

Mean-variance portfolio selection under partial information with drift uncertainty

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Abstract

In this paper, we study the mean-variance portfolio selection problem under partial information with drift uncertainty. First we show that the market model is complete even in this case while the information is not complete and the drift is uncertain. Then, the optimal strategy based on partial information is derived, which reduces to solving a related backward stochastic differential equation (BSDE). Finally, we propose an efficient numerical scheme to approximate the optimal portfolio that is the solution of the BSDE mentioned above. Malliavin calculus and the particle representation play important roles in this scheme.

Keywords: Mean-variance portfolio selection; Malliavin calculus; Partial information; Drift uncertainty

JEL Classification: C61, G11

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1 Introduction

The mean-variance portfolio selection model pioneered by Markowitz [26] has paved the foundation for modern portfolio theory and has been widely applied in financial economics. Markowitz proposed and solved the problem in a single period setting. For half of a century, however, the optimal dynamic mean-variance portfolio selection problem was not solved due to the non-separable structure of the variance minimization problem in the sense of dynamic programming. This difficulty was finally overcome by Li and Ng [19] and Zhou and Li [38] via an embedding scheme, for multi-period and continuous-time cases, respectively. Since then, many scholars have devoted their attentions to the study of the dynamic extensions of the Markowitz model, see, for example, Li et al. [20], Lim and Zhou [22], Zhou and Yin [39], Bielecki et al. [7], Li and Zhou [21], Chiu and Li [9] in continuous-time settings.

In the research mentioned above, the initial and terminal times are fixed as 0 and T , respectively. The optimal strategy u_t for $t \in [0, T]$ is determined at time 0. However, the optimal strategy u_t found at time 0 for $t \in [0, T]$ might not be optimal at time t if you look at it at time t . This is the so called time inconsistency problem. It is well recognized that the optimal strategy for mean variance problem is usually time inconsistent. How to resolve this problem is a very active research field and we refer the reader to the following reference for works in this subject [5, 16, 27, 38]. The time consistency problem for our partial information model will not be studied in this article, although we hope to deal with it in a future research.

On the other hand, all the works mentioned above assume that the Brownian motions that are driving the stock prices are completely observable to the investors. In reality, however, the driving Brownian motions are often not observable to the investors, and the stock prices are the only observable information based on which the investors make the decisions. This fact motivates the study of the so-called partial information portfolio selection problem. In 2007, Xiong and Zhou [32] established the separation principle to separate the filtering and optimization problems for the mean-variance portfolio selection problem with partial information. They also developed analytical and numerical approaches in obtaining the filter as well as solving the related backward stochastic differential equation. In this paper, we will propose another more efficient numerical scheme based on Malliavin calculus.

Here we mention a few works where separation principle is also derived: Detemple [10], Dothan and Feldman [12] and Gennotte [14]. We also indicate that Malliavin calculus is also used to study optimal portfolio problems under partial information (see [1, 11, 31]).

The optimal redeeming problem of stock loans under drift uncertainty has been studied by Xu and Yi [33]. In their model, the inherent uncertainty of the trend of the stock is modeled by a two-state random variable representing bull and bear trends, respectively; the

current trend of the stock is not known to the investor so that she/he has to make decisions based on partial information. They derive the optimal redeeming strategies based on the prediction of the stock trend.

In this paper, we study a mean-variance problem under partial information with drift uncertainty. Our contributions to the literature are summarized below: First, we show that the market model is complete even in this case while the information is not complete and the drift is uncertain. Second, the optimal strategy based on partial information is derived, which involves the optimal filter of the drift. Third, an efficient numerical approximation is suggested to solve the BSDE which arises from the mean-variance problem under drift uncertainty. This scheme is investigated in the context of the Malliavin calculus approach for the approximation of conditional expectations. We also prove the convergence of our numerical scheme, and study the L^2 error induced by the Euler approximation and by the strong law of large number (SLLN).

The rest of the paper is organized as follows. Some preliminary results on filtering and Malliavin calculus are given in Section 2. In Section 3, we derive the innovation process associated with the posteriori probability process of the drift uncertainty model and study its optimization problem under partial information. We also prove the completeness of the market under \mathcal{G} filtration (partial information). A new numerical scheme is proposed and the asymptotic behavior is studied in Section 4, a couple of numerical results are also presented.

2 Preliminaries

In this section, we state some elementary facts about stochastic filtering and Malliavin calculus for the convenience of the reader. We refer the reader to Sections 8.1-8.3 of Kallianpur [17] for more details about the general filtering problem and the stochastic equation of the optimal filter, and the book of Nualart and Nualart [28] about the Malliavin calculus.

Let T be a fixed positive constant representing the investment horizon. Let (Ω, \mathcal{F}, P) be a complete probability space and let \mathcal{F}_t , $t \in [0, T]$, be an increasing family of sub σ -fields of \mathcal{F} . Throughout this article, we always augment the σ -filtration with all P -null sets. The signal $h_t(\omega)$ and the observation $Z_t(\omega)$, $t \in [0, T]$, are assumed to be two N -dimensional processes defined on (Ω, \mathcal{F}, P) and further related as follows:

$$Z_t(\omega) = \int_0^t h_u(\omega) du + W_t(\omega), \quad (2.1)$$

where $\{W_t, t \in [0, T]\}$ is an N -dimensional Wiener process, and $h_t(\omega)$ is a \mathbb{R}^N -valued, (t, ω) -

measurable function satisfying

$$\int_0^T \mathbb{E}(|h_t|^2) dt < \infty, \quad (2.2)$$

where $|\cdot|$ denotes the Euclidean norm of N -dimensional vector. Further, for each $s \in [0, T]$, the σ -fields $\mathcal{F}_s^{h,W} := \sigma\{h_u, W_u, 0 \leq u \leq s\}$ and $\mathcal{F}_s^W := \sigma\{W_{u'} - W_u, s \leq u \leq u' \leq T\}$ are independent. Let $\{\mathcal{F}_t^Z\}_{0 \leq t \leq T}$ be the filtration generated by Z . This filtration is called the observation σ -field. Let $v_t := (v_t^1, \dots, v_t^N)'$, be an N -dimensional $\{\mathcal{F}_t^Z\}_{t \in [0, T]}$ -adapted innovation process, which is also an N -dimensional $\{\mathcal{F}_t^Z\}_{t \in [0, T]}$ -adapted Brownian motion.

We list three theorems for ready references. The following one appears in Section 8.3 of [17] (page 208).

Theorem 2.1. *Under conditions (2.1) and (2.2), every separable, square-integrable \mathcal{F}_t^Z -martingale Y_t is sample path continuous a.s. and has the representation*

$$Y_t - \mathbb{E}(Y_0) = \sum_{i=1}^N \int_0^t \Phi_s^i dv_s^i, \quad t \in [0, T], \quad (2.3)$$

where

$$\int_0^T \mathbb{E}(|\Phi_s|^2) ds < \infty \quad (2.4)$$

and $\Phi_s := (\Phi_s^1, \dots, \Phi_s^N)'$ is jointly measurable and adapted to \mathcal{F}_t^Z .

The next theorem is called the Clark-Ocone formula (see Theorem 6.1.1 of [28]). It expresses a square integrable random variable in terms of the conditional expectation of its Malliavin derivative. Let $B = (B_t)_{t \geq 0}$ be a multi-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of B and $\mathcal{F} = \vee_{t \geq 0} \mathcal{F}_t$. Denote by D the Malliavin derivative operator. We define the Sobolev space $\mathbb{D}^{1,2}$ of random variables as follows:

$$\mathbb{D}^{1,2} = \left\{ F \in L^0(\Omega, \mathcal{F}, P) : \|F\|_{1,2}^2 = \mathbb{E}(|F|^2) + \mathbb{E} \left[\int_0^\infty |D_t F|^2 dt \right] < \infty \right\},$$

where $L^0(\Omega, \mathcal{F}, P)$ denotes the set of \mathcal{F} -measurable random variables.

Theorem 2.2 (Clark-Ocone formula). *Let $F \in \mathbb{D}^{1,2} \cap L^0(\Omega, \mathcal{F}_T, P)$. Then, F admits the following representation*

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t F | \mathcal{F}_t) dB_t.$$

Let $\mathcal{M}(d, q, \mathbb{R})$ be a vector space of matrices with d rows and q columns of \mathbb{R} -valued entries, $\|\cdot\|$ be the canonical Euclidean norm.

