution still cannot be proved. Instead, we will solve the general problem (3.4) using the stochastic Perron's method in the next section. Note that our mathematical arguments based on stochastic perron's method for the model with drift uncertainty can be easily modified to cover the simpler benchmark case without model uncertainty. It is our goal to provide a streamlined proof for the general model in the present paper, which is motivated by some practical ambiguous returns in hedge fund investment.

Our ultimate goal is to show that the value function V defined in (2.12) is the unique viscosity solution of the HJB equation (3.4). To this end, we first need to be careful for the boundary conditions on $\partial\mathcal{O}^0$, which should be defined using semi-continuous envelope of viscosity solutions. To be precise, we denote the lower (resp. upper) semi-continuous envelope of $\mathcal{B}^i$, $i \in \{1, 2\}$, by $\mathcal{B}_*$ (resp. $\mathcal{B}^*$). On $\partial\mathcal{O}^0$, we will consider

$$\mathcal{B}_*[\varphi] := \begin{cases} \mathcal{B}^1[\varphi], & \text{on } \partial\mathcal{O}_1^0 \setminus \partial\mathcal{O}_2^0, \\ \mathcal{B}^2[\varphi], & \text{on } \partial\mathcal{O}_2^0 \setminus \partial\mathcal{O}_1^0, \\ \min\{\mathcal{B}^1[\varphi],\ \mathcal{B}^2[\varphi]\}, & \text{on } \partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0, \end{cases}$$

and $\mathcal{B}^*$ is defined in the same way by replacing $\mathcal{B}_* = \min\{\mathcal{B}^1, \mathcal{B}^2\}$ using $\mathcal{B}^* = \max\{\mathcal{B}^1, \mathcal{B}^2\}$ on the boundary $\partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0$. Furthermore, we denote

$$\begin{aligned}\mathrm{USC}_b(A) &:= \{\text{bounded u.s.c functions on } A\},\\ \mathrm{LSC}_b(A) &:= \{\text{bounded l.s.c functions on } A\}.\end{aligned}$$

The precise definition of viscosity sub/super solutions is given as below.

Definition 3.2 (Viscosity solution).

(i) $v \in \mathrm{USC}_b(\mathrm{cl}(\mathcal{O}))$ is a viscosity sub-solution of (3.4) if for any test function φ such that $\mathbf{z} \in \mathcal{O}$ is a strict maximum point of $v - \varphi$ at zero, we have

$$\begin{cases} \mathcal{F}[\varphi](\mathbf{z}) \leq 0, & \textit{on } \mathbf{z} \in \mathcal{O}^+,\\ \min\left\{\mathcal{F}[\varphi](\mathbf{z}),\ \mathcal{B}_*[\varphi](\mathbf{z})\right\} \leq 0, & \textit{on } \mathbf{z} \in \partial\mathcal{O}^0,\\ v(\mathbf{z}) \leq 1, & \textit{on } \mathbf{z} \in \partial\mathcal{O}_R,\\ v(\mathbf{z}) \leq 0, & \textit{on } \mathbf{z} \in \partial\mathcal{O}_{c/r}. \end{cases} \tag{3.6}$$

(ii) $v \in \mathrm{LSC}_b(\mathrm{cl}(\mathcal{O}))$ is a viscosity super-solution of (3.4) if for any test function φ such that $\mathbf{z} \in \mathcal{O}$ is a strict minimum point of $v - \varphi$ at zero, we have

$$\begin{cases} \mathcal{F}[\varphi](\mathbf{z}) \geq 0, & \textit{on } \mathbf{z} \in \mathcal{O}^+,\\ \max\left\{\mathcal{F}[\varphi](\mathbf{z}),\ \mathcal{B}^*[\varphi](\mathbf{z})\right\} \geq 0, & \textit{on } \mathbf{z} \in \partial\mathcal{O}^0,\\ v(\mathbf{z}) \geq 1, & \textit{on } \mathbf{z} \in \partial\mathcal{O}_R,\\ v(\mathbf{z}) \geq 0, & \textit{on } \mathbf{z} \in \partial\mathcal{O}_{c/r}. \end{cases} \tag{3.7}$$

(iii) v is a viscosity solution of (3.4) if v is both viscosity sub-solution and super-solution.

Remark 3.3. The definition of viscosity solutions is inextricably involved with min/max when the boundary conditions are given on derivatives. Consider $(p, X) \in \overline{\mathcal{J}}_{\mathcal{O}}^{2,\pm}\varphi(\mathbf{z})$ for some $\mathbf{z} \in \partial\mathcal{O}^0$, where $\overline{\mathcal{J}}_{\mathcal{O}}^{2,\pm}$ denote the closure of the second order superjet/subjet. Then there exists $(\mathbf{z}_n, p_n, X_n) \in \mathcal{J}_{\mathcal{O}}^{2,\pm}$ such that $(\mathbf{z}_n, p_n, X_n) \to (\mathbf{z}, p, X)$. However, in this case, we cannot guarantee that $\mathbf{z}_n \in \partial\mathcal{O}^0$ for any $n \in \mathbb{N}$. For more detailed discussion, readers can refer to Section 7 in [16].

Now, we are ready to state the main result of this paper.

Theorem 3.4 (The Main Theorem). *The value function V, defined at (2.12), is a unique viscosity solution of the HJB equation (3.4).*

The proof of the theorem is split into several steps, which will be provided in the next sections. In summary, the first step is to define *stochastic sub/super-solutions*. We continue to show that supremum (resp. infimum) of *stochastic sub-solutions* (resp. *stochastic super-solutions*) is a viscosity super-solution (resp. sub-solution). Then the main theorem can be concluded with the help of the following comparison principle of the HJB equation, whose proof is reported in the next section.

Proposition 3.5 (Comparison Principle). *Let u and v be a sub-solution and super-solution of (3.4), respectively. Then $u \leq v$ in $\mathrm{cl}(\mathcal{O})$.*

4. Stochastic Perron's Method and Proofs. This section contributes to the proof of Theorem 3.4 using stochastic Perron's method, which helps us to avoid the lengthy and technical proof of dynamic programming principle. To begin, we first need the concept of *random initial conditions* and *exit times*.

Definition 4.1. *We call (τ, ξ) a random initial condition if τ is a $\mathbb{G}$-stopping time valued in $[\![0, \tau_D]\!]$, $\xi = (\xi^X, \xi^{Y^1}, \xi^{Y^2})$ is a $\mathcal{G}_\tau$-measurable random variable valued in $\mathrm{cl}(\mathcal{O}) \cup \{\boldsymbol{\Delta}\}$, and $\xi = \boldsymbol{\Delta}$ if and only if $\tau = \tau_D$. We denote $\mathscr{R}$ as the set of all random initial conditions.*

Definition 4.2. *The exit time of $X^{\tau,\xi,\pi}$ from $\mathcal{O}$, denoted by $\tau_E^{\tau,\xi,\pi}$, is defined by*

$$\tau_E^{\tau,\xi,\pi} := \inf\{t \geq \tau : \; X_t^{\tau,\xi,\pi} \notin \mathcal{O}\}.$$

4.1. Stochastic Sub-solutions. This subsection first introduces the definition of *stochastic sub-solutions* of (3.4) and establishes the result that the stochastic envelope of *stochastic sub-solutions* is a viscosity super-solution of (3.4). In a nutshell, stochastic sub-solutions are functions that become $\mathbb{G}$-submartingales by operating on $Z = (X, Y)$. The purpose of defining the stochastic sub-solutions is to provide one direction of dynamic programming principle to some extent that

$$\inf_{\pi \in \mathscr{A}} \sup_{\mathbb{Q} \in \mathscr{L}} \mathbb{E}^{\mathbb{Q}}\left[V(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho \|\theta_t\|^2 \,\mathrm{d}t \;\middle|\; \mathcal{G}_\tau \right] \geq V(\xi), \tag{4.1}$$

for any *random initial condition* (τ, ξ) and $\mathbb{G}$-stopping time ρ such that $\tau \leq \rho$.

Definition 4.3 (Stochastic sub-solutions). *If $v \in \mathrm{LSC}_b(\mathrm{cl}(\mathcal{O}))$ satisfies*

(SB1) $v \leq 1$ on $\partial\mathcal{O}_R$ and $v \leq 0$ on $\partial\mathcal{O}_{c/r}$,

(SB2) for any $(\tau, \xi) \in \mathscr{R}$, $\pi \in \mathscr{A}$, and $\mathbb{G}$-stopping time $\rho \in [\tau, \tau_E^{\tau,\xi,\pi}]$, there exists a θ-measure $\mathbb{Q} \in \mathscr{L}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[v(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_t\|^2 \,\mathrm{d}t \;\middle|\; \mathcal{G}_\tau \right] \geq v(\xi), \tag{4.2}$$

where v in (4.2) is understood as its extension to $\mathrm{cl}(\mathcal{O}) \cup \{\mathbf{\Delta}\}$ by allocating $v(\mathbf{\Delta}) = 0$, then v is called a stochastic sub-solution of (3.4). In addition, we denote by $\mathcal{V}^-$ the class of all stochastic sub-solutions of (3.4).

For the remaining of the paper, stochastic sub-solutions means stochastic sub-solutions of (3.4). In addition, to understand the meaning of the extension up to $\mathbf{\Delta}$, we can simply consider $\tau = \tau_D$ and derive that $\tau = \tau_E = \tau_D = \rho$ and $\xi = \mathbf{\Delta}$. Then both sides in (4.2) equal to zero and the equation is trivially satisfied.

Remark 4.4. Note that we do not impose the oblique-type boundary condition $\mathcal{B}$ arising from the high-watermark fees in the definition of stochastic sub-solutions. The Dirichlet boundary conditions are from the associated financial problems, namely the ruin probability minimization problem. Such boundary conditions are invariant given the underlying processes, i.e., the same Dirichlet boundary conditions are imposed regardless of the SDE for $Z = (X, Y)$. However, the oblique-type boundary condition $\mathcal{B}$ comes from the structure of the process, the running maximum of the process, as $\mathcal{F}$ does. Therefore, we can deal with $\mathcal{B}$ and $\mathcal{F}$ together in the same manner in applying SPM. This in turn shows another advantage of stochastic Perron's method that is effective to handle control problem with high-watermark fee, especially with multiple hedge funds. Therefore, it is redundant to include the oblique-type boundary condition in Definition 4.3, which actually will make the argument more complicated because it is difficult to verify that $\mathcal{V}^-$ is closed under the maximum operation with condition $\mathcal{B}$.

