A discrete-time mean-field stochastic linear-quadratic optimal control problem with financial application

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ABSTRACT

This paper is concerned with a discrete-time mean-field stochastic linear-quadratic optimal control problem arising from financial application. Through matrix dynamical optimization method, a group of linear feedback controls is investigated. The problem is then reformulated as an operator stochastic linear-quadratic optimal control problem by a sequence of bounded linear operators over Hilbert space, the optimal control with six algebraic Riccati difference equations is obtained by backward induction. The two above approaches are proved to be coincided by the classical method of completing the square. Finally, after discussing the solution of the problem under multidimensional noises, a financial application example is given.

KEYWORDS

mean-field theory; Riccati difference equation; stochastic linear-quadratic optimal control problem

1. Introduction

In this paper, we consider a discrete-time mean-field stochastic linear-quadratic optimal control problem. Here the terms ‘mean-field’ and ‘linear-quadratic’ refer to a dynamic model exhibiting macroscopic behaviour of an attractive mean-field interaction and linear stochastic systems with quadratic performance criterion, respectively. In the combination of these two issues, the investigation of classical mean-field stochastic differential equation (SDE) problems can be traced back to 1960s, when McKean (1966) first discussed a similar connection between a series of Markov processes and certain non-linear parabolic equations. Then, many scientific results are emerging. Dawson (1983) investigated the dynamics and fluctuations of mean-field systems in the critical condition by adopting approach based on the theory in (Papanicolaou, Stroock & Varadhan, 1977) for Markov processes. Dawsont & Gärtner (1987) examined the conversion from the large deviations from the McKean-Vlasov limit to a

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generalization of the theory of Freidlin & Wentzell (1984). Gärtner (1988) systematically gave research results for a system of diffusions in a domain range of $\mathbb{R}^d$ with long-range weak interaction. Similar issues can refer to (Bossy & Talay, 1997; Chan, 1994; Dai Pra & den Hollander, 1996). Buckdahn et al. (2009) considered a special approximation on the solution of some decoupled forward-backward equations and gave the convergence speed. The problem was then investigated under a more general framework in (Buckdahn, Li & Peng, 2009).

In theoretical research, the field of stochastic optimal control has made great progress. Rockafellar & Wets (1990) considered some generalized stochastic linear-quadratic optimal control problems in discrete time. In a discrete-time system, *Riccati difference equation* plays an important role in the synthesis of the optimal control. Beghi & D'alessandro (1998) derived the optimal control for a discrete-time linear-quadratic problem with control-dependent noise. Moore, Zhou & Lim (1999) considered some partially observed stochastic models where the stochastic disturbances depend on both the states and the controls. Ait Rami, Chen & Zhou (2002) extended Beghi and D'alessandro's result through allowing indefinite weighting matrices in the cost functional. Huang, Zhang & Zhang (2008) discussed the problem with an infinite horizon, in which the concepts of stochastic stabilizability and exact observability are introduced.

Attention is also focused on the solution of optimal control problems with mean-field terms. By introducing the mean terms into the cost functional, the variations of the state process and the control process can be minimized so that they are not too sensitive to random events (Yong, 2013). Andersson, & Djehiche (2011) studied this problem under the assumption of convex action space. Such assumption is consensus in further research. Du, Li & Wei (2011) showed the existence of solution and obtained a theorem of comparison for one dimensional mean-field backward stochastic differential equations. Yong (2013) considered a linear-quadratic optimal control problem, which consists of continuous-time mean-field stochastic differential equations with deterministic coefficients. Elliott, Li & Ni (2013) considered a discrete-time optimal control problem and discussed different methods for solving the problem. Necessary and sufficient conditions for the solvability of the problem were presented.

For recent research, this problem is extended in two aspects: with indefinite weight matrices in the cost functional and with an infinite horizon, see (Ni, Li & Zhang, 2014; Ni, Elliott & Li, 2015; Ni, Zhang & Li, 2015; Ni, Li & Zhang, 2016). Sun & Yong (2016) showed that the non-emptiness of the admissible control set for all initial state is equivalent to the $\mathcal{L}^2$-stabilizability of the control system by concerning continuous-time model in an infinite horizon with constant coefficients. Li et al. summarized their recent research results for a linear mean-field stochastic differential equation with a quadratic cost functional in (Li, Sun & Yong, 2016). Readers may refer to literature such as (Andersson, & Djehiche, 2011; Bensoussan, Frehse & Yam, 2013; Buckdahn, Djehiche & Li, 2011; Meyer-Brandis, Oksendal & Zhou, 2012) for some other mean-field type control problems.

On the other hand, mean-field game is also a hot research topic in mean-field theory. Huang, Malhamé & Caines (2006) decomposed a class of stochastic games into optimal controls problems and designated the *Nash certainty equivalence principle* as a property of solvability. Bensoussan et al. (2016) studied the unique existence of an equilibrium strategies of linear-quadratic mean field games (MFGs) by adjoint equation method. For relevant literatures, readers can refer to (Bauso, Tembine & Başar, 2012; Bensoussan, Frehse & Yam, 2013; Carmona & Delarue, 2013; Carmona & Lacker, 2015; Guéant, Lasry & Lions, 2011).

Stochastic optimal control theory has been widely applied in varies practical problems since Wonham's work in 1968 (Wonham, 1968). The development of mathematical mean-field stochastic linear-quadratic optimal control theory has greatly promoted the research of related applications in recent works. Zhou & Yin (2003) studied a continuous-time regime-switching model for portfolio selection, where a Markov chain modulated diffusion formulation was used to model the problem. Xie, Li & Wang (2008) investigated mean-variance portfolio selection problems using general stochastic control technique. An incomplete market was studied with correlative multiple risky assets and a liability according to a Brownian motion with drift. By adopting the techniques in (Zhou & Yin, 2003), Chen, Yang & Yin (2008) investigated the feasibility and obtained the optimal strategy. The corresponding efficient frontier was also delineated, and hence the associated mutual fund theorem over a continuous-time Markov regime-switching model was established. Cui, Li & Li (2014) proposed a new mean-field framework that provides a more efficient modelling tool and accurate solution to solve sustainability problems. Dang, Forsyth & Li (2016) considered Markowitz's problem through method of transforming the problem into an equivalent one with bankruptcy prohibition but without portfolio constraints and then treated by martingale theory. Literatures can be referred to (Cui, Li & Li, 2015; Hou & Xu, 2016; Zhang & Chen, 2016; Ziemba, 2003).

The problem studied in this paper arises from a practical problem in finance. When making investment decisions, investors not only consider the current assets but also the liabilities of the investors. Most of the existing research works in financial applications only consider the investment entity's equity assets without considering debt. In order to adapt practical application, we consider a system consists of n assets and n liabilities in this study. The system state is adjusted to two linear stochastic difference equations with several cost-functional affected variables. Under a series of necessary and sufficient conditions for the solvability of the problem, Ricatti equations of the adjusted model are obtained and a more general framework from mean-field linear-quadratic controls theory to financial applications is provided. The remainder of the paper is organized as follows. In Section 2, we give the formulation for the problem. Preliminaries for the analyses are presented in Section 3. In Section 4, the closed-loop optimal control is obtained and then it is represented via Riccati equations. Some financial applications of the problem are presented in Section 5. The paper is then concluded in Section 6.

2. Problem Formulation

Let N be a positive integer. The system equation is the following set of linear stochastic difference equations with $k \in \{0, 1, 2, \cdots, N-1\} \equiv \mathbb{N}$,

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E} x_k + B_k u_k + \bar{B}_k \mathbb{E} u_k) \\ \qquad\quad +(C_k x_k + \bar{C}_k \mathbb{E} x_k + D_k u_k + \bar{D}_k \mathbb{E} u_k) w_k, \\ y_{k+1} = (F_k y_k + \bar{F}_k \mathbb{E} y_k) + (G_k y_k + \bar{G}_k \mathbb{E} y_k) v_k, \\ x_0 = \zeta^x, y_0 = \zeta^y, \end{cases} \tag{1}$$

where $x_k, y_k \in \mathbb{R}^n$. $A_k, \bar{A}_k, C_k, \bar{C}_k, F_k, \bar{F}_k, G_k, \bar{G}_k \in \mathbb{R}^{n\times n}$, and $B_k, \bar{B}_k, D_k, \bar{D}_k \in \mathbb{R}^{n\times m}$ are given deterministic matrices. $\mathbb{E}$ is the expectation operator. Denote the set $\{0, 1, 2, \cdots, N\}$ by $\bar{\mathbb{N}}$. In (1), $\{x_k, k \in \bar{\mathbb{N}}\}$ and $\{y_k, k \in \bar{\mathbb{N}}\}$ are the state processes and $\{u_k \in \mathbb{R}^m, k \in \mathbb{N}\}$ is a control process. $\{w_k, v_k, k \in \mathbb{N}\}$ are defined on probability space $(\Omega, \mathcal{F}, P)$, represent the stochastic distribution for the two state processes, and are assumed to be martingale difference sequences

$$\begin{aligned} &\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0,\ \mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1,\ \mathbb{E}[v_{k+1}|\mathcal{F}_k] = 0, \\ &\mathbb{E}[(v_{k+1})^2|\mathcal{F}_k] = 1,\ \mathbb{E}[w_{k+1}v_{k+1}|\mathcal{F}_k] = \rho, \end{aligned} \tag{2}$$

where $\mathcal{F}_k$ is the σ-algebra generated by $\{\zeta^x, w_l, l = 0, 1, \cdots, k\}$ and $\{\zeta^y, v_l, l = 0, 1, \cdots, k\}$. The cost functional associated with (1) is

$$\begin{aligned} J(\zeta^x, \zeta^y, u) = \ & \sum_{k=0}^{N-1} \mathbb{E}\Big((x_k - y_k)^T Q_k (x_k - y_k) + u_k^T R_k u_k \\ & +\mathbb{E}(x_k - y_k)^T \bar{Q}_k \mathbb{E}(x_k - y_k) + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E} u_k\Big) \\ & +\mathbb{E}\left((x_N - y_N)^T Q_N (x_N - y_N)\right) + \mathbb{E}(x_N - y_N)^T \bar{Q}_N \mathbb{E}(x_N - y_N), \end{aligned} \tag{3}$$

where $Q_k, \bar{Q}_k, k \in \bar{\mathbb{N}}$ and $R_k, \bar{R}_k, k \in \mathbb{N}$ are deterministic symmetric matrices with appropriate dimensions. We introduce the following admissible control set of $u = (u_0, u_1, \cdots, u_{N-1})$

$$\mathcal{U}_{ad} \equiv \left\{u \ \middle|\ u_k \in \mathbb{R}^m,\ \text{is } \mathcal{F}_k\text{-measurable},\ \mathbb{E}|u_k|^2 < \infty\right\}.$$

The optimal control problem considered in this paper is then stated as follows:

Problem (MF-LQ). *For any given square-integrable initial values ζ^x and ζ^y, find $u^o \in \mathcal{U}_{ad}$ such that*

$$J(\zeta^x, \zeta^y, u^o) = \inf_{u\in\mathcal{U}_{ad}} J(\zeta^x, \zeta^y, u). \tag{4}$$

We then call u^o an optimal control for Problem (MF-LQ).

3. Preliminaries

In this section, we convert Problem (MF-LQ) to a quadratic optimization problem in Hilbert space. After a series of statements of standard notation and definition, we give necessary and sufficient conditions for the solvability of the Problem (MF-LQ). Firstly, some spaces are introduced as follows: for $k \in \bar{\mathbb{N}}$,

$$\mathcal{Z}_k = L^2_{\mathcal{F}_k}(\mathbb{R}^n) = \left\{ \xi : \Omega \mapsto \mathbb{R}^n \,\middle|\, \xi \text{ is } \mathcal{F}_k\text{-measurable, } \mathbb{E}|\xi|^2 < \infty \right\},$$

$$\mathcal{Z}[0,k] = \left\{ (z_0, ..., z_k) \,\middle|\, z_k \in \mathcal{Z}_k, \text{ is } \mathcal{F}_k\text{-measurable, } \textstyle\sum_{l=0}^{k} \mathbb{E}|z_l|^2 < \infty \right\},$$

and for $l \in \mathbb{N}$,

$$\mathcal{U}_l = L^2_{\mathcal{F}_l}(\mathbb{R}^m) = \left\{ \eta : \Omega \mapsto \mathbb{R}^m \,\middle|\, \eta \text{ is } \mathcal{F}_l\text{-measurable, } \mathbb{E}|\eta|^2 < \infty \right\}.$$

Here, $\mathcal{Z}_k$, $\mathcal{U}_l$ and $\mathcal{Z}[0,k]$ are Hilbert spaces. There are two cases of the domain and range of expectation operator: $\mathbb{E}$ maps $\mathcal{Z}_k$ to $\mathbb{R}^n$ or $\mathcal{U}_l$ to $\mathbb{R}^m$ (Elliott, Li & Ni, 2013). Therefore, the notation $\mathbb{E}$ and adjoint operator $\mathbb{E}^*$ may differ from place to place. Let $\mathcal{H} = \mathcal{Z}_k, \mathcal{U}_l$. For illustration, we now use $\mathbb{E}_{\mathcal{H}}$ and $\mathbb{E}^*_{\mathcal{H}}$ to emphasize $\mathcal{H}$. $\mathbb{E}$ and $\mathbb{E}^*$ may appear in the form of $M\mathbb{E}_{\mathcal{H}}$ and $\mathbb{E}^*_{\mathcal{H}} N \mathbb{E}_{\mathcal{H}'}$ where M, N are matrices with appropriate dimensions and $\mathcal{H}$, $\mathcal{H}'$ can be different. To simplify the expressions in this paper, $\bar{A}\mathbb{E}z$, $\bar{B}\mathbb{E}u$, $\mathbb{E}^*\bar{Q}\mathbb{E}z$, $\mathbb{E}^*\bar{L}\mathbb{E}u$, $\mathbb{E}^*\bar{R}\mathbb{E}u$ are used to denote $\bar{A}\mathbb{E}_{\mathcal{Z}_k}(z)$, $\bar{B}\mathbb{E}_{\mathcal{U}_k}(u)$, $\mathbb{E}^*_{\mathcal{Z}_k}\bar{Q}\mathbb{E}_{\mathcal{Z}_k}(z)$, $\mathbb{E}^*_{\mathcal{Z}_k}\bar{L}\mathbb{E}_{\mathcal{U}_k}(u)$, $\mathbb{E}^*_{\mathcal{U}_k}\bar{R}\mathbb{E}_{\mathcal{U}_k}(u)$, respectively. Here, $z \in \mathcal{Z}_k$, $u \in \mathcal{U}_k$, $\bar{A}, \bar{Q} \in \mathbb{R}^{n\times n}$, $\bar{B}, \bar{L} \in \mathbb{R}^{n\times m}$, $\bar{R} \in \mathbb{R}^{m\times m}$.

Definition 3.1. (i). Problem (MF-LQ) is said to be finite for ζ^x and ζ^y if

$$\inf_{u\in\mathcal{U}_{ad}} J(\zeta^x, \zeta^y, u) > -\infty.$$

Problem (MF-LQ) is said to be finite if it is finite for any ζ^x and ζ^y.
(ii). Problem (MF-LQ) is said to be uniquely solvable for ζ^x and ζ^y if there exists a unique $u^o \in \mathcal{U}_{ad}$ such that (4) holds for ζ^x and ζ^y. Problem (MF-LQ) is said to be uniquely solvable if it is uniquely solvable for any ζ^x and ζ^y.