Denote by $L^p(0, T; \mathbb{R}^d)$ the set of all \mathbb{R}^d -valued $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted processes f in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose L^p norm are finite, namely

$$\|f\|_{L^p(0, T; \mathbb{R}^d)} := \left(\mathbb{E} \int_0^T |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

Let $L^p(\mathcal{F}, \mathbb{R}^d)$ be the set of all \mathbb{R}^d -valued random variables ξ with finite L^p norm

$$\|\xi\|_p := (\mathbb{E}|\xi|^p)^{\frac{1}{p}} < \infty.$$

The next theorem which appears in Section 7 of [29] (Theorem 7.2), states the error approximation of the Euler scheme for the solution $(X_t)_{t \in [0, T]}$ to a d -dimensional stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad (2.5)$$

where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d, q, \mathbb{R})$ are continuous functions, $W = (W_t)_{t \in [0, T]}$ denotes a q -dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_0 : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^d$ is a random vector, independent of W .

The discrete time Euler scheme with step $\frac{T}{n}$ is defined by

$$\begin{aligned} \bar{X}_{t_{k+1}^n} &= \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n}) \sqrt{\frac{T}{n}} Z_{k+1}^n, \quad \bar{X}_0 = X_0, \\ & \quad k = 0, \dots, n-1, \end{aligned}$$

where $t_k^n = \frac{kT}{n}, k = 0, \dots, n$ and $(Z_k^n)_{1 \leq k \leq n}$ denotes a sequence of independent and identically distributed random vectors given by

$$Z_k^n := \sqrt{\frac{n}{T}} (W_{t_k^n} - W_{t_{k-1}^n}), \quad k = 0, \dots, n.$$

Theorem 2.3 (Strong rate for the Euler scheme). *Suppose the coefficients b and σ of the SDE (2.5) satisfy the following regularity condition: there exist a real constant $C_{b, \sigma, T} > 0$ and an exponent $\beta \in (0, 1]$ such that for all $s, t \in [0, T], x, y \in \mathbb{R}^d$,*

$$|b(t, x) - b(s, x)| + \|\sigma(t, x) - \sigma(s, x)\| \leq C_{b, \sigma, T} (1 + |x|) |t - s|^\beta, \quad (2.6)$$

$$|b(t, x) - b(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq C_{b, \sigma, T} |y - x|. \quad (2.7)$$

Then for all $p > 0$, there exists a universal constant $\kappa_p > 0$, depending on p only, such that for every $n \geq T$,

$$\left\| \sup_{0 \leq k \leq n} |X_{t_k^n} - \bar{X}_{t_k^n}| \right\|_p \leq K(p, b, \sigma, T) (1 + \|X_0\|_p) \left(\frac{T}{n} \right)^{\beta \wedge \frac{1}{2}}, \quad (2.8)$$

where

$$K(p, b, \sigma, T) = \kappa_p C'_{b, \sigma, T} e^{\kappa_p (1 + C'_{b, \sigma, T})^2 T}$$

and

$$C'_{b, \sigma, T} = C_{b, \sigma, T} + \max_{t \in [0, T]} |b(t, 0)| + \max_{t \in [0, T]} \|\sigma(t, 0)\| < +\infty. \quad (2.9)$$

3 Problem formulation

3.1 Model setup

Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a complete filtered probability space, which represents the financial market. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, and P denotes the probability measure. In this probability space, there exists a standard one-dimensional Brownian motion W . The price process of the underlying stock is denoted by S_t , $t \in [0, T]$, which satisfies the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + S_t dW_t, \quad (3.1)$$

where μ is random and independent of the Brownian motion W , and it may only takes two possible values a and b that satisfy

$$\gamma := a - b > 0.$$

The stock is said to be in its bull trend when $\mu = a$, and in its bear trend when $\mu = b$. Note that the volatility for the model (3.1) is normalized to be one.

The observed information up to time t is given by

$$\mathcal{G}_t := \sigma(S_s : s \leq t), \quad t \in [0, T].$$

The *posteriori probability process* $\pi = (\pi_t)_{t \in [0, T]}$ is defined as

$$\pi_t := P(\mu = a | \mathcal{G}_t). \quad (3.2)$$

It is used to estimate the chance that the stock is in its bull trend given the observed information up to and including time t . Assume that $0 < \pi_0 < 1$. This means it is not clear whether the stock is in its bull trend or bear trend at time 0.

Let u_t , called a portfolio, be the amount invested in the stock at time t .

Definition 3.1. A portfolio (or trading strategy) is called **self-financing** if all the changes of the values of the portfolio are due to gains or losses realized on investment, that is, no funds are borrowed or withdrawn from the portfolio at any time. A portfolio process $\{u_t\}_{t \in [0, T]}$ is called **admissible** if it is $\{\mathcal{G}_t\}_{t \in [0, T]}$ -adapted, self-financing and

$$\int_0^T \mathbb{E}(u_t^2) dt < \infty.$$

Denote by Y_t the wealth process of an agent, and u_t an admissible trading strategy. Starting with an initial wealth $y_0 > 0$, Y_t satisfies the following *wealth equation*:

$$\begin{cases} dY_t = (\mu u_t + (Y_t - u_t)r) dt + u_t dW_t, & t \in [0, T], \\ Y_0 = y_0. \end{cases} \quad (3.3)$$

where r denotes the interest rate, which is a constant.

Our goal is to solve the following optimization

Problem (MV): To find the best admissible portfolio u_t to minimize $\text{Var}(Y_T)$ subject to the constraint $\mathbb{E}(Y_T) = z$, where Y_t is given by (3.3), and z is a given positive number.

Taken as observation, the log-price process $L = (\log S_t)_{t \in [0, T]}$, by Itô's lemma, satisfies the following SDE

$$dL_t = \left(\mu - \frac{1}{2} \right) dt + dW_t. \quad (3.4)$$

Then, the innovation process

$$\nu_t = L_t - \int_0^t \left(b - \frac{1}{2} + \gamma \pi_s \right) ds \quad (3.5)$$

is a Brownian motion with respect to the observation filtration \mathcal{G}_t . It is easy (see [17], Chapter 8.1) to verify that π_t satisfies the following SDE:

$$d\pi_t = \gamma \pi_t (1 - \pi_t) d\nu_t, \quad \pi_0 = P(\mu = a). \quad (3.6)$$

By (3.3) and (3.5), we get the ν_t -driven representation for Y :

$$\begin{cases} dY_t = ((b + \gamma \pi_t - r) u_t + r Y_t) dt + u_t d\nu_t, & t \in [0, T], \\ Y_0 = y_0. \end{cases} \quad (3.7)$$

3.2 Optimization under partial information

The optimization problem (MV) turns to minimize $\text{Var}(Y_T)$ with state equations (3.7) and the constraint $\mathbb{E}(Y_T) = z$.

Let

$$\rho_t := \exp \left(- \int_0^t (b - r + \gamma \pi_s) d\nu_s - \int_0^t \left(r + \frac{1}{2} (b - r + \gamma \pi_s)^2 \right) ds \right). \quad (3.8)$$

Applying Itô's formula to ρ_t , we get

$$d\rho_t = -r\rho_t dt - (b - r + \gamma \pi_t) \rho_t d\nu_t. \quad (3.9)$$

Further, applying Itô's formula to $Y_t \rho_t$, we have

$$d(Y_t \rho_t) = (Y_t (r \rho_t - \mu \rho_t) + u_t \rho_t) d\nu_t.$$

It is also easy to prove that $Y_t \rho_t$ is square integrable. Therefore, $Y_t \rho_t$ is a $\{\mathcal{G}_t\}_{t \in [0, t]}$ -martingale and hence,

$$\mathbb{E}(Y_t \rho_t) = y_0.$$

Denote Y_T by v . To find the optimal portfolio, we seek the best \mathcal{G}_T -measurable terminal wealth v to minimize the variance

$$\mathbb{E}(v - z)^2 \quad (3.10)$$

subject to constraints

$$\mathbb{E}v = z \quad \text{and} \quad \mathbb{E}(\rho_T v) = y_0. \quad (3.11)$$

Let $\mathbb{H} := L^2(\Omega, \mathcal{G}_T, P)$. For $X \in \mathbb{H}$, let

$$\|X\|_{\mathbb{H}} := \left(\mathbb{E}(X^2) \right)^{\frac{1}{2}}.$$

Then, \mathbb{H} is a Hilbert space endowed with the norm $\|\cdot\|_{\mathbb{H}}$. Note that

$$\mathbb{E}(v - z)^2 = \|v - 0\|_{\mathbb{H}}^2 - z^2.$$

Therefore, the optimal v is the projection of 0 onto the hyperplane

$$\{v \in \mathbb{H} : \mathbb{E}v = z, \mathbb{E}(v \rho_T) = y_0\}.$$

3.3 Completeness of the market

Denote by $L_{\mathcal{G}}^2(0, T; \mathbb{R})$ the set of all \mathbb{R} -valued, $\{\mathcal{G}_t\}_{t \in [0, T]}$ -adapted processes $f(t)$ such that

$$\mathbb{E} \int_0^T |f(t)|^2 dt < \infty.$$

Then $L_{\mathcal{G}}^2(0, T; \mathbb{R})$ becomes a Hilbert space endowed with the norm

$$\|f\|_{L_{\mathcal{G}}^2(0, T; \mathbb{R})} := \left(\mathbb{E} \int_0^T |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Definition 3.2. A contingent claim $v \in \mathbb{H}$ is called attainable if there is $\Phi_s \in L_{\mathcal{G}}^2(0, T; \mathbb{R})$ such that

$$v\rho_T = \mathbb{E}(v\rho_T) + \int_0^T \Phi_s d\nu_s. \quad (3.12)$$

Denote the collection of all attainable contingent claims by $AC(\mathcal{G})$. Then $AC(\mathcal{G})$ is a subspace of \mathbb{H} . Denote by \mathbb{H}_0 the closure of $AC(\mathcal{G})$ in \mathbb{H} under the norm $\|\cdot\|_{\mathbb{H}}$.

