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Better Than Optimal Mean-Variance Portfolio Policy in Multi-period Asset-Liability Management Problem

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Abstract

When the wealth is larger than some threshold in multi-period mean-variance asset-liability management, the pre-committed policy is no longer mean-variance efficient policy for the remaining investment horizon. To revise the pre-committed policy, by relaxing self-financing constraint and allowing to withdraw some wealth, we derive a new dominating policy, which is better than the corresponding pre-committed policy. The revised policy can achieve the same mean-variance pair attained by the pre-committed policy, and yields a nonnegative free cash flow stream over the investment horizon.

Key words: mean-variance model; free cash flow stream; wealth threshold; asset-liability management.

1. Introduction

Markowitz (1952) introduced mean-variance (MV) model around 70 years ago. It is a mathematical framework to find a portfolio of assets such that the expected return is maximized for a given level of risk. In mean-variance framework, one formulates the portfolio model into a bi-objective optimization problem according to the expected return and the variance, which represent the gain and risk, respectively. In order to trace out the efficient frontier for this bi-objective optimization problem, one typically puts weights on the two criteria, turning it into a single-objective optimization problem.

After Markowitz's seminal work for single-period, the dynamic version of MV model did not make any progress for decades, because there is an inherent nonseparable structure of the variance term in the sense of dynamic programming. In 1989, Richardson (1989) studied the optimal MV policy in a continuous-time setting, followed by Bajeuxbesnainou and Portrait (1998). Li and Ng (2000) and Zhou and Li (2000) adopted an embedding technique to derive the optimal MV policy in multi-period setting and continuous-time setting, respectively. Since then, the past twenty years have witnessed tremendous interests and research efforts in dealing with dynamic MV portfolio selection problems as well as their applications, such as Li et al. (2002); Zhou and Yin (2003); Hu and Zhou (2005); Bielecki et al. (2005); Chiu and Li (2006) in continuous-time settings, Leippold et al. (2004); Zhu et al. (2004); Liang et al. (2008); Cui et al. (2014); Ni et al. (2019) in multi-period settings, Černỳ and Kallsen (2007, 2009) in semi-martingale framework.

Meanwhile, in investment practices, financial institutions usually use liabilities to enhance their expected wealth. Therefore, the liability plays an important role in determining the investment portfolio. Sharpe and Tint (1990) first introduced the mean-variance asset-liability management problem in single-period setting. Leippold et al. (2004) extended it to multi-period setting and derived the closed-form optimal policies under the exogenous and endogenous liabilities. Li et al. (2017) employed the parameterized method to tackle the multi-period mean-variance asset-liability management problem, and derived the analytical optimal policies and the corresponding efficient frontier.

However, the global optimal MV policy, which is determined in the beginning of the investment and termed as *pre-committed policy*, is not always mean-variance efficient during the investment. In multi-period mean-variance framework, Zhu et al. (2004) and Cui et al. (2012) showed that if the wealth is larger than a threshold, the pre-committed policy is no longer mean-variance efficient policy for the remaining investment horizon. To avoid such irrational behavior, Cui et al. (2012) proposed a revised dominating policy by relaxing the self-financing constraint and allowing to withdraw some wealth. They proved that the dominating policy can achieve the same mean-variance pair attained by the pre-committed policy and obtain some positive free cash flows over the investment time horizon.

In this paper, we mainly further develop the result in Cui et al. (2012), which originally deals with for a multi-period mean-variance portfolio selection problem *without* liability, and derive the dominating policy for the multi-period asset-liability management problem. The organization of the rest of the paper is as follows. In Section 2, we present the pre-committed mean-variance policy for the multi-period asset-liability management problem. In Section 3, we provide the revised dominating policy and prove its properties. In Section 4, we consider a simple numerical example to illustrate our theoretical results. We conclude our paper in Section 5.

