

## SOCIAL OPTIMA IN LEADER-FOLLOWER MEAN FIELD LINEAR QUADRATIC CONTROL\*

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**Abstract.** This paper investigates a linear quadratic mean field leader-follower team problem, where the model involves one leader and a large number of weakly-coupled interactive followers. The leader and the followers cooperate to optimize the social cost. Specifically, for any strategy provided first by the leader, the followers would like to choose a strategy to minimize social cost functional. Using variational analysis and person-by-person optimality, we construct two auxiliary control problems. By solving sequentially, the auxiliary control problems with consistent mean field approximations, we can obtain a set of decentralized social optimality strategy with help of a class of forward-backward consistency systems. The relevant Stackelberg equilibrium is further proved under some proper conditions.

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### 1. INTRODUCTION

Mean field games have been studied by researchers from various aspects [7, 8, 10, 25]. They involve a large number of population and the interaction between each individual is negligible. The mean field game approach has been applied in many fields such as finance [14], economics [41], information technology [21], engineering [13, 24] and medicine [4]. The mean field linear quadratic (LQ) control problem is a special class of control problems, which can model many problems in applications and its solution exhibits elegant properties. For more work about the problem, readers can refer to [3, 9, 18, 19, 22, 26, 36, 37, 39, 44].

For the model with one major player and  $N$  minor players, the state of the major player has a significant influence on state equations and cost functionals of other minor individuals, which can be considered as a strong effect of the major player on minor ones. Mean field games (control) with major and minor players have been discussed in some literature, such as [23, 32] for LQ problems, [6, 33] for nonlinear problems, [12, 20] for probabilistic approaches and [11] for finite-state problems. In such types of problem, there is no hierarchical structure of decision making between the major player and the minor players.

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In contrast to the model discussed above, leader-follower (Stackelberg) problems contain at least two hierarchies of players. One hierarchy of the players is defined as the leaders with a major position and another hierarchy of the players is defined as the followers with a minor position. The leader has priority to announce a strategy first and then the followers seek their strategies to minimize their cost functionals with response to the given strategy of leader. According to the followers' optimal points, the leader will choose his optimal strategy to minimize its cost functional. Leader-follower problem also has been widely investigated. For example, the two-person leader-follower problem combines with stochastic LQ differential game had been studied by Yong in [43] and the problem of one leader and  $N$  followers who play a noncooperative game under LQ stochastic differential game had been studied by Moon and Basar in [31]. For further literature related to Stackelberg games, readers can refer to [1, 2, 27, 29, 34, 35, 38].

Different from noncooperative games, the social optimization (team optimization) problem is a joint decision problem which all the players have the same goal and work cooperatively. The aim of each player is to select an optimal strategy and maximize the total payoff. Team optimization problem has been studied for many years. Marschak [30] first considered team optimization based on game theory. Ho and Chu studied team decision theory in optimal control problems [17]. Groves did the research of viewing the incentive problem as a team problem which the information for decisions is incomplete [16]. The team theory and person-by-person optimization with binary decision was investigated by Bauso and Pesenti [5] and the team problems under stochastic information structure with suboptimal solutions was studied in [15].

In this paper, we investigate social optimality of the leader-follower mean field LQ control problem. Our model contains one leader and  $N$  followers. The leader's state appears in both state equation and cost functional of each follower. It shows that the dynamics and cost functionals of the  $N$  followers are directly influenced by the behavior of the leader. Unlike the model in [23, 31], our model has a population state average term in all state equations and cost functionals. This implies that such state dynamics and cost functionals are highly interactive and coupled. In reality, it is almost impossible for one player to obtain all the information of other players. Therefore, decentralized control which is based on the individual information set will be used instead of centralized control which is based on full information set and the information structure of each agent is different.

Compared with previous works, this paper mainly makes the following contributions:

- A social optimum problem is studied for mean field models with hierarchical structure. Unlike the problem in [31] where the leader and followers play a noncooperative game and try to minimize their own individual cost functional, all individuals in our models aim to minimize the social cost functional which equals the summation of cost functionals of all players. The  $N$  followers are coupled by the population state average term. Since the cost functional presents individual performance in the game problems, the order of magnitude of the perturbation is  $\frac{1}{N}$  which can be ignored. The population state average term may be approximated by a stochastic process directly (see [18]). However, in the team problems, the order of magnitude of the perturbation cannot be ignored after summing up all the cost functionals, which makes the problem very complicated. To overcome such difficulties, we approximate some terms as  $N$  goes to infinity and use a duality procedure combined with auxiliary equations to transform the variation of the social cost functional into a standard LQ control form. Then, we construct an auxiliary control problem and a forward-backward consistency system which contains four equations to help us obtain the decentralized form of the optimal controls for the  $N$  followers.
- The decentralized controls of the leader-follower problem are obtained and the solvability of a high-dimensional consistency condition system (CC system) is discussed. Since the leader's state equation and cost functional are fully coupled with the followers' state equations and cost functionals, it is more difficult to solve the leader's problem. Except constructing auxiliary problem by mean field approximation as in the former part, we need to construct six auxiliary equations and use duality relations to obtain the decentralized form of the optimal control for the leader. Unlike the problem for  $N$  followers, the final CC system of the leader's problem contains ten equations which becomes a high-dimensional problem. To solve such equations directly is very difficult since they are fully coupled and have high-dimensional characteristics. We transform the high-dimensional CC system to a simple form of linear forward-backward

stochastic differential equation (FBSDE; see [28, 42]) and discuss the solvability of the FBSDE through Ricatti equation method.

- The decentralized strategies of leader-follower problem are proved to be Stackelberg equilibrium by perturbation analysis. Different from [22, 23, 31, 40], we discuss the Stackelberg equilibrium for the team optimization problem. First, we need to prove the decentralized strategies for the followers have asymptotic social optimality. Because of the Stackelberg problem contains two hierarchies, we consider two coupled cost functionals (the leader and the followers) when using the standard method (see [22]). We prove the asymptotic optimality by decoupling them with two duality procedures and some arguments in error estimates. Second, we need to prove the decentralized strategies for the leader-follower problem is Stackelberg equilibrium. Also, some error estimates are very hard to be given directly since they are fully coupling. We decompose them by applying Ricatti equation method and then estimate them in proper order.

In the real world, our model can be used to describe some examples. For an automatic machine, the major part first gives an information to the system and the minor parts will adjust their parameters automatically such that the whole system keeps in the best state. For the economic environment, the small companies may hesitate to make decisions and often follow a monopoly company when facing to the volatility of the market. A monopoly company announces a decision first. Once the small companies try to make decisions according to their own situations, the monopoly company adjusts its decision such that the sum of the social wealth can be maximized. Moreover, the relationship between the employer and the employees or the federal government and the government in each state also can be described by our model.

The paper is organized as follows. The problem is formulated in Section 2. In Section 3, we solve the optimal controls for followers based on person-by-person optimality and obtain the CC system of the follower's problem. In Section 4, we seek the social optimal solution of the leader's problem and give the CC system of the leader's problem. Then the CC system is transformed to a simple form of linear FBSDE in Section 5 and its wellposedness is discussed. In Section 6, we give the details of proving the Stackelberg equilibrium. Also, some prior lemmas will be introduced and proved. In Section 7, a numerical example is provided to simulate the efficiency of decentralized control. Section 8 is the conclusion of the paper.

*Notation:* throughout this paper,  $\mathbb{R}^{n \times m}$  and  $\mathbb{S}^n$  denote the set of all  $(n \times m)$  real matrices and the set of all  $(n \times n)$  symmetric matrices, respectively.  $\|\cdot\|$  is the standard Euclidean norm and  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product. For given symmetric matrix  $S \geq 0$ , the quadratic form  $x^T S x$  may be defined as  $\|x\|_S^2$ , where  $x^T$  is the transpose of  $x$ .  $C^1([0, T]; \mathbb{R}^{n \times m})$  be the space of all  $\mathbb{R}^{n \times m}$ -valued continuously differentiable functions on  $[0, T]$ . For notation  $o(1)$ ,  $\lim_{n \rightarrow \infty} o(1) = 0$ . By [42], for sake of notation simplicity, we will use  $K$  to denote a generic constant in following discussion. The value of  $K$  may be different at different places and it only depends on the coefficients and initial values.

## 2. PROBLEM FORMULATION

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space which contains all  $\mathbb{P}$ -null sets in  $\mathcal{F}$ .  $\xi_i \in \mathbb{R}^n$  are the values of the initial states and  $W_i(\cdot)$  are  $d$ -dimensional standard Brownian motions, where  $i = 0, 1, \dots, N$ .  $\xi_i$  and  $W_i(\cdot)$  are defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Consider a large-population system which contains one leader and  $N$  followers. The state processes of the leader and the  $i$ th follower,  $i = 1, 2, \dots, N$ , are modeled by the following linear stochastic differential equations (SDE) on a finite time horizon  $[0, T]$ :

$$\begin{cases} dx_0(t) = [A_0(t)x_0(t) + B_0(t)u_0(t) + C_0(t)x^{(N)}(t)]dt + D_0(t)dW_0(t), & x_0(0) = \xi_0, \\ dx_i(t) = [A(t)x_i(t) + B(t)u_i(t) + C(t)x^{(N)}(t) + F(t)x_0(t)]dt + D(t)dW_i(t), & x_i(0) = \xi_i, \end{cases} \quad (2.1)$$

where  $x^{(N)}(t) := \frac{1}{N} \sum_{i=1}^N x_i(t)$  is the state average of the followers. Let  $\sigma$ -algebra  $\mathcal{F}_t^i = \sigma(W_i(s), 0 \leq s \leq t)$  and  $\mathcal{G}_t^i = \mathcal{F}_t^i \vee \sigma\{\xi_i, \xi_0, W_0(s), 0 \leq s \leq t\}$ , where  $0 \leq i \leq N$ .  $\mathcal{F}_t = \sigma(W_i(s), 0 \leq s \leq t, 0 \leq i \leq N)$  and  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{\xi_i, 0 \leq i \leq N\}$ .  $\mathbb{F}^i = \{\mathcal{F}_t^i\}_{0 \leq t \leq T}$  is the natural filtration generated by  $W_i(\cdot)$  and  $\mathbb{G}^i = \{\mathcal{G}_t^i\}_{0 \leq t \leq T}$ , where