Definition 3.3. The market is complete if $\mathbb{H}_0 = \mathbb{H}$.

The following statement is quite surprise to us. Namely, the market is complete although the information is not complete and the drift is uncertain. It is worth mentioning that completeness was left open in [32] for their model. Because of this lacking of completeness result, the authors of [32] turn to search the optimal solution v in the space \mathbb{H}_0 .

Theorem 3.1. The market is complete.

Proof. Since $\mathbb{H}_0 \subseteq \mathbb{H}$, it suffices to show $\mathbb{H} \subseteq \mathbb{H}_0$. For any $V \in \mathbb{H}$, let $V_n = V \min\{|V|^{-\frac{1}{n}}, 1\}$. Then

$$(V_n - V)^2 = V^2 \mathbf{1}_{|V| > 1} (|V|^{-\frac{1}{n}} - 1)^2 \leq V^2 \mathbf{1}_{|V| > 1} \leq V^2.$$

Since $V \in \mathbb{H}$, we have $\mathbb{E}|V|^2 < \infty$. By the dominated convergent theorem,

$$\lim_{n \rightarrow \infty} \|V_n - V\|_{\mathbb{H}}^2 = \lim_{n \rightarrow \infty} \mathbb{E}[(V_n - V)^2] = \mathbb{E} \left[\lim_{n \rightarrow \infty} V^2 \mathbf{1}_{|V| > 1} (|V|^{-\frac{1}{n}} - 1)^2 \right] = 0.$$

Therefore, if we can show $V_n \in AC(\mathcal{G})$, then V is in the closure of $AC(\mathcal{G})$ under the norm $\|\cdot\|_{\mathbb{H}}$, namely $V \in \mathbb{H}_0$, and the claim follows.

We now show $V_n \in AC(\mathcal{G})$ for any $n \geq 1$. Notice

$$\mathbb{E}|V_n|^{2+\frac{1}{n}} = \mathbb{E}[|V|^{(1-\frac{1}{n})(2+\frac{1}{n})} \mathbf{1}_{|V| > 1} + |V|^{2+\frac{1}{n}} \mathbf{1}_{|V| \leq 1}] \leq \mathbb{E}(|V|^2 \mathbf{1}_{|V| > 1} + 1) < \infty,$$

so $V_n \in L^{2+\frac{1}{n}}$. By Hölder's inequality, we have

$$\mathbb{E}|V_n \rho_T|^2 \leq \left(\mathbb{E}|V_n|^{2\frac{2n+1}{2n}} \right)^{\frac{2n}{2n+1}} \left(\mathbb{E}\rho_T^{2(2n+1)} \right)^{\frac{1}{2n+1}} < \infty,$$

as $\mathbb{E}\rho_T^p < \infty$, for all $p > 1$. Hence $\mathbb{E}(V_n \rho_T | \mathcal{G}_t)$ is a square integrable martingale. By Theorem 2.1, we have

$$\mathbb{E}(V_n \rho_T | \mathcal{G}_t) - \mathbb{E}(V_n \rho_T) = \int_0^t \Phi_s d\nu_s, \quad t \in [0, T], \quad (3.13)$$

for some $\Phi_s \in L^2_{\mathcal{G}}(0, T; \mathbb{R})$. When $t = T$, since $V_n \rho_T$ is \mathcal{G}_T measurable, the above equation reduces to

$$V_n \rho_T - \mathbb{E}(V_n \rho_T) = \int_0^T \Phi_s d\nu_s, \quad (3.14)$$

which implies $V_n \in AC(\mathcal{G})$. \square

It was shown in [32] that the optimal terminal v to the optimization problem for the model in [32] is given by

$$v = \frac{(z\langle \beta, \beta \rangle_{\mathbb{H}} - y_0\langle \alpha, \beta \rangle_{\mathbb{H}})\alpha + (-z\langle \alpha, \beta \rangle_{\mathbb{H}} + y_0\langle \alpha, \alpha \rangle_{\mathbb{H}})\beta}{\langle \alpha, \alpha \rangle_{\mathbb{H}}\langle \beta, \beta \rangle_{\mathbb{H}} - \langle \alpha, \beta \rangle_{\mathbb{H}}^2}, \quad (3.15)$$

where α, β are the orthogonal projections on \mathbb{H}_0 of 1 and ρ_T , respectively.

A numerical scheme was given in [32] under the completeness assumption. Although that numerical scheme can be extended to the current model, we will propose a more efficient numerical scheme for our model, which is one of the main contributions of the current article.

3.4 Find the optimal strategy

Similar to [32], the terminal problem can be solved as follows.

Lemma 3.1. *The optimal terminal wealth for the problem (3.10) is*

$$v = \frac{z\mathbb{E}(\rho_T^2) - y_0\mathbb{E}\rho_T + (y_0 - z\mathbb{E}\rho_T)\rho_T}{\text{Var}(\rho_T)}, \quad (3.16)$$

where ρ_T is given by (3.8).

After finding the optimal terminal wealth, we then seek a portfolio to realize it. Namely, for v given by (3.16), we need to find a solution of the following BSDE:

$$\begin{cases} dY_t = ((b + \gamma\pi_t - r)u_t + rY_t)dt + u_t d\nu_t, & t \in [0, T], \\ Y_T = v. \end{cases} \quad (3.17)$$

Theorem 3.2. *The optimal portfolio is given by*

$$u_t = (b - r + \gamma\pi_t)Y_t + \rho_t^{-1}\eta_t, \quad (3.18)$$

where $\eta_t \in L_G^2(0, T; \mathbb{R}^d)$ is uniquely determined by the martingale representation theorem

$$\mathbb{E}(\theta|\mathcal{G}_t) = \mathbb{E}(\theta) + \int_0^t \eta_s d\nu_s, \quad \forall t \in [0, T], \quad (3.19)$$

with $\theta = \rho_T Y_T$, and the explicit representation of the integrand η_s is given in Theorem 4.1 below, which is identified by Malliavin derivative using the Clark-Ocone formula.

Proof. As seen from the arguments above, we need to seek a solution to the following forward-backward SDE:

$$\begin{cases} dY_t = ((b + \gamma\pi_t - r)u_t + rY_t)dt + u_t d\nu_t, & Y_0 = y_0, \\ d\pi_t = \gamma\pi_t(1 - \pi_t)d\nu_t, \\ d\rho_t = -r\rho_t dt - (b - r + \gamma\pi_t)\rho_t d\nu_t, \\ \rho_0 = 1, \quad \pi_0 = c_0, \quad Y_T = c_1 + c_2\rho_T, \end{cases} \quad (3.20)$$

where $c_0 = P(\mu = a)$, $c_1 = \frac{z\mathbb{E}\rho_T^2 - y_0\mathbb{E}\rho_T}{\text{Var}(\rho_T)}$ and $c_2 = \frac{y_0 - z\mathbb{E}\rho_T}{\text{Var}(\rho_T)}$ are known constants.

To prove the invertibility of ρ_t , we define Φ_t to be the solution to the following SDE:

$$\begin{cases} d\Phi_t = (r + (b - r + \gamma\pi_t)^2)\Phi_t dt + (b - r + \gamma\pi_t)\Phi_t d\nu_t, \\ \Phi_0 = 1. \end{cases} \quad (3.21)$$

Apply Itô's formula to $\rho_t\Phi_t$, we have

$$d(\rho_t\Phi_t) = 0,$$

and hence, $\rho_t\Phi_t \equiv \rho_0\Phi_0 = 1$. Consequently, Φ_t is the inverse of ρ_t . Since $\rho_t Y_t$ is a martingale, then

$$Y_t = \rho_t^{-1}\mathbb{E}(\rho_T Y_T|\mathcal{G}_t) = \rho_t^{-1}\mathbb{E}(\theta|\mathcal{G}_t) = \Phi_t\mathbb{E}(\theta|\mathcal{G}_t). \quad (3.22)$$

Applying Itô's formula to (3.22) and comparing the result with (3.20), we obtain the expression (3.18) for the optimal portfolio. \square

Finding the numerical solution $(u_t, Y_t, \pi_t, \rho_t)$ of the FBSDE (3.20) is the object of the next section.