2. Pre-committed mean-variance policy for multi-period asset-liability management problem

Consider a capital market consists of one risk-free asset, n risky assets and one liability. The investment horizon is T (> 0). The variable r_t (> 1) is the given deterministic return of risk-free asset, the vector $\mathbf{e}_t = (e_t^1, \cdots, e_t^n)'$ is the random total returns of the n risky assets and q_t is the random changing rate of the liability at the time period t, where M' denotes the transpose of a vector or a matrix M. All the variables are defined on the probability space $(\Omega, \mathcal{F}, P)$. In addition, $(\mathbf{e}_t, q_t)$ is supposed to be statistically independent at different time periods. An investor with an initial wealth x_0 and initial liability l_0 enters the capital market at time 0. He/she can also adjust the portfolio at the beginning of each following $T-1$ consecutive periods. The global optimal pre-committed policy is denoted as $\boldsymbol{\pi}_t^* = ((\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*)'$, $t = 0, 1, \cdots, T-1$, which is the solution of the following dynamic stochastic optimization problem,

$$
\begin{cases}
\min & \mathrm{Var}(x_T - l_T) \equiv \mathbb{E}[(x_T - l_T - d)^2], \\
\text{s.t.} & \mathbb{E}[x_T - l_T] = d, \\
& x_{t+1} = r_t\left(x_t - \sum\limits_{i=1}^{n} \pi_t^i\right) + \sum\limits_{i=1}^{n} e_t^i \pi_t^i \\
& \qquad = r_t x_t + \mathbf{P}_t' \boldsymbol{\pi}_t, \\
& l_{t+1} = q_t l_t, \qquad\qquad t = 0, 1, \cdots, T-1,
\end{cases}
\tag{1}
$$

where $\mathbf{P}_t = (e_t^1 - r_t, \cdots, e_t^n - r_t)'$ denotes the excess returns of the n risky assets.

According to Li et al. (2017), the pre-committed policy is given by

$$
\boldsymbol{\pi}_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] r_t \left(x_t - \gamma^* \prod_{k=t}^{T-1} r_k^{-1} \right) + \left(\prod_{k=t+1}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1 - B_k) r_k} \right) \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t \mathbf{P}_t] l_t,
\tag{2}
$$

where the parameters are defined as follows,

$$
\begin{aligned}
B_t &\triangleq \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \\
\widehat{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \\
\widetilde{B}_t &\triangleq \mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t], \\
\gamma^* &\triangleq \frac{x_0 \prod\limits_{k=0}^{T-1} (1 - B_k) r_k - d - l_0 \prod\limits_{k=0}^{T-1} \left(\mathbb{E}[q_k] - \widehat{B}_k \right)}{\prod\limits_{k=0}^{T-1} (1 - B_k) - 1}.
\end{aligned}
$$

Furthermore, the mean-variance efficient frontier is given by

$$
\mathrm{Var}(x_T - l_T) = \frac{\prod\limits_{k=0}^{T-1} (1 - B_k)}{1 - \prod\limits_{k=0}^{T-1} (1 - B_k)} \left(d - x_0 \prod_{k=0}^{T-1} r_k + l_0 \prod_{k=0}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{1 - B_k} \right)^2 + l_0^2 C_0,
\tag{3}
$$

where

$$
C_0 = -\prod_{k=0}^{T-1} \frac{\left(\mathbb{E}[q_k] - \widehat{B}_k\right)^2}{1 - B_k} - \sum_{j=0}^{T-1} \left(\prod_{k=j+1}^{T-1} \frac{\left(\mathbb{E}[q_k] - \widehat{B}_k\right)^2}{1 - B_k} \right) \widetilde{B}_j \left(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \right) + \prod_{k=0}^{T-1} \mathbb{E}[q_k^2] \geq 0.
$$

Setting the expected terminal surplus $d = x_0 \prod\limits_{k=0}^{T-1} r_k - l_0 \prod\limits_{k=0}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{1 - B_k}$, we can obtain the global minimum variance as

$$
\mathrm{Var}_{\min}(x_T - l_T) := C_0 {l_0}^2.
$$

3. A dominating policy for multi-period asset-liability management problem

Under multi-period asset-liability management framework, we also show that if the wealth is larger than some threshold, the pre-committed policy is no longer mean-variance efficient policy for the remaining investment horizon. Then, we follow Cui et al. (2012), and propose a revised dominating policy by relaxing the self-financing constraint and allowing to withdraw some wealth.[1]

We first construct the dominating policy for simple two-period case in Section 3.1 and obtain the dominating policy for general T-period case in Section 3.2.