Our first task is to find one stochastic sub-solution so that $\mathcal{V}^-$ is not empty. One can think of (4.2) as an upper-bound, in other words, stochastic sub-solution can be found by considering a "better situation". If there is no fee in reaching the high-watermark, the case is clearly better for the investor. The minimal ruin probability in this frictionless market was already studied by [35, 10], which will turn out to be a stochastic sub-solution in our case. Put

$$\begin{aligned}\mathfrak{U}(x) &:= \begin{cases} \left(\frac{c-rx}{c-rR}\right)^{\kappa}, & R \le x \le c/r, \\ 0, & c/r < x, \end{cases} \\ \kappa &:= \frac{1}{2r}\left[(r+\lambda^D+R)+\sqrt{(r+\lambda^D+R)^2-4r\lambda^D}\right], \\ \Sigma &:= \frac{1}{2}\mu_\Delta^\top(\sigma\sigma^\top)^{-1}\mu_\Delta.\end{aligned}$$

Before proceeding, note that $\mathfrak{U}$ is a solution of the following differential equation:

$$\begin{cases} \lambda^D\mathfrak{U}(x)+\Sigma[\mathfrak{U}'(x)]^2/\mathfrak{U}''(x)+(c-rx)\mathfrak{U}'(x)=0, \quad R<x<c/r, \\ \mathfrak{U}(R)=1, \quad \mathfrak{U}(c/r)=0. \end{cases} \tag{4.3}$$

Lemma 4.5. *Let* $\psi^-(x,y) := \mathfrak{U}(x)$. *Then* $\psi^- \in \mathcal{V}^-$.

Proof. It is obvious that ψ^- is continuous and satisfies (SB1) in Definition 4.3. To prove that ψ^- is a stochastic sub-solution, let us consider an arbitrary *random initial condition* (τ,ξ), $\pi \in \mathscr{A}$, and a $\mathbb{G}$-stopping time $\rho \in [\tau, \tau_E^{\tau,\xi,\pi}]$. Then we will show that (SB2) is satisfied with the reference measure $\mathbb{P}$. In other words, we choose

$$\theta = 0 \tag{4.4}$$

in the representation of (2.10). For the rest of this proof, we omit the super-scripts τ,ξ,π for simplicity. Define a process $(\overline{X}_t)_{t\ge 0}$ given by $\overline{X}_\tau = X_\tau$, $\overline{X}_{\tau_D} := \mathbf{\Delta}$, and

$$\mathrm{d}\overline{X}_t = [r\overline{X}_t - c + \pi_t^\top\mu_\Delta]\,\mathrm{d}t + \pi_t^\top\sigma\,\mathrm{d}W_t, \qquad \text{for} \quad t<\tau_D.$$

In other words, $\overline{X}$ is a process without high-watermark fees, thus $X \le \overline{X}$ on $[\![\tau,\tau_E]\!]$. As $\mathfrak{U}$ is non-increasing in $[R,\infty)$,

$$\mathbb{E}[\psi^-(X_\rho,Y_\rho)|\mathcal{G}_\tau] = \mathbb{E}[\mathbb{1}_{\rho<\tau_D}\mathfrak{U}(X_\rho)|\mathcal{G}_\tau] \ge \mathbb{E}[\mathbb{1}_{\rho<\tau_D}\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] = \mathbb{E}[\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] \tag{4.5}$$

Then, it suffices to show $\mathbb{E}[\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] \ge \mathfrak{U}(\overline{X}_\tau)(=\psi^-(\xi))$. We first consider the event $U := \{\overline{X}_\tau \in [R,c/r)\} \in \mathcal{G}_\tau$ and let $\nu := \inf\{t\ge\tau\colon\ \overline{X}_t \ge c/r\}$. On the event U,

$$\mathbb{E}[\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] \ge \mathbb{E}[\mathfrak{U}(\overline{X}_{\rho\wedge\nu})|\mathcal{G}_\tau].$$

In addition, applying Itô's formula on the event U yields

$$\begin{aligned}\mathfrak{U}(\overline{X}_{\rho\wedge\nu}) =& \mathfrak{U}(\overline{X}_\tau) + \int_\tau^{\rho\wedge\nu}\left\{\mathfrak{U}'(\overline{X}_t)[(r\overline{X}_t-c)+\pi_t^\top\mu_\Delta] + \mathfrak{U}''(\overline{X}_t)\frac{1}{2}\|\sigma^\top\pi_t\| - \lambda^D\mathfrak{U}(\overline{X}_t)\right\}\mathrm{d}t \\ &+ \int_\tau^{\rho\wedge\nu}\mathfrak{U}'(\overline{X}_t)\pi_t^\top\sigma\,\mathrm{d}W_t - \int_\tau^{\rho\wedge\nu}\mathfrak{U}(\overline{X}_{t-})\,\mathrm{d}\mathcal{M}_t^D\end{aligned}$$

The $\mathrm{d}t$-integral term is non-negative. Moreover, $\mathfrak{U}$, $\mathfrak{U}'$, and π are bounded, so the local martingales terms are martingales. Therefore, we have

$$\mathbb{1}_U\mathbb{E}[\mathfrak{U}(\overline{X}_{\rho\wedge\nu}) \mid \mathcal{G}_\tau] \ge \mathbb{1}_U\mathfrak{U}(\overline{X}_\tau). \tag{4.6}$$

On the other hand, on the event $U^c = \{\overline{X}_\tau \in [c/r, \infty) \cup \{\mathbf{\Delta}\}\}$, it clearly follows that $\mathfrak{U}(\overline{X}_\tau) = 0 \leq \mathbb{E}[\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau]$. Therefore, thanks to (4.5)-(4.6), we obtain

$$\begin{aligned}\mathbb{E}[\psi^-(X_\rho, Y_\rho)|\mathcal{G}_\tau] &\geq \mathbb{E}[\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] = \mathbb{E}[\mathbb{1}_U \mathfrak{U}(\overline{X}_\rho) + \mathbb{1}_{U^c}\mathfrak{U}(\overline{X}_\rho)|\mathcal{G}_\tau] \\ &\geq \mathbb{1}_U \mathfrak{U}(\overline{X}_\tau) = \mathfrak{U}(\overline{X}_\tau) \\ &= \psi^-(\xi). \end{aligned} \tag{4.7}$$

Thus by (4.4), ψ^- satisfies (SB2). ∎

To show the stochastic envelope of stochastic sub-solutions is a viscosity super-solution, we first show $\mathcal{V}^-$ is closed under maximum operation.

LEMMA 4.6. *If $v^1, v^2 \in \mathcal{V}^-$, then $v^1 \vee v^2 \in \mathcal{V}^-$.*

Proof. It is easy to check that $v^1 \vee v^2 \in \mathrm{LSC}_b(\mathrm{cl}(\mathcal{O}))$ and $v^1 \vee v^2$ satisfies (SB1) in Definition 4.3. Let $(\tau, \xi) \in \mathscr{R}$, $\pi \in \mathscr{A}$, ρ be a $\mathbb{G}$-stopping time valued in interval $[\tau, \tau_E^{\tau,\xi,\pi}]$. Because v^1 and v^2 are stochastic sub-solutions, there exist $\mathbb{Q}^1$ and $\mathbb{Q}^2$ satisfying (SB2). We denote by θ^i, $i \in \{1, 2\}$, the processes that generate $\mathbb{Q}^i$. To find the measure satisfying (SB2) for $v^1 \vee v^2$, we define $\mathfrak{A} := \{v^1(\xi) > v^2(\xi)\} \in \mathcal{G}_\tau$, $\theta := \mathbb{1}_{[\![\tau,\infty]\!]}[\mathbb{1}_{\mathfrak{A}}\theta^1 + \mathbb{1}_{\mathfrak{A}^c}\theta^2]$, and let $\mathbb{Q}$ denote the measure generated by θ, i.e., on the stochastic interval $[\![\tau, \infty]\!]$,

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_\cdot} = \mathbb{1}_{\mathfrak{A}}\left.\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_\cdot} + 1_{\mathfrak{A}^c}\left.\frac{\mathrm{d}\mathbb{Q}^2}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_\cdot}.$$

Then as v^1 is a stochastic sub-solution and $\mathfrak{A} \in \mathcal{G}_\tau$, we have

$$\begin{aligned}\mathbb{1}_{\mathfrak{A}} v^1(\xi) \leq& \mathbb{1}_{\mathfrak{A}}\mathbb{E}^{\mathbb{Q}^1}\left[v^1(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s^1\|^2\,\mathrm{d}s \,\middle|\, \mathcal{G}_\tau\right] \\ =&\mathbb{E}\left[\mathbb{1}_{\mathfrak{A}}\left.\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_\rho}\left\{v^1(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s^1\|^2\,\mathrm{d}s\right\} \,\middle|\, \mathcal{G}_\tau\right] \\ =&\mathbb{E}\left[\mathbb{1}_{\mathfrak{A}}\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_\rho}\left\{v^1(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s\right\} \,\middle|\, \mathcal{G}_\tau\right] \\ \leq& \mathbb{1}_{\mathfrak{A}}\mathbb{E}^{\mathbb{Q}}\left[(v^1 \vee v^2)(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \,\middle|\, \mathcal{G}_\tau\right]. \end{aligned} \tag{4.8}$$

The second equality above is obtained by Definition 2.2 and boundness of v^1. Similarly, we obtain

$$\mathbb{1}_{\mathfrak{A}^c} v^2(\xi) \leq \mathbb{1}_{\mathfrak{A}^c}\mathbb{E}^{\mathbb{Q}}\left[(v^1 \vee v^2)(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \,\middle|\, \mathcal{G}_\tau\right]. \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$(v^1 \vee v^2)(\xi) \leq \mathbb{E}^{\mathbb{Q}}\left[(v^1 \vee v^2)(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \,\middle|\, \mathcal{G}_\tau\right].$$

Thus, $(v^1 \vee v^2)$ satisfies (SB2) with $\mathbb{Q}$. ∎

In the next theorem, we will use Lemma 4.6 to construct a "bump" function to argue by contradiction.