4 Numerical schemes based on Malliavin calculus

Pardoux and Peng [30] obtained the unique solvability results for the nonlinear BSDEs in 1990. Since then a growing literature investigates the numerical methods for BSDEs ([2, 3, 4, 13, 15, 23, 24, 25, 36, 37]). In [6], the Malliavin calculus approach and Monte Carlo approximation are employed to study the conditional expectation, in the Ph.D. thesis [35] of Zhang, some fine properties of the BSDEs by using Malliavin derivatives under weaker conditions were also studied. However, those works mentioned above are based on the setting that the drift coefficients of the BSDEs are deterministic. We can not apply these numerical schemes to our model directly.

In Xiong and Zhou's [32] model, the coefficients of u_t and Y_t which appear in the drift term are random in general. They proposed a numerical approximation (u_t^n, Y_t^n) to the solution (u_t, Y_t) to that kind of BSDE with random coefficients. However, due to technical difficulty, only the convergence of Y_t^n to the wealth process Y_t is proved, and leave the convergence problem of the portfolio unsolved. The rate of convergence of Y_t^n to Y_t is not established in that paper.

In this section, we propose an efficient numerical scheme for the BSDE (3.17) whose terminal value v takes the form $c_1 + c_2\rho_T$, where c_1, c_2 are constants and ρ_t is a diffusion process which is Malliavin differentiable (see Theorem 4.1 for detailed calculation). With the help of Malliavin calculus, we prove that our scheme for the portfolio and the wealth processes converge in the strong L^2 sense and derive the rate of convergence.

Denote $N(t) := \mathbb{E}(\theta|\mathcal{G}_t)$. We note that the main complexity in Xiong and Zhou's [32] numerical scheme for BSDEs results from the approximation of the integrand η_t in (3.19), which is difficult to calculate directly. They use the following procedure to approximate η_t : First they divide $[0, t]$ into n_1 subintervals and approximate the quadratic covariation process

$$A_t := \langle N, \nu \rangle_t = \int_0^t \eta_s ds$$

by the discrete version over the partition points. They further divide each subinterval mentioned above into n_2 smaller ones and obtain an approximation of η_s , $s \leq t$. This procedure is not computationally efficient because the double-partition increases error dramatically. This will be seen from the numerical examples in the subsequent section.

In order to overcome the aforementioned drawback of the above numerical scheme, we turn to use the Clark-Ocone formula from Malliavin calculus to get an explicit expression of η_t . In fact, it will be the conditional expectation of a Malliavin derivative. Our numerical scheme will be based on this representation.

Theorem 4.1. *We can represent η_t as $\mathbb{E}(D_t\theta|\mathcal{G}_t)$ where D_t is the Malliavin derivative operator evaluated at time t . Further,*

$$D_t\theta = (c_1 + 2c_2\rho_T)D_t\rho_T \quad (4.1)$$

and $D_t\rho_T$ is given by

$$D_t\rho_T = \rho_T \left[- \int_t^T \gamma(b-r+\gamma\pi_s)D_t\pi_s ds - (b-r+\gamma\pi_t) + \int_t^T \gamma D_t\pi_s d\nu_s \right], \quad (4.2)$$

with

$$D_t\pi_s = \gamma\pi_t(1-\pi_t) \exp \left(\int_t^s \gamma(1-\pi_r)d\nu_r - \frac{1}{2} \int_t^s \gamma^2(1-2\pi_r)^2 dr \right). \quad (4.3)$$

Proof. Note that

$$\theta = \rho_T Y_T = c_1\rho_T + c_2\rho_T^2,$$

and hence, (4.1) follows by applying the Malliavin derivative on both sides.

As

$$\rho_T = \exp \left(- \int_0^T [r + \frac{1}{2}(b-r+\gamma\pi_s)^2] ds - \int_0^T (b-r+\gamma\pi_s) d\nu_s \right),$$

a direct calculation yields (4.2).

Applying Malliavin derivative to both sides of the integral form of the identity (3.6), we get

$$D_t\pi_s = \gamma\pi_t(1-\pi_t) + \int_t^s \gamma(1-2\pi_r)D_t\pi_r d\nu_r. \quad (4.4)$$

Then, (4.3) follows by solving the linear SDE (4.4). Finally, the representation of η_t as $\mathbb{E}(D_t\theta|\mathcal{G}_t)$ follows from the Clark-Ocone formula given in Section 2. \square

Remark 4.1. *Our method is based on the Malliavin differentiability of ρ_T so that the solution of (3.17) can be represented explicitly. In this paper, our setting is a drift uncertainty model with μ only takes two values, nevertheless, the whole analysis can be generalized to a model with several risky assets, where the drift vector takes finite states, under which assumption we can still deduce the Malliavin differentiability of ρ_T .*

4.1 A numerical scheme and its analysis

Based on Theorem 4.1, it is easy to show that

$$\eta_t = \mathbb{E}(D_t\theta|\mathcal{G}_t) = N_1(t) + \gamma N_2(t) + \gamma N_3(t),$$

with $N_j(t) = \mathbb{E}(I_j|\mathcal{G}_t)$, $j = 1, 2, 3$, where

$$I_1 = -(c_1\rho_T + 2c_2\rho_T^2)(b - r + \gamma\pi_t), \quad (4.5)$$

$$I_2 = (c_1\rho_T + 2c_2\rho_T^2) \int_t^T D_t\pi_s d\nu_s, \quad (4.6)$$

$$I_3 = -(c_1\rho_T + 2c_2\rho_T^2) \int_t^T (b - r + \gamma\pi_s) D_t\pi_s ds, \quad (4.7)$$

and $D_t\pi_s$ is given by (4.3).

As in the proof of Theorem 3.2, the key to solve the optimal portfolio is the martingale representation of the \mathcal{G}_t -martingale $\mathbb{E}(\theta|\mathcal{G}_t)$. We will establish particle representation for this martingale.

The solution of (3.9) is given by

$$\rho_t = \exp\left(-\int_0^t (b - r + \gamma\pi_s) d\nu_s - \int_0^t \left(r + \frac{1}{2}(b - r + \gamma\pi_s)^2\right) ds\right), \quad (4.8)$$

Denote $\tilde{\rho}_t := \log \rho_t$, then

$$d\tilde{\rho}_t = -(b - r + \gamma\pi_t) d\nu_t - \left(r + \frac{1}{2}(b - r + \gamma\pi_t)^2\right) dt. \quad (4.9)$$

To approximate $\mathbb{E}(\pi_t|\mathcal{G}_{t'})$, we use the conditional SLLN such that π_t^i is given by (3.6) with ν_s be replaced by ν_s^i for $s \geq t'$, where ν^i , $i = 1, 2, \dots$ are independent copies of ν . More precisely, we define the following processes $\pi^i(t, t')$ with two time-indices as follows: For $t \leq t'$, $\pi^i(t, t') = \pi_t$, and for $t \geq t'$,

$$d\pi^i(t, t') = \gamma\pi^i(t, t')(1 - \pi^i(t, t')) d\nu_t^i, \quad \pi^i(t', t') = \pi_{t'}. \quad (4.10)$$

To approximate $\mathbb{E}(\rho_t|\mathcal{G}_{t'})$, we use $\mathbb{E}(\exp(\tilde{\rho}_t)|\mathcal{G}_{t'})$ instead. For $t \leq t'$, $\tilde{\rho}^i(t, t') = \tilde{\rho}_t$, and for $t \geq t'$,

$$d\tilde{\rho}^i(t, t') = -(b - r + \gamma\pi^i(t, t')) d\nu_t^i - \left(r + \frac{1}{2}(b - r + \gamma\pi^i(t, t'))^2\right) dt, \quad \tilde{\rho}^i(t', t') = \tilde{\rho}_{t'}. \quad (4.11)$$

The following proposition follows from the same argument (which we term it as conditional SLLN) in the proof of Theorem 2.3 of Kurtz and Xiong [18] (more specifically, two equations on lines 9 and 14 on page 115 there).