3.1. Two-period case

We first consider the two-period case. According to expression (2), the pre-committed policy at time 1 is given as follows,

$$\boldsymbol{\pi}_1^*(x_1) = -\mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[\mathbf{P}_1](r_1x_1 - \gamma^*) + \mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[q_1\mathbf{P}_1]l_1,$$

which is a linear function of the current wealth x_1. Before constructing the dominating policy, we need to know whether or not the pre-committed policy at time 1 is a good policy for the remaining one period.

Let us consider the one period mean-variance asset-liability management problem at time 1,

$$\begin{cases} \min & \mathrm{Var}_1(x_2 - l_2), \\ \text{s.t.} & \mathbb{E}_1[x_2 - l_2] = \bar{d}, \\ & x_2 = r_1x_1 + \mathbf{P}_1'\boldsymbol{\pi}_1, \\ & l_2 = q_1l_1. \end{cases}$$

According to expression (2) and expression (3), the one-period efficient policy and one-period efficient frontier are given by

$$\begin{aligned} \bar{\boldsymbol{\pi}}_1(x_1) &= -\mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[\mathbf{P}_1](r_1x_1 - \bar{\gamma}) + \mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[q_1\mathbf{P}_1]l_1, \\ \mathrm{Var}_1(x_2 - l_2) &= \frac{1 - B_1}{B_1}\left(\bar{d} - x_1r_1 + l_1\frac{\mathbb{E}[q_1] - \widehat{B}_1}{1 - B_1}\right)^2 + l_1^2C_1, \end{aligned}$$

where C_1 does not depend on x_1, l_1 and $\bar{\gamma}$,

$$\bar{\gamma} = \frac{x_1(1 - B_1)r_1 - \bar{d} - l_1(\mathbb{E}[q_1] - \widehat{B}_1)}{B_1}.$$

By matching $\boldsymbol{\pi}_1^*(x_1) = \bar{\boldsymbol{\pi}}_1(x_1)$, we have

$$\bar{d} = x_1(1 - B_1)r_1 + \gamma^*B_1 - l_1(\mathbb{E}[q_1] - \widehat{B}_1).$$

[1] Such investment environment is called *semi-self-financing* environment. Dang and Forsyth (2016) considered the portfolio selection problem in semi-self-financing environment.

Thus, when $\bar{d} < x_1 r_1 - l_1 \frac{\mathbb{E}[q_1]-\widehat{B}_1}{1-B_1}$, i.e.,

$$x_1 > \frac{\gamma^*}{r_1} + l_1 \frac{\mathbb{E}[q_1] - \widehat{B}_1}{r_1(1-B_1)}, \tag{4}$$

the pre-committed policy $\boldsymbol{\pi}_1^*(x_1)$ is no longer a one-period mean-variance efficient policy.

We propose the following revised portfolio policy, $\widehat{\boldsymbol{\pi}}_0^*(\widehat{x}_0)$, $\widehat{\boldsymbol{\pi}}_1^*(\widehat{x}_1)$,

- At time 0, $\widehat{\boldsymbol{\pi}}_0^*(\widehat{x}_0) = \boldsymbol{\pi}_0^*(x_0)$;
- At time 1,

$$\widehat{\boldsymbol{\pi}}_1^*(\widehat{x}_1) = -\mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[\mathbf{P}_1](r_1\widehat{x}_1 - \widehat{\gamma}_1) + \mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[q_1\mathbf{P}_1]l_1,$$

where

$$\widehat{x}_1 = \begin{cases} \bar{x}_1, & \text{if } \bar{x}_1 \le \bar{x}_1^*, \\ \bar{x}_1 - 2(\bar{x}_1 - \bar{x}_1^*)B_1, & \text{if } \bar{x}_1 > \bar{x}_1^*, \end{cases} \tag{5}$$