$0 \leq i \leq N$ . Correspondingly, we denote  $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ ,  $\mathbb{G} = \{\mathcal{G}_t\}_{0 \leq t \leq T}$ . Next we introduce the following spaces:

$$\begin{aligned} L^\infty(0, T; \mathbb{R}^{n \times m}) &= \{\varphi : [0, T] \rightarrow \mathbb{R}^{n \times m} \mid \varphi(\cdot) \text{ is bounded and measurable}\}, \\ L_{\mathbb{F}}^2(\Omega; \mathbb{R}^m) &= \{\xi : \Omega \rightarrow \mathbb{R}^m \mid \xi \text{ is } \mathbb{F}\text{-measurable, } \mathbb{E}\|\xi\|^2 < \infty\}, \\ L_{\mathbb{F}}^2(0, T; \mathbb{R}^m) &= \{x : [0, T] \times \Omega \rightarrow \mathbb{R}^m \mid x(\cdot) \text{ is } \mathbb{F}\text{-progressively} \\ &\quad \text{measurable, } \|x(t)\|_{L^2}^2 := \mathbb{E} \int_0^T \|x(t)\|^2 dt < \infty\}, \\ L_{\mathbb{F}}^2(\Omega; C([0, T]; \mathbb{R}^m)) &= \left\{x : [0, T] \times \Omega \rightarrow \mathbb{R}^m \mid x(\cdot) \text{ is } \mathbb{F}\text{-progressively} \right. \\ &\quad \left. \text{measurable, continuous, } \mathbb{E} \sup_{t \in [0, T]} \|x(t)\|^2 < \infty\right\}, \\ \mathcal{M}[0, T] &:= L_{\mathbb{F}}^2(\Omega; C([0, T]; \mathbb{R}^n)) \times L_{\mathbb{F}}^2(\Omega; C([0, T]; \mathbb{R}^m)) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^{m \times d}). \end{aligned}$$

The set of admissible controls for the leader is defined as follows:

$$\mathcal{U}_0 = \{u_0 \mid u_0(t) \in L_{\mathbb{G}^0}^2(0, T; \mathbb{R}^m)\},$$

and the set of admissible controls for the  $i$ th follower is defined as follows:

$$\mathcal{U}_i = \{u_i \mid u_i(t) \in L_{\mathbb{G}^i}^2(0, T; \mathbb{R}^m)\}, \quad 1 \leq i \leq N.$$

These are the decentralized control sets and we let  $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_N$ . For comparison, the centralized control set is given by

$$\mathcal{U}_c = \{(u_0, u_1, \dots, u_N) \mid u_i(t) \in L_{\mathbb{G}^i}^2(0, T; \mathbb{R}^m), 0 \leq i \leq N\}.$$

Now we introduce the cost functionals of the leader and the  $i$ th follower,  $1 \leq i \leq N$ . For the leader, the cost functional is defined as follows:

$$\begin{aligned} \mathcal{J}_0(u_0(\cdot); u(\cdot)) &= \mathbb{E} \left\{ \int_0^T [\|x_0(t) - \Theta_0(t)x^{(N)}(t) - \eta_0(t)\|_{Q_0(t)}^2 + \|u_0(t)\|_{R_0(t)}^2] dt \right. \\ &\quad \left. + \|x_0(T) - \hat{\Theta}_0 x^{(N)}(T) - \hat{\eta}_0\|_{G_0}^2 \right\}, \end{aligned} \tag{2.2}$$

where  $u(\cdot) = (u_1(t), \dots, u_N(t)) \in \mathcal{U}$ .  $Q_0(\cdot)$ ,  $R_0(\cdot)$  and  $G_0(\cdot)$  are weight matrices.  $Q_0(\cdot)$  and  $\Theta_0(\cdot)$  represent the coupling between the leader and the population state average. This implies that the states of the followers can influence the cost functional of the leader. For the  $i$ th follower, the individual cost functional is defined as follows:

$$\begin{aligned} \mathcal{J}_i(u_0(\cdot); u(\cdot)) &= \mathbb{E} \left\{ \int_0^T [\|x_i(t) - \Theta(t)x^{(N)}(t) - \Theta_1(t)x_0(t) - \eta(t)\|_{Q(t)}^2 + \|u_i(t)\|_{R(t)}^2] dt \right. \\ &\quad \left. + \|x_i(T) - \hat{\Theta} x^{(N)}(T) - \hat{\Theta}_1 x_0(T) - \hat{\eta}\|_G^2 \right\}, \end{aligned} \tag{2.3}$$

where  $Q(\cdot)$ ,  $R(\cdot)$  and  $G(\cdot)$  are weight matrices.  $Q(\cdot)$ ,  $\Theta(\cdot)$  and  $\Theta_1(\cdot)$  represent the coupling between the  $i$ th follower, the population state average and the leader. This implies that the cost functional of the  $i$ th follower

will be affected by the behavior of both the leader and the other followers. All the individuals in the system, including the leader and followers, aim to minimize the social cost functional, which is denoted by

$$\mathcal{J}_{soc}^{(N)}(u_0(\cdot); u(\cdot)) = \alpha N \mathcal{J}_0(u_0(\cdot); u(\cdot)) + \sum_{i=1}^N \mathcal{J}_i(u_0(\cdot); u(\cdot)), \quad \alpha > 0.$$

Similar to [23] and [32], we have a scaling factor  $\alpha N$  before  $\mathcal{J}_0(u_0(\cdot); u(\cdot))$  such that  $\mathcal{J}_0(u_0(\cdot); u(\cdot))$  and  $\mathcal{J}_i(u_0(\cdot); u(\cdot))$  have the same order of magnitude. Otherwise, if  $\alpha N = 1$ , then the performance of the leader will be insensitive when  $N$  becomes larger. Now we introduce our assumptions.

**(A1)** The coefficients of (1), (2) and (3) satisfy

$$\begin{cases} A_0(\cdot), C_0(\cdot), A(\cdot), C(\cdot), F(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n}), \\ B_0(\cdot), B(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times m}), \quad D_0(\cdot), D(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times d}). \end{cases}$$

$$\begin{cases} Q_0(\cdot), Q(\cdot) \in L^\infty(0, T; \mathbb{S}^n), \quad R_0(\cdot), R(\cdot) \in L^\infty(0, T; \mathbb{S}^m), \\ \Theta_0(\cdot), \Theta_1(\cdot), \Theta(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n}), \quad \eta_0(\cdot), \eta(\cdot) \in L^2(0, T; \mathbb{R}^n), \\ \hat{\Theta}_0, \hat{\Theta}_1, \hat{\Theta} \in \mathbb{R}^{n \times n}, \quad G_0, G \in \mathbb{S}^n, \quad \hat{\eta}_0, \hat{\eta} \in \mathbb{R}^n. \end{cases}$$

**(A2)**  $x_0(0)$  and  $W_0(\cdot)$  are mutually independent.  $\{x_i(0), 1 \leq i \leq N\}$  and  $\{W_i(\cdot), 1 \leq i \leq N\}$  are independent of each other.  $\mathbb{E}x_i(0) = \hat{\xi}$ ,  $1 \leq i \leq N$ . For some constant  $K$ , which is independent of  $N$ , such that  $\sup_{1 \leq i \leq N} \mathbb{E}\|x_i(0)\|^2 \leq K$ . Furthermore,  $x_0(0)$ ,  $W_0(\cdot)$  and  $\{x_i(0), 1 \leq i \leq N\}$ ,  $\{W_i(t), 1 \leq i \leq N\}$  are independent of each other.

**(A3)**  $Q_0(\cdot) \geq 0$ ,  $R_0(\cdot) > \delta I$ ,  $G_0 \geq 0$  and  $Q(\cdot) \geq 0$ ,  $R(\cdot) > \delta I$ ,  $G \geq 0$ , for some  $\delta > 0$ .

From now on, we may suppress the notation of time  $t$  if necessary. We introduce our leader-follower problem:

**Problem 2.1.** Under **(A1)**–**(A3)**, for any  $u_0 \in \mathcal{U}_0$ , to find a mapping  $\mathcal{M}: \mathcal{U}_0 \rightarrow \mathcal{U}$ , and a control  $\bar{u}_0 \in \mathcal{U}_0$  such that

$$\begin{cases} \mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0)) = \inf_{u \in \mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(u_0; u), \\ \mathcal{J}_{soc}^{(N)}(\bar{u}_0; \mathcal{M}(\bar{u}_0)) = \inf_{u_0 \in \mathcal{U}_0} \mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0)). \end{cases}$$

Note that the  $\mathcal{M}$  here is a mapping, which is different from the notation  $\mathcal{M}[0, T]$  we just introduced.

### 3. THE MEAN FIELD LQ CONTROL PROBLEM FOR THE $N$ FOLLOWERS

#### 3.1. Person-by-person optimality

Fix  $u_0 \in \mathcal{U}_0$ . The leader firstly announces his own open-loop strategy. Let  $\bar{u} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N\}$  be the centralized optimal control of the followers and  $\bar{x} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$  be the corresponding states. Now we perturb  $\bar{u}_i$  and fix other  $\bar{u}_j$ , where  $j \neq i$ . Then we denote  $\delta u_i = u_i - \bar{u}_i$ ,  $\delta x_i = x_i - \bar{x}_i$ , where  $u_i$  is the control after perturbing and  $x_i$  is its corresponding state. The Fréchet differential  $\delta \mathcal{J}_0(\delta u_i) = \mathcal{J}_0(u_0; u) - \mathcal{J}_0(u_0; \bar{u}) + o(\|\delta u_i\|)$  and  $\delta \mathcal{J}_i(\delta u_i) = \mathcal{J}_i(u_0; u) - \mathcal{J}_i(u_0; \bar{u}) + o(\|\delta u_i\|)$ , where  $i = 1, \dots, N$ . Therefore, the variations of the state

equations for the leader, the  $i$ th follower and the  $j$ th follower, where  $j \neq i$ , are

$$\begin{cases} d\delta x_0 = (A_0\delta x_0 + C_0\delta x^{(N)})dt, & \delta x_0(0) = 0, \\ d\delta x_i = (A\delta x_i + B\delta u_i + C\delta x^{(N)} + F\delta x_0)dt, & \delta x_i(0) = 0, \\ d\delta x_j = (A\delta x_j + C\delta x^{(N)} + F\delta x_0)dt, & \delta x_j(0) = 0, \quad j \neq i, \end{cases}$$

and the variations of their corresponding cost functionals are

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_0(\delta u_i) = & \mathbb{E}\left\{\int_0^T \langle Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x_0 - \Theta_0\delta x^{(N)} \rangle dt \right. \\ & \left. + \langle G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x_0(T) - \hat{\Theta}_0\delta x^{(N)}(T) \rangle\right\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_i(\delta u_i) = & \mathbb{E}\left\{\int_0^T \langle Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_i - \Theta\delta x^{(N)} - \Theta_1\delta x_0 \rangle + \langle R\bar{u}_i, \delta u_i \rangle dt \right. \\ & \left. + \langle G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_i(T) - \hat{\Theta}\delta x^{(N)}(T) - \hat{\Theta}_1\delta x_0(T) \rangle\right\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_j(\delta u_i) = & \mathbb{E}\left\{\int_0^T \langle Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_j - \Theta\delta x^{(N)} - \Theta_1\delta x_0 \rangle dt \right. \\ & \left. + \langle G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_j(T) - \hat{\Theta}\delta x^{(N)}(T) - \hat{\Theta}_1\delta x_0(T) \rangle\right\}, \end{aligned}$$

respectively. Consequently, we have the variation of the social cost functional as:

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = & \frac{1}{2}\left[\alpha N\delta\mathcal{J}_0(\delta u_i) + \sum_{j \neq i} \delta\mathcal{J}_j(\delta u_i) + \delta\mathcal{J}_i(\delta u_i)\right] \\ = & \mathbb{E}\left\{\int_0^T \alpha N \langle Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x_0 \rangle - \alpha N \langle \Theta_0^T Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x^{(N)} \rangle + \langle Q(\bar{x}_i - \Theta\bar{x}^{(N)} \right. \\ & - \Theta_1\bar{x}_0 - \eta), \delta x_i \rangle - \langle \Theta^T Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x^{(N)} \rangle - \langle \Theta_1^T Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_0 \rangle \\ & + \langle R\bar{u}_i, \delta u_i \rangle + \sum_{j \neq i} \langle Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_j \rangle - \sum_{j \neq i} \langle \Theta^T Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x^{(N)} \rangle \\ & - \sum_{j \neq i} \langle \Theta_1^T Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_0 \rangle dt + \alpha N \langle G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x_0(T) \rangle \\ & - \alpha N \langle \hat{\Theta}_0^T G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x^{(N)}(T) \rangle + \langle G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_i(T) \rangle \\ & - \langle \hat{\Theta}^T G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T) \rangle - \langle \hat{\Theta}_1^T G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \\ & \delta x_0(T) \rangle + \sum_{j \neq i} \langle G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_j(T) \rangle - \sum_{j \neq i} \langle \hat{\Theta}^T G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) \\ & - \hat{\eta}), \delta x^{(N)}(T) \rangle - \sum_{j \neq i} \langle \hat{\Theta}_1^T G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_0(T) \rangle\left\}. \end{aligned} \tag{3.1}$$

When  $N \rightarrow \infty$ , it follows that

$$\begin{aligned}
\frac{1}{2} \delta \mathcal{J}_{soc}^{(N)}(\delta u_i) &= \mathbb{E} \left\{ \int_0^T \alpha \langle Q_0(\bar{x}_0 - \Theta_0 \bar{x}^{(N)} - \eta_0), N \delta x_0 \rangle - \alpha \langle \Theta_0^T Q_0(\bar{x}_0 - \Theta_0 \bar{x}^{(N)} - \eta_0), N \delta x^{(N)} \rangle \right. \\
&+ \langle Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x_i \rangle + \langle R \bar{u}_i, \delta u_i \rangle + \left\langle \frac{1}{N} \sum_{j \neq i} Q(\bar{x}_j - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), N \delta x_j \right\rangle \\
&- \left\langle \frac{1}{N} \sum_{j \neq i} \Theta^T Q(\bar{x}_j - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), N \delta x^{(N)} \right\rangle - \left\langle \frac{1}{N} \sum_{j \neq i} \Theta_1^T Q(\bar{x}_j - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \right. \\
&N \delta x_0 \left. \right\rangle dt + \alpha \langle G_0(\bar{x}_0(T) - \hat{\Theta}_0 \bar{x}^{(N)}(T) - \hat{\eta}_0), N \delta x_0(T) \rangle - \alpha \langle \hat{\Theta}_0^T G_0(\bar{x}_0(T) - \hat{\Theta}_0 \bar{x}^{(N)}(T) - \hat{\eta}_0), \\
&N \delta x^{(N)}(T) \rangle + \langle G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_i(T) \rangle + \left\langle \frac{1}{N} \sum_{j \neq i} G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) \right. \\
&- \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), N \delta x_j(T) \left. \right\rangle - \left\langle \frac{1}{N} \sum_{j \neq i} \hat{\Theta}^T G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), N \delta x^{(N)}(T) \right\rangle \\
&- \left. \left\langle \frac{1}{N} \sum_{j \neq i} \hat{\Theta}_1^T G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), N \delta x_0(T) \right\rangle \right\} + o(1).
\end{aligned}$$

Note that  $\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x_0\|^2 = O(\frac{1}{N^2})$ ,  $\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x^{(N)}\|^2 = O(\frac{1}{N^2})$  and  $\langle \Theta^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x^{(N)} \rangle + \langle \Theta_1^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x_0 \rangle + \langle \hat{\Theta}^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T) \rangle + \langle \hat{\Theta}_1^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_0(T) \rangle = o(1)$  (the rigorous proof will be shown in Section 6). Let

$$\begin{cases} \delta x_0^\dagger = \lim_{N \rightarrow +\infty} (N \delta x_0), \\ \delta x^\dagger = \lim_{N \rightarrow +\infty} (N \delta x_j) = \lim_{N \rightarrow +\infty} \left( \sum_{j \neq i} \delta x_j \right), \quad j \neq i. \end{cases} \quad (3.2)$$

Here  $N \delta x_0$  converges to  $\delta x_0^\dagger$  such that  $\mathbb{E} \int_0^T \|N \delta x_0 - \delta x_0^\dagger\|^2 = O(\frac{1}{N^2})$ . Similarly,  $\sum_{j \neq i} \delta x_j$  and  $N \delta x_j$  converge to  $\delta x^\dagger$  (see Section 6 for the detailed proof). Then one can obtain

$$\begin{cases} d\delta x_0^\dagger = (A_0 \delta x_0^\dagger + C_0 \delta x_i + C_0 \delta x^\dagger) dt, & \delta x_0^\dagger(0) = 0, \\ d\delta x^\dagger = (A \delta x^\dagger + C \delta x_i + C \delta x^\dagger + F \delta x_0^\dagger) dt, & \delta x^\dagger(0) = 0. \end{cases} \quad (3.3)$$

When  $N \rightarrow \infty$ , by mean field approximation, we use  $\hat{x}$  to approximate  $\bar{x}^{(N)}$ . Note that  $\hat{x}$  will be affected by  $u_0$  which is given by the leader. Moreover, the influence of individual follower on  $\hat{x}$  may be negligible. Hence, by

straightforward computation, we simplified the social cost functional as follows:

$$\begin{aligned}
\frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) &= \mathbb{E}\left\{\int_0^T \langle \alpha Q_0 \Psi_1 - \Theta_1^T Q \Psi_3, \delta x_0^\dagger \rangle + \langle Q \Psi_2^i - \Theta^T Q \Psi_3 - \alpha \Theta_0^T Q_0 \Psi_1, \delta x_i \rangle \right. \\
&\quad + \langle R \bar{u}_i, \delta u_i \rangle + \langle Q \Psi_3 - \Theta^T Q \Psi_3 - \alpha \Theta_0^T Q_0 \Psi_1, \delta x^\dagger \rangle dt + \langle \alpha G_0 \Psi_4(T) \\
&\quad - \hat{\Theta}_1^T G \Psi_6(T), \delta x_0^\dagger(T) \rangle + \langle G \Psi_6(T) - \hat{\Theta}^T G \Psi_6(T) - \alpha \hat{\Theta}_0^T G_0 \Psi_4(T), \\
&\quad \left. \delta x^\dagger(T) \rangle + \langle G \Psi_5^i(T) - \hat{\Theta}^T G \Psi_6(T) - \alpha \hat{\Theta}_0^T G_0 \Psi_4(T), \delta x_i(T) \rangle \right\}, \tag{3.4}
\end{aligned}$$

where

$$\begin{cases} \Psi_1(\cdot) := \bar{x}_0 - \Theta_0 \hat{x} - \eta_0, & \Psi_2^i(\cdot) := \bar{x}_i - \Theta \hat{x} - \Theta_1 \bar{x}_0 - \eta, \\ \Psi_3(\cdot) := (I - \Theta) \hat{x} - \Theta_1 \bar{x}_0 - \eta, \end{cases}$$

are related to time  $t$ , and

$$\begin{cases} \Psi_4(T) := \bar{x}_0(T) - \hat{\Theta}_0 \hat{x}(T) - \hat{\eta}_0, & \Psi_5^i(T) := \bar{x}_i(T) - \hat{\Theta} \hat{x}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}, \\ \Psi_6(T) := (I - \hat{\Theta}) \hat{x}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}, \end{cases}$$

are related to time  $T$  which are terminal terms.

It is very important to formulate an auxiliary control problem to obtain the decentralized optimal control for analyzing the problem of social optimality. Usually, an auxiliary control problem is a standard LQ control problem (see [22, 40]). However, (3.4) contains  $\delta x_0^\dagger$  and  $\delta x^\dagger$ , which are the terms we do not want them appear in the social cost functional. Therefore, we need to use a duality procedure (see [44, Chapter 3]) to get off the dependence of  $\delta\mathcal{J}_{soc}^{(N)}(\delta u_i)$  on  $\delta x_0^\dagger$  and  $\delta x^\dagger$ . To this end, we introduce two auxiliary equations

$$\begin{cases} dk_1 = \alpha_1 dt + \beta_1 dW_0, & k_1(T) = \alpha G_0 \Psi_4(T) - \hat{\Theta}_1^T G \Psi_6(T), \\ dk_2 = \alpha_2 dt + \beta_2 dW_0, & k_2(T) = (I - \hat{\Theta}^T) G \Psi_6(T) - \alpha \hat{\Theta}_0^T G_0 \Psi_4(T). \end{cases} \tag{3.5}$$

Using Itô formula, we have the following duality relations

$$\begin{aligned}
&\mathbb{E}\langle \alpha G_0 \Psi_4(T) - \hat{\Theta}_1^T G \Psi_6(T), \delta x_0^\dagger(T) \rangle \\
&= \mathbb{E}\langle k_1(0), \delta x_0^\dagger(0) \rangle + \mathbb{E} \int_0^T \langle k_1, A_0 \delta x_0^\dagger + C_0 \delta x_i + C_0 \delta x^\dagger \rangle + \langle \alpha_1, \delta x_0^\dagger \rangle dt, \\
&\mathbb{E}\langle (I - \hat{\Theta}^T) G \Psi_6(T) - \alpha \hat{\Theta}_0^T G_0 \Psi_4(T), \delta x^\dagger(T) \rangle \\
&= \mathbb{E}\langle k_2(0), \delta x^\dagger(0) \rangle + \mathbb{E} \int_0^T \langle k_2, A \delta x^\dagger + C \delta x_i + C \delta x^\dagger + F \delta x_0^\dagger \rangle + \langle \alpha_2, \delta x^\dagger \rangle dt. \tag{3.6}
\end{aligned}$$