We express the system states explicitly in terms of k. Let

$$\begin{cases} \bar{\Phi}(k,l) = \prod_{i=l}^{k}(A_i + \bar{A}_i), \quad \Phi(k,l) = \prod_{i=l}^{k}(A_i + w_i C_i), \\ \bar{\Xi}(k,l) = \prod_{i=l}^{k}(F_i + \bar{F}_i), \quad \Xi(k,l) = \prod_{i=l}^{k}(F_i + v_i G_i) \end{cases}$$

for $k \geq l$ and $\bar{\Phi}(k,l) = \Phi(k,l) = \bar{\Xi}(k,l) = \Xi(k,l) = I$ for $k < l$. Define the following operators on $\zeta^x, \zeta^y \in \mathcal{Z}_0$, $u \in \mathcal{U}_{ad}$ for $k \in \bar{\mathbb{N}}$:

$$\begin{cases} \Gamma_k(\zeta^x) = & \Phi(k-1,0)\zeta^x, \\ \bar{\Gamma}_k(\zeta^x) = & \sum_{l=1}^{k-1} \big(\Phi(k-1,l)(\bar{A}_{l-1} + w_{l-1}\bar{C}_{l-1})\bar{\Phi}(l-2,0)\mathbb{E}\zeta^x\big), \\ \Psi_k(\zeta^y) = & \Xi(k-1,0)\zeta^y, \\ \bar{\Psi}_k(\zeta^y) = & \sum_{l=1}^{k-1} \big(\Xi(k-1,l)(\bar{F}_{l-1} + v_{l-1}\bar{G}_{l-1})\bar{\Xi}(l-2,0)\mathbb{E}\zeta^y\big), \\ L_k(u) = & \sum_{l=1}^{k-1} \Phi(k-1,l)(B_{l-1} + w_{l-1}D_{l-1})u_{l-1} + (B_{k-1} + w_{k-1}D_{k-1}u_{k-1}), \\ \bar{L}_k(u) = & \sum_{l=1}^{k-1} \Phi(k-1,l+1)(\bar{A}_{l-1} + w_{l-1}\bar{C}_{l-1}) \sum_{i=1}^{l-1} \bar{\Phi}(l-2,i)(B_{i-1} + \bar{B}_{i-1})\mathbb{E}u_{i-1} \\ & + \sum_{l=1}^{k-1} \Phi(k-1,l)(\bar{B}_{l-1} + w_{l-1}\bar{D}_{l-1})\mathbb{E}u_{l-1} + (\bar{B}_{k-1} + w_{k-1}\bar{D}_{k-1})\mathbb{E}u_{k-1}, \end{cases}$$

where $\Gamma_k, \bar{\Gamma}_k, \Psi_k, \bar{\Psi}_k : \mathcal{Z}_0 \mapsto \mathcal{Z}[0,k],\ k \in \bar{\mathbb{N}}$, $L_k, \bar{L}_k : \mathcal{U}_{ad} \mapsto \mathcal{Z}[0,k],\ k \in \bar{\mathbb{N}}$, are linear and bounded. Then the system states can be expressed as

$$\begin{cases} x_k = \Gamma_k(\zeta^x) + \bar{\Gamma}_k(\zeta^x) + L_k(u) + \bar{L}_k(u) \\ y_k = \Psi_k(\zeta^y) + \bar{\Psi}_k(\zeta^y). \end{cases}$$

The cost functional $J(\zeta^x, \zeta^y, u)$ has the following form of usual inner products

$$\begin{aligned} J = & \sum_{k=0}^{N-1} \Big(\langle Q_k x_k, x_k \rangle - 2\langle Q_k x_k, y_k \rangle + \langle Q_k y_k, y_k \rangle + \langle \bar{Q}\mathbb{E}x_k, \mathbb{E}x_k \rangle \\ & -2\langle \bar{Q}_k \mathbb{E}x_k, \mathbb{E}y_k \rangle + \langle \bar{Q}_k \mathbb{E}y_k, \mathbb{E}y_k \rangle + \langle R_k u_k, u_k \rangle + \langle \bar{R}_k \mathbb{E}u_k, \mathbb{E}u_k \rangle \Big) \\ & + \langle Q_N x_N, x_N \rangle - 2\langle Q_N x_N, y_N \rangle + \langle Q_N y_N, y_N \rangle + \langle \bar{Q}\mathbb{E}x_N, \mathbb{E}x_N \rangle \\ & -2\langle \bar{Q}_N \mathbb{E}x_N, \mathbb{E}y_N \rangle + \langle \bar{Q}_N \mathbb{E}y_N, \mathbb{E}y_N \rangle. \end{aligned}$$

Recall that $\langle Q_k x_k, y_k \rangle$ denotes $\mathbb{E}(y_k^T Q_k x_k)$ with similar meanings for related notation.

Here, if we let

$$
\begin{aligned}
\Theta_1 = & \sum_{k=0}^{N} \Big((\Gamma_k + \bar{\Gamma}_k)^* Q_k (L_k + \bar{L}_k) + (\Gamma_k + \bar{\Gamma}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (L_k + \bar{L}_k) \Big), \\
\Theta_2 = & \sum_{k=0}^{N-1} \Big(R_k + \mathbb{E}^* \bar{R}_k \mathbb{E} + (L_k + \bar{L}_k)^* Q_k (L_k + \bar{L}_k) + (L_k + \bar{L}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (L_k + \bar{L}_k) \Big) \\
& + (L_N + \bar{L}_N)^* Q_N (L_N + \bar{L}_N) + (L_N + \bar{L}_N)^* \mathbb{E}^* \bar{Q}_N \mathbb{E} (L_N + \bar{L}_N), \\
\Theta_3 = & \sum_{k=0}^{N} \Big((\Psi_k + \bar{\Psi}_k)^* Q_k (L_k + \bar{L}_k) + (\Psi_k + \bar{\Psi}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (L_k + \bar{L}_k) \Big), \\
\Lambda_1 = & \sum_{k=0}^{N} \Big((\Gamma_k + \bar{\Gamma}_k)^* Q_k (\Gamma_k + \bar{\Gamma}_k) + (\Gamma_k + \bar{\Gamma}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (\Gamma_k + \bar{\Gamma}_k) \Big), \\
\Lambda_2 = & -\sum_{k=0}^{N} \Big((\Psi_k + \bar{\Psi}_k)^* Q_k (\Gamma_k + \bar{\Gamma}_k) + (\Psi_k + \bar{\Psi}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (\Gamma_k + \bar{\Gamma}_k) \Big), \\
\Lambda_3 = & \sum_{k=0}^{N} \Big((\Psi_k + \bar{\Psi}_k)^* Q_k (\Psi_k + \bar{\Psi}_k) + (\Psi_k + \bar{\Psi}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E} (\Psi_k + \bar{\Psi}_k) \Big),
\end{aligned}
$$

then it can be shown that

$$
\begin{aligned}
J(\zeta^x, \zeta^y, u) = \; & 2\langle \Theta_1 u, \zeta^x \rangle + \langle \Theta_2 u, u \rangle + 2\langle \Theta_3 u, \zeta^y \rangle \\
& + \langle \Lambda_1 \zeta^x, \zeta^x \rangle + 2\langle \Lambda_2 \zeta^x, \zeta^y \rangle + \langle \Lambda_3 \zeta^y, \zeta^y \rangle.
\end{aligned} \tag{5}
$$

In this paper we use numerator layout for matrices calculus, i.e., for any matrix Y, $\frac{\partial}{\partial Y} Tr(AYB) = BA$ if AYB is meaningful. We then have the following result.

Proposition 3.2. *(i). If $J(\zeta^x, \zeta^y, u)$ has a minimum, then $\Theta_2 \geq 0$.*

(ii). Problem (MF-LQ) is (uniquely) solvable if and only if $\Theta_2 \geq 0$ and there exists a (unique) u such that

$$
u^T \Theta_2 + x^T \Theta_1 + y^T \Theta_3 = 0.
$$

(iii). If $\Theta_2 > 0$, then for any ζ^x and ζ^y, $J(\zeta^x, \zeta^y, u)$ admits a pathwise unique minimizer u^o given by

$$
u_k^o = -\big(\Theta_2^{-1} (\Theta_1^* \zeta^x + \Theta_3^* \zeta^y) \big)(k), \quad k \in \mathbb{N}. \tag{6}
$$

In addition, if

$$
Q_k, Q_k + \bar{Q}_k \geq 0, \;\; k \in \bar{\mathbb{N}}, \quad R_k, R_k + \bar{R}_k > 0, \;\; k \in \mathbb{N}, \tag{7}
$$

then $\Theta_2 > 0$.

Proof. The proofs of (i), (ii) and the first part of (iii) are well known and therefore omitted here (Moore, Zhou & Lim, 1999; Yong, 2013). We now prove the second part of (iii). From (7), for $k \in \bar{\mathbb{N}}$, we have

$$
\begin{aligned}
(L_k+\bar{L}_k)^* Q_k (L_k+\bar{L}_k) + (L_k+\bar{L}_k)^* \mathbb{E}^* \bar{Q}_k \mathbb{E}(L_k+\bar{L}_k) &\geq 0, \\
\langle R_k u_k, u_k\rangle + \langle \bar{R}_k \mathbb{E}u_k, \mathbb{E}u_k\rangle = \ & \mathbb{E}\big[u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k\big] \\
= \ & \mathbb{E}\big[(u_k-\mathbb{E}u_k)^T R_k (u_k-\mathbb{E}u_k)\big] \\
& +(\mathbb{E}u_k)^T (R_k+\bar{R}_k)\mathbb{E}u_k > 0, \ \ k\in\mathbb{N}
\end{aligned}
$$

for any non-zero $u \in \mathcal{U}_{ad}$, which implies $\Theta_2 > 0$. □

4. Closed-loop Optimal Control via Riccati Equations

In this section, we first find the optimal control within the class of linear state feedback controls by using matrix minimum principle. Secondly, several sequences of bounded linear operators are presented and problem (MF-LQ) is reformulated as an operator stochastic linear-quadratic optimal control problem. We then find the optimal control via Riccati equations.

4.1. Linear Feedback Control

The linear feedback controls a linear functional of the system states, which gives the control based on the current system states. Suppose that a control having the following form is in used:

$$
u_k = L_k^x x_k + \bar{L}_k^x \mathbb{E}x_k + L_k^y y_k + \bar{L}_k^y \mathbb{E}y_k, \quad k \in \mathbb{N}, \tag{8}
$$

where $L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y \in \mathbb{R}^{m\times n}$. Under (8), the closed-loop system (1) becomes

$$
\begin{cases}
x_{k+1} = A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k(L_k^x x_k + \bar{L}_k^x \mathbb{E}x_k + L_k^y y_k + \bar{L}_k^y \mathbb{E}y_k) \\
\qquad +\bar{B}_k\big[(L_k^x+\bar{L}_k^x)\mathbb{E}x_k + (L_k^y+\bar{L}_k^y)\mathbb{E}y_k\big] \\
\qquad +\big\{C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k(L_k^x x_k + \bar{L}_k^x \mathbb{E}x_k + L_k^y y_k + \bar{L}_k^y \mathbb{E}y_k) \\
\qquad +\bar{D}_k\big[(L_k^x+\bar{L}_k^x)\mathbb{E}x_k + (L_k^y+\bar{L}_k^y)\mathbb{E}y_k\big]\big\} w_k, \\
y_{k+1} = (F_k y_k + \bar{F}_k \mathbb{E}y_k) + (G_k y_k + \bar{G}_k \mathbb{E}y_k) v_k, \\
x_0 = \zeta^x, y_0 = \zeta^y,
\end{cases} \tag{9}
$$

and the cost functional (3) may be represented as

$$
\begin{aligned}
& J(\zeta^x, \zeta^y, u) \\
= & \sum_{k=0}^{N-1} \big\{ Tr\big[Q_k\big(\mathbb{E}(x_k x_k^T) - \mathbb{E}(x_k y_k^T) - \mathbb{E}(y_k x_k^T) + \mathbb{E}(y_k y_k^T)\big)\big] \\
& + Tr\big[\bar{Q}_k\big(\mathbb{E}x_k \mathbb{E}x_k^T - \mathbb{E}x_k \mathbb{E}y_k^T - \mathbb{E}y_k \mathbb{E}x_k^T + \mathbb{E}y_k \mathbb{E}y_k^T\big)\big] \\
& + Tr\big[(L_k^x)^T R_k L_k^x \mathbb{E}(x_k x_k^T) + \big((L_k^x)^T R_k \bar{L}_k^x + (\bar{L}_k^x)^T R_k L_k^x + (\bar{L}_k^x)^T R_k \bar{L}_k^x\big)\mathbb{E}x_k \mathbb{E}x_k^T\big] \\
& + Tr\big[(L_k^x)^T R_k L_k^y \mathbb{E}(y_k x_k^T) + \big((L_k^x)^T R_k \bar{L}_k^y + (\bar{L}_k^x)^T R_k L_k^y + (\bar{L}_k^x)^T R_k \bar{L}_k^y\big)\mathbb{E}y_k \mathbb{E}x_k^T\big] \\
& + Tr\big[(L_k^y)^T R_k L_k^x \mathbb{E}(x_k y_k^T) + \big((L_k^y)^T R_k \bar{L}_k^x + (\bar{L}_k^y)^T R_k L_k^x + (\bar{L}_k^y)^T R_k \bar{L}_k^x\big)\mathbb{E}x_k \mathbb{E}y_k^T\big] \\
& + Tr\big[(L_k^y)^T R_k L_k^y \mathbb{E}(y_k y_k^T) + \big((L_k^y)^T R_k \bar{L}_k^y + (\bar{L}_k^y)^T R_k L_k^y + (\bar{L}_k^y)^T R_k \bar{L}_k^y\big)\mathbb{E}y_k \mathbb{E}y_k^T\big] \\
& + Tr\big[(L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\mathbb{E}x_k \mathbb{E}x_k^T + (L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^y + \bar{L}_k^y)\mathbb{E}y_k \mathbb{E}x_k^T\big] \\
& + Tr\big[(L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\mathbb{E}x_k \mathbb{E}y_k^T + (L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^y + \bar{L}_k^y)\mathbb{E}y_k \mathbb{E}y_k^T\big]\big\} \\
& + Tr\big[Q_N\big(\mathbb{E}(x_N x_N^T) - \mathbb{E}(x_N y_N^T) - \mathbb{E}(y_N x_N^T) + \mathbb{E}(y_N y_N^T)\big)\big] \\
& + Tr\big[\bar{Q}_N\big(\mathbb{E}x_N \mathbb{E}x_N^T - \mathbb{E}x_N \mathbb{E}y_N^T - \mathbb{E}y_N \mathbb{E}x_N^T + \mathbb{E}y_N \mathbb{E}y_N^T\big)\big].
\end{aligned}
\tag{10}
$$

From the form (8) of the control, we may view $\{(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), k \in \mathbb{N}\}$ as the new control input. Write

$$
\begin{cases}
X_k = \mathbb{E}\left(x_k x_k^T\right), \quad \bar{X}_k = \mathbb{E}x_k(\mathbb{E}x_k)^T, \quad XY_k = \mathbb{E}\left(x_k y_k^T\right), \\
\bar{XY}_k = \mathbb{E}x_k(\mathbb{E}y_k)^T, \quad Y_k = \mathbb{E}\left(y_k y_k^T\right), \quad \bar{Y}_k = \mathbb{E}y_k(\mathbb{E}y_k)^T.
\end{cases}
$$