$$\bar{x}_1 = r_0\widehat{x}_0 + \mathbf{P}_0'\widehat{\boldsymbol{\pi}}_0^*,$$

$$\widehat{\gamma}_1 = \begin{cases} \widehat{\gamma}_0, & \text{if } \bar{x}_1 \le \bar{x}_1^*, \\ \widehat{\gamma}_0 + 2r_1(\bar{x}_1 - \bar{x}_1^*)(1-B_1), & \text{if } \bar{x}_1 > \bar{x}_1^*, \end{cases} \tag{6}$$

with $\widehat{\gamma}_0 = \gamma^*$, $\widehat{x}_0 = x_0$ and

$$\bar{x}_1^* = \frac{\gamma^*}{r_1} + l_1 \frac{\mathbb{E}[q_1] - \widehat{B}_1}{r_1(1-B_1)}. \tag{7}$$

Obviously, our new policy has the opportunity to receive a free cash flow stream (FCFS) at time 1. Furthermore, in the following theorem, we prove that the proposed revised policy can achieve the same mean-variance pair as one determined by the original pre-committed policy.

Theorem 1. *The proposed revised policy can achieve the same mean-variance pair as one determined by the original pre-committed policy, i.e.,*

$$\mathbb{E}[x_2 - l_2|x_0, l_0]|_{\widehat{\boldsymbol{\pi}}^*} = \mathbb{E}[x_2 - l_2|x_0, l_0]|_{\boldsymbol{\pi}^*},$$
$$\mathrm{Var}(x_2 - l_2|x_0, l_0)|_{\widehat{\boldsymbol{\pi}}^*} = \mathrm{Var}(x_2 - l_2|x_0, l_0)|_{\boldsymbol{\pi}^*}.$$

Proof. The pre-committed policy at time 1 is given by

$$\boldsymbol{\pi}_1^*(x_1) = -\mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[\mathbf{P}_1](r_1 x_1 - \gamma^*) + \mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[q_1\mathbf{P}_1]l_1,$$

which yields the following conditional mean-variance pair,

$$\mathbb{E}[x_2 - l_2|x_1, l_1]|_{\boldsymbol{\pi}^*} = r_1x_1 - B_1(r_1x_1 - \gamma^*) + (\widehat{B}_1 - \mathbb{E}[q_1])l_1,$$
$$\mathrm{Var}(x_2 - l_2|x_1, l_1)|_{\boldsymbol{\pi}^*} = (B_1 - B_1^2)(r_1x_1 - \gamma^*)^2 + 2B_1l_1(\widehat{B}_1 - \mathbb{E}[q_1])(r_1x_1 - \gamma^*) + C_1l_1^2,$$

where C_1 does not depend on x_1, l_1 and γ^*.

Let us consider another policy at time 1,

$$\widehat{\boldsymbol{\pi}}_1^*(\widehat{x}_1) = -\mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[\mathbf{P}_1](r_1\widehat{x}_1 - \widehat{\gamma}_1) + \mathbb{E}^{-1}[\mathbf{P}_1\mathbf{P}_1']\mathbb{E}[q_1\mathbf{P}_1]l_1,$$

which implies the following conditional mean-variance pair,

$$\begin{aligned}
&\mathbb{E}[x_2 - l_2|\widehat{x}_1, l_1]|_{\widehat{\boldsymbol{\pi}}^*} = r_1\widehat{x}_1 - B_1(r_1\widehat{x}_1 - \widehat{\gamma}_1) + (\widehat{B}_1 - \mathbb{E}[q_1])l_1,\\
&\mathrm{Var}(x_2 - l_2|\widehat{x}_1, l_1)|_{\widehat{\boldsymbol{\pi}}^*} = (B_1 - B_1^2)(r_1\widehat{x}_1 - \widehat{\gamma}_1)^2 + 2B_1l_1(\widehat{B}_1 - \mathbb{E}[q_1])(r_1\widehat{x}_1 - \widehat{\gamma}_1) + C_1l_1^2,
\end{aligned}$$

where C_1 does not depend on $\widehat{x}_1$, l_1 and $\widehat{\gamma}_1$.