Putting (3.4) and (3.6) together, we obtain

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = \mathbb{E} \left\{ \int_0^T \langle R\bar{u}_i, \delta u_i \rangle + \langle \alpha Q_0\Psi_1 - \Theta_1^T Q\Psi_3 + \alpha_1 + F^T k_2 + A_0^T k_1, \delta x_0^\dagger \rangle \right. \\ + \langle Q\Psi_3 - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2 + \alpha_2 + A^T k_2, \delta x^\dagger \rangle \\ + \langle Q\Psi_2^i - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2, \delta x_i \rangle dt \\ \left. + \langle G\Psi_5^i(T) - \hat{\Theta}^T G\Psi_6(T) - \alpha\hat{\Theta}_0^T G_0\Psi_4(T), \delta x_i(T) \rangle \right\}. \end{aligned}$$

Comparing the coefficients, it follows that

$$\begin{cases} \alpha_1 = -(\alpha Q_0\Psi_1 - \Theta_1^T Q\Psi_3 + F^T k_2 + A_0^T k_1), \\ \alpha_2 = -(Q\Psi_3 - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2 + A^T k_2). \end{cases}$$

Then, according to above discussion, the two auxiliary equations can be rewritten as:

$$\begin{cases} dk_1 = -(\alpha Q_0\Psi_1 - \Theta_1^T Q\Psi_3 + F^T k_2 + A_0^T k_1)dt + \beta_1 dW_0, \\ dk_2 = -(Q\Psi_3 - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2 + A^T k_2)dt + \beta_2 dW_0, \\ k_1(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^T G\Psi_6(T), \quad k_2(T) = (I - \hat{\Theta}^T)G\Psi_6(T) - \alpha\hat{\Theta}_0^T G_0\Psi_4(T), \end{cases}$$

and the variation of social cost functional is equivalent to

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = \mathbb{E} \left\{ \int_0^T \langle Q\bar{x}_i, \delta x_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle + \langle -Q(\Theta\hat{x} + \Theta_1\bar{x}_0 + \eta) \right. \\ - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2, \delta x_i \rangle dt + \langle G\bar{x}_i(T), \delta x_i(T) \rangle \\ \left. + \langle -G(\hat{\Theta}\hat{x}(T) + \hat{\Theta}_1\bar{x}_0(T) + \hat{\eta}) - \hat{\Theta}^T G\Psi_6(T) - \alpha\hat{\Theta}_0^T G_0\Psi_4(T), \delta x_i(T) \rangle \right\}. \end{aligned} \quad (3.7)$$

### 3.2. Decentralized strategy design for followers

As discussed in previous subsection, when  $N$  is sufficiently large, a stochastic process  $\hat{x}$  can be used to approximate  $x^{(N)}$ . Now, we can introduce the following auxiliary control problem for the  $i$ th follower.

**Problem 3.1.** (P2) Minimize  $\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i)$  over  $u_i \in \mathcal{U}_i$ , where

$$dx_i = [Ax_i + Bu_i + C\hat{x} + F\bar{x}_0(u_0)]dt + DdW_i, \quad x_i(0) = \xi_i, \quad i = 1, 2, \dots, N, \quad (3.8)$$

$$\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i) = \mathbb{E} \left\{ \int_0^T \|x_i\|_Q^2 + \|u_i\|_R^2 + 2\langle \chi_1, x_i \rangle dt + \|x_i(T)\|_G^2 + 2\langle \chi_2, x_i(T) \rangle \right\}, \quad (3.9)$$

with

$$\begin{cases} \chi_1 = -Q(\Theta\hat{x} + \Theta_1\bar{x}_0(u_0) + \eta) - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2, \\ \chi_2 = -G(\hat{\Theta}\hat{x}(T) + \hat{\Theta}_1\bar{x}_0(u_0)(T) + \hat{\eta}) - \hat{\Theta}^T G\Psi_6(T) - \alpha\hat{\Theta}_0^T G_0\Psi_4(T). \end{cases}$$

Here,  $\bar{x}_0(u_0)$  means  $\bar{x}_0$  is related to  $u_0$ .  $\bar{x}_0$ ,  $\hat{x}$ ,  $k_1$  and  $k_2$  are determined by

$$\begin{cases} d\bar{x}_0 = [A_0\bar{x}_0 + B_0u_0 + C_0\hat{x}]dt + D_0dW_0, & \bar{x}_0(0) = \xi_0, \\ d\hat{x} = [A\hat{x} + B\hat{u} + C\hat{x} + F\bar{x}_0(u_0)]dt, & \hat{x}(0) = \hat{\xi}, \\ dk_1 = -(\alpha Q_0\Psi_1 - \Theta_1^T Q\Psi_3 + F^T k_2 + A_0^T k_1)dt + \beta_1 dW_0, \\ dk_2 = -(Q\Psi_3 - \Theta^T Q\Psi_3 - \alpha\Theta_0^T Q_0\Psi_1 + C_0^T k_1 + C^T k_2 + A^T k_2)dt + \beta_2 dW_0, \\ k_1(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^T G\Psi_6(T), \quad k_2(T) = (I - \hat{\Theta}^T)G\Psi_6(T) - \alpha\hat{\Theta}_0^T G_0\Psi_4(T), \end{cases} \quad (3.10)$$

where  $\hat{x}$  and  $\hat{u}$  are the approximations of  $x^{(N)}$  and  $\frac{1}{N}\sum_{i=1}^N u_i$ , respectively.

In what follows, we let  $\bar{u} = \mathcal{M}(u_0) = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N\} \in \mathcal{U}$ . Note that  $\bar{u}$  here represents the decentralized optimal control, which is different from the same notation in the beginning of Section 3.

**Proposition 3.2.** *Assume that (A1)–(A3) hold. For given  $u_0 \in \mathcal{U}_0$ , (P2) has a unique optimal control*

$$\bar{u}_i = -R^{-1}B^T p_i,$$

where  $p_i$  is an adaptive solution to the following backward stochastic differential equation (BSDE)

$$dp_i = -(A^T p_i + Q\bar{x}_i + \chi_1)dt + \zeta_0 dW_0 + \zeta_i dW_i, \quad p_i(T) = G\bar{x}_i(T) + \chi_2.$$

*Proof.* The variation of the state equation in (3.8) is

$$d\delta x_i = (A\delta x_i + B\delta u_i)dt, \quad \delta x_i(0) = 0, \quad i = 1, 2, \dots, N,$$

and the variation of the corresponding cost functional is

$$\frac{1}{2}\delta\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i) = \mathbb{E}\left\{\int_0^T \langle Q\bar{x}_i, \delta x_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle + \langle \chi_1, \delta x_i \rangle dt + \langle G\bar{x}_i(T), \delta x_i(T) \rangle + \langle \chi_2, \delta x_i(T) \rangle\right\}. \quad (3.11)$$

Using a similar argument from (3.5) to (3.7), we construct the following auxiliary equation

$$dp_i = -(A^T p_i + Q\bar{x}_i + \chi_1)dt + \zeta_0 dW_0 + \zeta_i dW_i, \quad p_i(T) = G\bar{x}_i(T) + \chi_2, \quad (3.12)$$

and have the following duality relation between  $p_i$  and  $\delta x_i$  by using Itô formula

$$\mathbb{E}\langle p_i(T), \delta x_i(T) \rangle = \mathbb{E}\langle p_i(0), \delta x_i(0) \rangle + \mathbb{E}\int_0^T \langle A^T p_i - (A^T p_i + Q\bar{x}_i + \chi_1), \delta x_i \rangle + \langle B^T p_i, \delta u_i \rangle dt.$$

For given  $u_0 \in \mathcal{U}_0$ , since  $Q \geq 0$  and  $R > \delta I$ , for some  $\delta > 0$ , (3.9) is uniformly convex and it has a unique optimal control. Combining above equation with (3.11), we have

$$\frac{1}{2}\delta\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i) = \mathbb{E}\int_0^T \langle R\bar{u}_i + B^T p_i, \delta u_i \rangle dt.$$

(3.11) equal zero is equivalent to

$$R\bar{u}_i + B^T p_i = 0.$$

Thus, we have

$$\bar{u}_i = -R^{-1}B^T p_i. \quad (3.13)$$

The proposition follows.  $\square$

Substituting (3.13) into (3.8) and combining (3.12), we have the following FBSDE

$$\begin{cases} d\bar{x}_i = [A\bar{x}_i - BR^{-1}B^T p_i + C\hat{x} + F\bar{x}_0]dt + DdW_i, & x_i(0) = \xi_i, \quad i = 1, 2, \dots, N, \\ dp_i = -(A^T p_i + Q\bar{x}_i + \chi_1)dt + \zeta_0 dW_0 + \zeta_i dW_i, & p_i(T) = Gx_i(T) + \chi_2. \end{cases} \quad (3.14)$$

By taking limits, the above FBSDE can be rewritten as:

$$\begin{cases} d\hat{x} = [(A + C)\hat{x} + F\bar{x}_0 - BR^{-1}B^T \hat{p}]dt, & \hat{x}(0) = \hat{\xi}, \\ d\hat{p} = -(A^T \hat{p} + Q\hat{x} + \chi_1)dt + \zeta_0 dW_0, & \hat{p}(T) = G\hat{x}(T) + \chi_2. \end{cases} \quad (3.15)$$

### 3.3. The consistency condition of the follower problem

Let

$$\begin{cases} \Xi_1 := (I - \Theta^T)Q(I - \Theta) + \alpha\Theta_0^T Q_0 \Theta_0, & \Xi_1^G := (I - \hat{\Theta}^T)G(I - \hat{\Theta}) + \alpha\hat{\Theta}_0^T G_0 \hat{\Theta}_0, \\ \Xi_2 := (I - \Theta^T)Q\Theta_1 + \alpha\Theta_0^T Q_0, & \Xi_2^G := (I - \hat{\Theta}^T)G\hat{\Theta}_1 + \alpha\hat{\Theta}_0^T G_0, \\ \Xi_3 := (I - \Theta^T)Q\eta - \alpha\Theta_0^T Q_0 \eta_0, & \Xi_3^G := (I - \hat{\Theta}^T)G\hat{\eta} - \alpha\hat{\Theta}_0^T G_0 \hat{\eta}_0, \\ \Xi_4 := \Theta_1^T Q\Theta_1 + \alpha Q_0, & \Xi_4^G := \hat{\Theta}_1^T G\hat{\Theta}_1 + \alpha G_0, \\ \Xi_5 := \Theta_1^T Q\eta - \alpha Q_0 \eta_0, & \Xi_5^G := \hat{\Theta}_1^T G\hat{\eta} - \alpha G_0 \hat{\eta}_0. \end{cases}$$

Combining (3.10) and (3.15), we can obtain the CC system

$$\begin{cases} d\hat{x} = [(A + C)\hat{x} + F\bar{x}_0 - BR^{-1}B^T k_2]dt, & \hat{x}(0) = \hat{\xi}, \\ d\bar{x}_0 = [A_0\bar{x}_0 + B_0 u_0 + C_0 \hat{x}]dt + D_0 dW_0, & \bar{x}_0(0) = \xi_0, \\ dk_1 = -[\Xi_4 \bar{x}_0 - \Xi_2^T \hat{x} + A_0^T k_1 + F^T k_2 + \Xi_5]dt + \beta_1 dW_0, \\ dk_2 = -[\Xi_1 \hat{x} - \Xi_2 \bar{x}_0 + C_0^T k_1 + (A + C)^T k_2 - \Xi_3]dt + \beta_2 dW_0, \\ k_1(T) = \Xi_4^G \bar{x}_0(T) - (\Xi_2^G)^T \hat{x}(T) + \Xi_5^G, & k_2(T) = \Xi_1^G \hat{x}(T) - \Xi_2^G \bar{x}_0(T) - \Xi_3^G, \end{cases} \quad (3.16)$$

where  $\hat{p} = k_2$  can be easily verified.