Then by (9), we may express the new system states as

$$
\begin{cases}
X_{k+1} = \mathcal{X}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), \quad \bar{X}_{k+1} = \bar{\mathcal{X}}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), \\
XY_{k+1} = \mathcal{XY}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), \quad \bar{XY}_{k+1} = \bar{\mathcal{XY}}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), \\
Y_{k+1} = \mathcal{Y}_k, \quad \bar{Y}_{k+1} = \bar{\mathcal{Y}}_k.
\end{cases}
\tag{11}
$$

The derivations are straightforward an hence omitted here. After some calculations, $J(\zeta^x, \zeta^y, u)$ with u defined in (8) may be represented in terms of X, $\bar{X}$, XY, $\bar{XY}$, Y

and $\bar{Y}$ as follows:

$$
\begin{aligned}
& J(\zeta^x, \zeta^y, u) \\
= & \sum_{k=0}^{N-1} \{Tr[Q_k(X_k - XY_k - (XY_k)^T + Y_k)] \\
& +Tr[\bar{Q}_k(\bar{X}_k - \bar{XY}_k - (\bar{XY}_k)^T + \bar{Y}_k)] \\
& +Tr[(L_k^x)^T R_k L_k^x X_k + ((L_k^x)^T R_k \bar{L}_k^x + (\bar{L}_k^x)^T R_k L_k^x + (\bar{L}_k^x)^T R_k \bar{L}_k^x)\bar{X}_k] \\
& +Tr[(L_k^x)^T R_k L_k^y (XY_k)^T + ((L_k^x)^T R_k \bar{L}_k^y + (\bar{L}_k^x)^T R_k L_k^y + (\bar{L}_k^x)^T R_k \bar{L}_k^y)(\bar{XY}_k)^T] \\
& +Tr[(L_k^y)^T R_k L_k^x XY_k + ((L_k^y)^T R_k \bar{L}_k^x + (\bar{L}_k^y)^T R_k L_k^x + (\bar{L}_k^y)^T R_k \bar{L}_k^x)\bar{XY}_k] \\
& +Tr[(L_k^y)^T R_k L_k^y Y_k + ((L_k^y)^T R_k \bar{L}_k^y + (\bar{L}_k^y)^T R_k L_k^y + (\bar{L}_k^y)^T R_k \bar{L}_k^y)\bar{Y}_k] \\
& +Tr[(L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\bar{X}_k + (L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^y + \bar{L}_k^y)(\bar{XY}_k)^T] \\
& +Tr[(L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\bar{XY}_k + (L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^y + \bar{L}_k^y)\bar{Y}_k]\} \\
& +Tr[Q_N(X_N - XY_N - (XY_N)^T + Y_N)] \\
& +Tr[\bar{Q}_N(\bar{X}_N - \bar{XY}_N - (\bar{XY}_N)^T + \bar{Y}_N)] \\
& \equiv \mathcal{J}(X_0, \bar{X}_0, XY_0, \bar{XY}_0, Y_0, \bar{Y}_0, \mathcal{L}),
\end{aligned}
\tag{12}
$$

where $\mathcal{L} \equiv \{(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y), k \in \mathbb{N}\}$. Therefore, Problem (MF-LQ) is equivalent to the following problem:

$$
\begin{cases}
\min\limits_{L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y \in \mathbb{R}^{m\times n}, k\in\mathbb{N}} \mathcal{J}(X_0, \bar{X}_0, XY_0, \bar{XY}_0, Y_0, \bar{Y}_0, \mathcal{L}), \\
\text{subject to (11)}.
\end{cases}
\tag{13}
$$

Clearly, this is a matrix dynamical optimization problem. A natural way to deal with this class of problems is by the matrix minimum principle (Athans, 1967). Following the framework above, we can obtain the optimal control of form (8). Define the optimal feedback gains $L_k^o = (L_k^{xo}, L_k^{yo}, \bar{L}_k^{xo}, \bar{L}_k^{yo}),\ k \in \mathbb{N}$. We first introduce some notation and then present the result. Let

$$
\begin{cases}
\bar{W}_k^{(1)} = & R_k + B_k^T P_{k+1}^x B_k + D_k^T P_{k+1}^x D_k, \\
\bar{W}_k^{(2)} = & R_k + \bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) \\
& +(D_k + \bar{D}_k)^T P_{k+1}^x (D_k + \bar{D}_k), \\
\bar{H}_k^{(1)} = & A_k^T P_{k+1}^x B_k + C_k^T P_{k+1}^x D_k, \\
\bar{H}_k^{(2)} = & (A_k + \bar{A}_k)^T (P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) + (C_k + \bar{C}_k)^T P_{k+1}^x (D_k + \bar{D}_k), \\
\bar{H}_k^{(3)} = & F_k^T P_{k+1}^{xy} B_k + \rho G_k^T P_{k+1}^{xy} D_k, \\
\bar{H}_k^{(4)} = & (F_k + \bar{F}_k)^T (P_{k+1}^{xy} + \bar{P}_{k+1}^{xy})(B_k + \bar{B}_k) + \rho(G_k + \bar{G}_k)^T P_{k+1}^{xy} (D_k + \bar{D}_k),
\end{cases}
\tag{14}
$$

with

$$
\begin{cases}
P_k^x = & Q_k + (L_k^{xo})^T R_k L_k^{xo} + (A_k + B_k L_k^{xo})^T P_{k+1}^x (A_k + B_k L_k^{xo}) \\
& +(C_k + D_k L_k^{xo})^T P_{k+1}^x (C_k + D_k L_k^{xo}), \\
P_N^x = & Q_N,
\end{cases}
\tag{15}
$$

$$
\begin{cases}
\bar{P}_k^x = & \bar{Q}_k + (L_k^{xo})^T R_k \bar{L}_k^{xo} + (\bar{L}_k^{xo})^T R_k L_k^{xo} + (\bar{L}_k^{xo})^T R_k \bar{L}_k^{xo} \\
& +(L_k^{xo} + \bar{L}_k^{xo})^T \bar{R}_k (L_k^{xo} + \bar{L}_k^{xo}) \\
& +(A_k + B_k L_k^{xo})^T P_{k+1}^x \big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big]^T P_{k+1}^x (A_k + B_k L_k^{xo}) \\
& +\big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big]^T P_{k+1}^x \big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[A_k + \bar{A}_k + (B_k + \bar{B}_k)(L_k^{xo} + \bar{L}_k^{xo})\big]^T \bar{P}_{k+1}^x \big[A_k + \bar{A}_k + (B_k + \bar{B}_k)(L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +(C_k + D_k L_k^{xo})^T P_{k+1}^x \big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big]^T P_{k+1}^x (C_k + D_k L_k^{xo}) \\
& +\big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big]^T P_{k+1}^x \big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big], \\
\bar{P}_N^x = & \bar{Q}_N,
\end{cases}
\tag{16}
$$

$$
\begin{cases}
P_k^{xy} = & -Q_k + (L_k^{yo})^T R_k L_k^{xo} + (B_k L_k^{yo})^T P_{k+1}^x (A_k + B_k L_k^{xo}) \\
& +(D_k L_k^{yo})^T P_{k+1}^x (C_k + D_k L_k^{xo}) + F_k^T P_{k+1}^{xy} (A_k + B_k L_k^{xo}) \\
& +\rho G_k^T P_{k+1}^{xy} (C_k + D_k L_k^{xo}), \\
P_N^{xy} = & -Q_N,
\end{cases}
\tag{17}
$$

$$
\begin{cases}
\bar{P}_k^{xy} = & -\bar{Q}_k + (L_k^{yo})^T R_k \bar{L}_k^{xo} + (\bar{L}_k^{yo})^T R_k L_k^{xo} + (\bar{L}_k^{yo})^T R_k \bar{L}_k^{xo} \\
& +(L_k^{yo}+\bar{L}_k^{yo})^T \bar{R}_k (L_k^{xo}+\bar{L}_k^{xo}) + \big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo}+\bar{L}_k^{yo})\big]^T P_{k+1}^x (A_k + B_k L_k^{xo}) \\
& +(B_k L_k^{yo})^T P_{k+1}^x \big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x \big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big]^T \bar{P}_{k+1}^x \big[(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x (C_k + D_k L_k^{xo}) \\
& +(D_k L_k^{yo})^T P_{k+1}^x \big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x \big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +F_k^T P_{k+1}^{xy} \big[\bar{A}_k + B_k \bar{L}_k^{xo} + \bar{B}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\bar{F}_k^T P_{k+1}^{xy} \big[A_k + \bar{A}_k + (B_k + \bar{B}_k)(L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +(F_k + \bar{F}_k)^T \bar{P}_{k+1}^{xy} \big[A_k + \bar{A} + (B_k + \bar{B}_k)(L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\rho G_k^T P_{k+1}^{xy} \big[\bar{C}_k + D_k \bar{L}_k^{xo} + \bar{D}_k (L_k^{xo} + \bar{L}_k^{xo})\big] \\
& +\rho \bar{G}_k^T P_{k+1}^{xy} \big[C_k + \bar{C}_k + (D_k + \bar{D}_k)(L_k^{xo} + \bar{L}_k^{xo})\big], \\
\bar{P}_N^{xy} = & -Q_N,
\end{cases}
\tag{18}
$$

$$
\begin{cases}
P_k^y = & Q_k + (L_k^{yo})^T R_k L_k^{yo} + (B_k L_k^{yo})^T P_{k+1}^x B_k L_k^{yo} + (D_k L_k^{yo})^T P_{k+1}^x D_k L_k^{yo} \\
& +F_k^T P_{k+1}^{xy} B_k L_k^{yo} + \rho G_k^T P_{k+1}^{xy} D_k L_k^{yo} + (B_k L_k^{yo})^T (P_{k+1}^{xy})^T F_k \\
& +\rho (D_k L_k^{yo})^T (P_{k+1}^{xy})^T G_k + F_k^T P_{k+1}^y F_k + G_k^T P_{k+1}^y G_k, \\
P_N^y = & Q_N,
\end{cases}
\tag{19}
$$

and

$$
\begin{cases}
\begin{aligned}
\bar{P}_k^y = \ & \bar{Q}_k + (L_k^{yo})^T R_k \bar{L}_k^{yo} + (\bar{L}_k^{yo})^T R_k L_k^{yo} + (\bar{L}_k^{yo})^T R_k \bar{L}_k^{yo} \\
& +(L_k^{yo} + \bar{L}_k^{yo})^T \bar{R}_k (L_k^{yo} + \bar{L}_k^{yo}) + (B_k L_k^{yo})^T P_{k+1}^x \big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +\big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x B_k L_k^{yo} \\
& +\big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x \big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +\big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big]^T \bar{P}_{k+1}^x \big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +(D_k L_k^{yo})^T P_{k+1}^x \big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +\big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x D_k L_k^{xo} \\
& +\big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big]^T P_{k+1}^x \big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +F_k^T P_{k+1}^{xy} \big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k + \bar{L}_k^{yo})\big] + \bar{F}_k^T P_{k+1}^{xy} \big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +(F_k + \bar{F}_k)^T \bar{P}_{k+1}^{xy} \big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +\rho G_k^T P_{k+1}^{xy} \big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k + \bar{L}_k^{yo})\big] + \rho \bar{G}_k^T P_{k+1}^{xy} \big[(D_k + \bar{D}_k)(L_k^{yo} + \bar{L}_k^{yo})\big] \\
& +\big[B_k \bar{L}_k^{yo} + \bar{B}_k (L_k + \bar{L}_k^{yo})\big]^T (P_{k+1}^{xy})^T F_k + \big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big]^T (P_{k+1}^{xy})^T \bar{F}_k \\
& +\big[(B_k + \bar{B}_k)(L_k^{yo} + \bar{L}_k^{yo})\big]^T (\bar{P}_{k+1}^{xy})^T (F_k + \bar{F}_k) \\
& +\rho \big[D_k \bar{L}_k^{yo} + \bar{D}_k (L_k + \bar{L}_k^{yo})\big]^T (P_{k+1}^{xy})^T G_k \\
& +\rho \big[(D_k + \bar{D}_k)(L_k^{yo} + \bar{L}_k^{yo})\big]^T (P_{k+1}^{xy})^T \bar{G}_k \\
& +F_k^T P_{k+1}^y \bar{F}_k + \bar{F}_k^T P_{k+1}^y F_k + \bar{F}_k^T P_{k+1}^y \bar{F}_k \\
& +(F_k + \bar{F}_k)^T \bar{P}_{k+1}^y (F_k + \bar{F}_k) + G_k^T P_{k+1}^y \bar{G}_k + \bar{G}_k^T P_{k+1}^y G_k + \bar{G}_k^T P_{k+1}^y \bar{G}_k,
\end{aligned} \\
\bar{P}_N^y = \bar{Q}_N.
\end{cases} \tag{20}
$$

The optimal control can be obtained by the following theorem.

Theorem 4.1. *For Problem (MF-LQ), under the condition*

$$
Q_k, Q_k + \bar{Q}_k \geq 0,\ k \in \bar{\mathbb{N}},\ R_k, R_k + \bar{R}_k > 0,\ k \in \mathbb{N}, \tag{21}
$$

the unique optimal control within the class of controls of form (8) is

$$
\begin{aligned}
u_k^o = & \big[-(\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(1)})^T\big] x_k + \big[-(\bar{W}_k^{(2)})^{-1}(\bar{H}_k^{(2)})^T + (\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(1)})^T\big] \mathbb{E}x_k \\
& + \big[-(\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(3)})^T\big] y_k + \big[-(\bar{W}_k^{(2)})^{-1}(\bar{H}_k^{(4)})^T + (\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(3)})^T\big] \mathbb{E}y_k, \\
\equiv & L_k^{xo} x_k + \bar{L}_k^{xo} \mathbb{E}x_k + L_k^{yo} y_k + \bar{L}_k^{yo} \mathbb{E}y_k,\ k \in \mathbb{N},
\end{aligned} \tag{22}
$$

with the property $P_k^x,\ P_k^x + \bar{P}_k^x \geq 0,\ k \in \bar{\mathbb{N}}$.

Proof. See the Appendix.