Proposition 4.1. Denote $\rho^i(T, t) = \exp(\tilde{\rho}^i(T, t))$. Then, we have a.s.,

$$\begin{aligned} N_1(t) &= -(b - r + \gamma\pi_t) \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m (c_1 \rho^i(T, t) + 2c_2 (\rho^i(T, t))^2), \\ N_2(t) &= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m (c_1 \rho^i(T, t) + 2c_2 (\rho^i(T, t))^2) \int_t^T D_t \pi^i(s, t) d\nu_s^i, \\ N_3(t) &= \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m -(c_1 \rho^i(T, t) + 2c_2 (\rho^i(T, t))^2) \int_t^T (b - r + \gamma\pi^i(s, t)) D_t \pi^i(s, t) ds. \end{aligned}$$

In order to approximate $N_k(t)$, ($k = 1, 2, 3$), we use the discrete Euler scheme to approximate π_t . We divide the time interval $[0, T]$ into n small intervals and let $\delta = \frac{T}{n}$.

Note that the diffusion coefficient in the SDE (3.6) is $\sigma(x) = \gamma x(1 - x)$, which does not satisfy the global Lipschitz condition (2.7). To overcome this hurdle, we define $\bar{\sigma}(x)$ as following

$$\bar{\sigma}(x) = \begin{cases} \gamma x(1 - x), & x \in [0, 1], \\ 0, & x \notin [0, 1]. \end{cases}$$

Using the fact that $\pi_t \in [0, 1]$ for all $t \in [0, T]$, it is easy to see that π_t is a solution of the following SDE

$$d\pi_t = \bar{\sigma}(\pi_t) d\nu_t. \quad (4.12)$$

This SDE satisfies the global Lipschitz condition (2.7), so π_t is the unique solution. Therefore, we approximate π_t by applying the Euler scheme to (4.12) instead of the SDE (3.6).

$$\pi_{k\delta}^\delta := \pi_{(k-1)\delta}^\delta + \bar{\sigma}(\pi_{(k-1)\delta}^\delta) (\nu_{k\delta} - \nu_{(k-1)\delta})$$

with $\pi_0^\delta = \pi_0$, ($k = 0, \dots, n$). Similarly, we use $\rho_{k\delta}^\delta$ and $\Phi_{k\delta}^\delta$ to approximate $\rho_{k\delta}$ and $\Phi_{k\delta}$, which are defined as following:

$$\tilde{\rho}_{k\delta}^\delta := \tilde{\rho}_{(k-1)\delta}^\delta - \delta \left(r + \frac{1}{2} (b - r + \gamma (\pi_{(k-1)\delta}^\delta)^2) \right) - (b - r + \gamma \pi_{(k-1)\delta}^\delta) (\nu_{k\delta} - \nu_{(k-1)\delta}),$$

with $\tilde{\rho}_0^\delta = 0$. Then we have $\rho_{n\delta}^\delta = \exp(\tilde{\rho}_{n\delta}^\delta)$.

$$\begin{aligned} \tilde{\Phi}_{k\delta}^\delta &:= \tilde{\Phi}_{(k-1)\delta}^\delta + \delta \left(r + \frac{1}{2} (b - r + \gamma \pi^\delta((k-1)\delta, k\delta))^2 \right) \\ &\quad + (b - r + \gamma \pi^\delta((k-1)\delta, k\delta)) (\nu_{k\delta} - \nu_{(k-1)\delta}), \end{aligned}$$

with $\tilde{\Phi}_0^\delta = 0$. Then $\Phi_{k\delta}^\delta = \exp(\tilde{\Phi}_{k\delta}^\delta)$.

Note that we aim to approximate the $N_k(t)$, ($k = 1, 2, 3$) by using particle representation as well as the discrete Euler scheme, thus we define $\pi^{i,\delta}(l\delta, k\delta)$, $l, l' \geq 1$, in two steps.

For $l \leq k$, let

$$\pi^{i,\delta}(l\delta, k\delta) := \pi^{i,\delta}((l-1)\delta, k\delta) + \bar{\sigma}(\pi^{i,\delta}((l-1)\delta, k\delta)) (\nu_{l\delta} - \nu_{(l-1)\delta})$$

with $\pi^{i,\delta}(0, k\delta) = \pi_0$; for $l > k$, let

$$\pi^{i,\delta}(l\delta, k\delta) := \pi^{i,\delta}((l-1)\delta, k\delta) + \bar{\sigma}(\pi^{i,\delta}((l-1)\delta, k\delta)) (\nu_{l\delta}^i - \nu_{(l-1)\delta}^i).$$

For $l \leq k$, let

$$\begin{aligned} \tilde{\rho}^{i,\delta}(l\delta, k\delta) &:= \tilde{\rho}^{i,\delta}((l-1)\delta, k\delta) - \delta(r + \frac{1}{2}(b-r + \gamma\pi^{i,\delta}((l-1)\delta, k\delta))^2) \\ &\quad - (b-r + \gamma\pi^{i,\delta}((l-1)\delta, k\delta)) (\nu_{l\delta} - \nu_{(l-1)\delta}); \end{aligned} \quad (4.13)$$

with $\tilde{\rho}_0^{i,\delta} = 0$; for $l > k$, let

$$\begin{aligned} \tilde{\rho}^{i,\delta}(l\delta, k\delta) &:= \tilde{\rho}^{i,\delta}((l-1)\delta, k\delta) - \delta(r + \frac{1}{2}(b-r + \gamma\pi^{i,\delta}((l-1)\delta, k\delta))^2) \\ &\quad - (b-r + \gamma\pi^{i,\delta}((l-1)\delta, k\delta)) (\nu_{l\delta}^i - \nu_{(l-1)\delta}^i). \end{aligned} \quad (4.14)$$

Then $\rho^{i,\delta}(l\delta, k\delta) = \exp(\tilde{\rho}^{i,\delta}(l\delta, k\delta))$.

Next we approximate $N_i(t)$ by $N_i^{m,\delta}(k\delta)$, ($i = 1, 2, 3$; m is related to the SLLN, which will be chosen later). For all $s \in [t, T]$, $t \in [0, T]$, let $k = [nt]$, $j = [ns]$. Then $t \in [k\delta, (k+1)\delta)$ and $s \in [j\delta, (j+1)\delta)$. We define $N_i^{m,\delta}(k\delta)$, ($i = 1, 2, 3$) as follows:

$$\begin{aligned} N_1^{m,\delta}(k\delta) &= -(b-r + \gamma\pi^\delta(k\delta)) \frac{1}{m} \sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + 2c_2(\rho^{i,\delta}(T, k\delta))^2), \\ N_2^{m,\delta}(k\delta) &= \frac{1}{m} \sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + 2c_2(\rho^{i,\delta}(T, k\delta))^2) S_2^{i,\delta}(T, k\delta), \\ N_3^{m,\delta}(k\delta) &= \frac{1}{m} \sum_{i=1}^m -(c_1\rho^{i,\delta}(T, k\delta) + 2c_2(\rho^{i,\delta}(T, k\delta))^2) S_3^{i,\delta}(T, k\delta), \end{aligned}$$

where

$$\begin{aligned} S_2^{i,\delta}(T, k\delta) &= \sum_{l=1}^{n-k} D_{k\delta} \pi^{i,\delta}((l+k-1)\delta, k\delta) (\nu_{l\delta}^i - \nu_{(l-1)\delta}^i), \\ S_3^{i,\delta}(T, k\delta) &= \sum_{l=1}^{n-k} \delta(b-r + \gamma\pi^{i,\delta}((l+k-1)\delta, k\delta)) D_{k\delta} \pi^{i,\delta}((l+k-1)\delta, k\delta). \end{aligned}$$

In the above, $D_{k\delta}\pi^{i,\delta}(j\delta, k\delta)$, $(j = k, \dots, n-1)$ are still stochastic integrals. By (4.4), we define $D_{k\delta}\pi^{i,\delta}(j\delta, k\delta)$ only in one step. Namely, for $j \geq k$, we define

$$D_{k\delta}\pi^{i,\delta}(j\delta, k\delta) := D_{k\delta}\pi^{i,\delta}((j-1)\delta, k\delta) + \gamma(1 - 2\pi^{i,\delta}((j-1)\delta, k\delta)) D_{k\delta}\pi^{i,\delta}((j-1)\delta, k\delta) (\nu_{j\delta}^i - \nu_{(j-1)\delta}^i)$$

with $D_{k\delta}\pi^{i,\delta}(k\delta, k\delta) = \gamma\pi_{k\delta}(1 - \pi_{k\delta})$.