## 4. THE OPTIMAL CONTROL PROBLEM FOR THE LEADER

Now, let **(P2)** have a unique solution. Then, for  $u_0 \in \mathcal{U}_0$  given by leader, the followers choose their optimal control  $\bar{u} = \mathcal{M}(u_0) = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N\} \in \mathcal{U}$ , where  $\bar{u}_i$  is shown in (3.13). Now we consider the optimal control of the leader to further minimize the social cost functional. In the infinite population system,  $x^{(N)}$  may be approximated by  $\hat{x}$ . Hence, we can construct the following auxiliary optimal control problem for the leader.

**Problem 4.1. (P3)** Minimize  $\hat{\mathcal{J}}_{soc}^{(N)}(u_0; \bar{u})$  over  $u_0 \in \mathcal{U}_0$ , where

$$\begin{aligned} dx_0 &= [A_0 x_0 + B_0 u_0 + C_0 \hat{x}]dt + D_0 dW_0, \quad x_0(0) = \xi_0, \\ \hat{\mathcal{J}}_{soc}^{(N)}(u_0; \bar{u}) &= \alpha N \hat{\mathcal{J}}_0(u_0; \bar{u}) + \sum_{i=1}^N \hat{\mathcal{J}}_i(u_0; \bar{u}). \end{aligned} \quad (4.1)$$

(P3) is based on (P2). Therefore, combining (3.13), the equations below (3.4) and the equations (2), (3) with mean field approximations, the cost functionals of the leader and the  $i$ th follower are

$$\begin{aligned} \hat{\mathcal{J}}_0(u_0; \bar{u}) &= \mathbb{E} \left\{ \int_0^T \langle Q_0 \Psi_1, \Psi_1 \rangle + \langle R_0 u_0, u_0 \rangle dt + \langle G_0 \Psi_4, \Psi_4 \rangle \right\}, \\ \hat{\mathcal{J}}_i(u_0; \bar{u}) &= \mathbb{E} \left\{ \int_0^T \langle Q \Psi_2^i, \Psi_2^i \rangle + \langle B^T p_i, R^{-1} B^T p_i \rangle dt + \langle G \Psi_5^i, \Psi_5^i \rangle \right\}, \end{aligned}$$

where  $\hat{x}$ ,  $\bar{x}_0$ ,  $k_1$ ,  $k_2$ ,  $\bar{x}_i$ ,  $p_i$  are determined by (3.16) and (3.14).

Using a similar argument in Section 3, we let  $\bar{u}_0$  be the optimal strategy of the leader and perturb  $u_0$  in (4.1), where  $\delta u_0 = u_0 - \bar{u}_0$ . Since  $\bar{x}_0$ ,  $\hat{x}$ ,  $\bar{x}_i$  and  $p_i$  are determined by  $u_0$ , we denote their corresponding perturbations as:  $\delta \bar{x}_0 = \bar{x}_0(u_0) - \bar{x}_0(\bar{u}_0)$ ,  $\delta \hat{x} = \hat{x}(u_0) - \hat{x}(\bar{u}_0)$ ,  $\delta \bar{x}_i = \bar{x}_i(u_0) - \bar{x}_i(\bar{u}_0)$  and  $\delta p_i = p_i(u_0) - p_i(\bar{u}_0)$ . For sake of notation simplicity, we drop  $(\bar{u}_0)$  in the following  $\bar{x}_0(\bar{u}_0)$ ,  $\hat{x}(\bar{u}_0)$ ,  $\bar{x}_i(\bar{u}_0)$  and  $p_i(\bar{u}_0)$ , etc. Then, one can obtain

$$d\delta \bar{x}_0 = [A_0 \delta \bar{x}_0 + B_0 \delta u_0 + C_0 \delta \hat{x}]dt, \quad \delta \bar{x}_0(0) = 0,$$

and the variations of corresponding cost functionals

$$\begin{aligned} \frac{1}{2} \delta \hat{\mathcal{J}}_0(\delta u_0) &= \mathbb{E} \left\{ \int_0^T \langle Q_0 \Psi_1, \delta \bar{x}_0 - \Theta_0 \delta \hat{x} \rangle + \langle R_0 \bar{u}_0, \delta u_0 \rangle dt + \langle G_0 \Psi_4, \delta \bar{x}_0(T) - \hat{\Theta}_0 \delta \hat{x}(T) \rangle \right\}, \\ \frac{1}{2} \sum_{i=1}^N \delta \hat{\mathcal{J}}_i(\delta u_0) &= \sum_{i=1}^N \mathbb{E} \left\{ \int_0^T \langle Q \Psi_2^i, \delta \bar{x}_i - \Theta \delta \hat{x} - \Theta_1 \delta \bar{x}_0 \rangle + \langle R^{-1} B^T p_i, B^T \delta p_i \rangle dt \right. \\ &\quad \left. + \langle G \Psi_5^i, \delta \bar{x}_i(T) - \hat{\Theta} \delta \hat{x}(T) - \hat{\Theta}_1 \delta \bar{x}_0(T) \rangle \right\}. \end{aligned}$$

Here  $\Psi_1$ ,  $\Psi_2^i$ ,  $\Psi_4(T)$ ,  $\Psi_5^i(T)$ , are related to  $\bar{u}_0$ . In what follows,  $\Psi_1$ ,  $\Psi_2^i$ ,  $\Psi_3$ ,  $\Psi_4(T)$ ,  $\Psi_5^i(T)$ ,  $\Psi_6(T)$  will be related to  $\bar{u}_0$ . Therefore, the variation of the social cost functional is

$$\begin{aligned} &\frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) \\ &= \alpha N \mathbb{E} \int_0^T \langle Q_0 \Psi_1, \delta \bar{x}_0 \rangle - \langle \Theta_0^T Q_0 \Psi_1, \delta \hat{x} \rangle + \langle R_0 \bar{u}_0, \delta u_0 \rangle dt \\ &\quad + \sum_{i=1}^N \mathbb{E} \int_0^T \langle Q \Psi_2^i, \delta \bar{x}_i \rangle - \langle \Theta^T Q \Psi_2^i, \delta \hat{x} \rangle - \langle \Theta_1^T Q \Psi_2^i, \delta \bar{x}_0 \rangle + \langle B R^{-1} B^T p_i, \delta p_i \rangle dt \\ &\quad + \alpha N \langle G_0 \Psi_4(T), \delta \bar{x}_0(T) \rangle - \alpha N \langle \hat{\Theta}_0^T G_0 \Psi_4(T), \delta \hat{x}(T) \rangle - \sum_{i=1}^N \langle \hat{\Theta}_1^T G \Psi_5^i(T), \delta \bar{x}_0(T) \rangle \\ &\quad - \sum_{i=1}^N \langle \hat{\Theta}^T G \Psi_5^i(T), \delta \hat{x}(T) \rangle + \sum_{i=1}^N \langle G \Psi_5^i(T), \delta \bar{x}_i(T) \rangle. \end{aligned} \quad (4.2)$$

Similarly, the variations of those equations in (3.14) and (3.16) are given by

$$\begin{cases} d\delta\bar{x}_i = [A\delta\bar{x}_i - BR^{-1}B^T\delta p_i + C\delta\hat{x} + F\delta\bar{x}_0]dt, & \delta\bar{x}_i(0) = 0, \quad i = 1, 2, \dots, N, \\ d\delta p_i = -(A^T\delta p_i + Q\delta x_i + [\Xi_1 - Q]\delta\hat{x} - \Xi_2\delta\bar{x}_0 + C_0^T\delta k_1 + C^T\delta k_2)dt + \delta\zeta_0 dW_0 + \delta\zeta_i dW_i, \\ \delta p_i(T) = G\delta x_i(T) + [\Xi_1^G - G]\delta\hat{x}(T) - \Xi_2^G\delta\bar{x}_0(T), \end{cases}$$

and

$$\begin{cases} d\delta\hat{x} = [(A + C)\delta\hat{x} + F\delta\bar{x}_0 - BR^{-1}B^T\delta k_2]dt, & \delta\hat{x}(0) = 0, \\ d\delta k_1 = -[\Xi_4\delta\bar{x}_0 - \Xi_2^T\delta\hat{x} + A_0^T\delta k_1 + F^T\delta k_2]dt + \delta\beta_1 dW_0, \\ d\delta k_2 = -[\Xi_1\delta\hat{x} - \Xi_2\delta\bar{x}_0 + C_0^T\delta k_1 + (A + C)^T\delta k_2]dt + \delta\beta_2 dW_0, \\ \delta k_1(T) = \Xi_4^G\delta\bar{x}_0(T) - (\Xi_2^G)^T\delta\hat{x}(T), \quad \delta k_2(T) = \Xi_1^G\delta\hat{x}(T) - \Xi_2^G\delta\bar{x}_0(T). \end{cases}$$

Since (4.2) contains many terms that we do not want them appear in the social cost functional, we will use a similar argument in Section 3 to get off the dependence of  $\delta\hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0)$  on those terms. Therefore, we need to construct six auxiliary equations to help us obtain the optimal control of the leader. We introduce the first three auxiliary equations:

$$\begin{cases} dq_i = m_i dt + n_i^0 dW_0 + n_i dW_i, & q_i(0) = 0, \quad i = 1, 2, \dots, N, \\ dl_1 = s_1 dt + r_1 dW_0, & l_1(0) = 0, \\ dl_2 = s_2 dt + r_2 dW_0, & l_2(0) = 0. \end{cases}$$

where

$$\begin{cases} m_i = -(BR^{-1}B^T p_i - BR^{-1}B^T y_i - Aq_i), \\ s_1 = C_0 l_2 + A_0 l_1 - C_0 q_i, \quad s_2 = (A + C)l_2 - BR^{-1}B^T \hat{y}^i + Fl_1 - Cq_i, \\ n_i = 0, \quad n_i^0 = 0, \quad r_1 = 0, \quad r_2 = 0. \end{cases}$$