4.2. Operator Linear-quadratic Problem

In this subsection, the optimal control via operator linear-quadratic theory shall be derived. Firstly, we reformulate the discrete-time operator LQ problem. A linear controlled system in abstract form rewritten from (1) is

$$\begin{cases} x_{k+1} = (\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k, \\ y_{k+1} = \mathcal{F}_k y_k + \mathcal{G}_k y_k v_k, \\ x_0 = \zeta^x, y_0 = \zeta^y, \end{cases} \tag{23}$$

with several sequences of operators

$$\begin{cases} \mathcal{A}_k z = A_k z + \bar{A}_k \mathbb{E} z, \quad \mathcal{B}_k u = B_k u + \bar{B}_k \mathbb{E} u, \\ \mathcal{C}_k z = C_k z + \bar{C}_k \mathbb{E} z, \quad \mathcal{D}_k u = D_k u + \bar{D}_k \mathbb{E} u, \\ \mathcal{F}_k z = F_k z + \bar{F}_k \mathbb{E} z, \quad \mathcal{G}_k z = G_k z + \bar{G}_k \mathbb{E} z, \end{cases} \tag{24}$$

where $z \in \mathcal{Z}_k$, $u \in \mathcal{U}_k$, $\mathcal{A}_k$,$\mathcal{C}_k$,$\mathcal{F}_k$,$\mathcal{G}_k$ are from $\mathcal{Z}_k$ to $\mathcal{Z}_k$ and $\mathcal{B}_k$, $\mathcal{D}_k$ are from $\mathcal{U}_k$ to $\mathcal{U}_k$, $k \in \mathbb{N}$. Furthermore, the performance functional in the form of inner products (5) has been presented in previous section. We now define operators

$$\begin{cases} \mathcal{Q}_k z = (Q_k + \mathbb{E}^* \bar{Q}_k \mathbb{E}) z, \quad z \in \mathcal{Z}_k, \ k \in \bar{\mathbb{N}}, \\ \mathcal{R}_k u = (R_k + \mathbb{E}^* \bar{R}_k \mathbb{E}) u, \quad u \in \mathcal{U}_k, \ k \in \mathbb{N}. \end{cases} \tag{25}$$

Recall that $\mathbb{E}\big[y_k^T (Q_k + \mathbb{E}^* \bar{Q}_k \mathbb{E}) x_k\big] = \langle \mathcal{Q}_k x_k, y_k \rangle$ with similar meanings for related notation. Hence,

$$\begin{aligned} J(\zeta^x, \zeta^y, u) = \ & \sum_{k=0}^{N-1} \Big(\langle \mathcal{Q}_k x_k, x_k \rangle - 2 \langle \mathcal{Q}_k x_k, y_k \rangle + \langle \mathcal{Q}_k y_k, y_k \rangle + \langle \mathcal{R}_k u_k, u_k \rangle \Big) \\ & + \langle \mathcal{Q}_N x_N, x_N \rangle - 2 \langle \mathcal{Q}_N x_N, y_N \rangle + \langle \mathcal{Q}_N y_N, y_N \rangle. \end{aligned} \tag{26}$$

Problem (MF-LQ) in abstract form can be represented as

$$\begin{cases} \text{Minimize (26)}, \\ \text{subject to } u \in \mathcal{U}_{ad}, \text{ with } (x_\cdot, y_\cdot, u_\cdot), \text{ satisfying (23)}. \end{cases} \tag{27}$$

We then use u^* to represent the optimal control for Problem (MF-LQ) in abstract form. Suppose we have a sequence of self-adjoint linear operators $\big\{\mathcal{P}_k^x : \mathcal{Z}_k \mapsto \mathcal{Z}_k; \ k \in \bar{\mathbb{N}}\big\}$, $\big\{\mathcal{P}_k^y : \mathcal{Z}_k \mapsto \mathcal{Z}_k; \ k \in \bar{\mathbb{N}}\big\}$ and linear operator $\big\{\mathcal{P}_k^{xy} : \mathcal{Z}_k \mapsto \mathcal{Z}_k; \ k \in \bar{\mathbb{N}}\big\}$ determined by

$$\begin{cases} \mathcal{P}_k^x \mathcal{Z}_l \equiv \{\mathcal{P}_k^x z \mid z \in \mathcal{Z}_l\} \subseteq \mathcal{Z}_l, \quad l \le k, \\ \mathcal{P}_k^{xy} \mathcal{Z}_l \equiv \big\{\mathcal{P}_k^{xy} z \mid z \in \mathcal{Z}_l\big\} \subseteq \mathcal{Z}_l, \quad l \le k, \\ \mathcal{P}_k^y \mathcal{Z}_l \equiv \big\{\mathcal{P}_k^y z \mid z \in \mathcal{Z}_l\big\} \subseteq \mathcal{Z}_l, \quad l \le k. \end{cases}$$

where $\mathcal{Z}_l \subseteq \mathcal{Z}_k$, $l \leq k$. Clearly,

$$
\begin{aligned}
&\langle \mathcal{P}^x_{k+1} x_{k+1}, x_{k+1} \rangle \\
&= \big\langle \mathcal{P}^x_{k+1} \big[(\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k\big], \big[(\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k\big] \big\rangle \\
&= \mathbb{E}\big[(\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T \mathcal{P}^x_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k)\big] + \mathbb{E}\big[(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T \mathcal{P}^x_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k)\big] \\
&= \big\langle \mathcal{P}^x_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k), (\mathcal{A}_k x_k + \mathcal{B}_k u_k) \big\rangle + \big\langle \mathcal{P}^x_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k), (\mathcal{C}_k x_k + \mathcal{D}_k u_k) \big\rangle,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&\langle \mathcal{P}^{xy}_{k+1} x_{k+1}, y_{k+1} \rangle \\
&= \big\langle \mathcal{P}^{xy}_{k+1} \big[(\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k\big], \big[\mathcal{F}_k y_k + \mathcal{G}_k y_k v_k\big] \big\rangle \\
&= \mathbb{E}\big[(\mathcal{F}_k y_k)^T \mathcal{P}^{xy}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k)\big] + \rho \mathbb{E}\big[(\mathcal{G}_k y_k)^T \mathcal{P}^{xy}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k)\big] \\
&= \big\langle \mathcal{P}^{xy}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k), \mathcal{F}_k y_k \big\rangle + \rho \big\langle \mathcal{P}^{xy}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k), \mathcal{G}_k y_k \big\rangle,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\langle \mathcal{P}^y_{k+1} y_{k+1}, y_{k+1} \rangle \\
&= \big\langle \mathcal{P}^y_{k+1} (\mathcal{F}_k y_k + \mathcal{G}_k y_k v_k), (\mathcal{F}_k y_k + \mathcal{G}_k y_k v_k) \big\rangle \\
&= \mathbb{E}\big[(\mathcal{F}_k y_k)^T \mathcal{P}^y_{k+1} \mathcal{F}_k y_k\big] + \mathbb{E}\big[(\mathcal{G}_k y_k)^T \mathcal{P}^y_{k+1} \mathcal{G}_k y_k\big] \\
&= \big\langle \mathcal{P}^y_{k+1} \mathcal{F}_k y_k, \mathcal{F}_k y_k \big\rangle + \big\langle \mathcal{P}^y_{k+1} \mathcal{G}_k y_k, \mathcal{G}_k y_k \big\rangle.
\end{aligned}
\tag{30}
$$

We now consider the problem by backward recursion with only $k \in \{N-1, N\}$. By (28)–(30), we have

$$
\begin{aligned}
\langle \mathcal{P}^x_N x_N, x_N \rangle =& \ \big\langle \mathcal{P}^x_N (\mathcal{A}_{N-1} x_{N-1} + \mathcal{B}_{N-1} u_{N-1}), (\mathcal{A}_{N-1} x_{N-1} + \mathcal{B}_{N-1} u_{N-1}) \big\rangle \\
& + \big\langle \mathcal{P}^x_N (\mathcal{C}_{N-1} x_{N-1} + \mathcal{D}_{N-1} u_{N-1}), (\mathcal{C}_{N-1} x_{N-1} + \mathcal{D}_{N-1} u_{N-1}) \big\rangle, \\
\langle \mathcal{P}^{xy}_N x_N, y_N \rangle =& \ \big\langle \mathcal{P}^{xy}_N (\mathcal{A}_{N-1} x_{N-1} + \mathcal{B}_{N-1} u_{N-1}), \mathcal{F}_{N-1} y_{N-1} \big\rangle \\
& + \rho \big\langle \mathcal{P}^{xy}_N (\mathcal{C}_{N-1} x_{N-1} + \mathcal{D}_{N-1} u_{N-1}), \mathcal{G}_{N-1} y_{N-1} \big\rangle, \\
\langle \mathcal{P}^y_N y_N, y_N \rangle =& \ \big\langle \mathcal{P}^y_N \mathcal{F}_{N-1} y_{N-1}, \mathcal{F}_{N-1} y_{N-1} \big\rangle + \big\langle \mathcal{P}^y_N \mathcal{G}_{N-1} y_{N-1}, \mathcal{G}_{N-1} y_{N-1} \big\rangle.
\end{aligned}
\tag{31}
$$

Let $\mathcal{P}^x_N = -\mathcal{P}^{xy}_N = \mathcal{P}^y_N = \mathcal{Q}_N$. By taking (31) into J^N_{N-1}, we have

$$
\begin{aligned}
&J^N_{N-1}(x_{N-1}, y_{N-1}, u) \\
&= \langle \mathcal{Q}_{N-1} x_{N-1}, x_{N-1} \rangle - 2 \langle \mathcal{Q}_{N-1} x_{N-1}, y_{N-1} \rangle + \langle \mathcal{Q}_{N-1} y_{N-1}, y_{N-1} \rangle \\
&\quad + \langle \mathcal{R}_{N-1} u_{N-1}, u_{N-1} \rangle + \langle \mathcal{Q}_N x_N, x_N \rangle - 2 \langle \mathcal{Q}_N x_N, y_N \rangle + \langle \mathcal{Q}_N y_N, y_N \rangle \\
&= 2 \langle \Theta_{1,N-1} u_{N-1}, x_{N-1} \rangle + \langle \Theta_{2,N-1} u_{N-1}, u_{N-1} \rangle + 2 \langle \Theta_{3,N-1} u_{N-1}, y_{N-1} \rangle \\
&\quad + \langle \Lambda_{1,N-1} x_{N-1}, x_{N-1} \rangle + 2 \langle \Lambda_{2,N-1} x_{N-1}, y_{N-1} \rangle + \langle \Lambda_{3,N-1} y_{N-1}, y_{N-1} \rangle,
\end{aligned}
\tag{32}
$$

where

$$
\begin{cases}
\Theta_{1,N-1} = \mathcal{A}_{N-1}^T \mathcal{P}_N^x \mathcal{B}_{N-1} + \mathcal{C}_{N-1}^T \mathcal{P}_N^x \mathcal{D}_{N-1}, \\
\Theta_{2,N-1} = \mathcal{R}_{N-1} + \mathcal{B}_{N-1}^T \mathcal{P}_N^x \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^T \mathcal{P}_N^x \mathcal{D}_{N-1}, \\
\Theta_{3,N-1} = \mathcal{F}_{N-1}^T \mathcal{P}_N^{xy} \mathcal{B}_{N-1} + \rho \mathcal{G}_{N-1}^T \mathcal{P}_N^{xy} \mathcal{D}_{N-1}, \\
\Lambda_{1,N-1} = \mathcal{Q}_{N-1} + \mathcal{A}_{N-1}^T \mathcal{P}_N^x \mathcal{A}_{N-1} + \mathcal{C}_{N-1}^T \mathcal{P}_N^x \mathcal{C}_{N-1}, \\
\Lambda_{2,N-1} = -\mathcal{Q}_{N-1} + \mathcal{F}_{N-1}^T \mathcal{P}_N^{xy} \mathcal{A}_{N-1} + \rho \mathcal{G}_{N-1}^T \mathcal{P}_N^{xy} \mathcal{C}_{N-1}, \\
\Lambda_{3,N-1} = \mathcal{Q}_{N-1} + \mathcal{F}_{N-1}^T \mathcal{P}_N^y \mathcal{F}_{N-1} + \mathcal{G}_{N-1}^T \mathcal{P}_N^y \mathcal{G}_{N-1}.
\end{cases}
\tag{33}
$$

If $\Theta_{2,N-1}$ is positive definite and self-adjoint, (32) can then be rewritten as

$$
\begin{aligned}
& J_{N-1}^N(x_{N-1}, y_{N-1}, u) \\
&= \big\langle (\Lambda_{1,N-1} - \Theta_{1,N-1}\Theta_{2,N-1}^{-1}\Theta_{1,N-1}^*) x_{N-1}, x_{N-1} \big\rangle \\
&\quad + 2\big\langle (\Lambda_{2,N-1} - \Theta_{3,N-1}\Theta_{2,N-1}^{-1}\Theta_{1,N-1}^*) x_{N-1}, y_{N-1} \big\rangle \\
&\quad + \big\langle (\Lambda_{3,N-1} - \Theta_{3,N-1}\Theta_{2,N-1}^{-1}\Theta_{3,N-1}^*) y_{N-1}, y_{N-1} \big\rangle \\
&\quad + \big\langle \Theta_{2,N-1}(u_{N-1} + \Theta_{2,N-1}^{-1}\Theta_{1,N-1}^* x_{N-1} + \Theta_{2,N-1}^{-1}\Theta_{3,N-1}^* \cdot y_{N-1}), \\
&\qquad (u_{N-1} + \Theta_{2,N-1}^{-1}\Theta_{1,N-1}^* x_{N-1} + \Theta_{2,N-1}^{-1}\Theta_{3,N-1}^* y_{N-1}) \big\rangle.
\end{aligned}
\tag{34}
$$

Let

$$
\begin{cases}
\mathcal{P}_{N-1}^x = \Lambda_{1,N-1} - \Theta_{1,N-1}\Theta_{2,N-1}^{-1}\Theta_{1,N-1}^*, \\
\mathcal{P}_{N-1}^{xy} = \Lambda_{2,N-1} - \Theta_{3,N-1}\Theta_{2,N-1}^{-1}\Theta_{1,N-1}^*, \\
\mathcal{P}_{N-1}^y = \Lambda_{3,N-1} - \Theta_{3,N-1}\Theta_{2,N-1}^{-1}\Theta_{3,N-1}^*,
\end{cases}
\tag{35}
$$

and the cost functional (34) may achieve its minimum if we select

$$
u_{N-1}^* = -\Theta_{2,N-1}^{-1}\Theta_{1,N-1}^* x_{N-1} - \Theta_{2,N-1}^{-1}\Theta_{3,N-1}^* y_{N-1}, \tag{36}
$$

where the minimum is

$$
\begin{aligned}
& J_{N-1}^N(x_{N-1}, y_{N-1}, u^*) \\
= \quad & \langle \mathcal{P}_{N-1}^x x_{N-1}, x_{N-1} \rangle + 2\langle \mathcal{P}_{N-1}^{xy} x_{N-1}, y_{N-1} \rangle + \langle \mathcal{P}_{N-1}^y y_{N-1}, y_{N-1} \rangle.
\end{aligned}
\tag{37}
$$

For this, we reach to the following lemma, which gives a compact form of $\mathcal{P}_{N-1}^x$, $\mathcal{P}_{N-1}^{xy}$ and $\mathcal{P}_{N-1}^y$.