THEOREM 4.7. *The lower stochastic envelope of $\mathcal{V}^-$,*

$$v^- := \sup_{v \in \mathcal{V}^-} v, \tag{4.10}$$

is a viscosity super-solution of (3.4).

Proof. Lemma 4.5 already asserts that $v^- \geq \psi^-$. Therefore, we have $v^- \geq 1$ on $\mathcal{O}_R$ and $v^- \geq 0$ on $\mathcal{O}_{c/r}$. It remains to show that for this v^- and any test function φ such that $\mathbf{z} \in \mathcal{O}$ is a minimum point of $v^- - \varphi$ at zero, we have

$$\begin{cases} \mathcal{F}[\varphi](\mathbf{z}) \geq 0, & \text{on } \ \mathbf{z} \in \mathcal{O}^+, \\ \max\left\{\mathcal{F}[\varphi](\mathbf{z}),\ \mathcal{B}^*[\varphi](\mathbf{z})\right\} \geq 0, & \text{on } \ \mathbf{z} \in \partial\mathcal{O}^0. \end{cases}$$

We first show the claim above holds on the boundary part $\partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0$.

Let us consider the region $B_a(\mathbf{z}_0)$ of a ball with center $\mathbf{z}_0 \in \mathrm{cl}(\mathcal{O})$ and the radius a intersecting with $\mathcal{O}$ that

$$B_a(\mathbf{z}_0) := \{\mathbf{z} \in \mathrm{cl}(\mathcal{O}) \colon \ \|\mathbf{z} - \mathbf{z}_0\| < a\}.$$

To argue by contradiction, we suppose that there exist $\mathbf{z}_0 = (x_0, 0, 0) \in \partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0$ and some $\varphi \in C^2(\mathcal{O})$ such that $v^- - \varphi$ attains its strict minimum of zero at $\mathbf{z}_0$ and

$$\max\left\{\mathcal{F}[\varphi](\mathbf{z}_0),\ \mathcal{B}^1[\varphi](\mathbf{z}_0),\ \mathcal{B}^2[\varphi](\mathbf{z}_0)\right\} < 0. \tag{4.11}$$

Therefore it follows that there exists a constant $\theta^\varphi \in \mathcal{L}$ such that

$$\lambda^D \varphi(\mathbf{z}_0) - (rx_0 - c)\varphi_x(\mathbf{z}_0) - \inf_{\pi \in \mathcal{K}} \mathcal{A}^{\pi,\theta^\varphi}[\varphi](\mathbf{z}_0) < 0. \tag{4.12}$$

Using φ, we will construct a bump function that still is in $\mathcal{V}^-$, in which it contradicts to (4.10). By continuity of $\mathcal{F}$ and $\mathcal{B}^i$, $i \in \{1, 2\}$, we can choose a small ball $B_{2a}(\mathbf{z}_0)$, $a > 0$, such that for any $\mathbf{z} \in \mathrm{cl}(B_{2a}(\mathbf{z}_0))$,

$$\max\left\{\lambda^D \varphi(\mathbf{z}) - (rx - c)\varphi_x(\mathbf{z}) - \inf_{\pi \in \mathcal{K}} \mathcal{A}^{\pi,\theta^\varphi}[\varphi](\mathbf{z}) < 0,\ \mathcal{B}^1[\varphi](\mathbf{z}),\ \mathcal{B}^2[\varphi](\mathbf{z})\right\} < 0. \tag{4.13}$$

As $v^- - \varphi$ is l.s.c and $\mathrm{cl}(B_{2a}(\mathbf{z}_0)) \setminus B_a(\mathbf{z}_0)$ is compact, there exists $\delta > 0$ satisfying

$$v^- - \varphi \geq \delta, \quad \text{on} \quad \mathrm{cl}(B_{2a}(\mathbf{z}_0)) \setminus B_a(\mathbf{z}_0).$$

As a result of Proposition 4.1 in [7] and Lemma 4.6, we can choose a non-decreasing sequence $\{v_n\} \subseteq \mathcal{V}^-$ such that $v_n \nearrow v^-$. By Lemma 2.4 in [9], we can pick $v := v_N$ such that

$$v - \varphi \geq \delta/2 \ \text{ on } \ \mathrm{cl}(B_{2a}(\mathbf{z}_0)) \setminus B_a(\mathbf{z}_0).$$

Then we further choose $0 < \eta < \delta/2$ small enough such that $\varphi^\eta := \varphi + \eta$ satisfies

$$\max\left\{\lambda^D \varphi^\eta(\mathbf{z}) - (rx - c)\varphi^\eta_x(\mathbf{z}) - \inf_{\pi \in \mathcal{K}} \mathcal{A}^{\pi,\theta^\varphi}[\varphi^\eta](\mathbf{z}),\ \mathcal{B}^1[\varphi^\eta](\mathbf{z}),\ \mathcal{B}^2[\varphi^\eta](\mathbf{z})\right\} < 0, \tag{4.14}$$

on $\mathrm{cl}(B_{2a}(\mathbf{z}_0))$. By this construction, we have

$$\varphi^\eta \leq \varphi + \delta/2 \leq v \ \text{ on } \ \mathrm{cl}(B_{2a}(\mathbf{z}_0)) \setminus B_a(\mathbf{z}_0), \tag{4.15}$$

$$\varphi^\eta(\mathbf{z}_0) = \varphi(\mathbf{z}_0) + \eta = v^-(\mathbf{z}_0) + \eta > v^-(\mathbf{z}_0). \tag{4.16}$$

Let us define

$$v^\eta := \begin{cases} v \vee \varphi^\eta, & \mathrm{cl}(B_{2a}(\mathbf{z}_0)), \\ v, & \text{otherwise.} \end{cases}$$

Then we will show that $v^\eta \in \mathcal{V}^-$ and this is a contradiction by (4.10) and (4.16).

To this end, we consider an arbitrary $(\tau,\xi) \in \mathscr{R}$, $\pi \in \mathscr{A}$, and a $\mathbb{G}$-stopping time $\rho \in [\![\tau, \tau_E^{\tau,\xi,\pi}]\!]$. Our goal is to find a probability measure satisfying (SB2) for v^η. As v is a stochastic sub-solution, for any strategy π we can find $(\theta_t^{v,\pi})_{t\ge 0}$ producing a probability measure $\mathbb{Q}^{v,\pi} \in \mathscr{L}$ that satisfies (SB2) for v. Define

$$\Gamma \coloneqq \big\{\xi \in B_a(\mathbf{z}_0) \text{ and } v(\xi) < \varphi^\eta(\xi)\big\} \in \mathcal{G}_\tau,$$

and let τ_a (resp. ξ_a) denote the exit time (resp. exit position) of the ball $B_a(\mathbf{z}_0)$, i.e.,

$$\begin{aligned} \tau_a &\coloneqq \inf\{t \in [\tau, \tau_E^{\tau,\xi,\pi}]: \;\; Z_t^{\tau,\xi,\pi} \notin B_a(\mathbf{z}_0)\}, \\ \xi_a &\coloneqq Z_{\tau_a}^{\tau,\xi,\pi}. \end{aligned}$$

By $(\theta_t^{v,\pi})_{t\ge 0}$ and θ^φ in (4.14), define $(\widetilde{\theta}_t)_{t\ge 0}$ as

$$\widetilde{\theta}_t^\pi \coloneqq \mathbb{1}_{t\ge\tau}(\theta^\varphi \mathbb{1}_\Gamma + \theta_t^{v,\pi}\mathbb{1}_{\Gamma^c})$$

Note that $\xi_a \in \partial B_a(\mathbf{z}_0) \cup \{\boldsymbol{\Delta}\}$ and $(\tau_a,\xi_a) \in \mathscr{R}$. Therefore, for (τ_a,ξ_a) and $\pi \in \mathscr{A}$, there exists $\theta^{v,a,\pi}$ producing $\mathbb{Q}^{v,a,\pi}$ given by (2.10) that satisfies (SB2) for v. Then define

$$\theta^\pi \coloneqq \mathbb{1}_{[\![0,\tau_a]\!]}\widetilde{\theta}^\pi + \mathbb{1}_{]\!]\tau_a,\infty]\!]}\theta^{v,a,\pi}, \tag{4.17}$$

and $\mathbb{Q}^\pi$ be the measure by θ^π. Then for any $\pi \in \mathscr{R}$, we show that $\mathbb{Q}^\pi$ is the measure for v^η to satisfy (SB2) from which we obtain the contradiction.