Here  $q_i$ ,  $l_1$  and  $l_2$  are used to free  $\delta\hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0)$  from the dependence on  $p_i$ ,  $k_1$  and  $k_2$ , respectively. By a similar argument from (3.5) to (3.7), we can rewrite the variation of the social cost functional as follows:

$$\begin{aligned} & \frac{1}{2}\delta\hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) \\ &= \mathbb{E} \int_0^T \sum_{i=1}^N \langle \alpha Q_0 \Psi_1 - \Theta_1^T Q \Psi_2^i, \delta\bar{x}_0 \rangle + \sum_{i=1}^N \langle -\alpha \Theta_0^T Q_0 \Psi_1 - \Theta^T Q \Psi_2^i, \delta\hat{x} \rangle + \sum_{i=1}^N \langle Q \Psi_2^i, \delta\bar{x}_i \rangle \\ &+ N \langle \alpha R_0 \bar{u}_0, \delta u_0 \rangle + \sum_{i=1}^N \langle BR^{-1}B^T p_i, \delta p_i \rangle + \sum_{i=1}^N \langle l_1, -(A_0^T \delta k_1 + F^T \delta k_2 + \Xi_4 \delta\bar{x}_0 - \Xi_2^T \delta\hat{x}) \rangle \\ &+ \sum_{i=1}^N \langle s_1, \delta k_1 \rangle + \sum_{i=1}^N \langle s_2, \delta k_2 \rangle + \sum_{i=1}^N \langle l_2, -((A + C)^T \delta k_2 + C_0^T \delta k_1 + \Xi_1 \delta\hat{x} - \Xi_2 \delta\bar{x}_0) \rangle \\ &+ \sum_{i=1}^N \langle -(A^T \delta p_i + Q \delta\bar{x}_i + [\Xi_1 - Q] \delta\hat{x} - \Xi_2 \delta\bar{x}_0 + C_0^T \delta k_1 + C^T \delta k_2), q_i \rangle + \sum_{i=1}^N \langle \delta p_i, m_i \rangle \\ &+ \sum_{i=1}^N [\langle \delta\beta_1, r_1 \rangle + \langle \delta\beta_2, r_2 \rangle] + \sum_{i=1}^N [\langle \delta\zeta_0, n_i^0 \rangle + \langle \delta\zeta_i, n_i \rangle] dt + \sum_{i=1}^N \langle G \Psi_5^i(T) - G q_i(T), \delta\bar{x}_i(T) \rangle \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \langle \alpha G_0 \Psi_4(T) - \hat{\Theta}_1^T G \Psi_5^i(T) - (\Xi_4^G)^T l_1(T) + (\Xi_2^G)^T l_2(T) + (\Xi_2^G)^T q_i(T), \delta \bar{x}_0(T) \rangle \\
& - \sum_{i=1}^N \langle \alpha \hat{\Theta}_0^T G_0 \Psi_4(T) + \hat{\Theta}^T G \Psi_5^i(T) - (\Xi_2^G)^T l_1(T) + (\Xi_1^G)^T l_2(T) + (\Xi_1^G - G)^T q_i(T), \delta \hat{x}(T) \rangle.
\end{aligned}$$

Next, we introduce another three auxiliary equations:

$$\left\{ \begin{array}{l}
d\hat{y}^i = \hat{\alpha} dt + \hat{\beta} dW_0 + \sum_{i=1}^N \hat{\beta}_i dW_i, \\
\hat{y}^i(T) = \alpha \hat{\Theta}_0^T G_0 \Psi_4(T) + \hat{\Theta}^T G \Psi_5^i(T) - (\Xi_2^G)^T l_1(T) + (\Xi_1^G)^T l_2(T) + (\Xi_1^G - G)^T q_i(T), \\
dy_0^i = \hat{\alpha}_0 dt + \hat{\beta}_0 dW_0 + \sum_{i=1}^N \hat{\beta}_i^0 dW_i, \\
y_0^i(T) = \alpha G_0 \Psi_4(T) - \hat{\Theta}_1^T G \Psi_5^i(T) - (\Xi_4^G)^T l_1(T) + (\Xi_2^G)^T l_2(T) + (\Xi_2^G)^T q_i(T), \\
dy_i = \alpha_i dt + \beta_0 dW_0 + \beta_i dW_i, \quad y_i(T) = G \Psi_5^i(T) - G q_i(T), \quad i = 1, 2, \dots, N.
\end{array} \right.$$

where

$$\left\{ \begin{array}{l}
\hat{\alpha}_0 = -(\alpha Q_0 \Psi_1 - \Theta_1^T Q \Psi_2^i + F^T y_i - F^T \hat{y}^i + A_0^T y_0^i - \Xi_4^T l_1 + \Xi_2^T l_2 + \Xi_2^T q_i), \\
\hat{\alpha} = -\alpha \Theta_0^T Q_0 \Psi_1 - \Theta^T Q \Psi_2^i + C^T y_i - (A + C)^T \hat{y}^i + C_0^T y_0^i + \Xi_2 l_1 - \Xi_1^T l_2 - (\Xi_1 - Q)^T q_i, \\
\alpha_i = -(Q \Psi_2^i + A^T y_i - Q^T q_i).
\end{array} \right.$$

Here  $\hat{y}^i$ ,  $y_0^i$  and  $y_i$  are used to free  $\delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0)$  from the dependence on  $\delta \hat{x}$ ,  $\delta \bar{x}_0$  and  $\delta \bar{x}_i$ , respectively. Similarly, by Itô formula and the duality relations, the variation of the social cost functional can be further rewritten as follows:

$$\begin{aligned}
& \frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) \\
& = \mathbb{E} \int_0^T \sum_{i=1}^N \langle \alpha Q_0 \Psi_1 - \Theta_1^T Q \Psi_2^i + F^T y_i - F^T \hat{y}^i + A_0^T y_0^i + \hat{\alpha}_0 - \Xi_4^T l_1 + \Xi_2^T l_2 \\
& \quad + \Xi_2^T q_i, \delta \bar{x}_0 \rangle + \sum_{i=1}^N \langle -\alpha \Theta_0^T Q_0 \Psi_1 - \Theta^T Q \Psi_2^i + C^T y_i - (A + C)^T \hat{y}^i + C_0^T y_0^i \\
& \quad - \hat{\alpha} + \Xi_2 l_1 - \Xi_1^T l_2 - (\Xi_1 - Q)^T q_i, \delta \hat{x} \rangle + \sum_{i=1}^N \langle Q \Psi_2^i + A^T y_i + \alpha_i - Q q_i, \delta \bar{x}_i \rangle \\
& \quad + \sum_{i=1}^N \langle BR^{-1} B^T p_i - BR^{-1} B^T y_i + m_i - A q_i, \delta p_i \rangle + \sum_{i=1}^N [\langle n_i, \delta \zeta_i \rangle + \langle n_i^0, \delta \zeta_0 \rangle] \\
& \quad + \sum_{i=1}^N \langle s_1 - C_0 l_2 - A_0 l_1 + C_0 q_i, \delta k_1 \rangle + \sum_{i=1}^N \langle s_2 - (A + C) l_2 + BR^{-1} B^T \hat{y}^i - F l_1 \\
& \quad + C q_i, \delta k_2 \rangle + \sum_{i=1}^N [\langle r_1, \delta \beta_1 \rangle + \langle r_2, \delta \beta_2 \rangle] + \left\langle \alpha N R_0 \bar{u}_0 + \sum_{i=1}^N B_0^T y_0^i, \delta u_0 \right\rangle dt,
\end{aligned}$$

which implies,

$$\frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) = \mathbb{E} \int_0^T \left\langle \alpha N R_0 \bar{u}_0 + \sum_{i=1}^N B_0^T y_0^i, \delta u_0 \right\rangle dt.$$

Thus,  $\frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) = 0$  is equivalent to

$$\alpha N R_0 \bar{u}_0 + \sum_{i=1}^N B_0^T y_0^i = 0.$$

Then, we have the centralized form of the optimal control for the leader

$$\bar{u}_0 = -\frac{\alpha^{-1}}{N} R_0^{-1} B_0^T \sum_{i=1}^N y_0^i := u_0^{(N)}, \quad (4.3)$$

where  $\bar{u}_0$  relies on  $N$  and the following FBSDE

$$\left\{ \begin{array}{l} dy_i = -(A^T y_i - Q^T q_i + Q \Psi_2^i) dt + \beta_0 dW_0 + \beta_i dW_i, \quad y_i(T) = G \Psi_5^i(T) - G q_i(T), \\ dq_i = (BR^{-1} B^T y_i + A q_i - BR^{-1} B^T p_i) dt, \quad q_i(0) = 0, \quad i = 1, 2, \dots, N, \\ d\hat{y}^i = (-\alpha \Theta_0^T Q_0 \Psi_1 - \Theta^T Q \Psi_2^i + C^T y_i - (A + C)^T \hat{y}^i + C_0^T y_0^i + \Xi_2 l_1 - \Xi_1^T l_2 \\ \quad - (\Xi_1 - Q)^T q_i) dt + \hat{\beta} dW_0 + \sum_{i=1}^N \hat{\beta}_i dW_i, \\ \hat{y}^i(T) = \alpha \hat{\Theta}_0^T G_0 \Psi_4(T) + \hat{\Theta}^T G \Psi_5^i(T) - (\Xi_2^G)^T l_1(T) + (\Xi_1^G)^T l_2(T) + (\Xi_1^G - G)^T q_i(T), \\ dy_0^i = -(\alpha Q_0 \Psi_1 - \Theta_1^T Q \Psi_2^i + F^T y_i - F^T \hat{y}^i + A_0^T y_0^i - \Xi_4^T l_1 + \Xi_2^T l_2 + \Xi_2^T q_i) dt \\ \quad + \hat{\beta}_0 dW_0 + \sum_{i=1}^N \hat{\beta}_i^0 dW_i, \\ y_0^i(T) = \alpha G_0 \Psi_4(T) - \hat{\Theta}_1^T G \Psi_5^i(T) - (\Xi_4^G)^T l_1(T) + (\Xi_2^G)^T l_2(T) + (\Xi_2^G)^T q_i(T), \\ dl_1 = (A_0 l_1 + C_0 l_2 - C_0 q_i) dt, \quad l_1(0) = 0, \\ dl_2 = [F l_1 + (A + C) l_2 - BR^{-1} B^T \hat{y}^i - C q_i] dt, \quad l_2(0) = 0. \end{array} \right. \quad (4.4)$$