Lemma 4.2. *If*

$$
Q_{N-1}, Q_{N-1} + \bar{Q}_{N-1} \geq 0, \quad Q_N, Q_N + \bar{Q}_N \geq 0, \quad R_{N-1}, R_{N-1} + \bar{R}_{N-1} > 0, \tag{38}
$$

then $\mathcal{P}^x_{N-1}$, $\mathcal{P}^{xy}_{N-1}$, $\mathcal{P}^y_{N-1}$ *defined in (31) have the following form*

$$\begin{cases}\mathcal{P}^x_{N-1} = (I-\mathbb{E}^*)S^x_{N-1}(I-\mathbb{E}) + \mathbb{E}^* T^x_{N-1}\mathbb{E},\\ \mathcal{P}^{xy}_{N-1} = (I-\mathbb{E}^*)S^{xy}_{N-1}(I-\mathbb{E}) + \mathbb{E}^* T^{xy}_{N-1}\mathbb{E},\\ \mathcal{P}^y_{N-1} = (I-\mathbb{E}^*)S^y_{N-1}(I-\mathbb{E}) + \mathbb{E}^* T^y_{N-1}\mathbb{E}.\end{cases} \tag{39}$$

where

$$\begin{cases}S^x_{N-1} = & Q_{N-1} + A^T_{N-1}Q_N A_{N-1} + C^T_{N-1}Q_N C_{N-1} - H^{(1)}_{N-1}(W^{(1)}_{N-1})^{-1}(H^{(1)}_{N-1})^T,\\ T^x_{N-1} = & Q_{N-1} + \bar{Q}_{N-1} + (A_{N-1}+\bar{A}_{N-1})^T(Q_N+\bar{Q}_N)(A_{N-1}+\bar{A}_{N-1})\\ & +(C_{N-1}+\bar{C}_{N-1})^T Q_N (C_{N-1}+\bar{C}_{N-1}) - H^{(2)}_{N-1}(W^{(2)}_{N-1})^{-1}(H^{(2)}_{N-1})^T,\\ S^{xy}_{N-1} = & -Q_{N-1} + F^T_{N-1}Q_N A_{N-1} + \rho G^T_{N-1}Q_N C_{N-1} - H^{(3)}_{N-1}(W^{(1)}_{N-1})^{-1}(H^{(1)}_{N-1})^T,\\ T^{xy}_{N-1} = & -Q_{N-1} - \bar{Q}_{N-1} + (F_{N-1}+\bar{F}_{N-1})^T(Q_N+\bar{Q}_N)(A_{N-1}+\bar{A}_{N-1})\\ & +\rho(G_{N-1}+\bar{G}_{N-1})^T Q_N (C_{N-1}+\bar{C}_{N-1}) - H^{(4)}_{N-1}(W^{(2)}_{N-1})^{-1}(H^{(2)}_{N-1})^T,\\ S^y_{N-1} = & Q_{N-1} + F^T_{N-1}Q_N F_{N-1} + G^T_{N-1}Q_N G_{N-1} - H^{(3)}_{N-1}(W^{(1)}_{N-1})^{-1}(H^{(3)}_{N-1})^T,\\ T^y_{N-1} = & Q_{N-1} + \bar{Q}_{N-1} + (F_{N-1}+\bar{F}_{N-1})^T(Q_N+\bar{Q}_N)(F_{N-1}+\bar{F}_{N-1})\\ & +(G_{N-1}+\bar{G}_{N-1})^T Q_N (G_{N-1}+\bar{G}_{N-1}) - H^{(4)}_{N-1}(W^{(2)}_{N-1})^{-1}(H^{(4)}_{N-1})^T,\\ S^x_N = & Q_N,\ T^x_N = Q_N + \bar{Q}_N,\ S^{xy}_N = -Q_N,\\ T^{xy}_N = & -Q_N - \bar{Q}_N,\ S^y_N = Q_N,\ T^y_N = Q_N + \bar{Q}_N.\end{cases}$$

with

$$\begin{cases}W^{(1)}_{N-1} = R_{N-1} + B^T_{N-1}Q_N B_{N-1} + D^T_{N-1}Q_N D_{N-1},\\ W^{(2)}_{N-1} = R_{N-1} + \bar{R}_{N-1} + (B_{N-1}+\bar{B}_{N-1})^T(Q_N+\bar{Q}_N)(B_{N-1}+\bar{B}_{N-1})\\ \qquad +(D_{N-1}+\bar{D}_{N-1})^T Q_N (D_{N-1}+\bar{D}_{N-1}),\\ H^{(1)}_{N-1} = A^T_{N-1}Q_N B_{N-1} + C^T_{N-1}Q_N D_{N-1},\\ H^{(2)}_{N-1} = (A_{N-1}+\bar{A}_{N-1})^T(Q_N+\bar{Q}_N)(B_{N-1}+\bar{B}_{N-1})\\ \qquad +(C_{N-1}+\bar{C}_{N-1})^T Q_N (D_{N-1}+\bar{D}_{N-1}),\\ H^{(3)}_{N-1} = F^T_{N-1}Q_N B_{N-1} + \rho G^T_{N-1}Q_N D_{N-1},\\ H^{(4)}_{N-1} = (F_{N-1}+\bar{F}_{N-1})^T(Q_N+\bar{Q}_N)(B_{N-1}+\bar{B}_{N-1})\\ \qquad +\rho(G_{N-1}+\bar{G}_{N-1})^T Q_N (D_{N-1}+\bar{D}_{N-1}),\end{cases}$$

Proof. See the Appendix.

We now express the optimal control using Riccati difference equations by the following theorem.

Theorem 4.3. *Let*

$$Q_k, Q_k+\bar{Q}_k \geq 0,,\ \ k\in\bar{\mathbb{N}},\ \ R_k, R_k+\bar{R}_k > 0, \tag{40}$$

and introduce Riccati equations

$$\begin{cases} S_k^x = Q_k + A_k^T S_{k+1}^x A_k + C_k^T S_{k+1}^x C_k - H_k^{(1)}(W_k^{(1)})^{-1}(H_k^{(1)})^T,\\ T_k^x = Q_k + \bar{Q}_k + (A_k+\bar{A}_k)^T T_{k+1}^x (A_k+\bar{A}_k) + (C_k+\bar{C}_k)^T S_{k+1}^x (C_k+\bar{C}_k)\\ \qquad -H_k^{(2)}(W_k^{(2)})^{-1}(H_k^{(2)})^T,\\ S_k^{xy} = -Q_k + F_k^T S_{k+1}^{xy} A_k + \rho G_k^T S_{k+1}^{xy} C_k - H_k^{(3)}(W_k^{(1)})^{-1}(H_k^{(1)})^T,\\ T_k^{xy} = -Q_k - \bar{Q}_k + (F_k+\bar{F}_k)^T T_{k+1}^{xy}(A_k+\bar{A}_k) + \rho(G_k+\bar{G}_k)^T S_{k+1}^{xy}(C_k+\bar{C}_k)\\ \qquad -H_k^{(4)}(W_k^{(2)})^{-1}(H_k^{(2)})^T,\\ S_k^y = Q_k + F_k^T S_{k+1}^y F_k + G_k^T S_{k+1}^y G_k - H_k^{(3)}(W_k^{(1)})^{-1}(H_k^{(3)})^T,\\ T_k^y = Q_k + \bar{Q}_k + (F_k+\bar{F}_k)^T T_{k+1}^y (F_k+\bar{F}_k) + (G_k+\bar{G}_k)^T S_{k+1}^y (G_k+\bar{G}_k)\\ \qquad -H_k^{(4)}(W_k^{(2)})^{-1}(H_k^{(4)})^T,\\ S_N^x = Q_N,\ T_N^x = Q_N+\bar{Q}_N,\ S_N^{xy} = -Q_N,\\ T_N^{xy} = -Q_N - \bar{Q}_N,\ S_N^y = Q_N,\ T_N^y = Q_N+\bar{Q}_N. \end{cases} \tag{41}$$

with

$$\begin{cases} W_k^{(1)} = R_k + B_k^T S_{k+1}^x B_k + D_k^T S_{k+1}^x D_k,\\ W_k^{(2)} = R_k + \bar{R}_k + (B_k+\bar{B}_k)^T T_{k+1}^x (B_k+\bar{B}_k) + (D_k+\bar{D}_k)^T S_{k+1}^x (D_k+\bar{D}_k),\\ H_k^{(1)} = A_k^T S_{k+1}^x B_k + C_k^T S_{k+1}^x D_k,\\ H_k^{(2)} = (A_k+\bar{A}_k)^T T_{k+1}^x (B_k+\bar{B}_k) + (C_k+\bar{C}_k)^T S_{k+1}^x (D_k+\bar{D}_k),\\ H_k^{(3)} = F_k^T S_{k+1}^{xy} B_k + \rho G_k^T S_{k+1}^{xy} D_k,\\ H_k^{(4)} = (F_k+\bar{F}_k)^T T_{k+1}^{xy}(B_k+\bar{B}_k) + \rho(G_k+\bar{G}_k)^T S_{k+1}^{xy}(D_k+\bar{D}_k). \end{cases} \tag{42}$$

The unique optimal control for Problem (MF-LQ) is

$$\begin{aligned} u_k^* = &-(W_k^{(1)})^{-1}(H_k^{(1)})^T(x_k - \mathbb{E}x_k) - (W_k^{(2)})^{-1}(H_k^{(2)})^T\mathbb{E}x_k\\ &-(W_k^{(1)})^{-1}(H_k^{(3)})^T(y_k - \mathbb{E}y_k) - (W_k^{(2)})^{-1}(H_k^{(4)})^T\mathbb{E}y_k,\ \ k\in\mathbb{N}. \end{aligned} \tag{43}$$

Proof. See the Appendix.

5. An Example

Basing upon the general theory in previous sections, in this section, we consider an example extended from financial application in asset-liability management with numerical results.

5.1. Example Setting

Suppose an investment market and a loan market consisting of m risky investment acceptable assets, one risk-free asset and one loan product over a time horizon N. Let $B_k = (B_k^1, ..., B_k^m)$ be the row vector of random excess returns of the m risky assets, a_k and f_k are given return of the risk-free asset and repayment of loan at time period k respectively. We assume that vectors $B_k,\ k = 0, 1, ..., N-1$ are statistically independent and the only information known about the random excess return vector B_k is its first two moments: its mean $\mathbb{E}(B_k)$ and covariance $Cov(B_k)$.

Let x_k and y_k be the total asset and liability at the beginning of the k-th period, respectively. Let $u_k^i, i = 1, 2, ..., m$, be the amount invested in the i-th risky asset at period k. The system combined with assets and liabilities at the beginning of the $(k+1)$-th period is given by

$$\begin{cases} x_{k+1} = a_k x_k + B_k u_k, \\ y_{k+1} = f_k y_k, \\ x_0 = \zeta^x,\ y_0 = \zeta^y, \end{cases} \tag{44}$$

Define

$$\begin{cases} D_k^i = (0, ..., 0, 1, 0, ..., 0), \text{where 1 is the } i\text{th entry}, \\ w_k^i = B_k^i - \mathbb{E}(B_k^i),\ w_k = (w_k^1, w_k^2, ..., w_k^m)^T, \quad i = 1, ..., m, \quad k = 1, ..., N-1. \end{cases}$$

These lead to

$$\begin{cases} x_{k+1} = a_k x_k + \mathbb{E}B_k u_k + \sum_{i=1}^{m} D_k^i u_k w_k^i, \\ y_{k+1} = f_k y_k, \\ x_0 = \zeta^x,\ y_0 = \zeta^y. \end{cases} \tag{45}$$

Clearly, $x_k, y_k \in \mathbb{R},\ k \in \mathbb{N}$. $a_k, f_k \in \mathbb{R}$ and $\mathbb{E}B_k, D_k^i \in \mathbb{R}^{1\times m}$ are deterministic. By taking expectation of the system state, we have

$$\begin{cases} \mathbb{E}x_{k+1} = a_k \mathbb{E}x_k + \mathbb{E}B_k \mathbb{E}u_k, \\ \mathbb{E}y_{k+1} = f_k \mathbb{E}y_k, \\ \mathbb{E}x_0 = \mathbb{E}\zeta^x,\ \mathbb{E}y_0 = \mathbb{E}\zeta^y. \end{cases}$$

Hence,

$$\begin{cases} x_{k+1} - \mathbb{E}x_{k+1} = a_k(x_k - \mathbb{E}x_k) + \mathbb{E}B_k(u_k - \mathbb{E}u_k) + \sum_{i=1}^{m} D_k^i u_k w_k^i, \\ y_{k+1} - \mathbb{E}y_{k+1} = f_k(y_k - \mathbb{E}y_k), \\ x_0 = \zeta^x,\ y_0 = \zeta^y. \end{cases} \tag{46}$$

Define $\mathcal{F}'_k$ by the information set at the beginning of period k which is generated by $\{\zeta^x, w_l, l = 0, 1, ..., k\}$. Recall that w_k is a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathbb{E}\big[w_{k+1}w_{k+1}^T|\mathcal{F}'_k\big] = \alpha_{k+1} = Cov(B_{k+1})$. The cost functional (an extension of variance function) associated with (44) is

$$J(\zeta^x, \zeta^y, u) = \sum_{k=0}^{N-1} \mathbb{E}\big(u_k^T R_k u_k\big) + \mathbb{E}\big(q_N(x_N - y_N)^2\big) + \bar{q}_N\big(\mathbb{E}(x_N - y_N)\big)^2, \tag{47}$$

where $q_N, \bar{q}_N, R_k, k \in \mathbb{N}$ are deterministic symmetric matrices with appropriate dimensions. In this paper, we consider the case where short-selling of stock is allowed, i.e., $u_k^i, i = 1, ..., k$, could take values in $\mathbb{R}$. Hence, the admissible policy set of $u = (u_0, u_1, \cdots, u_{N-1})$ in this section

$$\mathcal{U}_{ad} \equiv \left\{u \;\middle|\; u_k \in \mathbb{R}^m, \text{ is } \mathcal{F}'_k\text{-measurable }, \mathbb{E}|u_k|^2 < \infty\right\}.$$

Problem (MF-LQ) extended from asset-liability management is represented as follows:

Problem (MF-AL). *For any given square-integrable initial values ζ^x and ζ^y, find $u^o \in \mathcal{U}_{ad}$ such that*

$$J(\zeta^x, \zeta^y, u^o) = \inf_{u \in \mathcal{U}_{ad}} J(\zeta^x, \zeta^y, u). \tag{48}$$

We then call u^o an optimal control for Problem (MF-AL).

To proceed, we recall the following lemma (Dunne & Stone, 1993).