In particular, we can obtain a contradiction from the place where the measure by $\theta^{\varphi,\pi}$ is taken. Itô's formula on the event Γ yields

$$\begin{aligned} \varphi^\eta(Z_{\rho\wedge\tau_a}^{\tau,\xi,\pi}) - \varphi^\eta(Z_\tau^{\tau,\xi,\pi}) = &\int_\tau^{\rho\wedge\tau_a} \big[\mathcal{A}^{\pi,\theta^\pi}[\varphi^\eta] + \frac{1}{2\varepsilon}\|\theta_t^\pi\|^2 - \lambda^D\varphi^\eta + (rX_t^{\tau,\xi,\pi} - c)\varphi_x^\eta\big](Z_t^{\tau,\xi,\pi})\,\mathrm{d}t \\ &- \sum_{i=1,2}\int_\tau^{\rho\wedge\tau_a} \mathcal{B}^i[\varphi^\eta](Z_t^{\tau,\xi,\pi})\,\mathrm{d}M_t^i \\ &- \int_\tau^{\rho\wedge\tau_a} \varphi^\eta(Z_{t-}^{\tau,\xi,\pi})\,\mathrm{d}\mathcal{M}_t^D \\ &+ \int_\tau^{\rho\wedge\tau_a} \big[\varphi_x^\eta(Z_t^{\tau,\xi,\pi})\pi_t^\top\sigma - \nabla_y\varphi^\eta(Z_t^{\tau,\xi,\pi})^\top \operatorname{diag}(\pi_t)\sigma_\Delta^B\big]\,\mathrm{d}W_t^{\mathbb{Q}^\pi}. \end{aligned} \tag{4.18}$$

On the compact set $\mathrm{cl}(B_a(\mathbf{z}_0))$, φ^η and $\nabla\varphi^\eta$ are bounded. Therefore, $\varphi^\eta(Z^{\tau,\xi,\pi})$ and $\nabla\varphi^\eta(Z^{\tau,\xi,\pi})$ are bounded on $[\![\tau, \rho\wedge\tau_a]\!]$. Moreover, π is valued in the compact set $\mathcal{K}$. Therefore, the last two terms in (4.18) are $\mathbb{G}$-martingales. Then by (4.11), we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^\pi}\big[\mathbb{1}_\Gamma v^\eta\big(Z_{\rho\wedge\tau_a}^{\tau,\xi,\pi}\big)\big|\mathcal{G}_\tau\big] &\ge \mathbb{E}^{\mathbb{Q}^\pi}\bigg[\mathbb{1}_\Gamma\Big\{\varphi^\eta\big(Z_{\rho\wedge\tau_a}^{\tau,\xi,\pi}\big) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a\wedge\rho}\|\theta_t^\pi\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_\tau\bigg] \\ &\ge \mathbb{1}_\Gamma\varphi^\eta\big(Z_\tau^{\tau,\xi,\pi}\big) = \mathbb{1}_\Gamma\varphi^\eta(\xi) \\ &= \mathbb{1}_\Gamma v^\eta(\xi). \end{aligned}$$

Note that at the last equality, we do not exclude the case that $\tau = \tau^D$, i.e., $\xi = \boldsymbol{\Delta}$. Recall that on Γ^c, we have $v(\xi) = v^\eta(\xi)$ and $\theta^\pi = \theta^{v,\pi}$ which is the (τ,ξ)-optimal control of v. Let $\mathbb{Q}^{v,\pi}$ denote the

$\theta^{v,\pi}$-measure. By (SB2), it follows that

$$\begin{aligned}\mathbb{1}_{\Gamma^c} v^\eta(\xi) = \mathbb{1}_{\Gamma^c} v(\xi) &\le \mathbb{E}^{\mathbb{Q}^{v,\pi}}\left[\mathbb{1}_{\Gamma^c}\Big\{v\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a\wedge\rho} \|\theta^{v,\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_\tau\right]\\ &\le \mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Gamma^c}\Big\{v^\eta\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a\wedge\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_\tau\right].\end{aligned}$$

Hence, we obtain that

$$v^\eta(\xi) \le \mathbb{E}^{\mathbb{Q}^{\pi}}\left[v^\eta\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a\wedge\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big|\mathcal{G}_\tau\right]. \tag{4.19}$$

Now, to replace $\rho\wedge\tau_a$ with ρ in (4.19), we first consider the event $\Lambda := \{\rho > \tau_a\} \in \mathcal{G}_{\tau_a\wedge\rho}$. Since $v = v^\eta$ at $\partial B_a(\mathbf{z}_0)$ and on $\rrbracket\tau_a, \rho\rrbracket \cap (\Lambda\times\mathbb{R}_+)$, we have $\theta^\pi = \theta^{v,a,\pi}$. Then denoting by $\mathbb{Q}^{v,a,\pi}$ the $\theta^{v,a,\pi}$-measure,

$$\begin{aligned}\mathbb{1}_{\Lambda} v^\eta(\xi_a) = \mathbb{1}_{\Lambda} v(\xi_a) &\le \mathbb{E}^{\mathbb{Q}^{v,a,\pi}}\left[\mathbb{1}_{\Lambda}\Big\{v(Z^{\tau,\xi,\pi}_{\rho}) - \frac{1}{2\varepsilon}\int_{\tau_a}^{\rho} \|\theta^{v,a,\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_{\tau_a}\right]\\ &\le \mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Lambda}\big\{v^\eta(Z^{\tau,\xi,\pi}_{\rho}) - \frac{1}{2\varepsilon}\int_{\tau_a}^{\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_{\tau_a}\right].\end{aligned} \tag{4.20}$$

Moreover, by (4.19) together with (4.20), we can get

$$\begin{aligned}v^\eta(\xi) &\le \mathbb{E}^{\mathbb{Q}^{\pi}}\left[v^\eta(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a\wedge\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big|\mathcal{G}_\tau\right]\\ &= \mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Lambda^c}\Big\{v^\eta(Z^{\tau,\xi,\pi}_{\rho}) - \frac{1}{2\varepsilon}\int_\tau^{\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\} + \mathbb{1}_{\Lambda}\Big\{v^\eta(\xi_a) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_\tau\right].\end{aligned} \tag{4.21}$$

By (4.20), we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Lambda}\Big\{v^\eta(\xi_a) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_\tau\right] &= \mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Lambda}\Big\{v^\eta(\xi_a) - \frac{1}{2\varepsilon}\int_\tau^{\tau_a} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_{\tau_a}\right]\Big|\mathcal{G}_\tau\right]\\ &\le \mathbb{E}^{\mathbb{Q}^{\pi}}\left[\mathbb{1}_{\Lambda}\Big\{v^\eta(Z^{\tau,\xi,\pi}_{\rho}) - \frac{1}{2\varepsilon}\int_\tau^{\rho} \|\theta^{\pi}_t\|^2\,\mathrm{d}t\Big\}\Big|\mathcal{G}_\tau\right].\end{aligned} \tag{4.22}$$

Therefore, in view of (4.21) and (4.22), we deduce that $v^\eta \in \mathcal{V}^-$, which clearly contradicts (4.10). Hence, it follows that v^- is a viscosity super-solution of (3.4) at $\mathbf{z}_0 \in \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2$.

We can deal with points in other regions $\mathbf{z}_0 \notin \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2$ in similar ways. To be more precise, for $\mathbf{z}_0 \in \mathcal{O}^+$ (resp. $\mathbf{z}_0 \in \partial\mathcal{O}_i$, $i \in \{1,2\}$), we suppose that there exist a function $\varphi \in C^2(\mathcal{O})$ such that $v^- - \varphi$ attains its strict minimum of zero at $\mathbf{z}_0$ and

$$\mathcal{F}[\varphi](\mathbf{z}_0) < 0$$
$$\big(\text{resp. } \max\{\mathcal{F}[\varphi](\mathbf{z}_0),\ \mathcal{B}^i[\varphi](\mathbf{z}_0)\} < 0\big).$$

Then, by employing similar contradiction arguments, we can conclude that v^- is indeed a viscosity super-solution of (3.4). ∎

4.2. Stochastic Super-solutions. Roughly speaking, *stochastic super-solutions* can be defined to facilitate the derivation of the other direction of DPP as

$$\inf_{\pi\in\mathscr{A}} \sup_{\mathbb{Q}\in\mathscr{L}} \mathbb{E}^{\mathbb{Q}}\left[V(Z^{\tau,\xi,\pi}_\rho) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \;\Big|\; \mathcal{G}_\tau\right] \le V(\xi).$$

Note that the item (SP2) in the next definition is precisely motivated by the inequality above.

DEFINITION 4.8 (Stochastic super-solutions). *If* $v \in \mathrm{USC}_b(\mathrm{cl}(\mathcal{O}))$ *satisfies*

(SP1) $v \geq 1$ *on* $\partial\mathcal{O}_R$ *and* $v \geq 0$ *on* $\partial\mathcal{O}_{c/r}$*,*

(SP2) for any random initial condition (τ, ξ)*, there exists* $\pi \in \mathscr{A}$ *such that for any* $\mathbb{G}$*-stopping time* $\rho \in [\tau, \tau_E^{\tau,\xi,\pi}]$ *and* $\mathbb{Q} \in \mathscr{L}$*,*

$$\mathbb{E}^{\mathbb{Q}}\left[v(Z_\rho^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon}\int_\tau^\rho e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \;\middle|\; \mathcal{G}_\tau\right] \leq v(\xi), \tag{4.23}$$

where v *in (4.2) is understood as its extension to* $\mathrm{cl}(\mathcal{O}) \cup \{\mathbf{\Delta}\}$ *by allocating* $v(\mathbf{\Delta}) = 0$*, then* v *is called a stochastic super-solution of (3.4). In addition, we let* $\mathcal{V}^+$ *denote the class of all stochastic super-solutions of (3.4).*

We can find a stochastic super-solution by considering a "worse scenario". Consider a situation that the investor does not invest in the hedge funds, i.e., $\pi = 0$. Then, the investor's wealth follows $\mathrm{d}X_t = [rX_t - c]\,\mathrm{d}t,\ X_0 = x$. We thus, can obtain that

$$\mathfrak{p}(x) := \mathbb{P}(\tau_R^{x,y,0} < \tau_D) = \left(\frac{c - rx}{c - rR}\right)^{\frac{\lambda^D}{r}}.$$

LEMMA 4.9. *Let* $\psi^+(x, y) := \mathfrak{p}(x)$*. Then* $\psi^+ \in \mathcal{V}^+$*.*

Proof. It is obvious that $\psi^+ \in \mathrm{USC}_b(\mathrm{cl}(\mathcal{O}))$ and satisfies (SP1). Let (τ, ξ) be a *random initial condition* and we choose $\pi = 0$ for the strategy. Thus, for $\tau < \tau_D$,

$$\mathrm{d}X_t^{\tau,\xi,\pi} = [rX_t^{\tau,\xi,\pi} - c]\,\mathrm{d}t.$$

Consider $\rho \in [\tau, \tau_E^{\tau,\xi,\pi}]$ as a $\mathbb{G}$-stopping time. In the rest of the proof, we suppress the superscripts τ, ξ, π. By Itô's formula, we have

$$\begin{aligned}\mathfrak{p}(X_\rho) - \mathfrak{p}(X_\tau) &= \int_\tau^\rho \left\{\mathfrak{p}'(X_t)[rX_t - c] - \lambda^D\mathfrak{p}(X_t)\right\} - \int_\tau^\rho \mathfrak{p}(X_{s-})\,\mathrm{d}\mathcal{M}_s^D \\ &= -\int_\tau^\rho \mathfrak{p}(X_{s-})\,\mathrm{d}\mathcal{M}_s^D\end{aligned}$$

As for any equivalent probability measure $\mathbb{Q}$ given by (2.10), $\mathcal{M}^D$ is $(\mathbb{Q}, \mathbb{G})$-martingale, it follows that $\mathbb{E}^{\mathbb{Q}}[\mathfrak{p}(X_\rho)|\mathcal{G}_\tau] = \mathfrak{p}(X_\tau)$ for any $\mathbb{Q} \in \mathscr{L}$. Therefore, for any θ-measure $\mathbb{Q} \in \mathscr{L}$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\psi^+(Z_\rho) - \frac{1}{2\varepsilon}\int_\tau^\rho \|\theta_t\|^2\,\mathrm{d}t\middle|\mathcal{G}_\tau\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathfrak{p}(Z_\rho) - \frac{1}{2\varepsilon}\int_\tau^\rho \|\theta_t\|^2\,\mathrm{d}t\middle|\mathcal{G}_\tau\right] \\ &\leq \mathbb{E}^{\mathbb{Q}}[\mathfrak{p}(X_\rho)|\mathcal{G}_\tau] = \mathfrak{p}(X_\tau) = \psi^+(\xi).\end{aligned}$$

Therefore, ψ^+ satisfies (SP2), and we can deduce that $\psi^+ \in \mathcal{V}^+$. ◻

As in the previous section, we need to show $\mathcal{V}^+$ is stable under minimum operation. The proof follows closely the argument to prove Lemma 4.6, so we omit it.