Denote

$$\left\{ \begin{array}{l} y^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N y_i, \quad \hat{y}^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \hat{y}^i, \quad y_0^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N y_0^i, \\ q^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N q_i, \quad l_1^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N l_1, \quad l_2^* = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N l_2. \end{array} \right.$$

Here, using a similar argument of (3.2), we can easily prove that  $\frac{1}{N} \sum_{i=1}^N y_i$ ,  $\frac{1}{N} \sum_{i=1}^N \hat{y}^i$ ,  $\frac{1}{N} \sum_{i=1}^N y_0^i$ ,  $\frac{1}{N} \sum_{i=1}^N q_i$ ,  $\frac{1}{N} \sum_{i=1}^N l_1$  and  $\frac{1}{N} \sum_{i=1}^N l_2$  converge to  $y^*$ ,  $\hat{y}^*$ ,  $y_0^*$ ,  $q^*$ ,  $l_1^*$  and  $l_2^*$ , respectively. Thus, combining (3.16) and (4.4),

when  $N \rightarrow \infty$ , we can obtain the CC system for the leader-follower problem

$$\left\{ \begin{array}{l} d\hat{x} = [(A + C)\hat{x} + F\bar{x}_0 - BR^{-1}B^T k_2]dt, \quad \hat{x}(0) = \hat{\xi}, \\ d\bar{x}_0 = [A_0\bar{x}_0 + C_0\hat{x} - B_0(\alpha R_0)^{-1}B_0^T y_0^*]dt + D_0dW_0, \quad \bar{x}_0(0) = \xi_0, \\ dk_1 = -[\Xi_4\bar{x}_0 - \Xi_2^T \hat{x} + A_0^T k_1 + F^T k_2 + \Xi_5]dt + \beta_1 dW_0, \\ dk_2 = -[\Xi_1\hat{x} - \Xi_2\bar{x}_0 + C_0^T k_1 + (A + C)^T k_2 - \Xi_3]dt + \beta_2 dW_0, \\ k_1(T) = \Xi_4^G \bar{x}_0(T) - (\Xi_2^G)^T \hat{x}(T) + \Xi_5^G, \quad k_2(T) = \Xi_1^G \hat{x}(T) - \Xi_2^G \bar{x}_0(T) - \Xi_3^G, \\ dy^* = -(A^T y^* - Q^T q^* + Q\Psi_3)dt + \beta^* dW_0, \quad y^*(T) = G\Psi_6(T) - Gq^*(T), \\ dq^* = (BR^{-1}B^T y^* + Aq^* - BR^{-1}B^T k_2)dt, \quad q^*(0) = 0, \\ d\hat{y}^* = [-\alpha\hat{\Theta}_0^T Q_0\Psi_1 - \hat{\Theta}^T Q\Psi_3 + C^T y^* - (A + C)^T \hat{y}^* + C_0^T y_0^* + \Xi_2 l_1^* - \Xi_1^T l_2^* \\ \quad - (\Xi_1 - Q)^T q^*]dt + \hat{\beta}^* dW_0, \\ \hat{y}^*(T) = \alpha\hat{\Theta}_0^T G_0\Psi_4(T) + \hat{\Theta}^T G\Psi_6(T) - (\Xi_2^G)^T l_1^*(T) + (\Xi_1^G)^T l_2^*(T) + (\Xi_1^G - G)^T q^*(T), \\ dy_0^* = -(\alpha Q_0\Psi_1 - \hat{\Theta}_1^T Q\Psi_3 + F^T y^* - F^T \hat{y}^* + A_0^T y_0^* - \Xi_4^T l_1^* + \Xi_2^T l_2^* + \Xi_2^T q^*)dt \\ \quad + \hat{\beta}_0^* dW_0, \\ y_0^*(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^T G\Psi_6(T) - (\Xi_4^G)^T l_1^*(T) + (\Xi_2^G)^T l_2^*(T) + (\Xi_2^G)^T q^*(T), \\ dl_1^* = (A_0 l_1^* + C_0 l_2^* - C_0 q^*)dt, \quad l_1^*(0) = 0, \\ dl_2^* = [F l_1^* + (A + C)l_2^* - BR^{-1}B^T \hat{y}^* - C q^*]dt, \quad l_2^*(0) = 0. \end{array} \right. \quad (4.5)$$

and the decentralized optimal control for the leader

$$u_0^* = -(\alpha R_0)^{-1}B_0^T y_0^*. \quad (4.6)$$

The final CC system is highly coupled with five forward equations and five backward equations. The existence and uniqueness of (4.5) is very important for obtaining the optimal control, however it is very difficult to solve such high-dimensional system. We need to simplify the CC system to a FBSDE using block matrices and these will be discussed in next section.

## 5. WELL-POSEDNESS OF THE CC SYSTEM

Note that in (4.5), the equations of  $(\hat{x}, \bar{x}_0, k_1, k_2)$  form a coupled FBSDE and  $(y^*, q^*, \hat{y}^*, y_0^*, l_1^*, l_2^*)$  form another coupled FBSDE. The two FBSDEs are also fully coupled with each other. Therefore, we try to look at the above FBSDEs in a different way. To this end, we set

$$\mathbb{X} = \begin{pmatrix} \hat{x} \\ \bar{x}_0 \\ q^* \\ l_1^* \\ l_2^* \end{pmatrix}, \quad \mathbb{Y} = \begin{pmatrix} y^* \\ \hat{y}^* \\ y_0^* \\ k_1 \\ k_2 \end{pmatrix}, \quad \mathbb{X}(0) = \begin{pmatrix} \hat{\xi} \\ \xi_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{Y}(T) = \begin{pmatrix} G\Psi_6 - Gq^*(T) \\ \alpha\hat{\Theta}_0^T G_0\Psi_4 - \hat{\Theta}^T G\Psi_6 - (\Xi_2^G)^T l_1^*(T) + (\Xi_1^G)^T l_2^*(T) + (\Xi_1^G - G)^T q^*(T) \\ \alpha G_0\Psi_4 - \hat{\Theta}_1^T G\Psi_6 - (\Xi_4^G)^T l_1^*(T) + (\Xi_2^G)^T l_2^*(T) + (\Xi_2^G)^T q^*(T) \\ \Xi_4^G \bar{x}_0(T) - (\Xi_2^G)^T \hat{x}(T) + \Xi_5^G \\ \Xi_1^G \hat{x}(T) - \Xi_2^G \bar{x}_0(T) - \Xi_3^G \end{pmatrix}.$$

Then (4.5) is equivalent to

$$\begin{cases} d\mathbb{X} = [\mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{Y} + b]dt + \mathbb{D}dW_0, & \mathbb{X}(0) = (\hat{\xi}^T \quad \xi_0^T \quad 0 \quad 0 \quad 0)^T, \\ d\mathbb{Y} = [\hat{\mathbb{A}}\mathbb{X} + \hat{\mathbb{B}}\mathbb{Y} + \hat{b}]dt + \hat{\mathbb{D}}dW_0, & \mathbb{Y}(T) = \mathbb{G}\mathbb{X}(T) + g, \end{cases} \quad (5.1)$$



with

$$\mathbb{A} = \begin{pmatrix} A+C & F & 0 & 0 & 0 \\ C_0 & A_0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & -C_0 & A_0 & C_0 \\ 0 & 0 & -C & F & (A+C) \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & -BR^{-1}B^T \\ 0 & 0 & -B_0(\alpha R_0)^{-1}B_0^T & 0 & 0 \\ BR^{-1}B^T & 0 & 0 & 0 & -BR^{-1}B^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -BR^{-1}B^T & 0 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} 0 \\ D_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\mathbb{A}} = \begin{pmatrix} -Q(I-\Theta) & Q\Theta_1 & Q^T & 0 & 0 \\ \Xi_1 - Q(I-\Theta) & -\Xi_2 + Q\Theta_1 & -(\Xi_1 - Q)^T & \Xi_2 & -\Xi_1^T \\ \Xi_2^T & -\Xi_4 & -\Xi_2^T & \Xi_4^T & -\Xi_2^T \\ \Xi_2^T & -\Xi_4 & 0 & 0 & 0 \\ -\Xi_1 & \Xi_2 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} Q\eta \\ -\Xi_3 + Q\eta \\ -\Xi_5 \\ -\Xi_5 \\ \Xi_3 \end{pmatrix}, \quad \hat{\mathbb{D}} = \begin{pmatrix} \beta^* \\ \hat{\beta}^* \\ \hat{\beta}_0^* \\ \beta_1 \\ \beta_2 \end{pmatrix}, \quad \hat{\mathbb{B}} = \begin{pmatrix} -A^T & 0 & 0 & 0 & 0 \\ C^T & -(A+C)^T & C_0^T & 0 & 0 \\ -F^T & F^T & -A_0^T & 0 & 0 \\ 0 & 0 & 0 & -A_0^T & -F^T \\ 0 & 0 & 0 & -C_0^T & -(A+C)^T \end{pmatrix},$$

$$g = \begin{pmatrix} -G\hat{\eta} \\ \Xi_3^G - G\hat{\eta} \\ \Xi_5^G \\ \Xi_5^G \\ -\Xi_3^G \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} G(I-\hat{\Theta}) & -G\hat{\Theta}_1 & -G & 0 & 0 \\ -\Xi_1^G + G(I-\hat{\Theta}) & \Xi_2^G - G\hat{\Theta}_1 & (\Xi_1^G - G)^T & -(\Xi_2^G)^T & (\Xi_1^G)^T \\ -(\Xi_2^G)^T & \Xi_4^G & (\Xi_2^G)^T & -(\Xi_4^G)^T & (\Xi_2^G)^T \\ -(\Xi_2^G)^T & \Xi_4^G & 0 & 0 & 0 \\ \Xi_1^G & -\Xi_2^G & 0 & 0 & 0 \end{pmatrix}.$$

Denote

$$\bar{\mathbb{A}} = \begin{pmatrix} \mathbb{A} + \mathbb{B}\mathbb{G} & \mathbb{B} \\ \hat{\mathbb{A}} - \mathbb{G}\mathbb{A} + \hat{\mathbb{B}}\mathbb{G} - \mathbb{G}\hat{\mathbb{B}}\mathbb{G} & \hat{\mathbb{B}} - \mathbb{G}\mathbb{B} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b \\ \hat{b} - \mathbb{G}b \end{pmatrix}, \quad \bar{\mathbb{D}} = \begin{pmatrix} \mathbb{D} \\ \hat{\mathbb{D}} - \mathbb{G}\mathbb{D} \end{pmatrix}, \quad \bar{\mathbb{Y}} = \mathbb{Y} - \mathbb{G}\mathbb{X}.$$

Then (5.1) can be rewritten as:

$$\begin{cases} d \begin{pmatrix} \mathbb{X} \\ \bar{\mathbb{Y}} \end{pmatrix} = \left\{ \bar{\mathbb{A}} \begin{pmatrix} \mathbb{X} \\ \bar{\mathbb{Y}} \end{pmatrix} + \bar{b} \right\} dt + \bar{\mathbb{D}} dW_0, \\ \mathbb{X}(0) = (\xi^T \quad \xi_0^T \quad 0 \quad 0 \quad 0)^T, \quad \bar{\mathbb{Y}}(T) = g. \end{cases} \quad (5.2)$$

This is a fully coupled FBSDE. From the Theorem 3.7 of Chapter 2 in [28], the FBSDE (5.2) is solvable for all  $g \in L_{\mathbb{F}}^2(\Omega; \mathbb{R}^{5n})$  if and only if the following condition holds:

$$\det \left\{ (0, I) e^{\bar{\mathbb{A}}t} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} > 0, \quad \forall t \in [0, T]. \quad (5.3)$$

In the case, (5.1) admits a unique solution for any given  $g \in L_{\mathbb{F}}^2(\Omega; \mathbb{R}^{5n})$ .