We can choose suitable $\widehat{x}_1$ and $\widehat{\gamma}_1$ such that

$$\begin{cases}
\mathbb{E}[x_2 - l_2|x_1, l_1]|_{\boldsymbol{\pi}^*} = \mathbb{E}[x_2 - l_2|\widehat{x}_1, l_1]|_{\widehat{\boldsymbol{\pi}}^*},\\
\mathrm{Var}(x_2 - l_2|x_1, l_1)|_{\boldsymbol{\pi}^*} = \mathrm{Var}(x_2 - l_2|\widehat{x}_1, l_1)|_{\widehat{\boldsymbol{\pi}}^*},
\end{cases}$$

which leads to the desired expressions in (5) and (6).

We further demonstrate that $\mathbb{E}[x_2 - l_2|x_0, l_0]$ and $\mathrm{Var}(x_2 - l_2|x_0, l_0)$ are the same under both policies. First of all, we have

$$\begin{aligned}
\mathbb{E}[x_2 - l_2|x_0, l_0]|_{\boldsymbol{\pi}^*} &= \mathbb{E}\big[\mathbb{E}[x_2 - l_2|x_1, l_1]|_{\boldsymbol{\pi}_1^*}\big|x_0, l_0\big]\big|_{\boldsymbol{\pi}_0^*}\\
&= \mathbb{E}\big[\mathbb{E}[x_2 - l_2|\widehat{x}, l_1]|_{\widehat{\boldsymbol{\pi}}_1^*}\big|x_0, l_0\big]\big|_{\widehat{\boldsymbol{\pi}}_0^*}\\
&= \mathbb{E}[x_2 - l_2|x_0, l_0]|_{\widehat{\boldsymbol{\pi}}^*},
\end{aligned}$$

and then we get

$$\begin{aligned}
&\mathrm{Var}(x_2 - l_2|x_0, l_0)|_{\boldsymbol{\pi}^*}\\
=\ &\mathbb{E}[\mathrm{Var}(x_2 - l_2|x_1, l_1)|_{\boldsymbol{\pi}_1^*}|x_0, l_0]|_{\boldsymbol{\pi}_0^*} + \mathrm{Var}(\mathbb{E}[x_2 - l_2|x_1, l_1]|_{\boldsymbol{\pi}_1^*}|x_0, l_0)|_{\boldsymbol{\pi}_0^*}\\
=\ &\mathbb{E}[\mathrm{Var}(x_2 - l_2|x_1, l_1)|_{\widehat{\boldsymbol{\pi}}_1^*}|x_0, l_0]|_{\widehat{\boldsymbol{\pi}}_0^*} + \mathrm{Var}(\mathbb{E}[x_2 - l_2|x_1, l_1]|_{\widehat{\boldsymbol{\pi}}_1^*}|x_0, l_0)|_{\widehat{\boldsymbol{\pi}}_0^*}\\
=\ &\mathrm{Var}(x_2 - l_2|x_0, l_0)|_{\widehat{\boldsymbol{\pi}}^*}.
\end{aligned}$$

□

3.2. General T-period case

We propose the following revised policy for general T-period case,

- At period 0, $\widehat{\boldsymbol{\pi}}_0^*(\widehat{x}_0) = \boldsymbol{\pi}_0^*(x_0)$;
- At period t,

$$\widehat{\boldsymbol{\pi}}_t^*(\widehat{x}_t) = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]r_t\left(\widehat{x}_t - \widehat{\gamma}_t\prod_{j=t}^{T-1} r_j^{-1}\right) + \left(\prod_{j=t+1}^{T-1}\frac{\mathbb{E}[q_j] - \widehat{B}_j}{(1 - B_j)r_j}\right)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t]l_t,$$

where

$$\widehat{x}_t = \begin{cases} \bar{x}_t, & \text{if } \bar{x}_t \le \bar{x}_t^*, \\ \bar{x}_t - 2(\bar{x}_t - \bar{x}_t^*)D_t, & \text{if } \bar{x}_t > \bar{x}_t^*, \end{cases} \tag{8}$$

$$\bar{x}_{t+1} = r_t \widehat{x}_t + \mathbf{P}_t' \widehat{\boldsymbol{\pi}}_t^*,$$