Finally, we obtain

$$\eta_{k\delta}^{\delta,m} = N_1^{m,\delta}(k\delta) + \gamma N_2^{m,\delta}(k\delta) + \gamma N_3^{m,\delta}(k\delta). \quad (4.15)$$

To summarize, we can approximate Y_t and u_t , $k\delta \leq t < (k+1)\delta$, by $Y_{k\delta}^{\delta,m}$ and $u_{k\delta}^{\delta,m}$, where

$$Y_{k\delta}^{\delta,m} = \Phi_{k\delta}^\delta \frac{1}{m} \sum_{i=1}^m \left(c_1 \rho^{i,\delta}(T, k\delta) + c_2 (\rho^{i,\delta}(T, k\delta))^2 \right) \quad (4.16)$$

and

$$u_{k\delta}^{\delta,m} = (b - r + \gamma\pi_{k\delta}^\delta) Y_{k\delta}^{\delta,m} + \Phi_{k\delta}^\delta \eta_{k\delta}^{\delta,m}. \quad (4.17)$$

In the proof of the following main theorem, we will need the following simple lemma which we cannot find a reference so we present it here for the convenience of the reader.

Lemma 4.1. *Let \mathcal{G} be the sub- σ -field. Suppose that given \mathcal{G} , ξ , ξ_1 , ξ_2 , \dots , ξ_m are i.i.d. random variables with $\mathbb{E}(\xi^4) < \infty$. Then,*

$$\mathbb{E} \left(\left| \frac{1}{m} \sum_{i=1}^m \xi_i - \mathbb{E}(\xi|\mathcal{G}) \right|^4 \right) \leq \frac{16}{m^2} \mathbb{E}(\xi^4).$$

Proof. Without loss of generality, we assume that $\mathbb{E}(\xi|\mathcal{G}) = 0$. Then,

$$\begin{aligned} \mathbb{E} \left(\left| \frac{1}{m} \sum_{n=1}^m \xi_i \right|^4 \right) &= \mathbb{E} \frac{1}{m^4} \sum_{i,j=1, i \neq j}^m \mathbb{E}(\xi_i^2 \xi_j^2 | \mathcal{G}) + \mathbb{E} \frac{1}{m^4} \sum_{i=1}^m \mathbb{E}(\xi_i^4 | \mathcal{G}) \\ &= \frac{m^2 - m}{m^4} \mathbb{E}(\mathbb{E}(\xi_1^2 | \mathcal{G}) \mathbb{E}(\xi_2^2 | \mathcal{G})) + \frac{m}{m^4} \mathbb{E} \xi_1^4 \\ &\leq \frac{1}{m^2} \mathbb{E}(\xi^4). \end{aligned} \quad (4.18)$$

For the general case $\mathbb{E}(\xi|\mathcal{G}) \neq 0$, since given \mathcal{G} , ξ , ξ_1 , ξ_2 , \dots , ξ_m are i.i.d. random variables, then $\mathbb{E}[(\xi_i - \mathbb{E}(\xi|\mathcal{G})) | \mathcal{G}] = 0$ holds for $i = 1, \dots, n$, thus by Jensen's inequality and applying (4.18), we have

$$\mathbb{E} \left(\left| \frac{1}{m} \sum_{n=1}^m \xi_i - \mathbb{E}(\xi|\mathcal{G}) \right|^4 \right) \leq \frac{1}{m^2} \mathbb{E}((\xi - \mathbb{E}(\xi|\mathcal{G}))^4) \leq \frac{16}{m^2} \mathbb{E}(\xi^4).$$

□

Theorem 4.2. *There exists a constant C such that for any $k\delta \leq T$, we have*

$$\|u_{k\delta} - u_{k\delta}^{\delta,m}\|_2 \leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right)$$

and

$$\|Y_{k\delta} - Y_{k\delta}^{\delta,m}\|_2 \leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right).$$

Proof. In the following, $C > 0$ will denote a generic constant, which may change from line to line.

Since we apply the Euler scheme for the new equation (4.12) which satisfies all the conditions in Theorem 2.3. Thus,

$$\|\pi_{k\delta} - \pi_{k\delta}^\delta\|_4 \leq C\sqrt{\delta}.$$

We claim that

$$\|\Phi_{k\delta} - \Phi_{k\delta}^\delta\|_4 \leq C\sqrt{\delta}, \quad \|D_{k\delta}\rho_T - D_{k\delta}\rho_T^\delta\|_4 \leq C\sqrt{\delta}.$$

In the following, we will verify the first inequality since the second one can be deduced similarly.

For easy computation, we rewrite the SDE (3.21) for Φ_t

$$\begin{cases} d\Phi_t = c(\pi_t)\Phi_t dt + h(\pi_t)\Phi_t d\nu_t, \\ \Phi_0 = 1, \end{cases}$$

where $c(x), h(x)$ are bounded functions ($x \in [0, 1]$) and Lipschitz continuous in x . By Jensen's inequality, the Burkholder-Davis-Gundy inequality and Grönwall's inequality, we get

$$\|\Phi_t\|_8 \leq Ce^{CT}, \quad \text{for all } t \in [0, T]. \quad (4.19)$$

Note that

$$\Phi_{k\delta}^\delta = 1 + \sum_{i=1}^k (c(\pi_{(i-1)\delta}^\delta)\Phi_{(i-1)\delta}^\delta) \delta + \sum_{i=1}^k h(\pi_{(i-1)\delta}^\delta) \Phi_{(i-1)\delta}^\delta (\nu_{i\delta} - \nu_{(i-1)\delta}),$$

then

$$\begin{aligned} & \mathbb{E}|\Phi_{k\delta} - \Phi_{k\delta}^\delta|^4 \\ & \leq C\mathbb{E} \left| \int_0^{k\delta} c(\pi_s)\Phi_s ds - \sum_{i=1}^k \delta c(\pi_{(i-1)\delta}^\delta)\Phi_{(i-1)\delta}^\delta \right|^4 \\ & \quad + C\mathbb{E} \left| \int_0^{k\delta} h(\pi_s)\Phi_s d\nu_s - \sum_{i=1}^k h(\pi_{(i-1)\delta}^\delta)\Phi_{(i-1)\delta}^\delta (\nu_{i\delta} - \nu_{(i-1)\delta}) \right|^4 \\ & = I_1 + I_2, \end{aligned} \quad (4.20)$$

by Jensen's and Hölder's inequality, we get

$$\begin{aligned}
I_1 &= C\mathbb{E} \left| \int_0^{k\delta} c(\pi_s)\Phi_s ds - \int_0^{k\delta} c(\pi_s^\delta)\Phi_s^\delta ds \right|^4 \\
&= C\mathbb{E} \left| \int_0^{k\delta} (c(\pi_s)\Phi_s - c(\pi_s^\delta)\Phi_s^\delta) ds \right|^4 \\
&\leq C\mathbb{E} \left| \int_0^{k\delta} (c(\pi_s) - c(\pi_s^\delta))\Phi_s ds \right|^4 + C\mathbb{E} \left| \int_0^{k\delta} c(\pi_s^\delta)(\Phi_s - \Phi_s^\delta) ds \right|^4 \\
&\leq C\mathbb{E} \int_0^{k\delta} (c(\pi_s) - c(\pi_s^\delta))^4 \Phi_s^4 ds + C\mathbb{E} \int_0^{k\delta} (c(\pi_s^\delta))^4 (\Phi_s - \Phi_s^\delta)^4 ds \\
&\leq C \int_0^{k\delta} \left(\mathbb{E} (c(\pi_s) - c(\pi_s^\delta))^8 \right)^{\frac{1}{2}} \|\Phi_s\|_8^4 ds \\
&\quad + C\mathbb{E} \int_0^{k\delta} (c(\pi_s^\delta))^4 (\Phi_s - \Phi_s^\delta)^4 ds.
\end{aligned}$$