LEMMA 4.10. *If* $v^1, v^2 \in \mathcal{V}^+$*, then* $v^1 \wedge v^2 \in \mathcal{V}^+$*.*

Then Lemma 4.10 will be used to construct a bump function in the following theorem.

THEOREM 4.11. *The lower stochastic envelope of* $\mathcal{V}^+$*,*

$$v^+ := \inf_{v \in \mathcal{V}^+} v, \tag{4.24}$$

is a viscosity sub-solution of (3.4)

Proof. By Lemma 4.9, $v^+ \le \psi^+$. Therefore, we have $v^+ \le 1$ on $\mathcal{O}_R$ and $v^+ \le$ on $\mathcal{O}_{c/r}$. As in the proof of Theorem 4.7, it is sufficient to verify the sub-solution property of v^+ only on the boundary part $\partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0$. Using the same notation of balls that intersect $\mathcal{O}$, we again will prove by contradiction. Suppose that there exist $\mathbf{z}_0 = (x_0, 0, 0) \in \partial\mathcal{O}_1^0 \cap \partial\mathcal{O}_2^0$ and $\varphi \in C^2(\mathcal{O})$ such that $v^- - \varphi$ attains its strict maximum of zero at $\mathbf{z}_0$ and

$$\min\left\{\mathcal{F}[\varphi](\mathbf{z}_0),\ \mathcal{B}^1[\varphi](\mathbf{z}_0),\ \mathcal{B}^2[\varphi](\mathbf{z}_0)\right\} > 0. \tag{4.25}$$

Again, as in the construction of a bump function in Theorem 4.7, we can choose constants $\pi^\varphi \in \mathcal{K}$, $\eta > 0, a > 0$, and a stochastic super-solution $v \in \mathcal{V}^+$ such that

$$\begin{cases} \varphi^\eta = \varphi + \eta \ge v, & \text{on } \ \mathrm{cl}(B_{2a}(\mathbf{z}_0)) \setminus B_a(\mathbf{z}_0), \\ \lambda^D \varphi^\eta - (rx - c)\varphi^\eta_x - \sup\limits_{\theta \in \mathcal{L}} \mathcal{A}^{\pi^\varphi, \theta}[\varphi^\eta] > 0, & \text{on } \ \mathrm{cl}(B_{2a}(\mathbf{z}_0)), \\ \min\left\{\mathcal{B}^1[\varphi^\eta],\ \mathcal{B}^2[\varphi^\eta]\right\} > 0, & \text{on } \ \mathrm{cl}(B_{2a}(\mathbf{z}_0)), \\ \varphi^\eta(\mathbf{z}_0) < v^-(\mathbf{z}_0), \end{cases} \tag{4.26}$$

and we define

$$v^\eta := \begin{cases} v \wedge \varphi^\eta, & \mathrm{cl}(B_{2a}(\mathbf{z}_0)), \\ v, & \text{otherwise}. \end{cases} \tag{4.27}$$

Then we will show that $v^\eta \in \mathcal{V}^+$. To show that v^η satisfies (SP2), let $(\tau, \xi) \in \mathscr{R}$. Since $v \in \mathcal{V}^+$, we can choose $(\pi^v_t)_{t\ge 0}$ for v to satisfy (SP2). Then with π^φ in (4.26), we define $\tilde{\pi}_{t\ge 0}$ as

$$\tilde{\pi}_t := \mathbb{1}_{t \ge \tau}(\pi^\varphi \mathbb{1}_\Gamma + \pi^v_t \mathbb{1}_{\Gamma^c}).$$

Let us denote

$$\Gamma := \left\{\xi \in B_a(\mathbf{z}_0) \text{ and } v(\xi) < \varphi^\eta(\xi)\right\},$$

and let τ_a (resp. ξ_a) denote the exit time (resp. exit position) of the ball $B_a(\mathbf{z}_0)$. Since $(\tau_a, \xi_a) \in \mathscr{R}$ and $v \in \mathcal{V}^+$, we can choose $\pi^{v,a} \in \mathscr{A}$ such that for any $\mathbb{Q} \in \mathscr{L}$ and $\mathbb{G}$-stopping time valued in $[\![\tau_a, \tau_E^{\tau,\xi,\pi^{v,a}}]\!]$, v satisfies (SP2). Finally, we let

$$\pi := \mathbb{1}_{[\![0,\tau_a]\!]}\tilde{\pi} + \mathbb{1}_{]\!]\tau_a,\infty]\!]}\pi^{v,a}. \tag{4.28}$$

We will show that v^η, with π, satisfies (SP2). Consider an arbitrary $\mathbb{G}$-stopping time $\rho \in [\tau, \tau_E^{\tau,\xi,\pi}]$ and θ-measure $\mathbb{Q} \in \mathscr{L}$. Applying Itô's formula on the event Γ yields, for any θ-measure $\mathbb{Q}$,

$$\begin{aligned} \varphi^\eta(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}) - \varphi^\eta(Z^{\tau,\xi,\pi}_\tau) = & \int_\tau^{\rho\wedge\tau_a} \left[\mathcal{A}^{\pi,\theta}[\varphi^\eta] + \frac{1}{2\varepsilon}\|\theta_t\|^2 - \lambda^D\varphi^\eta + (rX^{\tau,\xi,\pi}_t - c)\varphi^\eta_x\right](Z^{\tau,\xi,\pi}_t)\,\mathrm{d}t \\ & - \sum_{i=1,2}\int_\tau^{\rho\wedge\tau_a} \mathcal{B}^i[\varphi^\eta](Z^{\tau,\xi,\pi}_t)\,\mathrm{d}M^i_t \\ & - \int_\tau^{\rho\wedge\tau_a} \varphi^\eta(Z^{\tau,\xi,\pi}_{t-})\,\mathrm{d}\mathcal{M}^D_t \\ & + \int_\tau^{\rho\wedge\tau_a} \left[\varphi^\eta_x(Z^{\tau,\xi,\pi}_t)\pi_t^\top\sigma - \nabla_y\varphi^\eta(Z^{\tau,\xi,\pi}_t)^\top \operatorname{diag}(\pi_t)\sigma^B_\Delta\right]\mathrm{d}W^{\mathbb{Q}}_t. \end{aligned} \tag{4.29}$$

Therefore, by (4.26) and (4.27), we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Gamma}\Big\{v^{\eta}\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a\wedge\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg] &\le \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Gamma}\Big\{\varphi^{\eta}\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a\wedge\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg]\\ &\le \mathbb{1}_{\Gamma}\varphi^{\eta}\big(Z^{\tau,\xi,\pi}_{\tau}\big)=\mathbb{1}_{\Gamma}\varphi^{\eta}(\xi)\\ &= \mathbb{1}_{\Gamma}v^{\eta}(\xi).\end{aligned}$$

Recall that on Γ^c, we have $v(\xi)=v^{\eta}(\xi)$ and $\pi=\pi^{v}$. Since v is a stochastic super-solution by its construction, we have

$$\begin{aligned}\mathbb{1}_{\Gamma^c}v^{\eta}(\xi)=\mathbb{1}_{\Gamma^c}v(\xi) &\ge \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Gamma^c}\Big\{v\big(Z^{\tau,\xi,\pi^{v}}_{\rho\wedge\tau_a}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a\wedge\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg]\\ &\ge \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Gamma^c}\Big\{v^{\eta}\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a\wedge\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg].\end{aligned}$$

Thus, we deduce that

$$v^{\eta}(\xi)\ge \mathbb{E}^{\mathbb{Q}}\bigg[v^{\eta}\big(Z^{\tau,\xi,\pi}_{\rho\wedge\tau_a}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a\wedge\rho}\|\theta_t\|^2\,\mathrm{d}t\bigg|\mathcal{G}_{\tau}\bigg]. \tag{4.30}$$

To replace $\rho\wedge\tau_a$ with ρ, consider $\Lambda:=\{\rho>\tau_a\}\in\mathcal{G}_{\tau_a\wedge\rho}$. Recall that $v=v^{\eta}$ at $\partial B_a(\mathbf{z}_0)$ and on $]\!]\tau_a,\rho]\!]\cap(\Lambda\times\mathbb{R}_+)$, we have $\pi=\pi^{v,a}$. It then follows that

$$\begin{aligned}\mathbb{1}_{\Lambda}v^{\eta}(\xi_a)=\mathbb{1}_{\Lambda}v(\xi_a) &\ge \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Lambda}\Big\{v(Z^{\tau,\xi,\pi^{v,a}}_{\rho})-\frac{1}{2\varepsilon}\int_{\tau_a}^{\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau_a}\bigg]\\ &\ge \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Lambda}\big\{v^{\eta}(Z^{\tau,\xi,\pi}_{\rho})-\frac{1}{2\varepsilon}\int_{\tau_a}^{\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau_a}\bigg].\end{aligned} \tag{4.31}$$