Lemma 5.1. *Let $M \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$. If $c \in Range(M)$, then*

$$(M \pm cc^T)^\dagger = M - \frac{M^\dagger cc^T M^\dagger}{c^T M^\dagger c}.$$

By Theorem 4.3, we have the following result. Suppose that

$$R_k > 0,\ k \in \mathbb{N},\ \ q_N, q_N + \bar{q}_N \geq 0. \tag{49}$$

The unique optimal strategy for Problem (MF-AL) is given by

$$\begin{aligned} u_k^o = &-(W_k^{(1)})^{-1}(H_k^{(1)})^T(x_k - \mathbb{E}x_k) - (W_k^{(2)})^{-1}(H_k^{(2)})^T\mathbb{E}x_k \\ &-(W_k^{(1)})^{-1}(H_k^{(3)})^T(y_k - \mathbb{E}y_k) - (W_k^{(2)})^{-1}(H_k^{(4)})^T\mathbb{E}y_k,\ k \in \mathbb{N}. \end{aligned} \tag{50}$$

where

$$\begin{cases} W_k^{(1)} = R_k + S_{k+1}^x \mathbb{E}(B_k^T B_k), \\ W_k^{(2)} = R_k + T_{k+1}^x \mathbb{E}B_k^T \mathbb{E}B_k + S_{k+1}^x Cov(B_k), \\ H_k^{(1)} = a_k S_{k+1}^x \mathbb{E}B_k, \\ H_k^{(2)} = a_k T_{k+1}^x \mathbb{E}B_k, \\ H_k^{(3)} = f_k S_{k+1}^{xy} \mathbb{E}B_k, \\ H_k^{(4)} = f_k T_{k+1}^{xy} \mathbb{E}B_k, \end{cases}$$

and

$$\begin{cases} S_k^x = a_k^2 S_{k+1}^x \big[1 - S_{k+1}^x \mathbb{E}B_k \big(R_k + S_{k+1}^x \mathbb{E}(B_k^T B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ T_k^x = a_k^2 T_{k+1}^x \big[1 - T_{k+1}^x \mathbb{E}B_k \big(R_k + T_{k+1}^x \mathbb{E}B_k^T \mathbb{E}B_k + S_{k+1}^x Cov(B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ S_k^{xy} = a_k f_k S_{k+1}^{xy} \big[1 - S_{k+1}^x \mathbb{E}B_k \big(R_k + S_{k+1}^x \mathbb{E}(B_k^T B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ T_k^{xy} = a_k f_k T_{k+1}^{xy} \big[1 - T_{k+1}^x \mathbb{E}B_k \big(R_k + T_{k+1}^x \mathbb{E}B_k^T \mathbb{E}B_k + S_{k+1}^x Cov(B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ S_k^y = f_k^2 \big[S_{k+1}^y - (S_{k+1}^{xy})^2 \mathbb{E}B_k \big(R_k + S_{k+1}^x \mathbb{E}(B_k^T B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ T_k^y = f_k^2 \big[T_{k+1}^y - (T_{k+1}^{xy})^2 \mathbb{E}B_k \big(R_k + T_{k+1}^x \mathbb{E}B_k^T \mathbb{E}B_k + S_{k+1}^x Cov(B_k)\big)^{-1} \mathbb{E}B_k^T\big], \\ S_N^x = q_N,\ T_N^x = q_N + \bar{q}_N,\ S_N^{xy} = -q_N, \\ T_N^{xy} = -q_N - \bar{q}_N,\ S_N^y = q_N,\ T_N^y = q_N + \bar{q}_N. \end{cases}$$

Under the optimal strategy (50), the optimal solution of cost functional is

$$\begin{aligned} J(\zeta^x, \zeta^y, u^o) = \mathbb{E}\big[&S_0^x(\zeta^x - \mathbb{E}\zeta^x)^2 + T_0^x(\mathbb{E}\zeta^x)^2 + 2S_0^{xy}(\zeta^x - \mathbb{E}\zeta^x)(\zeta^y - \mathbb{E}\zeta^y) \\ &+ T_0^{xy}\mathbb{E}\zeta^x\mathbb{E}\zeta^y + S_0^y(\zeta^y - \mathbb{E}\zeta^y)^2 + T_0^y(\mathbb{E}\zeta^y)^2\big], \end{aligned}$$

and its related expectation of system state in N-th period is

$$\mathbb{E}(x_N - y_N) = \prod_{k=1}^{N-1} N_k \mathbb{E}\zeta^x + \Big(\sum_{k=0}^{N-1} \prod_{j=k+1}^{N-1} N_j M_k \prod_{j=1}^{k-1} f_k - \prod_{k=1}^{N-1} f_k\Big)\mathbb{E}\zeta^y$$

with $N_k = a_k\big(1 - T_{k+1}^x \mathbb{E}B_k (W_k^{(2)})^{-1} \mathbb{E}B_k^T\big)$ and $M_k = -f_k T_{k+1}^{xy} \mathbb{E}B_k (W_k^{(2)})^{-1} \mathbb{E}B_k^T$.

5.2. Numerical Results

Consider a 3-period numerical example. Coefficients are given as follows:
$a_k = 0.5,\ f_k = 0.6,\ \mathbb{E}B_k = \big(0.2,\ 0.3,\ 0.4\big),\ R_k = I,\ \bar{R}_k = 0,\ q_3 = 1,\ \bar{q}_3 = -1,$
$Cov(B_k) = \begin{pmatrix} 1 & 0.2 & 0.3 \\ 0.2 & 1 & 0.6 \\ 0.3 & 0.6 & 1 \end{pmatrix}.$
By simple calculation, we have

$$\mathbb{E}B_k^T\mathbb{E}B_k = \begin{pmatrix} 0.040 & 0.060 & 0.080 \\ 0.060 & 0.090 & 0.120 \\ 0.080 & 0.120 & 0.160 \end{pmatrix}, \quad \mathbb{E}(B_k^T B_k) = \begin{pmatrix} 1.040 & 0.260 & 0.380 \\ 0.260 & 1.090 & 0.720 \\ 0.380 & 0.720 & 1.160 \end{pmatrix}.$$

Based on the result in Section 5.1, the Riccati solutions for S_k and T_k for $k = 0, 1, 2, 3$ are given by

$$\begin{aligned} &S_3^x = 1, && S_3^{xy} = -1, && S_3^y = 1, \\ &S_2^x = 0.2260, && S_2^{xy} = -0.2712, && S_2^y = 0.3254, \\ &S_1^x = 0.0540, && S_1^{xy} = -0.0777, && S_1^y = 0.1119, \\ &S_0^x = 0.0133, && S_0^{xy} = -0.0230, && S_0^y = 0.0397, \end{aligned}$$

and $T_k^x = T_k^{xy} = T_k^y = 0$, which lead $N_k = a_k$ and $M_k = 0$. We also obtain the optimal control, that is $u_k^o = O_k^x(x_k - \mathbb{E}x_k) + O_k^y(y_k - \mathbb{E}y_k),\ k = 0, 1, 2,$ where

$$\begin{aligned} O_2^x &= \begin{pmatrix} -0.0300 & -0.0429 & -0.0730 \end{pmatrix}, \\ O_2^y &= \begin{pmatrix} 0.0359 & 0.0515 & 0.0876 \end{pmatrix}, \\ O_1^x &= \begin{pmatrix} -0.0150 & -0.0223 & -0.0319 \end{pmatrix}, \\ O_1^y &= \begin{pmatrix} 0.0216 & 0.0321 & 0.0460 \end{pmatrix}, \\ O_0^x &= \begin{pmatrix} -0.0048 & -0.0072 & -0.0098 \end{pmatrix}, \\ O_0^y &= \begin{pmatrix} 0.0069 & 0.0104 & 0.0141 \end{pmatrix}. \end{aligned}$$

6. Conclusion

In this paper, we first formulate the Problem (MF-LQ) and give necessary and sufficient conditions for the solvability of the problem. Two approaches, dynamical optimization by matrix minimum principle and operator linear-quadratic method, are investigated to derive the optimal control, where six Riccati equations are obtained accordingly. Also, after concerning with the solution of this problem under multidimensional noise assumption, we give an financial application with numerical results.

The current research may be extended in the following ways. In this paper, we find that the solvability of the problem is related to the definiteness of the system coefficient matrices. For future research, we may study a model with relaxation of conditions such as indefinite mean-field stochastic linear-quadratic optimal control problems. We may also expend the problem from finite horizon to infinite horizon, where the stability of system should be considered first.

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Appendix

Proof of Theorem 4.1

From Proposition 3.2, we know that Problem (MF-LQ) admits a unique minimizer under condition (21). Thus, the optimal control uniquely exists. We now introduce the Lagrangian function associated with Problem (13),

$$\begin{aligned}\mathfrak{L} = & \sum_{k=0}^{N-1} \mathfrak{L}_k + Tr\big[Q_N\big(X_N - XY_N - (XY_N)^T + Y_N\big)\big] \\ & + Tr\big[\bar{Q}_N\big(\bar{X}_N - \bar{XY}_N - (\bar{XY}_N)^T + \bar{Y}_N\big)\big]\end{aligned}$$

where

$$\mathfrak{L}_k = Tr\big[Q_k\big(X_k - XY_k - (XY_k)^T + Y_k\big)\big] + Tr\big[\bar{Q}_k\big(\bar{X}_k - \bar{XY}_k - (\bar{XY}_k)^T + \bar{Y}_k\big)\big]$$
$$+Tr\big[(L_k^x)^T R_k L_k^x X_k + \big((L_k^x)^T R_k \bar{L}_k^x + (\bar{L}_k^x)^T R_k L_k^x + (\bar{L}_k^x)^T R_k \bar{L}_k^x\big)\bar{X}_k\big]$$
$$+Tr\big[(L_k^x)^T R_k L_k^y (XY_k)^T + \big((L_k^x)^T R_k \bar{L}_k^y + (\bar{L}_k^x)^T R_k L_k^y + (\bar{L}_k^x)^T R_k \bar{L}_k^y\big)(\bar{XY}_k)^T\big]$$
$$+Tr\big[(L_k^y)^T R_k L_k^x XY_k + \big((L_k^y)^T R_k \bar{L}_k^x + (\bar{L}_k^y)^T R_k L_k^x + (\bar{L}_k^y)^T R_k \bar{L}_k^x\big)\bar{XY}_k\big]$$
$$+Tr\big[(L_k^y)^T R_k L_k^y Y_k + \big((L_k^y)^T R_k \bar{L}_k^y + (\bar{L}_k^y)^T R_k L_k^y + (\bar{L}_k^y)^T R_k \bar{L}_k^y\big)\bar{Y}_k\big]$$
$$+Tr\big[(L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\bar{X}_k + (L_k^x + \bar{L}_k^x)^T \bar{R}_k (L_k^y + \bar{L}_k^y)(\bar{XY}_k)^T\big]$$
$$+Tr\big[(L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^x + \bar{L}_k^x)\bar{XY}_k + (L_k^y + \bar{L}_k^y)^T \bar{R}_k (L_k^y + \bar{L}_k^y)\bar{Y}_k\big]$$
$$+Tr\left[\begin{pmatrix} P_{k+1}^x \\ \bar{P}_{k+1}^x \\ 2P_{k+1}^{xy} \\ 2\bar{P}_{k+1}^{xy} \\ P_{k+1}^y \\ \bar{P}_{k+1}^y \end{pmatrix}^T \cdot \begin{pmatrix} \mathcal{X}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y) - X_{k+1} \\ \bar{\mathcal{X}}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y) - \bar{X}_{k+1} \\ \mathcal{XY}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y) - XY_{k+1} \\ \bar{\mathcal{XY}}_k(L_k^x, \bar{L}_k^x, L_k^y, \bar{L}_k^y) - \bar{XY}_{k+1} \\ \mathcal{Y}_k - Y_{k+1} \\ \bar{\mathcal{Y}}_k - \bar{Y}_{k+1} \end{pmatrix}\right],$$

and $P_{k+1}^x\ \bar{P}_{k+1}^x\ P_{k+1}^{xy}\ \bar{P}_{k+1}^{xy}\ P_{k+1}^y\ \bar{P}_{k+1}^y,\ k \in \mathbb{N}$ are the Lagrangian multipliers. Denote $\mathbb{P}_{k+1} = \big(P_{k+1}^x\ \bar{P}_{k+1}^x\ 2P_{k+1}^{xy}\ 2\bar{P}_{k+1}^{xy}\ P_{k+1}^y\ \bar{P}_{k+1}^y\big)$ and $\mathbb{X}_k = \big(X_k\ \bar{X}_k\ XY_k\ \bar{XY}_k\ Y_k\ \bar{Y}_k\big)$. Clearly, by the matrix minimum principle (Athans, 1967), the optimal feedback gains L_k^o and Lagrangian multipliers $\mathbb{P}_k$ satisfy the following first-order necessary conditions

$$\begin{cases} \dfrac{\partial \mathfrak{L}_k}{\partial L_k^o} = 0, \quad \mathbb{P}_k = \dfrac{\partial \mathfrak{L}_k}{\partial \mathbb{X}_k}, \quad k \in \mathbb{N}, \\ \mathbb{P}_N = \big(Q_N \quad \bar{Q}_N \quad -2Q_N \quad -2\bar{Q}_N \quad Q_N \quad \bar{Q}_N\big), \end{cases}$$

i.e.,

$$\begin{cases} \dfrac{\partial \mathfrak{L}_k}{\partial L_k^{xo}} = 0,\ \dfrac{\partial \mathfrak{L}_k}{\partial \bar{L}_k^{xo}} = 0,\ \dfrac{\partial \mathfrak{L}_k}{\partial L_k^{yo}} = 0,\ \dfrac{\partial \mathfrak{L}_k}{\partial \bar{L}_k^{yo}} = 0, \\ P_k^x = \dfrac{\partial \mathfrak{L}_k}{\partial X_k},\ \bar{P}_k^x = \dfrac{\partial \mathfrak{L}_k}{\partial \bar{X}_k},\ P_k^{xy} = \dfrac{1}{2}\dfrac{\partial \mathfrak{L}_k}{\partial XY_k}, \\ \bar{P}_k^{xy} = \dfrac{1}{2}\dfrac{\partial \mathfrak{L}_k}{\partial \bar{XY}_k},\ P_k^y = \dfrac{\partial \mathfrak{L}_k}{\partial Y_k},\ \bar{P}_k^y = \dfrac{\partial \mathfrak{L}_k}{\partial \bar{Y}_k},\ k \in \mathbb{N} \\ P_N^x = Q_N,\ \bar{P}_N^x = \bar{Q}_N,\ P_N^{xy} = -Q_N, \\ \bar{P}_N^{xy} = -\bar{Q}_N,\ P_N^y = Q_N,\ \bar{P}_N^y = \bar{Q}_N. \end{cases}$$

Now, we shall calculate several gradient matrices. Firstly, we have

$$\begin{aligned}
\frac{\partial \mathfrak{L}_k}{\partial L_k^x} &= 2X_k\big[(L_k^x)^T(R_k + B_k^T P_{k+1}^x B_k + D_k^T P_{k+1}^x D_k) + A_k^T P_{k+1}^x B_k + C_k^T P_{k+1}^x D_k\big] \\
&\quad + 2\bar{X}_k\big\{(\bar{L}_k^x)^T\big[R_k + \bar{R}_k + 2XY_k\big[(L_k^y)^T(R_k + B_k^T P_{k+1}^x B_k + D_k^T P_{k+1}^x D_k) \\
&\quad + F_k^T P_{k+1}^{xy} B_k + \rho G_k^T P_{k+1}^{xy} D_k\big] + 2\bar{XY}_k\big\{(\bar{L}_k^y)^T\big[R_k + \bar{R}_k \\
&\quad + (B_k + \bar{B}_k)^T(P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) \qquad + (D_k + \bar{D}_k)^T P_{k+1}^x (D_k + \bar{D}_k)\big] \\
&\quad + (L_k^y)^T[\bar{R}_k + (B_k + \bar{B}_k)^T(P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) - B_k^T P_{k+1}^x B_k \\
&\quad + (D_k + \bar{D}_k)^T P_{k+1}^x (D_k + \bar{D}_k) - D_k^T P_{k+1}^x D_k\big] \\
&\quad + (F_k + \bar{F}_k)^T(P_{k+1}^{xy} + \bar{P}_{k+1}^{xy})(B_k + \bar{B}_k) - F_k^T P_{k+1}^{xy} B_k \\
&\quad + \rho(G_k + \bar{G}_k)^T P_{k+1}^{xy}(D_k + \bar{D}_k) - \rho G_k^T P_{k+1}^{xy} D_k\big\} \\
&= 2(X_k - \bar{X}_k)\big((L_k^x)^T \bar{W}_k^{(1)} + \bar{H}_k^{(1)}\big) + 2\bar{X}_k\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) \\
&\quad + 2(XY_k - \bar{XY}_k)\big((L_k^y)^T \bar{W}_k^{(1)} + \bar{H}_k^{(3)}\big) + 2\bar{XY}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\frac{\partial \mathfrak{L}_k}{\partial \bar{L}_k^x} &= 2\bar{X}_k\big\{(L_k^x + \bar{L}_k^x)^T\big[R_k + \bar{R}_k + (B_k + \bar{B}_k)^T(P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) \\
&\quad + (D_k + \bar{D}_k)^T P_{k+1}^x (D_k + \bar{D}_k)\big] + (A_k + \bar{A}_k)^T(P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) \\
&\quad + (C_k + \bar{C}_k)^T P_{k+1}^x (D_k + \bar{D}_k)\big\} \\
&\quad + 2\bar{XY}_k\big\{(L_k^y + \bar{L}_k^y)^T\big[R_k + \bar{R}_k + (B_k + \bar{B}_k)^T(P_{k+1}^x + \bar{P}_{k+1}^x)(B_k + \bar{B}_k) \\
&\quad + (D_k + \bar{D}_k)^T P_{k+1}^x (D_k + \bar{D}_k)\big] + (F_k + \bar{F}_k)^T(P_{k+1}^{xy} + \bar{P}_{k+1}^{xy})(B_k + \bar{B}_k) \\
&\quad + \rho(G_k + \bar{G}_k)^T P_{k+1}^{xy}(D_k + \bar{D}_k)\big\} \\
&= 2\bar{X}_k\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) + 2\bar{XY}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big),
\end{aligned} \tag{52}$$