$$\widehat{\gamma}_t = \begin{cases} \widehat{\gamma}_{t-1}, & \text{if } \bar{x}_t \le \bar{x}_t^*, \\ \widehat{\gamma}_{t-1} + 2\prod_{j=t}^{T-1} r_j(\bar{x}_t - \bar{x}_t^*)(1-D_t), & \text{if } \bar{x}_t > \bar{x}_t^*, \end{cases} \tag{9}$$

with $\widehat{x}_0 = x_0$, $\widehat{\gamma}_0 = \gamma^*$ and

$$D_t = 1 - \prod_{j=t}^{T-1}(1-B_j),$$

$$\bar{x}_t^* = \frac{\widehat{\gamma}_{t-1}}{\prod_{j=t}^{T-1} r_j} + l_t \prod_{j=t}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{r_j(1-B_j)}. \tag{10}$$

One major feature of the revised policy is that, when the wealth level $\bar{x}_t > \bar{x}_t^*$, we withdraw free cash flow, $2(\bar{x}_t - \bar{x}_t^*)D_t$, out of the market and apply the efficient mean-variance policy for the remaining amount in the market, $\widehat{x}_t = \bar{x}_t - 2(\bar{x}_t - \bar{x}_t^*)D_t$. Similar to Cui et al. (2012), both risk attitude parameter $\widehat{\gamma}_t$ and the wealth threshold $\bar{x}_t^*$ ($t = 1, 2, \cdots, T-1$) are path dependent. However, the wealth threshold for the mean-variance asset-liability management problem does not only depend on the investment opportunity part $\widehat{\gamma}_t$ but also depend on the liability part l_t.

In the next theorem, we extend the result in Theorem 1 and prove that the proposed revised policy can achieve the same mean-variance pair as one determined by the original pre-committed policy for general T-period case.

Theorem 2. *The proposed revised policy can achieve the same mean-variance pair as one determined by the original pre-committed policy, i.e.,*

$$\mathbb{E}[x_T - l_T | x_0, l_0]|_{\widehat{\boldsymbol{\pi}}^*} = \mathbb{E}[x_T - l_T | x_0, l_0]|_{\boldsymbol{\pi}^*},$$
$$\mathrm{Var}(x_T - l_T | x_0, l_0)|_{\widehat{\boldsymbol{\pi}}^*} = \mathrm{Var}(x_T - l_T | x_0, l_0)|_{\boldsymbol{\pi}^*}.$$

Proof. The case with $T = 2$ is proved in Theorem 1. We assume that the conclusion is true for $T = t$ and prove the conclusion is also true for $T = t+1$.

When $\bar{x}_1 = x_1 \le \bar{x}_1^*$, the truncated pre-committed optimal policy $\boldsymbol{\pi}_j^*(x_j)$ ($j = 1, 2, \cdots, t$) specified in (2) is t-period mean-variance efficient policy, which implies the conditional mean-variance pair as follows,

$$\big(\mathbb{E}[x_{t+1} - l_{t+1} | x_1, l_1]|_{\boldsymbol{\pi}^*}, \mathrm{Var}(x_{t+1} - l_{t+1} | x_1, l_1)|_{\boldsymbol{\pi}^*}\big).$$

Meanwhile, we consider the t-period revised policy $\widehat{\boldsymbol{\pi}}_j^*(\widehat{x}_j)(j = 1, 2, \cdots, t)$ with $\widehat{x}_1 = \bar{x}_1 = x_1$ and $\widehat{\gamma}_1 = \widehat{\gamma}_0 = \gamma^*$. According to the assumption of the mathematical induction, we have

$$\begin{cases} \mathbb{E}[x_{t+1} - l_{t+1} | x_1, l_1]|_{\boldsymbol{\pi}^*} = \mathbb{E}[x_{t+1} - l_{t+1} | \widehat{x}_1, l_1]|_{\widehat{\boldsymbol{\pi}}^*}, \\ \mathrm{Var}(x_{t+1} - l_{t+1} | x_1, l_1)|_{\boldsymbol{\pi}^*} = \mathrm{Var}(x_{t+1} - l_{t+1} | \widehat{x}_1, l_1)|_{\widehat{\boldsymbol{\pi}}^*}. \end{cases}$$