By (4.31), one can derive that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Lambda}\Big\{v^{\eta}(\xi_a)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg] &= \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Lambda}\Big\{v^{\eta}(\xi_a)-\frac{1}{2\varepsilon}\int_{\tau}^{\tau_a}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau_a}\bigg]\bigg|\mathcal{G}_{\tau}\bigg]\\ &\ge \mathbb{E}^{\mathbb{Q}}\bigg[\mathbb{1}_{\Lambda}\Big\{v^{\eta}(Z^{\tau,\xi,\pi}_{\rho})-\frac{1}{2\varepsilon}\int_{\tau}^{\rho}\|\theta_t\|^2\,\mathrm{d}t\Big\}\bigg|\mathcal{G}_{\tau}\bigg].\end{aligned} \tag{4.32}$$

Therefore, thanks to (4.31) and (4.32), the inequality holds that

$$\mathbb{1}_{\Lambda}v^{\eta}(\xi)\ge \mathbb{1}_{\Lambda}\mathbb{E}^{\mathbb{Q}}\bigg[v^{\eta}\big(Z^{\tau,\xi,\pi}_{\rho}\big)-\frac{1}{2\varepsilon}\int_{\tau}^{\rho}\|\theta_t\|^2\,\mathrm{d}t\bigg|\mathcal{G}_{\tau}\bigg]. \tag{4.33}$$

We can obtain the inequality on Λ^c in the similar fashion as in the proof of Theorem 4.7. Hence, it can be shown that $v^{\eta}\in\mathcal{V}^{+}$, which contradicts (4.26) and our claim holds. ∎

4.3. Proof of Comparison Principle. Comparison principle with either Neumann or oblique-type boundary conditions was already studied; see, for example, [2, 3]. However, because we have both Dirichlet and oblique-type boundary conditions in our problem, some tailor made arguments need to be developed here.

We plan to apply a typical doubling argument, nevertheless, the additional difficulty by considering oblique-type conditions is that we need to construct a test function with care. We will choose a test function in a way that $\mathcal{B}\neq 0$ in a viscosity sense. Then by the definition of viscosity solution, the test function should satisfy $\mathcal{F} = 0$ and this in turn will provide a contradiction. In what follows, we denote $\mathbf{q}^1 := [q^1, \ -1 - q^1, 0]^\top$ and $\mathbf{q}^2 := [q^2, \ 0, \ -1 - q^2]^\top$.

To explain the idea to choose a test function, let $\mathbf{z}, \ \mathbf{z}' \in \mathbb{R}^3$. As always, to push the variables into a diagonal entry, we need $\|\mathbf{z} - \mathbf{z}'\|^2/\alpha$ for some $\alpha > 0$, in the test function. Moreover, since the domain $\mathcal{O}$ is not bounded, for the test function to have a maximum in a compact set, one may want to put $\beta(\|\mathbf{z}\|^2 + \|\mathbf{z}'\|^2)/2$ for some $\beta > 0$. If we stop here, the test function may or may not satisfy $\mathcal{B}^i$, $i \in \{1, 2\}$. To be more precise, for $\mathbf{z} \in \partial\mathcal{O}^0$ or $\mathbf{z}' \in \partial\mathcal{O}^0$, we cannot guarantee that

$$\nabla\Big[\frac{1}{\alpha}\|\mathbf{z} - \mathbf{z}'\|^2 + \frac{\beta}{2}(\|\mathbf{z}\|^2 + \|\mathbf{z}'\|^2)\Big] \cdot \mathbf{q}^i > 0, \quad i \in \{1, 2\}. \tag{4.34}$$

To eliminate the possibility to satisfy $\mathcal{B}^i$, i.e., to focus on $\mathcal{F}$, we seek to remedy the test function to meet (4.34). To this end, pick any $\nu^i > 0$, $i \in \{1, 2\}$, and choose $\mathbf{z}_\nu := (R, \nu^1, \nu^2)$. Then for any $\mathbf{z} = (x, 0, y^2) \in \partial\mathcal{O}^0_1$, we have $(\mathbf{z} - \mathbf{z}_\nu) \cdot \mathbf{q}^1 = (x - R)q^1 + \nu^1(1 + q^1) > 0$. Likewise, we also have $(\mathbf{z} - \mathbf{z}_\nu) \cdot \mathbf{q}^2 > 0$ for any $\mathbf{z} \in \partial\mathcal{O}^0_2$. Therefore, instead of $\beta(\|\mathbf{z}\|^2 + \|\mathbf{z}'\|^2)/2$, we put

$$\chi_\beta(\mathbf{z}, \mathbf{z}') := \frac{\beta}{2}\|\mathbf{z} - \mathbf{z}_\nu\|^2 + \frac{\beta}{2}\|\mathbf{z}' - \mathbf{z}_\nu\|^2.$$

However, the effect of (4.34) is offset by the derivative of $\|\mathbf{z} - \mathbf{z}'\|^2/\alpha$. Thus, to remove the derivative, we add additional terms and define

$$\begin{aligned}
\zeta_\alpha(\mathbf{z}, \mathbf{z}') :=& \frac{\|\mathbf{z} - \mathbf{z}'\|^2}{2\alpha} + \sum_{i \in \{1,2\}} \Big\{ C^i_\alpha(\mathbf{z}, \mathbf{z}')[d^i(\mathbf{z}) - d^i(\mathbf{z}')] + \frac{\|\mathbf{q}^i\|^2}{2\alpha(\mathbf{n}^i \cdot \mathbf{q}^i)^2}[d^i(\mathbf{z}) - d^i(\mathbf{z}')]^2 \Big\}, \\
&+ \frac{q^1 q^2}{2\alpha(1 + q^1)(1 + q^2)}\Big[\sum_{i \in \{1,2\}} \{d^i(\mathbf{z}) - d^i(\mathbf{z}')\}\Big]^2, \\
C^i_\alpha(\mathbf{z}, \mathbf{z}') :=& (\mathbf{z} - \mathbf{z}') \cdot \mathbf{q}^i/(\alpha\mathbf{n}^i \cdot \mathbf{q}^i), \\
d^i(\mathbf{z}) :=& \mathrm{dist}(\mathbf{z}, \partial\mathcal{O}^0_i), \\
\mathbf{n}^1 :=& [0, -1, 0]^\top, \quad \mathbf{n}^2 := [0, 0, -1]^\top.
\end{aligned}$$

Note that $\nabla d^i = -\mathbf{n}^i$, $i \in \{1, 2\}$, $\mathbf{n}^1 \cdot \mathbf{q}^2 = \mathbf{n}^2 \cdot \mathbf{q}^1 = 0$, and

$$\mathbf{q}^i \cdot \mathbf{q}^j = \begin{cases} q^1 q^2, & i \neq j, \\ \|q^i\|^2, & i = j, \end{cases} \tag{4.35}$$

$$\mathbf{n}^i \cdot \mathbf{q}^j = \begin{cases} 0, & i \neq j, \\ 1 + q^i, & i = j, \end{cases} \tag{4.36}$$

$$C^i_\alpha(\mathbf{z}, \mathbf{z}')\mathbf{n}^i \cdot \mathbf{q}^j = \begin{cases} 0, & i \neq j, \\ \alpha^{-1}(\mathbf{z} - \mathbf{z}') \cdot \mathbf{q}^j, & i = j. \end{cases} \tag{4.37}$$

Then we define $\Psi_{\alpha,\beta} \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ as

$$\Psi_{\alpha,\beta}(\mathbf{z}, \mathbf{z}') := u(\mathbf{z}) - v(\mathbf{z}') - \psi_{\alpha,\beta}(\mathbf{z}, \mathbf{z}'), \tag{4.38}$$

$$\psi_{\alpha,\beta}(\mathbf{z}, \mathbf{z}') := \zeta_\alpha(\mathbf{z}, \mathbf{z}') + \chi_\beta(\mathbf{z}, \mathbf{z}'). \tag{4.39}$$

Now, we check some properties of ψ by straightforward calculations. First, we can derive that

$$\begin{aligned}
\nabla_{\mathbf{z}}\psi_{\alpha,\beta}(\mathbf{z},\mathbf{z}') = &\ \alpha^{-1}(\mathbf{z}-\mathbf{z}') + \sum_{i\in\{1,2\}} \Big\{ -C^i_\alpha(\mathbf{z},\mathbf{z}')\mathbf{n}^i + \mathbf{q}^i(\alpha\mathbf{n}^i\cdot\mathbf{q}^i)^{-1}[d^i(\mathbf{z})-d^i(\mathbf{z}')] \\
&- \frac{\|\mathbf{q}^i\|^2}{\alpha(\mathbf{n}^i\cdot\mathbf{q}^i)^2}[d^i(\mathbf{z})-d^i(\mathbf{z}')]\mathbf{n}^i\Big\} + \beta(\mathbf{z}-\mathbf{z}_\nu) \\
&- \frac{q^1q^2}{\alpha(1+q^1)(1+q^2)}\Big[\sum_{i\in\{1,2\}}\{d^i(\mathbf{z})-d^i(\mathbf{z}')\}\Big][\mathbf{n}^1+\mathbf{n}^2], \qquad (4.40)
\end{aligned}$$

$$\begin{aligned}
\nabla_{\mathbf{z}'}\psi_{\alpha,\beta}(\mathbf{z},\mathbf{z}') = &\ \alpha^{-1}(\mathbf{z}'-\mathbf{z}) + \sum_{i\in\{1,2\}} \Big\{ C^i_\alpha(\mathbf{z},\mathbf{z}')\mathbf{n}^i - \mathbf{q}^i(\alpha\mathbf{n}^i\cdot\mathbf{q}^i)^{-1}[d^i(\mathbf{z})-d^i(\mathbf{z}')] \\
&+ \frac{\|\mathbf{q}^i\|^2}{\alpha(\mathbf{n}^i\cdot\mathbf{q}^i)^2}[d^i(\mathbf{z})-d^i(\mathbf{z}')]\mathbf{n}^i\Big\} + \beta(\mathbf{z}'-\mathbf{z}_\nu) \\
&+ \frac{q^1q^2}{\alpha(1+q^1)(1+q^2)}\Big[\sum_{i\in\{1,2\}}\{d^i(\mathbf{z})-d^i(\mathbf{z}')\}\Big][\mathbf{n}^1+\mathbf{n}^2]. \qquad (4.41)
\end{aligned}$$