Similarly, it can be shown that

$$\begin{aligned}
\frac{\partial \mathfrak{L}_k}{\partial L_k^y} &= 2(Y_k - \bar{Y}_k)\big((L_k^y)^T \bar{W}_k^{(1)} + \bar{H}_k^{(3)}\big) + 2\bar{Y}_k\big(\big(L_k^y + \bar{L}_k^y\big)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) \\
&\quad + 2(XY_k - \bar{XY}_k)^T\big((L_k^x)^T \bar{W}_k^{(1)} + \bar{H}_k^{(1)}\big) + 2(\bar{XY}_k)^T\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big),
\end{aligned} \tag{53}$$

and

$$\frac{\partial \mathfrak{L}_k}{\partial \bar{L}_k^y} = 2\bar{Y}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) + (\bar{XY}_k)^T\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big). \tag{54}$$

Here, $\bar{W}_k^{(i)}, i = 1,2, \bar{H}_k^{(j)}, j = 1,2,3,4$, are defined in (14). Combining (51)–(54), $L_k^{xo}, \bar{L}_k^{xo}, L_k^{yo}$ and $\bar{L}_k^{yo}$ must satisfy

$$\begin{cases} (X_k - \bar{X}_k)\big((L_k^x)^T \bar{W}_k^{(1)} + \bar{H}_k^{(1)}\big) + \bar{X}_k\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) \\ +(XY_k - \bar{XY}_k)\big((L_k^y)^T \bar{W}_k^{(1)} + \bar{H}_k^{(3)}\big) + \bar{XY}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) = 0, \\ \bar{X}_k\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) + \bar{XY}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) = 0, \\ (Y_k - \bar{Y}_k)\big((L_k^y)^T \bar{W}_k^{(1)} + \bar{H}_k^{(3)}\big) + \bar{Y}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) \\ +(XY_k - \bar{XY}_k)^T\big((L_k^x)^T \bar{W}_k^{(1)} + \bar{H}_k^{(1)}\big) + (\bar{XY}_k)^T\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) = 0, \\ \bar{Y}_k\big((L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)}\big) + (\bar{XY}_k)^T\big((L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)}\big) = 0. \end{cases} \tag{55}$$

Note that (55) holds for any initial values $X_0 - \bar{X}_0$, $\bar{X}_0, XY_0 - \bar{XY}_0$, $\bar{XY}_0$, $Y_0 - \bar{Y}_0$ and $\bar{Y}_0$. Hence, (55) reduces to

$$\begin{cases} (L_k^x)^T \bar{W}_k^{(1)} + \bar{H}_k^{(1)} = 0, \\ (L_k^x + \bar{L}_k^x)^T \bar{W}_k^{(2)} + \bar{H}_k^{(2)} = 0, \\ (L_k^y)^T \bar{W}_k^{(1)} + \bar{H}_k^{(3)} = 0, \\ (L_k^y + \bar{L}_k^y)^T \bar{W}_k^{(2)} + \bar{H}_k^{(4)} = 0, \end{cases}$$

which are obtained by letting coefficients be zero in (55). Clearly, we obtain the optimal feedback gains within the class of controls (8)

$$\begin{cases} L_k^{xo} = -(\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(1)})^T, \\ \bar{L}_k^{xo} = -(\bar{W}_k^{(2)})^{-1}(\bar{H}_k^{(2)})^T + (\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(1)})^T, \\ L_k^{yo} = -(\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(3)})^T, \\ \bar{L}_k^{yo} = -(\bar{W}_k^{(2)})^{-1}(\bar{H}_k^{(4)})^T + (\bar{W}_k^{(1)})^{-1}(\bar{H}_k^{(3)})^T. \end{cases}$$

We now derive the expressions of P_k^x, $\bar{P}_k^x$, P_k^{xy}, $\bar{P}_k^{xy}$, $P_k^y, \bar{P}_k^y$. By (51), we have

$$\begin{cases} P_k^x = \dfrac{\partial \mathfrak{L}_k}{\partial X_k}\Big|_{L_k^x = L_k^{xo}}, \\ \bar{P}_k^x = \dfrac{\partial \mathfrak{L}_k}{\partial \bar{X}_k}\Big|_{L_k^x = L_k^{xo}, \bar{L}_k^x = \bar{L}_k^{xo}}, \\ P_k^{xy} = \dfrac{1}{2}\dfrac{\partial \mathfrak{L}_k}{\partial XY_k}\Big|_{L_k^x = L_k^{xo}, L_k^y = L_k^{yo}}, \\ \bar{P}_k^{xy} = \dfrac{1}{2}\dfrac{\partial \mathfrak{L}_k}{\partial \bar{XY}_k}\Big|_{L_k^x = L_k^{xo}, \bar{L}_k^x = \bar{L}_k^{xo}, L_k^y = L_k^{yo}, \bar{L}_k^y = \bar{L}_k^{yo}}, \\ P_k^y = \dfrac{\partial \mathfrak{L}_k}{\partial Y_k}\Big|_{L_k^y = L_k^{yo}}, \\ \bar{P}_k^y = \dfrac{\partial \mathfrak{L}_k}{\partial \bar{Y}_k}\Big|_{L_k^y = L_k^{yo}, \bar{L}_k^y = \bar{L}_k^{yo}}, \end{cases}$$

which are (15)–(20). The final step is to assure that for any $k \in \bar{\mathbb{N}}$, $P_k^x,\ P_k^x + \bar{P}_k^x \geq 0$. We prove this by backward induction. Clearly, $P_N^x,\ P_N^x + \bar{P}_N^x \geq 0$ by definition. For $k = N-1$, P_{N-1}^x is positive semi-definite and

$$\begin{aligned}
P_{N-1}^x + \bar{P}_{N-1}^x = {} & Q_{N-1} + \bar{Q}_{N-1} + (L_{N-1}^{xo} + \bar{L}_{N-1}^{xo})^T R_{N-1} (L_{N-1}^{xo} + \bar{L}_{N-1}^{xo}) \\
& + \big[A_{N-1} + \bar{A}_{N-1} + (B_{N-1} + \bar{B}_{N-1})(L_{N-1}^{xo} + \bar{L}_{N-1}^{xo})\big]^T (P_N^x + \bar{P}_N^x) \\
& \cdot \big[A_{N-1} + \bar{A}_{N-1} + (B_{N-1} + \bar{B}_{N-1})(L_{N-1}^{xo} + \bar{L}_{N-1}^{xo})\big] \\
& + \big[C_{N-1} + \bar{C}_{N-1} + (D_{N-1} + \bar{D}_{N-1})(L_{N-1}^{xo} + \bar{L}_{N-1}^{xo})\big]^T P_N^x \\
& \cdot \big[C_{N-1} + \bar{C}_{N-1} + (D_{N-1} + \bar{D}_{N-1})(L_{N-1}^{xo} + \bar{L}_{N-1}^{xo})\big] \geq 0.
\end{aligned}$$

After simple calculation, we have

$$\begin{aligned}
& J_{N-1}^N(x_{N-1}, y_{N-1}, u^o) \\
& = \mathbb{E}\big((x_{N-1} - y_{N-1})^T Q_{N-1} (x_{N-1} - y_{N-1})\big) + (\mathbb{E}(x_{N-1} - y_{N-1}))^T \bar{Q}_{N-1} \mathbb{E}(x_{N-1} - y_{N-1}) \\
& \quad + \mathbb{E}\big(u_{N-1}^T R_{N-1} u_{N-1}\big) + (\mathbb{E}u_{N-1})^T \bar{R}_{N-1} \mathbb{E}u_{N-1} + \mathbb{E}\big((x_N - y_N)^T Q_N (x_N - y_N)\big) \\
& \quad + (\mathbb{E}(x_N - y_N))^T \bar{Q}_N \mathbb{E}(x_N - y_N) \\
& = \mathbb{E}\big(x_{N-1}^T P_{N-1}^x x_{N-1}\big) + \big(\mathbb{E}x_{N-1}\big)^T \bar{P}_{N-1}^x \mathbb{E}x_{N-1} + 2\mathbb{E}\big(y_{N-1}^T P_{N-1}^{xy} x_{N-1}\big) \\
& \quad + 2\big(\mathbb{E}y_{N-1}\big)^T \bar{P}_{N-1}^{xy} \mathbb{E}x_{N-1} + \mathbb{E}\big(y_{N-1}^T P_{N-1}^y y_{N-1}\big) + \big(\mathbb{E}y_{N-1}\big)^T \bar{P}_{N-1}^y \mathbb{E}y_{N-1}.
\end{aligned}$$

We can easily induce that $P_k^x,\ P_k^x + \bar{P}_k^x \geq 0$. Note that

$$\begin{aligned}
& J_k^N(x_k, y_k, u^o|_{\{k,k+1,\cdots,N-1\}}) \\
& = \mathbb{E}\big((x_k - y_k)^T Q_k (x_k - y_k)\big) + (u_k^o)^T R_k u_k^o + \mathbb{E}(x_k - y_k)^T \bar{Q}_k \mathbb{E}(x_k - y_k) \\
& \quad + (\mathbb{E}u_k^o)^T \bar{R}_k \mathbb{E}u_k^o + J_{k+1}^N(x_{k+1}, y_{k+1}, u^o|_{\{k+1,k+2,\cdots,N-1\}}).
\end{aligned}$$

□

Proof of Lemma 4.2

We now reformulate equations (24) and (25) in terms of $\mathbb{E}$, $I - \mathbb{E}$ as follows

$$\begin{cases}
\mathcal{A}_k z = \big[A_k(I - \mathbb{E}) + (A_k + \bar{A}_k)\mathbb{E}\big] z, \\
\mathcal{C}_k z = \big[C_k(I - \mathbb{E}) + (C_k + \bar{C}_k)\mathbb{E}\big] z, \\
\mathcal{F}_k z = \big[F_k(I - \mathbb{E}) + (F_k + \bar{F}_k)\mathbb{E}\big] z, \\
\mathcal{G}_k z = \big[G_k(I - \mathbb{E}) + (G_k + \bar{G}_k)\mathbb{E}\big] z, \\
\mathcal{B}_k u = \big[B_k(I - \mathbb{E}) + (B_k + \bar{B}_k)\mathbb{E}\big] u, \\
\mathcal{D}_k u = \big[D_k(I - \mathbb{E}) + (D_k + \bar{D}_k)\mathbb{E}\big] u,
\end{cases}$$

and

$$\begin{cases}\mathcal{Q}_k z = \big[(I-\mathbb{E}^*)Q_k(I-\mathbb{E}) + \mathbb{E}^*(Q_k+\bar{Q}_k)\mathbb{E}\big]z,\\ \mathcal{R}_k u = \big[(I-\mathbb{E}^*)R_k(I-\mathbb{E}) + \mathbb{E}^*(R_k+\bar{R}_k)\mathbb{E}\big]u.\end{cases}$$

Obviously, from (24)–(33), we have

$$\begin{aligned}\Theta_{2,N-1} &= \mathcal{R}_{N-1} + \mathcal{B}_{N-1}^T\mathcal{P}_N^x\mathcal{B}_{N-1} + \mathcal{D}_{N-1}^T\mathcal{P}_N^x\mathcal{D}_{N-1}\\ &= (I-\mathbb{E}^*)\big[R_{N-1} + B_{N-1}^TQ_NB_{N-1} + D_{N-1}^TQ_ND_{N-1}\big](I-\mathbb{E})\\ &\quad +\mathbb{E}^*\big[R_{N-1}+\bar{R}_{N-1} + (B_{N-1}+\bar{B}_{N-1})^T(Q_N+\bar{Q}_N)(B_{N-1}+\bar{B}_{N-1})\\ &\quad +(D_{N-1}+\bar{D}_{N-1})^TQ_N(D_{N-1}+\bar{D}_{N-1})\big]\mathbb{E}\\ &= (I-\mathbb{E}^*)W_{N-1}^{(1)}(I-\mathbb{E}) + \mathbb{E}^*W_{N-1}^{(2)}\mathbb{E}.\end{aligned}$$

Similarly,

$$\begin{cases}\Theta_{1,N-1} = (I-\mathbb{E}^*)H_{N-1}^{(1)}(I-\mathbb{E}) + \mathbb{E}^*H_{N-1}^{(2)}\mathbb{E},\\ \Theta_{2,N-1} = (I-\mathbb{E}^*)W_{N-1}^{(1)}(I-\mathbb{E}) + \mathbb{E}^*W_{N-1}^{(2)}\mathbb{E},\\ \Theta_{3,N-1} = (I-\mathbb{E}^*)H_{N-1}^{(3)}(I-\mathbb{E}) + \mathbb{E}^*H_{N-1}^{(4)}\mathbb{E},\end{cases} \tag{56}$$

and

$$\begin{cases}\Lambda_{1,N-1} = (I-\mathbb{E}^*)\big(Q_{N-1} + A_{N-1}^TQ_NA_{N-1} + C_{N-1}^TQ_NC_{N-1}\big)(I-\mathbb{E})\\ \qquad\qquad +\mathbb{E}^*\big[Q_{N-1}+\bar{Q}_{N-1} + (A_{N-1}+\bar{A}_{N-1})^T(Q_N+\bar{Q}_N)(A_{N-1}+\bar{A}_{N-1})\\ \qquad\qquad +(C_{N-1}+\bar{C}_{N-1})^TQ_N(C_{N-1}+\bar{C}_{N-1})\big]\mathbb{E},\\ \Lambda_{2,N-1} = (I-\mathbb{E}^*)\big(-Q_{N-1} + F_{N-1}^TQ_NA_{N-1} + \rho G_{N-1}^TQ_NC_{N-1}\big)(I-\mathbb{E})\\ \qquad\qquad +\mathbb{E}^*\big[-Q_{N-1}-\bar{Q}_{N-1} + (F_{N-1}+\bar{F}_{N-1})^T(Q_N+\bar{Q}_N)(A_{N-1}+\bar{A}_{N-1})\\ \qquad\qquad +\rho(G_{N-1}+\bar{G}_{N-1})^TQ_N(C_{N-1}+\bar{C}_{N-1})\big]\mathbb{E},\\ \Lambda_{3,N-1} = (I-\mathbb{E}^*)\big(Q_{N-1} + F_{N-1}^TQ_NF_{N-1} + G_{N-1}^TQ_NG_{N-1}\big)(I-\mathbb{E})\\ \qquad\qquad +\mathbb{E}^*\big[Q_{N-1}+\bar{Q}_{N-1} + (F_{N-1}+\bar{F}_{N-1})^T(Q_N+\bar{Q}_N)(F_{N-1}+\bar{F}_{N-1})\\ \qquad\qquad +(G_{N-1}+\bar{G}_{N-1})^TQ_N(G_{N-1}+\bar{G}_{N-1})\big]\mathbb{E}.\end{cases} \tag{57}$$