Using the fact that c is bounded, Lipschitz continuous, and (4.19), we have

$$\begin{aligned}
I_1 &\leq C \int_0^{k\delta} \|\pi_s - \pi_s^\delta\|_8^4 ds + C \int_0^{k\delta} \mathbb{E} |\Phi_s - \Phi_s^\delta|^4 ds \\
&\leq Ck\delta \|\pi_s - \pi_s^\delta\|_8^4 + C \int_0^{k\delta} \mathbb{E} |\Phi_s - \Phi_s^\delta|^4 ds \\
&\leq C\delta^2 + C \int_0^{k\delta} \mathbb{E} |\Phi_s - \Phi_s^\delta|^4 ds, \tag{4.21}
\end{aligned}$$

the last inequality follows by using the Theorem 2.3. Similarly, I_2 can be estimated by the Burkholder-Davis-Gundy inequality and Hölder's inequality, thus we obtain

$$I_2 \leq C\delta^2 + C \int_0^{k\delta} \mathbb{E} |\Phi_s - \Phi_s^\delta|^4 ds, \tag{4.22}$$

plugging inequalities (4.21) and (4.22) into (4.20), we have

$$\mathbb{E} |\Phi_{k\delta} - \Phi_{k\delta}^\delta|^4 \leq C\delta^2 + C \int_0^{k\delta} \mathbb{E} |\Phi_s - \Phi_s^\delta|^4 ds,$$

an application of Grönwall's inequality, we finally obtain the following estimation:

$$\mathbb{E} |\Phi_{k\delta} - \Phi_{k\delta}^\delta|^4 \leq C\delta^2 e^{CT},$$

which implies

$$\|\Phi_{k\delta} - \Phi_{k\delta}^\delta\|_4 \leq C\sqrt{\delta}.$$

From the representation (3.18) and the approximation (4.17), we first estimate the error between $u_{k\delta}$ and $u_{k\delta}^{\delta,m}$,

$$\begin{aligned}
& \left\| u_{k\delta} - u_{k\delta}^{\delta,m} \right\|_2 \\
& \leq \left\| (b-r + \gamma\pi_{k\delta})\Phi_{k\delta}\mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) \right. \\
& \quad \left. - (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta\frac{1}{m}\sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + c_2(\rho^{i,\delta}(T, k\delta))^2) \right\|_2 \\
& \quad + \left\| \Phi_{k\delta}\mathbb{E}(c_1D_{k\delta}\rho_T + 2c_2\rho_TD_{k\delta}\rho_T|\mathcal{G}_{k\delta}) \right. \\
& \quad \left. - \Phi_{k\delta}^\delta\frac{1}{m}\sum_{i=1}^m (c_1D_{k\delta}\rho^{i,\delta}(T, k\delta) + 2c_2\rho^{i,\delta}(T, k\delta)D_{k\delta}\rho^{i,\delta}(T, k\delta)) \right\|_2 \\
& := J_1 + J_2. \tag{4.23}
\end{aligned}$$

We firstly estimate

$$\begin{aligned}
J_1 &= \left\| (b-r + \gamma\pi_{k\delta})\Phi_{k\delta}\mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) \right. \\
& \quad \left. - (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta\frac{1}{m}\sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + c_2(\rho^{i,\delta}(T, k\delta))^2) \right\|_2 \\
& \leq \left\| (b-r + \gamma\pi_{k\delta})\Phi_{k\delta}\mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) - (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta\mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) \right\|_2 \\
& \quad + \left\| (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta\mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) \right. \\
& \quad \left. - (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta\frac{1}{m}\sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + c_2(\rho^{i,\delta}(T, k\delta))^2) \right\|_2 \\
& \leq \left\| (b-r + \gamma\pi_{k\delta})\Phi_{k\delta} - (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) \right\|_4 \\
& \quad + \left\| (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1\rho_T + c_2\rho_T^2|\mathcal{G}_{k\delta}) - \mathbb{E}(c_1\rho_T^\delta + c_2(\rho_T^\delta)^2|\mathcal{G}_{k\delta}) \right\|_4 \\
& \quad + \left\| (b-r + \gamma\pi_{k\delta}^\delta)\Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1\rho_T^\delta + c_2(\rho_T^\delta)^2|\mathcal{G}_{k\delta}) - \frac{1}{m}\sum_{i=1}^m (c_1\rho^{i,\delta}(T, k\delta) + c_2(\rho^{i,\delta}(T, k\delta))^2) \right\|_4 \\
& \leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right), \tag{4.24}
\end{aligned}$$

where in the last inequalities of (4.24) we apply Lemma 4.1. Additionally,

$$\begin{aligned}
J_2 &\leq \left\| \Phi_{k\delta} - \Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1 D_{k\delta} \rho_T + 2c_2 \rho_T D_{k\delta} \rho_T | \mathcal{G}_{k\delta}) \right\|_4 \\
&\quad + \left\| \Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1 D_{k\delta} \rho_T + 2c_2 \rho_T D_{k\delta} \rho_T | \mathcal{G}_{k\delta}) - \mathbb{E}(c_1 D_{k\delta} \rho_T^\delta + 2c_2 \rho_T^\delta D_{k\delta} \rho_T^\delta | \mathcal{G}_{k\delta}) \right\|_4 \\
&\quad + \left\| \Phi_{k\delta}^\delta \right\|_4 \times \left\| \mathbb{E}(c_1 D_{k\delta} \rho_T^\delta + 2c_2 \rho_T^\delta D_{k\delta} \rho_T^\delta | \mathcal{G}_{k\delta}) \right. \\
&\quad \quad \left. - \frac{1}{m} \sum_{i=1}^m (c_1 D_{k\delta} \rho^{i,\delta}(T, k\delta) + 2c_2 \rho^{i,\delta}(T, k\delta) D_{k\delta} \rho^{i,\delta}(T, k\delta)) \right\|_4 \\
&\leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right). \tag{4.25}
\end{aligned}$$

Then by (4.23), (4.24) and (4.25), we have

$$\left\| u_{k\delta} - u_{k\delta}^{\delta,m} \right\|_2 \leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right).$$

Similarly, we can prove

$$\left\| Y_{k\delta} - Y_{k\delta}^{\delta,m} \right\|_2 \leq C \left(\sqrt{\delta} + \frac{1}{\sqrt{m}} \right). \tag{4.26}$$

□

Remark 4.2. *The optimal convergence rate under our scheme is $\frac{1}{\sqrt{m}}$ which is obtained by taking $m = n$. The error term $\sqrt{\delta}$ in last theorem comes from Euler scheme which can be improved by employing more efficient schemes. The term \sqrt{m} comes from conditional SLLN which seems cannot be improved.*

Remark 4.3. *In Xiong and Zhou's [32] model, only the convergence of Y_t^n to Y_t is proved, however, due to technical difficulty, the rate of convergence of Y_t^n to Y_t is not established in that paper. The above theorem not only proves the convergence of Y_t^n to Y_t , but also gives the rate of convergence. Note that the errors in our numerical scheme consist of the error from the Euler approximation and that from the SLLN only. From this point of view, under the drift uncertainty model, the numerical scheme we proposed is more efficient than that of [32].*

4.2 Numerical results

In last subsection, we proposed an efficient numerical scheme with respect to BSDEs of the form (3.17) with random coefficients and control variable. Now we use Matlab to compare our method with that of [32]. However, there is no explicit solution for (3.17), thus we are not

able to compare these two different numerical schemes for the mean variance problem directly. To overcome this difficulty, we consider a BSDE whose drift term is random, and the solution $(X(t), Z(t))$ can be computed explicitly. Then we obtain the numerical approximations for $(X(t), Z(t))$ by applying two different schemes to this BSDE. Next, we compare these two approximations with the actual solution. Finally, we apply the new algorithm to simulate the drift uncertainty model.

For convenience, we denote the numerical method proposed by Xiong and Zhou [32] as the “old algorithm”, the one proposed by us as the “new algorithm” and the explicit solution as the “true value”.

For the appearance of presentation, we now write W_t as $W(t)$ (especially when t has a complicated expression). Let

$$H(t) = \int_0^t (1 + W(s))dW(s) - \int_0^t (W(s)^2 + 2W(s))ds.$$

We consider a BSDE with random coefficients as following

$$\begin{cases} dX(t) = \left(-\frac{1}{2}(1 - 2W(t) - W(t)^2)X(t) - (1 + W(t))Z(t) \right)dt + Z(t)dW(t), \\ X(T) = \exp(H(T) - 2T), \quad t \in [0, T]. \end{cases} \quad (4.27)$$

Following the steps (see Theorem 2.2, Chapter 7 of [34]), we can solve the above linear BSDE (4.27) explicitly as

$$X(t) = e^{H(t)-2t}, \quad Z(t) = (1 + W(t))X(t). \quad (4.28)$$

Note that the drift term of the above example (4.27) has random coefficient, which is a feature shared by the BSDE (3.17). Since we do not have explicit solution to (3.17), as a compromise, we compare the “new algorithm” with the “old algorithm” by using the example (4.27) which has explicit solution (4.28).

For simplicity, let $T = 1$. We discrete $[0, 1]$ into $n_1 n_2$ small intervals. Denote $\delta_1 = \frac{1}{n_1}$ and $\delta_2 = \frac{1}{n_1 n_2}$.