When $\bar{x}_1 = x_1 > \bar{x}_1^*$, the truncated pre-committed optimal policy $\boldsymbol{\pi}_j^*(x_j)$ $(j = 1, 2, \cdots, t)$ is no longer t-period mean-variance efficient policy, which generates the conditional mean-variance pair as follows,

$$\mathbb{E}[x_{t+1} - l_{t+1}|x_1, l_1]|_{\boldsymbol{\pi}^*} = \prod_{j=1}^{t} r_j x_1 - \left(1 - \prod_{j=1}^{t}(1 - B_j)\right)\left(\prod_{j=1}^{t} r_j x_1 - \gamma^*\right) - l_1 \prod_{j=1}^{t} \left(\mathbb{E}[q_j] - \widehat{B}_j\right),$$

$$\begin{aligned}&\mathrm{Var}(x_{t+1} - l_{t+1}|x_1, l_1)|_{\boldsymbol{\pi}^*}\\ =&\frac{\prod_{j=1}^{t}(1 - B_j)}{1 - \prod_{j=1}^{t}(1 - B_j)}\left(\mathbb{E}[x_{t+1} - l_{t+1}|x_1, l_1]|_{\boldsymbol{\pi}^*} - \prod_{j=1}^{t} r_j x_1 + l_1 \prod_{j=1}^{t} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j}\right)^2 + l_1^2 C_1,\end{aligned}$$

where C_1 does not depend on x_1, l_1 and γ^*.

Let us consider t-period mean-variance efficient policy $\boldsymbol{\pi}_j^{t-e}(x_j)$ $(j = 1, 2, \cdots, t)$ with initial wealth $\widehat{x}_1$ and risk attitude parameter $\widehat{\gamma}_1$, which leads to the t-period efficient mean-variance pair as follows,

$$\mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\boldsymbol{\pi}^{t-e}} = \prod_{j=1}^{t} r_j \widehat{x}_1 + \left(1 - \prod_{j=1}^{t}(1 - B_j)\right)\left(\prod_{j=1}^{t} r_j \widehat{x}_1 - \widehat{\gamma}_1\right) - l_1 \prod_{j=1}^{t} \left(\mathbb{E}(q_j) - \widehat{B}_j\right),$$

$$\begin{aligned}&\mathrm{Var}(x_{t+1} - l_{t+1}|\widehat{x}_1, l_1)|_{\boldsymbol{\pi}^{t-e}}\\ =&\frac{\prod_{j=1}^{t}(1 - B_j)}{1 - \prod_{j=1}^{t}(1 - B_j)}\left(\mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\boldsymbol{\pi}^{t-e}} - \prod_{j=1}^{t} r_j \widehat{x}_1 + l_1 \prod_{j=1}^{t} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j}\right)^2 + l_1^2 C_1.\end{aligned}$$

We can choose suitable $\widehat{x}_1$ and $\widehat{\gamma}_1$ such that

$$\begin{cases} \mathbb{E}[x_{t+1} - l_{t+1}|x_1, l_1]|_{\boldsymbol{\pi}^*} = \mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\boldsymbol{\pi}^{t-e}}, \\ \mathrm{Var}(x_{t+1} - l_{t+1}|x_1, l_1)|_{\boldsymbol{\pi}^*} = \mathrm{Var}(x_{t+1} - l_{t+1}|\widehat{x}_1, l_1)|_{\boldsymbol{\pi}^{t-e}}, \end{cases}$$

which results in the solutions in (8) and (9).

Next, we consider the t-period revised policy $\widehat{\boldsymbol{\pi}}_j^*(\widehat{x}_j)(j = 1, 2, \cdots, t)$ with $\widehat{x}_1$ and $\widehat{\gamma}_1$. According to the assumption of the mathematical induction, we have

$$\begin{cases} \mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\boldsymbol{\pi}^{t-e}} = \mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\widehat{\boldsymbol{\pi}}^*}, \\ \mathrm{Var}(x_{t+1} - l_{t+1}|\widehat{x}_1, l_1)|_{\boldsymbol{\pi}^{t-e}} = \mathrm{Var}(x_{t+1} - l_{t+1}|\widehat{x}_1, l_1)|_{\widehat{\boldsymbol{\pi}}^*}. \end{cases}$$