To proceed, we need $\Theta_{2,N-1}^{-1}$. Under condition (38), $\mathcal{P}_N^x \geq 0$ and

$$\begin{aligned}
&\langle \mathcal{R}_{N-1} u_{N-1}, u_{N-1}\rangle \\
&= \mathbb{E}(u_{N-1}^T R_{N-1} u_{N-1}) + (\mathbb{E}u_{N-1})^T \bar{R}_{N-1}\mathbb{E}u_{N-1} \\
&= \mathbb{E}\big[(u_{N-1} - \mathbb{E}u_{N-1})^T R_{N-1}(u_{N-1} - \mathbb{E}u_{N-1})\big] \\
&\quad +(\mathbb{E}u_{N-1})^T (R_{N-1} + \bar{R}_{N-1})\mathbb{E}u_{N-1} \\
&\geq \lambda_1^{(N-1)}\mathbb{E}|u_{N-1} - \mathbb{E}u_{N-1}|_m^2 + \lambda_2^{(N-1)}|\mathbb{E}u_{N-1}|_m^2 \\
&= \lambda_1^{(N-1)}\big(\mathbb{E}|u_{N-1}|_m^2 - |\mathbb{E}u_{N-1}|_m^2\big) + \lambda_2^{(N-1)}|\mathbb{E}u_{N-1}|_m^2 \\
&\geq \lambda^{(N-1)}\big(\mathbb{E}|u_{N-1}|_m^2 - |\mathbb{E}u_{N-1}|_m^2 + |\mathbb{E}u_{N-1}|_m^2\big) \\
&= \lambda^{(N-1)}\|u_{N-1}\|_m^2.
\end{aligned} \tag{58}$$

Here $|\cdot|_m$ denotes the norm in $\mathbb{R}^m$; $\lambda_1^{(N-1)}, \lambda_2^{(N-1)}$ are the smallest eigenvalues of matrices R_{N-1} and $R_{N-1} + \bar{R}_{N-1}$, respectively, and $\lambda^{(N-1)} = \min\left\{\lambda_1^{(N-1)}, \lambda_2^{(N-1)}\right\}$; $\|\cdot\|_m$ is the norm induced by inner product in $\mathcal{U}_{N-1}$. Hence, $\Theta_{2,N-1}$ must be positive definite and self-adjoint. So far, (35)–(37) are established. Furthermore, the technique of operator pseudo-inverse is used to compute $\Theta_{2,N-1}^{-1}$ (Beutler, 1965; Elliott, Li & Ni, 2013). Clearly, $(I-\mathbb{E})(I-\mathbb{E})^\dagger = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{E}\mathbb{E}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, $(I-\mathbb{E})(I-\mathbb{E})^\dagger + \mathbb{E}\mathbb{E}^\dagger = I$ and $\mathbb{E}(I-\mathbb{E})^\dagger = 0, (I-\mathbb{E})\mathbb{E}^\dagger = 0$. From (56) and (57), we get

$$\begin{aligned}
\Theta_{2,N-1}^{-1} &= (\mathcal{R}_{N-1} + \mathcal{B}_{N-1}^T \mathcal{P}_N^x \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^T \mathcal{P}_N^x \mathcal{D}_{N-1})^{-1} \\
&= (I-\mathbb{E})^\dagger (W_{N-1}^{(1)})^{-1}(I-\mathbb{E}^*)^\dagger + \mathbb{E}^\dagger (W_{N-1}^{(2)})^{-1}(\mathbb{E}^*)^\dagger.
\end{aligned}$$

In fact, we have

$$\begin{aligned}
&\big[(I-\mathbb{E}^*)W_{N-1}^{(1)}(I-\mathbb{E}) + \mathbb{E}^* W_{N-1}^{(2)}\mathbb{E}\big]\big[(I-\mathbb{E})^\dagger (W_{N-1}^{(1)})^{-1}(I-\mathbb{E}^*)^\dagger + \mathbb{E}^\dagger (W_{N-1}^{(2)})^{-1}(\mathbb{E}^*)^\dagger\big] \\
&= (I-\mathbb{E}^*)(I-\mathbb{E})(I-\mathbb{E})^\dagger (I-\mathbb{E}^*)^\dagger + \mathbb{E}^*\mathbb{E}\mathbb{E}^\dagger(\mathbb{E}^*)^\dagger \\
&= I.
\end{aligned}$$

Hence, by simple calculation, we get (39). And we can easily derive that $\mathcal{P}_{N-1}^x$ is positive definite. □

Proof of Theorem 4.3

Suppose that $\mathcal{P}^x_{k+1} \geq 0$. By combining (28)–(30), we get

$$\begin{aligned}
\langle \mathcal{P}^x_{k+1} x_{k+1}, x_{k+1} \rangle = &\ \langle \mathcal{P}^x_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k), (\mathcal{A}_k x_k + \mathcal{B}_k u_k) \rangle \\
&+ \langle \mathcal{P}^x_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k), (\mathcal{C}_k x_k + \mathcal{D}_k u_k) \rangle, \\
\langle \mathcal{P}^{xy}_{k+1} x_{k+1}, y_{k+1} \rangle = &\ \langle \mathcal{P}^{xy}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k), \mathcal{F}_k y_k \rangle \\
&+ \rho \langle \mathcal{P}^{xy}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k), \mathcal{G}_k y_k \rangle, \\
\langle \mathcal{P}^y_{k+1} y_{k+1}, y_{k+1} \rangle = &\ \langle \mathcal{P}^y_{k+1} \mathcal{F}_k y_k, \mathcal{F}_k y_k \rangle + \langle \mathcal{P}^y_{k+1} \mathcal{G}_k y_k, \mathcal{G}_k y_k, \rangle.
\end{aligned}$$

We may isolate the following term from (26) in terms of k:

$$\begin{aligned}
&\langle \mathcal{Q}_k x_k, x_k \rangle - 2 \langle \mathcal{Q}_k x_k, y_k \rangle + \langle \mathcal{Q}_k y_k, y_k \rangle + \langle \mathcal{R}_k u_k, u_k \rangle \\
&\quad + \langle \mathcal{P}^x_{k+1} x_{k+1}, x_{k+1} \rangle - 2 \langle \mathcal{P}^{xy}_{k+1} x_{k+1}, y_{k+1} \rangle + \langle \mathcal{P}^y_{k+1} y_{k+1}, y_{k+1} \rangle \\
&= 2 \langle \Theta_{1,k} u_k, x_k \rangle + \langle \Theta_{2,k} u_k, u_k \rangle + 2 \langle \Theta_{3,k} u_k, y_k \rangle + \langle \Lambda_{1,k} x_k, x_k \rangle + 2 \langle \Lambda_{2,k} x_k, y_k \rangle + \langle \Lambda_{3,k} y_k, y_k \rangle \\
&= \big\langle (\Lambda_{1,k} - \Theta_{1,k} \Theta^{-1}_{2,k} \Theta^*_{1,k}) x_k, x_k \big\rangle + 2 \big\langle (\Lambda_{2,k} - \Theta_{3,k} \Theta^{-1}_{2,k} \Theta^*_{1,k}) x_k, y_k \big\rangle + \big\langle (\Lambda_{3,k} - \Theta_{3,k} \Theta^{-1}_{2,k} \Theta^*_{3,k}) y_k, y_k \big\rangle \\
&\quad + \big\langle \Theta_{2,k} (u_k + \Theta^{-1}_{2,k} \Theta^*_{1,k} x_k + \Theta^{-1}_{2,k} \Theta^*_{3,k} y_k), (u_k + \Theta^{-1}_{2,k} \Theta^*_{1,k} x_k + \Theta^{-1}_{2,k} \Theta^*_{3,k} y_k) \big\rangle,
\end{aligned}$$

where

$$\begin{cases}
\Theta_{1,k} = & \mathcal{A}^T_k \mathcal{P}^x_{k+1} \mathcal{B}_k + \mathcal{C}^T_k \mathcal{P}^x_{k+1} \mathcal{D}_k \\
\qquad = & (I - \mathbb{E}^*) H^{(1)}_k (I - \mathbb{E}) + \mathbb{E}^* H^{(2)}_k \mathbb{E}, \\
\Theta_{2,k} = & \mathcal{R}_k + \mathcal{B}^T_k \mathcal{P}^x_{k+1} \mathcal{B}_k + \mathcal{D}^T_k \mathcal{P}^x_{k+1} \mathcal{D}_k \\
\qquad = & (I - \mathbb{E}^*) W^{(1)}_k (I - \mathbb{E}) + \mathbb{E}^* W^{(2)}_k \mathbb{E}, \\
\Theta_{3,k} = & \mathcal{F}^T_k \mathcal{P}^{xy}_{k+1} \mathcal{B}_k + \rho \mathcal{G}^T_k \mathcal{P}^{xy}_{k+1} \mathcal{D}_k \\
\qquad = & (I - \mathbb{E}^*) H^{(3)}_k (I - \mathbb{E}) + \mathbb{E}^* H^{(4)}_k \mathbb{E},
\end{cases}$$

$$
\begin{cases}
\Lambda_{1,k} = & \mathcal{Q}_k + \mathcal{A}_k^T \mathcal{P}_{k+1}^x \mathcal{A}_k + \mathcal{C}_k^T \mathcal{P}_{k+1}^x \mathcal{C}_k \\
= & (I - \mathbb{E}^*)\big(Q_k + A_k^T S_{k+1}^x A_k + C_k^T S_{k+1}^x C_k\big)(I - \mathbb{E}) \\
& + \mathbb{E}^*\big[Q_k + \bar{Q}_k + (A_k + \bar{A}_k)^T T_{k+1}^x (A_k + \bar{A}_k) \\
& + (C_k + \bar{C}_k)^T S_{k+1}^x (C_k + \bar{C}_k)\big]\mathbb{E}, \\
\Lambda_{2,k} = & -\mathcal{Q}_k + \mathcal{F}_k^T \mathcal{P}_{k+1}^{xy} \mathcal{A}_k + \rho \mathcal{G}_k^T \mathcal{P}_{k+1}^{xy} \mathcal{C}_k \\
= & (I - \mathbb{E}^*)\big(-Q_k + F_k^T S_{k+1}^{xy} A_k + \rho G_k^T S_{k+1}^{xy} C_k\big)(I - \mathbb{E}) \\
& + \mathbb{E}^*\big[-Q_k - \bar{Q}_k + (F_k + \bar{F}_k)^T T_{k+1}^{xy} (A_k + \bar{A}_k) \\
& + \rho(G_k + \bar{G}_k)^T S_{k+1}^{xy} (C_k + \bar{C}_k)\big]\mathbb{E}, \\
\Lambda_{3,k} = & \mathcal{Q}_k + \mathcal{F}_k^T \mathcal{P}_{k+1}^y \mathcal{F}_k + \mathcal{G}_k^T \mathcal{P}_{k+1}^y \mathcal{G}_k \\
= & (I - \mathbb{E}^*)\big(Q_k + F_k^T S_{k+1}^y F_k + G_k^T S_{k+1}^y G_k\big)(I - \mathbb{E}) \\
& + \mathbb{E}^*\big[Q_k + \bar{Q}_k + (F_k + \bar{F}_k)^T T_{k+1}^y (F_k + \bar{F}_k) \\
& + (G_k + \bar{G}_k)^T S_{k+1}^y (G_k + \bar{G}_k)\big]\mathbb{E}.
\end{cases}
$$

Similar to (58), we have a positive-definite and self-adjoint $\Theta_{2,k}$ under condition (40) and hence

$$
\Theta_{2,k}^{-1} = (I - \mathbb{E})^\dagger (W_k^{(1)})^{-1} (I - \mathbb{E}^*)^\dagger + \mathbb{E}^\dagger (W_k^{(2)})^{-1} (\mathbb{E}^*)^\dagger.
$$

Let

$$
\begin{cases}
\mathcal{P}_k^x = \Lambda_{1,k} - \Theta_{1,k}\Theta_{2,k}^{-1}\Theta_{1,k}^* = (I - \mathbb{E}^*) S_k^x (I - \mathbb{E}) + \mathbb{E}^* T_k^x \mathbb{E}, \\
\mathcal{P}_k^{xy} = \Lambda_{2,k} - \Theta_{3,k}\Theta_{2,k}^{-1}\Theta_{1,k}^* = (I - \mathbb{E}^*) S_k^{xy} (I - \mathbb{E}) + \mathbb{E}^* T_k^{xy} \mathbb{E}, \\
\mathcal{P}_k^y = \Lambda_{3,k} - \Theta_{3,k}\Theta_{2,k}^{-1}\Theta_{3,k}^* = (I - \mathbb{E}^*) S_k^y (I - \mathbb{E}) + \mathbb{E}^* T_k^y \mathbb{E}.
\end{cases}
$$

Then, in a backward recursion,

$$
\begin{aligned}
& J_k^N(x_k, y_k, u^o|_{\{k,k+1,\cdots,N-1\}}) = \langle \mathcal{P}_k^x x_k, x_k\rangle + 2\langle \mathcal{P}_k^{xy} x_k, y_k\rangle + \langle \mathcal{P}_k^y x_k, x_k\rangle \\
& + \sum_{i=k}^{N-1} \big\langle \Theta_{2,i}(u_i + \Theta_{2,i}^{-1}\Theta_{1,i}^* x_i + \Theta_{2,i}^{-1}\Theta_{3,i}^* y_i), (u_i + \Theta_{2,i}^{-1}\Theta_{1,i}^* x_i + \Theta_{2,i}^{-1}\Theta_{3,i}^* y_i)\big\rangle.
\end{aligned}
$$

We can prove $\mathcal{P}_k^x \geq 0$ by induction. Consequently,

$$
\begin{aligned}
& J(\zeta^x, \zeta^y, u^o) = \langle \mathcal{P}_0^x x_0, x_0\rangle + 2\langle \mathcal{P}_0^{xy} x_0, y_0\rangle + \langle \mathcal{P}_0^y x_0, x_0\rangle \\
& + \sum_{k=0}^{N-1} \big\langle \Theta_{2,k}(u_k + \Theta_{2,k}^{-1}\Theta_{1,k}^* x_k + \Theta_{2,k}^{-1}\Theta_{3,k}^* y_k), (u_k + \Theta_{2,k}^{-1}\Theta_{1,k}^* x_k + \Theta_{2,k}^{-1}\Theta_{3,k}^* y_k)\big\rangle,
\end{aligned}
$$

and the optimal control

$$\begin{aligned}
u_k^* = & \; -\Theta_{2,k}^{-1}\Theta_{1,k}^* x_k - \Theta_{2,k}^{-1}\Theta_{3,k}^* y_k \\
= & \; -(I-\mathbb{E})^\dagger (W_k^{(1)})^{-1}(H_k^{(1)})^T(x_k - \mathbb{E}x_k) - \mathbb{E}^\dagger (W_k^{(2)})^{-1}(H_k^{(2)})^T \mathbb{E}x_k \\
& \; -(I-\mathbb{E})^\dagger (W_k^{(1)})^{-1}(H_k^{(3)})^T(y_k - \mathbb{E}y_k) - \mathbb{E}^\dagger (W_k^{(2)})^{-1}(H_k^{(4)})^T \mathbb{E}y_k, \quad k \in \mathbb{N},
\end{aligned}$$

which is (8) by computing $u_k^* = (I+\mathbb{E})u_k^* + \mathbb{E}u_k^*$. □