Moreover, we can observe that

$$\begin{aligned}
\nabla_{\mathbf{z}}\zeta_\alpha(\mathbf{z},\mathbf{z}') &= -\nabla_{\mathbf{z}'}\zeta_\alpha(\mathbf{z},\mathbf{z}'), \\
\nabla_{\mathbf{z}}\psi_{\alpha,\beta}(\mathbf{z},\mathbf{z}') &= -\nabla_{\mathbf{z}'}\psi_{\alpha,\beta}(\mathbf{z},\mathbf{z}') + \beta(\mathbf{z}-\mathbf{z}_\nu) + \beta(\mathbf{z}'-\mathbf{z}_\nu).
\end{aligned}$$

Hence, recalling (4.35)-(4.37), for any $\mathbf{z}\in\mathcal{O}$, $i\neq j$, $\mathbf{z}_j\in\partial\mathcal{O}^0_j$, we have

$$\nabla_{\mathbf{z}}\psi_{\alpha,\beta}(\mathbf{z}_j,\mathbf{z})\cdot\mathbf{q}^j = \beta(\mathbf{z}_j-\mathbf{z}_\nu)\cdot\mathbf{q}^j + \frac{q^1q^2}{\alpha(1+q^i)}d^j(\mathbf{z}) > 0, \qquad (4.42)$$

$$\nabla_{\mathbf{z}'}(-\psi_{\alpha,\beta})(\mathbf{z}_j,\mathbf{z})\cdot\mathbf{q}^j = -\beta(\mathbf{z}_j-\mathbf{z}_\nu)\cdot\mathbf{q}^j - \frac{q^1q^2}{\alpha(1+q^i)}d^j(\mathbf{z}) < 0. \qquad (4.43)$$

(4.42)-(4.43) will be used later in the proof of Proposition 3.5. In addition, from (4.40) - (4.41) , the second order derivative of ψ is obtained. Let

$$A := \mathbf{I}_3 + \sum_{i\in\{1,2,\}}\Big\{\frac{\|\mathbf{q}^i\|^2\mathbf{n}^i(\mathbf{n}^i)^\top}{(\mathbf{n}^i\cdot\mathbf{q}^i)^2} - \frac{\mathbf{n}^i(\mathbf{q}^i)^\top+\mathbf{q}^i(\mathbf{n}^i)^\top}{(\mathbf{n}^i\cdot\mathbf{q}^i)}\Big\} + \frac{q^1q^2}{(1+q^1)(1+q^2)}[\mathbf{n}^1+\mathbf{n}^2][\mathbf{n}^1+\mathbf{n}^2]^\top,$$

where $\mathbf{I}_3$ is the 3×3-identity matrix. If q^i, $i\in\{1,2\}$, are not too big, we clearly have $A\succeq 0$. Then we can write

$$\nabla^2\psi_{\alpha,\beta}(\mathbf{z},\mathbf{z}') = \frac{1}{\alpha}\begin{bmatrix} A & -A \\ -A & A\end{bmatrix} + \beta\begin{bmatrix}\mathbf{I}_3 & 0 \\ 0 & \mathbf{I}_3\end{bmatrix}.$$

We are ready to prove the comparison principle.

Proof of Proposition 3.5. We argue by contradiction. To this end, we suppose that for some $\mathbf{z}_e\in\mathrm{cl}(\mathcal{O})$, $u(\mathbf{z}_e)-v(\mathbf{z}_e)=\delta>0$. Let us choose β small enough such that $\delta>\chi_\beta(\mathbf{z}_e,\mathbf{z}_e)$, and choose $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n\downarrow 0$. Denote $\Psi_n := \Psi_{\alpha_n,\beta}$. As u and v are bounded, χ_β dominates $u-v$ outside a compact set. Therefore, for each $n\in\mathbb{N}$, Ψ_n has its maximum on $\mathrm{cl}(\mathcal{O})\times\mathrm{cl}(\mathcal{O})$ in a compact set and we denote the maximal point by $(\mathbf{z}_n,\mathbf{z}'_n)$, i.e.,

$$\Psi_n(\mathbf{z}_n,\mathbf{z}'_n) = \sup_{(\mathbf{z},\mathbf{z}')\in\mathrm{cl}(\mathcal{O})\times\mathrm{cl}(\mathcal{O})}\Psi_n(\mathbf{z},\mathbf{z}').$$

The maximal point $(\mathbf{z}_n, \mathbf{z}'_n)$ actually depends on β but we drop it for simplicity. As $\{(\mathbf{z}_n, \mathbf{z}'_n)\}_{n\geq 1}$ lie in a compact set, we choose a convergent subsequence, still denoted by $(\mathbf{z}_n, \mathbf{z}'_n)$, such that

$$(\mathbf{z}_n, \mathbf{z}'_n) \to (\overline{\mathbf{z}}, \overline{\mathbf{z}}') = (\overline{x}, \overline{y}, \overline{x}', \overline{y}').$$

As $u \leq v$ on $\partial\mathcal{O}_R \cup \partial\mathcal{O}_{c/r}$ by the definition of viscosity sub/super solution, $(\overline{\mathbf{z}}, \overline{\mathbf{z}}')$ must be in $\mathcal{O} \times \mathcal{O}$. The previous assumption yields that

$$\Psi_n(\mathbf{z}_n, \mathbf{z}'_n) \geq \sup_{\mathbf{z}\in \mathrm{cl}(\mathcal{O})} [u(\mathbf{z}) - v(\mathbf{z}) - \chi_\beta(\mathbf{z}, \mathbf{z})] \geq \delta - \chi_\beta(\mathbf{z}_e, \mathbf{z}_e) > 0.$$

Therefore, it follows that

$$\zeta_{\alpha_n}(\mathbf{z}_n, \mathbf{z}'_n) \leq u(\mathbf{z}_n) - v(\mathbf{z}'_n) - \chi_\beta(\mathbf{z}_n, \mathbf{z}'_n) - \sup_{\mathbf{z}\in \mathrm{cl}(\mathcal{O})} [u(\mathbf{z}) - v(\mathbf{z}) - \chi_\beta(\mathbf{z}, \mathbf{z})].$$

In view that the right hand side is bounded above but $\alpha_n \to 0$ as $n \to \infty$, $(\overline{x}, \overline{y}) = (\overline{x}', \overline{y}')$. Moreover, the fact that $u - v$ is u.s.c implies that

$$0 \leq \limsup_{n\to\infty} \zeta_{\alpha_n}(\mathbf{z}_n, \mathbf{z}'_n) \leq u(\overline{\mathbf{z}}) - v(\overline{\mathbf{z}}') - \chi_\beta(\overline{\mathbf{z}}, \overline{\mathbf{z}}') - \sup_{\mathbf{z}\in \mathrm{cl}(\mathcal{O})} [u(\mathbf{z}) - v(\mathbf{z}) - \chi_\beta(\mathbf{z}, \mathbf{z})] \leq 0.$$

Hence, $\lim_{n\to\infty} \zeta_{\alpha_n}(\mathbf{z}_n, \mathbf{z}'_n) = 0$.

By Crandall-Ishii's lemma, for large $n \in \mathbb{N}$, there exist $\mathbf{A}_n, \mathbf{B}_n \in \mathcal{S}^3$ such that

$$(\nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n) \in \overline{\mathcal{J}}^{2,+}_{\mathcal{O}} u(\mathbf{z}_n), \qquad (-\nabla_{\mathbf{z}'}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{B}_n) \in \overline{\mathcal{J}}^{2,-}_{\mathcal{O}} v(\mathbf{z}'_n)$$

and that

$$-\frac{10}{\alpha_n}\begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & \mathbf{I}_3 \end{bmatrix} \prec \begin{bmatrix} \mathbf{A}_n & 0 \\ 0 & -\mathbf{B}_n \end{bmatrix} \prec \frac{10}{\alpha_n}\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} + 2\beta \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & \mathbf{I}_3 \end{bmatrix}. \tag{4.44}$$

We can calculate that

$$\begin{aligned} \nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n) &= \nabla_{\mathbf{z}}\zeta_{\alpha_n}(\mathbf{z}_n, \mathbf{z}'_n) + \beta(\mathbf{z}_n - \mathbf{z}_\nu) \\ -\nabla_{\mathbf{z}'}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n) &= \nabla_{\mathbf{z}}\zeta_{\alpha_n}(\mathbf{z}_n, \mathbf{z}'_n) - \beta(\mathbf{z}'_n - \mathbf{z}_\nu). \end{aligned}$$

Let F be the function such that $\mathcal{F}[\varphi](\mathbf{z}) = F(\mathbf{z}, \varphi(\mathbf{z}), \nabla\varphi(\mathbf{z}), \nabla^2\varphi(\mathbf{z}))$. Then we have

$$\begin{aligned} \lambda^D(u(\mathbf{z}_n) - v(\mathbf{z}'_n)) &= F(\mathbf{z}_n, u(\mathbf{z}_n), \nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n) - F(\mathbf{z}_n, v(\mathbf{z}'_n), \nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n) \\ &\leq F(\mathbf{z}'_n, v(\mathbf{z}_n), \nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{B}_n) - F(\mathbf{z}_n, v(\mathbf{z}'_n), \nabla_{\mathbf{z}}\psi_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n) \\ &\leq F\big(\mathbf{z}'_n, v(\mathbf{z}_n), \nabla_{\mathbf{z}}\zeta_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{B}_n + 2\beta\mathbf{I}_3\big) \\ &\qquad - F\big(\mathbf{z}_n, v(\mathbf{z}'_n), \nabla_{\mathbf{z}}\zeta_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n - 2\beta\mathbf{I}_3\big) + c(\beta), \end{aligned} \tag{4.45}$$

where $c(\beta)$ is the modulus of continuity of F. The last inequality of (4.45) is obtained by the compactness of $\mathcal{K}$. By (4.44), we moreover, have $\mathbf{A}_n - 2\beta\mathbf{I}_3 \prec \mathbf{B}_n + 2\beta\mathbf{I}_3$. Therefore, we obtain

$$\begin{aligned} F\big(\mathbf{z}'_n, v(\mathbf{z}_n), &\nabla_{\mathbf{z}}\zeta_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{B}_n - 2\beta\mathbf{I}_3\big) \\ &\leq F\big(\mathbf{z}_n, v(\mathbf{z}'_n), \nabla_{\mathbf{z}}\zeta_{\alpha_n,\beta}(\mathbf{z}_n, \mathbf{z}'_n), \mathbf{A}_n + 2\beta\mathbf{I}_3\big). \end{aligned} \tag{4.46}$$