Let $\theta = \exp\left(2 \int_0^T (1 + W(s))dW(s) - 2 \int_0^T (1 + W(s))^2 ds\right)$ and $\Phi(t) = e^{-H(t)}$. Using the old algorithm, the approximation for $(X(t), Z(t))$ is given by

$$\begin{cases} X^{m, \delta_1}(t) = \Phi^{\delta_1}(t)N^{m, \delta_1}(t), \\ Z^{m, \delta_1}(t) = -X^{m, \delta_1}(t) + \Phi^{\delta_1}(t)\eta_1^{m, \delta_1}(t), \end{cases}$$

where

$$N(t) = \mathbb{E}(\theta | \mathcal{F}_t^W),$$

(\mathcal{F}_t^W is the σ -field generated by the values of the Brownian motion W up to and including time t) and $\Phi^{\delta_1}(t)$ is the approximation of $\Phi(t)$ generated by the Euler scheme, $N^{m,\delta_1}(t)$ is the approximation of $N(t)$ generated by the particle representation as well as the Euler scheme, and

$$\begin{aligned} \eta_1^{m,\delta_1} \left(\frac{k}{n_1} \right) &:= n_1 \sum_{j=1}^{n_2} \left(N^{m,\delta_2} \left(\frac{k-1}{n_1} + \frac{j}{n_1 n_2} \right) - N^{m,\delta_2} \left(\frac{k-1}{n_1} + \frac{j-1}{n_1 n_2} \right) \right) \\ &\quad \times \left(W^{\delta_2} \left(\frac{k-1}{n_1} + \frac{j}{n_1 n_2} \right) - W^{\delta_2} \left(\frac{k-1}{n_1} + \frac{j-1}{n_1 n_2} \right) \right), \end{aligned} \quad (4.29)$$

$$k = 1, 2, \dots, n_1 - 1, \quad n_1, n_2 = 1, 2, \dots.$$

On the other hand, since θ is Malliavin differentiable, we get

$$\begin{aligned} \eta_2(t) &:= \mathbb{E} (D_t \theta | \mathcal{F}_t^W) \\ &= \mathbb{E} \left(e^{2 \int_t^T (1+W(s)) dW(s) - 2 \int_t^T (1+W(s))^2 ds} \left[2W(T) - 4 \int_t^T (1+W(s)) ds + 2 \right] \middle| \mathcal{F}_t^W \right). \end{aligned} \quad (4.30)$$

By the new algorithm, the approximation for $(X(t), Z(t))$ is given by

$$\begin{cases} X^{m,\delta_2}(t) = \Phi^{\delta_2}(t) N^{m,\delta_2}(t), \\ Z^{m,\delta_2}(t) = -X^{m,\delta_2}(t) + \Phi^{\delta_2}(t) \eta_2^{m,\delta_2}(t), \end{cases}$$

where $\eta_2^{m,\delta_2}(t)$ is the approximation of $\eta_2(t)$ generated by the particle representation as well as the Euler scheme.

Using the aforementioned algorithms, we generate the following figures.

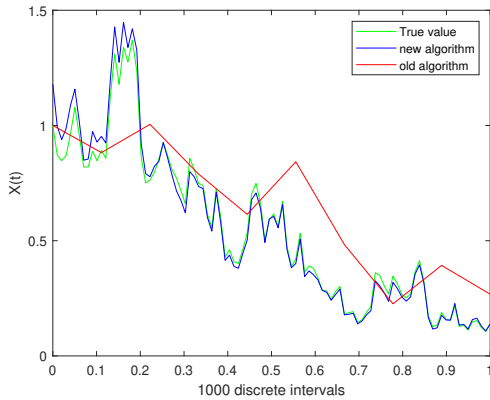


Figure 1: $X(t)$ with 100 discrete intervals

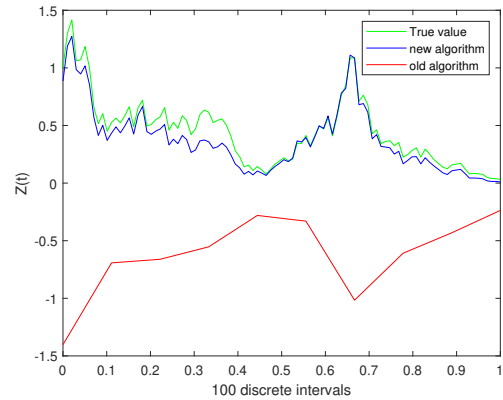
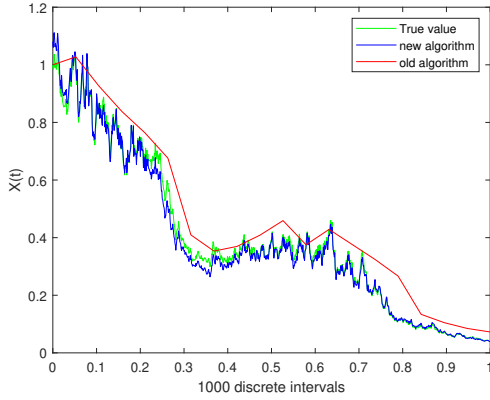
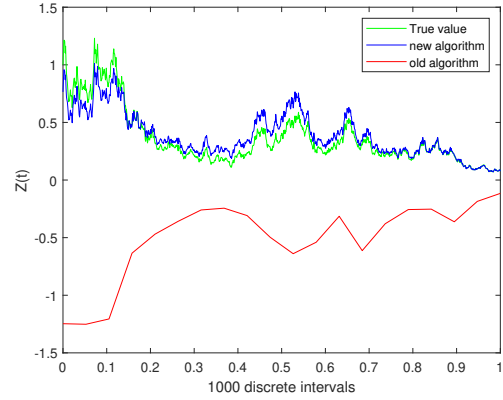


Figure 2: $Z(t)$ with 100 discrete intervals

Figure 3: $X(t)$ with 10^3 discrete intervalsFigure 4: $Z(t)$ with 10^3 discrete intervals

It can be seen from Figures 1, 2, 3, 4 that our new numerical scheme well approximated the true processes $X(t)$ and $Z(t)$. The curves of $X(t)$ and $Z(t)$ generated by the new scheme are almost the same as the true processes. In contrast, the paths generated by the old scheme are relatively far off due to the double-partition which is the main drawback in that scheme. More precisely, the inefficiency of the old algorithm results from the double-partition when generating η^{m,δ_1} , as we can see from (4.29), to obtain η^{m,δ_1} , the interval is partitioned into $n_1 \times n_2$ subintervals, this procedure increases the error dramatically. While the new algorithm overcomes this drawback by using the Malliavin calculus, the term η_2 can be represented explicitly by (4.30). Since $Z(t)$ contains the term η , therefore, the improvement of the approximation to the second component $Z(t)$ of the solution of the BSDE by the new algorithm over the old algorithm appears to be more pronounced than the respective improvement of the approximation to the first component $X(t)$.

Finally, we apply our efficient numerical scheme to simulate the wealth process Y_t and the optimal control u_t for the drift uncertainty model.

We choose the parameters as following: $n = 1000$, $\delta = \frac{1}{1000}$, $m = 1000$, $r = 0.03$, $a = 0.04$, $b = 0.032$, $y_0 = 100$, $z = y_0 \cdot (1 + r + 0.03)$, and $\pi_0 = 0.1$.

Figure 5 below is the numerical results for the innovation process ν_t , the wealth process Y_t and the optimal control u_t .

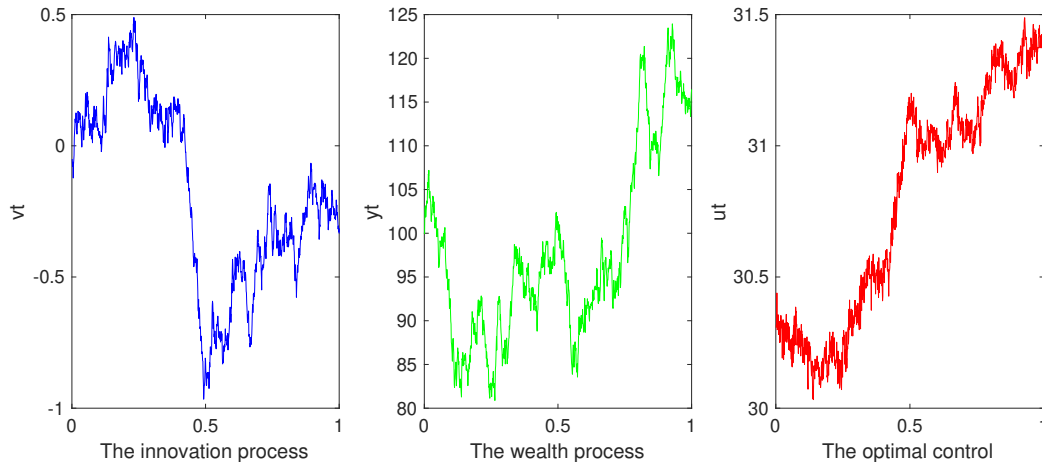


Figure 5: drift uncertainty model with 10^3 discrete intervals.

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