which implies that

$$\begin{cases} \mathbb{E}[x_{t+1} - l_{t+1}|x_1, l_1]|_{\boldsymbol{\pi}^*} = \mathbb{E}[x_{t+1} - l_{t+1}|\widehat{x}_1, l_1]|_{\widehat{\boldsymbol{\pi}}^*}, \\ \mathrm{Var}(x_{t+1} - l_{t+1}|x_1, l_1)|_{\boldsymbol{\pi}^*} = \mathrm{Var}(x_{t+1} - l_{t+1}|\widehat{x}_1, l_1)|_{\widehat{\boldsymbol{\pi}}^*}. \end{cases}$$

Finally, using the similar analysis in Theorem 1, we can further obtain

$$\mathbb{E}[x_{t+1} - l_{t+1}|x_0, l_0]|_{\boldsymbol{\pi}^*} = \mathbb{E}[x_{t+1} - l_{t+1}|x_0, l_0]|_{\widehat{\boldsymbol{\pi}}^*}$$

and

$$\mathrm{Var}(x_{t+1} - l_{t+1}|x_0, l_0)|_{\boldsymbol{\pi}^*} = \mathrm{Var}(x_{t+1} - l_{t+1}|x_0, l_0)|_{\widehat{\boldsymbol{\pi}}^*}.$$

□

4. Numerical example

In this section, we consider a numerical example to illustrate the performance of our revised dominating policy. There are one risky asset, one risk-free asset and one liability in the mean-variance asset-liability management problem. The investment horizon is $T = 5$. The risk-free rate is $r_t = 1.05$. The excess return of the risky asset and the changing rate of the liability, $(\mathbf{P}_t, q_t)$, follow 2-dimensional normal distribution with mean $(0.12, 1.1)'$ and covariance matrix $(0.0576, 0.012; 0.012, 0.04)'$. The initial wealth $x_0 = 3$ and the initial liability $l_0 = 1$.

We choose the terminal targets d from 2 to 5, and the step size is equal to 0.01. For each target, we simulate 10000 paths for $(\mathbf{P}_t, q_t)$ and compute the probability of getting a positive free cash flow stream and the expected value of the free cash flow stream.The results are reported in Figures 1 and 2. We can see that the dominating policy can receive a sizeable FCFS.

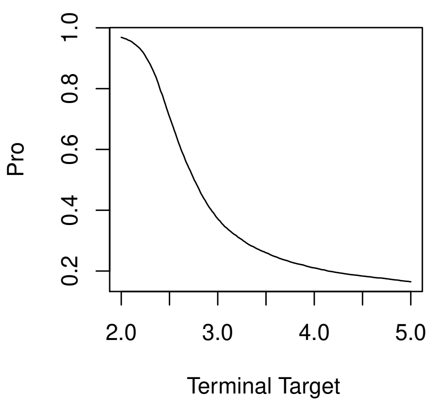

Figure 1: The probability of getting a positive free cash flow stream

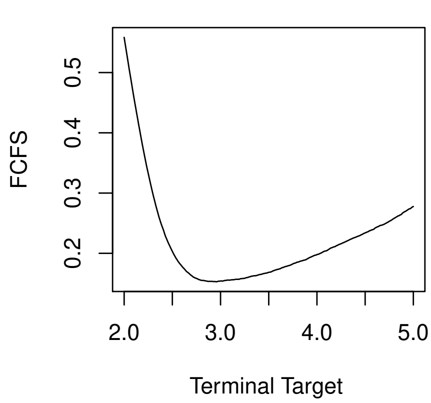

Figure 2: The expected value of the free cash flow stream

5. Conclusion

If the investor adopts the pre-committed policy, he/she may choose the inefficient policy during the multi-period mean-variance asset-liability management. We propose a revised dominating policy by relaxing the self-financing constraint and allowing to withdraw some wealth, and prove that the revised policy can achieve the same mean-variance pair as one determined by the pre-committed optimal mean-variance policy and receive a nonnegative free cash flow stream over the investment time horizon.

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