By (4.45) and (4.46), taking $n \uparrow \infty$ leads to $\lambda^D\delta \leq c(\beta)$. Again taking $\beta \downarrow 0$, we have the desired contradiction, which completes the proof. □

4.4. Proof of Theorem 3.4. Finally, we are ready to prove our main result of Theorem 3.4.

Proof of Theorem 3.4. Theorem 4.7, Theorem 4.11, together with Proposition 3.5 imply that $v^+ \leq v^-$. Therefore, it suffices to show $v^- \leq V \leq v^+$. To show the first inequality, let us consider an arbitrary $\phi \in \mathcal{V}^-$. It is obvious that $\phi \leq V$ on $\partial\mathcal{O}_R \cup \partial\mathcal{O}_{c/r}$. Let $(x, y) \in \mathcal{O}$ and take the *random initial condition* as $\tau = 0$ and $\xi = (x, y)$. We fix some $\pi \in \mathscr{R}$ and the hitting time defined by

$$\tau_{c/r}^{\tau,\xi,\pi} := \inf\{t \geq 0\colon X_t^{\tau,\xi,\pi} \geq c/r\}.$$

As there exists θ-generated measure $\mathbb{Q}$ for ϕ to satisfy (SB2), it follows that

$$\begin{aligned}\phi(x, y) &\leq \mathbb{E}^{\mathbb{Q}}\left[\phi(Z^{\tau,\xi,\pi}_{\tau_E^{\tau,\xi,\pi}}) - \frac{1}{2a}\int_\tau^{\tau_E^{\tau,\xi,\pi}} e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \;\middle|\; \mathcal{G}_\tau\right] \\ &\leq \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau_E^{\tau,\xi,\pi} = \tau_R^{x,y,\pi}} - \frac{1}{2a}\int_\tau^{\tau_E^{\tau,\xi,\pi}} e^{-\lambda^D s}\|\theta_s\|^2\,\mathrm{d}s \;\middle|\; \mathcal{G}_\tau\right]. \end{aligned} \tag{4.47}$$

Moreover, we have

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\tau_E^{\tau,\xi,\pi} = \tau_R^{x,y,\pi}}] = \mathbb{Q}[\tau_R^{\tau,\xi,\pi} < \tau_D \wedge \tau_{c/r}^{x,y,\pi}] \leq \mathbb{Q}[\tau_R^{x,y,\pi} < \tau_D]. \tag{4.48}$$

By combining (4.47) and (4.48), we have $\phi(x, y) \leq V(x, y)$, together with (4.10) yield $v^- \leq V$. In a similar fashion, we can show $V \leq v^+$ as well. Because v^- is a viscosity super-solution, by Proposition 3.5, we have $v^+ \leq v^-$. It follows that $v^- \leq V \leq v^+ \leq v^-$, which readily implies our desired equality $v^- = V = v^+$ and hence the value function is the unique viscosity solution of the HJB equation (3.4). ∎

Acknowledgments. The first and third authors acknowledge the support from the Singapore MOE AcRF grants R-146-000-271-112 and R-146-000-255-114 as well as the French Ministry of Foreign Affairs and the Merlion programme. The second author is partially supported by the Hong Kong Early Career Scheme under grant no. 25302116 and the Hong Kong Polytechnic University central research grant under no.15304317. In addition, the first author received the financial support from the Singapore MOE AcRF grant R-146-000-243-114 and the third author received the financial support from the NSFC Grant 11871364.

REFERENCES

[1] V. Agarwal, N. Daniel, and N. Y. Naik. Role of managerial incentives and discretion in hedge fund performance. *Journal of Finance*, 64(5):2221–2256, 2009.

[2] G. Barles. Nonlinear neumann boundary conditions for quasilinear degenerate elliptic equations and applications. *Journal of Differential Equations*, 154(1):191–224, 1999.

[3] G. Barles, C. Georgelin, and E. R. Jakobsen. On neumann and oblique derivatives boundary conditions for nonlocal elliptic equations. *Journal of Differential Equations*, 256(4):1368–1394, 2014.

[4] E. Bayraktar, A. Cosso, and H. Pham. Robust feedback switching control: Dynamic programming and viscosity solutions. *SIAM Journal on Control and Optimization*, 54(5):2594–2628, 2016.

[5] E. Bayraktar and J. Li. Stochastic perron for stochastic target problems. *Journal of Optimization Theory and Applications*, 170(3):1026–1054, 2016.

[6] E. Bayraktar and J. Li. On the controller-stopper problems with controlled jumps. *Applied Mathematics & Optimization*, pages 1–28, 2017.

[7] E. Bayraktar and M. Sîrbu. Stochastic perron's method and verification without smoothness using viscosity comparison: the linear case. *Proceedings of the American Mathematical Society*, 140(10):3645–3654, 2012.

[8] E. Bayraktar and M. Sîrbu. Stochastic perron's method for hamilton–jacobi–bellman equations. *SIAM Journal on Control and Optimization*, 51(6):4274–4294, 2013.

[9] E. Bayraktar and M. Sîrbu. Stochastic perron's method and verification without smoothness using viscosity comparison: obstacle problems and dynkin games. *Proceedings of the American Mathematical Society*, 142(4):1399–1412, 2014.

[10] E. Bayraktar and V. Young. Minimizing the probability of lifetime ruin under borrowing constraints. *Insurance: Mathematics and Economics*, 41(1):196–221, 2007.
[11] E. Bayraktar and Y. Zhang. Minimizing the probability of lifetime ruin under ambiguity aversion. *SIAM Journal on Control and Optimization*, 53(1):58–90, 2015.
[12] E. Bayraktar and Y. Zhang. Stochastic perron's method for the probability of lifetime ruin problem under transaction costs. *SIAM Journal on Control and Optimization*, 53(1):91–113, 2015.
[13] B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM Journal on Control and Optimization*, 49(3):948–962, 2011.
[14] H. N. Chau and M. Rásonyi. Robust utility maximization in markets with transaction costs. *Finance and Stochastics*, 23(3):677–696, 2019.
[15] Y. Chow, X. Yu, and C. Zhou. On dynamic programming principle for stochastic control under expectation constraints. *Journal of Optimization Theory and Applications*, 185(3):803–818, 2020.
[16] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67, 1992.
[17] J. Cvitanic and I. Karatzas. On portfolio optimization under drawdown constraints. *IMA Volumes in Mathematics and its Applications*, 65:35–46, 1995.
[18] S. Deng, X. Tan, and X. Yu. Utility maximization with proportional transaction costs under model uncertainty. *Mathematics of Operations Research*, 45(4):1210–1236, 2020.
[19] R. Elie and N. Touzi. Optimal lifetime consumption and investment under a drawdown constraint. *Finance and Stochastics*, 12(3):299–330, 2008.
[20] R. J. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. *Mathematical Finance*, 10(2):179–195, 2000.
[21] W. N. Goetzmann, J. E. Ingersoll, and S. A. Ross. High-water marks and hedge fund management contracts. *Journal of Finance*, 58(4):1685–1718, 2003.
[22] S. J. Grossman and Z. Zhou. Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3(3):241–276, 1993.
[23] P. Guasoni and J. Obłój. The incentives of hedge fund fees and high-water marks. *Mathematical Finance*, 26(2):269–295, 2016.
[24] P. Guasoni and G. Wang. Hedge and mutual funds' fees and the separation of private investments. *Finance and Stochastics*, 19(3):473–507, 2015.
[25] Y. Huang and X. Yu. Optimal stopping under model ambiguity: a time-consistent equilibrium approach. *Preprint, arxiv:1906.01232*, 2019.
[26] K. Janeček and M. Sîrbu. Optimal investment with high-watermark performance fee. *SIAM Journal on Control and Optimization*, 50(2):790–819, 2012.
[27] K. Janeček, Z. Li, and M. Sîrbu. Optimal investment with high-watermark fee in a multi-dimensional jump diffusion model. *SIAM Journal on Financial Mathematics*, 11(3):750–787, 2020.
[28] A. Neufeld and M. Sikic. Robust utility maximization in discrete-time markets with friction. *SIAM Journal on Control and Optimization*, 56(3):1912–1937, 2018.
[29] S. Panageas and M. M. Westerfield. High-water marks: High risk appetites? convex compensation, long horizon, and portfolio choice. *Journal of Finance*, 64(1):1–36, 2009.
[30] D. B. Rokhlin. Verification by stochastic perron's method in stochastic exit time control problems. *Journal of Mathematical Analysis and Applications*, 419(1):433–446, 2014.
[31] M. Sîrbu. A note on the strong formulation of stochastic control problems with model uncertainty. *Electronic Communications in Probability*, 19(81):1–10, 2014.
[32] M. Sîrbu. Stochastic perron's method and elementary strategies for zero-sum differential games. *SIAM Journal on Control and Optimization*, 52(3):1693–1711, 2014.
[33] J. Sirignano and K. Spiliopoulos. Dgm: A deep learning algorithm for solving partial differential equations. *Journal of computational physics*, 375:1339–1364, 2018.
[34] Y. Yang and X. Yu. Optimal entry and consumption under habit formation. *Preprint, arXiv: 1903.04257*, 2019.
[35] V. Young. Optimal investment strategy to minimize the probability of lifetime ruin. *North American Actuarial Journal*, 8(4):106–126, 2004.
[36] V. R. Young and Y. Zhang. Lifetime ruin under ambiguous hazard rate. *Insurance: Mathematics and Economics*, 70:125–134, 2016.