Under the condition (5.3), we may decouple the FBSDE (5.2) by

$$\bar{\mathbb{Y}} = \mathbb{K}\mathbb{X} + \kappa, \quad t \in [0, T],$$

where  $\mathbb{K} \in C^1([0, T]; \mathbb{S}^{5n})$  is a solution of the following Riccati equation

$$\dot{\mathbb{K}} + \mathbb{K}(A + \mathbb{B}G) + \mathbb{K}B\mathbb{K} - (\hat{\mathbb{B}} - \mathbb{G}\mathbb{B})\mathbb{K} - (\hat{A} - \mathbb{G}A + \hat{\mathbb{B}}G - \mathbb{G}\hat{\mathbb{B}}G) = 0, \quad t \in [0, T], \quad \mathbb{K}(T) = 0,$$

and  $\kappa \in C^1([0, T]; \mathbb{R}^{5n})$  satisfies

$$\dot{\kappa} + (\mathbb{K}\mathbb{B} - (\hat{\mathbb{B}} - \mathbb{G}\mathbb{B}))\kappa + \mathbb{K}b - (\hat{b} - \mathbb{G}b) = 0, \quad t \in [0, T], \quad \kappa(T) = g. \quad (5.4)$$

By the Theorems 3.7 and 4.3 of Chapter 2 in [28], if (5.3) hold, then the Riccati equation admits a unique solution  $\mathbb{K}(\cdot)$  which has the following representation:

$$\mathbb{K} = - \left[ (0, I) e^{\hat{\mathbb{A}}(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \left[ (0, I) e^{\hat{\mathbb{A}}(T-t)} \begin{pmatrix} I \\ 0 \end{pmatrix} \right], \quad t \in [0, T]. \quad (5.5)$$

**Example 5.1.** Consider the system (5.2) with parameters  $A_0 = 0.1$ ,  $B_0 = 1$ ,  $C_0 = 0.01$ ,  $D_0 = 1$ ,  $A = 0.05$ ,  $B = 1$ ,  $C = 0.05$ ,  $D = 1$ ,  $F = 0.3$ ,  $\Theta_0 = 1$ ,  $Q_0 = 1$ ,  $R_0 = 10$ ,  $G_0 = 0$ ,  $\Theta = 0.1$ ,  $\Theta_1 = 1$ ,  $Q = 0.9$ ,  $R = 15$ ,  $G = 0$ ,  $\alpha = 1.02$ ,  $T = 12$ ,  $\eta_0 = \eta = 0$ . Then, we have

$$\begin{aligned} \mathbb{A} &= \begin{pmatrix} 0.10 & 0.30 & 0 & 0 & 0 \\ 0.01 & 0.10 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 \\ 0 & 0 & -0.01 & 0.10 & 0.01 \\ 0 & 0 & -0.05 & 0.30 & 0.10 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.0667 \\ 0 & 0 & -0.0980 & 0 & 0 \\ 0.0667 & 0 & 0 & 0 & -0.0667 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0667 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{\mathbb{A}} &= \begin{pmatrix} -0.81 & 0.90 & 0.90 & 0 & 0 \\ 0.939 & -0.93 & -2.649 & 1.83 & -1.749 \\ 1.83 & -1.92 & -1.83 & 1.92 & -1.83 \\ 1.83 & -1.92 & 0 & 0 & 0 \\ -1.749 & 1.83 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbb{B}} = \begin{pmatrix} -0.05 & 0 & 0 & 0 & 0 \\ 0.05 & -0.10 & 0.01 & 0 & 0 \\ -0.30 & 0.30 & -0.10 & 0 & 0 \\ 0 & 0 & 0 & -0.10 & -0.30 \\ 0 & 0 & 0 & -0.01 & -0.10 \end{pmatrix}. \end{aligned}$$

Hence, according to the simulation through Matlab software, for any  $t \in [0, T]$ , we obtain

$$\bar{\mathbb{A}} = \begin{pmatrix} \mathbb{A} + \mathbb{B}G & \mathbb{B} \\ \hat{\mathbb{A}} - \mathbb{G}A + \hat{\mathbb{B}}G - \mathbb{G}\hat{\mathbb{B}}G & \hat{\mathbb{B}} - \mathbb{G}\mathbb{B} \end{pmatrix}, \quad \det \left\{ (0, I) e^{\bar{\mathbb{A}}t} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} > 0,$$

(e.g. for  $t = 6$ ,  $\det \left\{ (0, I) e^{\bar{\mathbb{A}}t} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} = 12.7053 > 0$ ). By the argument above, FBSDE (5.1) is solvable.

For further analysis, we make the following assumption:

(A4) The equation (5.2) has a unique solution and the solution  $(\mathbb{X}, \bar{\mathbb{Y}}, \bar{\mathbb{D}})$  belongs to  $\mathcal{M}[0, T]$ .

For the following equation

$$\begin{cases} d\bar{x}_i = [A\bar{x}_i - BR^{-1}B^T p_i + C\hat{x} + F\bar{x}_0]dt + DdW_i, & x_i(0) = \xi_i, \quad i = 1, 2, \dots, N, \\ dp_i = -[A^T p_i + Q\bar{x}_i + \chi_1]dt + \zeta_0 dW_0 + \zeta_i dW_i, & p_i(T) = Gx_i(T) + \chi_2, \end{cases} \quad (5.6)$$

where  $\chi_1$  and  $\chi_2$  are related to  $\bar{u}_0$ . We let  $p_i = \bar{P}\bar{x}_i + \bar{\varphi}$ ,  $t \in [0, T]$ , where  $\bar{P} \in C^1([0, T]; \mathbb{S}^n)$  is a solution of the following Riccati equation and  $\bar{\varphi} \in C^1([0, T]; \mathbb{R}^n)$  satisfies

$$\begin{cases} \dot{\bar{P}} + \bar{P}A - \bar{P}BR^{-1}B^T\bar{P} + A^T\bar{P} + Q = 0, & t \in [0, T], \quad \bar{P}(T) = G, \\ \dot{\bar{\varphi}} + (A^T - \bar{P}BR^{-1}B^T)\bar{\varphi} + \chi_1 + \bar{P}C\hat{x} + \bar{P}F\bar{x}_0 = 0, & t \in [0, T], \quad \bar{\varphi}(T) = \chi_2. \end{cases}$$

Since the Ricatti equation is standard, it has a unique solution. Hence, the FBSDE (5.6) is uniquely solvable and the solution belongs to  $\mathcal{M}[0, T]$ .

## 6. ASYMPTOTICALLY SOCIAL OPTIMALITY

In this section, we discuss that if the leader announces  $u_0^*$  obtained in (4.6) to the  $N$  followers, then the set of the optimal decentralized controls for the leader and the followers will constitute an approximated Stackelberg equilibrium. First, for the open-loop decentralized strategy  $(u_0^*, u^*)$  in (4.6) and (3.13), we have the realized decentralized state  $x_0^*$  and  $x_i^*$ , satisfies

$$\begin{cases} dx_0^*(t) = [A_0 x_0^*(t) - B_0(\alpha R_0)^{-1} B_0^T y_0^*(t) + C_0(x^*)^{(N)}(t)]dt + D_0 dW_0(t), \\ dx_i^*(t) = [Ax_i^*(t) - BR^{-1}B^T p_i(t) + C(x^*)^{(N)}(t) + Fx_0^*(t)]dt + DdW_i(t), \\ x_0^*(0) = \xi_0, \quad x_i^*(0) = \xi_i, \quad i = 1, 2, \dots, N, \end{cases} \quad (6.1)$$

where  $y_0^*, p_i$  satisfy (4.5) and (5.6), respectively. Then, by [2] and [31], we give the definition of the Stackelberg equilibrium.

**Definition 6.1.** A set of control laws  $\mathcal{M}(\check{u}_0) \in \mathcal{U}$  has asymptotic social optimality if

$$\left| \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N} \inf_{(\check{u}_0, \check{u}) \in \mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) \right| = O\left(\frac{1}{\sqrt{N}}\right),$$

where  $\mathcal{M}$  is a mapping and  $\mathcal{M} : \mathcal{U}_0 \rightarrow \mathcal{U}$ .  $\mathcal{U}_c$  is defined in Section 3 as a set of centralized information-based control.

**Definition 6.2.** A set of control laws  $(u_0^*, u^*) \in \mathcal{U}_0 \times \mathcal{U}$ , where  $u^* = \mathcal{M}(u_0^*)$ , is an Stackelberg equilibrium with respect to  $\mathcal{J}_{soc}^{(N)}(u_0, u)$  if the following two properties hold:

1.  $\mathcal{M}(\check{u}_0)$  has asymptotic social optimality under  $\check{u}_0$ .
2. The following equation is satisfied

$$\left| \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N} \inf_{\check{u}_0 \in \mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) \right| = O\left(\frac{1}{\sqrt{N}}\right).$$

We first need to introduce some lemmas before proving the Stackelberg equilibrium.

**Lemma 6.3.** Assume that (A1)–(A4) hold. Then

$$\mathbb{E} \int_0^T \|(x^*)^{(N)} - \hat{x}\|^2 dt + \mathbb{E} \int_0^T \|p^{(N)} - \hat{p}\|^2 dt + \mathbb{E} \int_0^T \|x_0^* - \bar{x}_0\|^2 dt = O\left(\frac{1}{N}\right).$$

*Proof.* See Appendix A. □

**Lemma 6.4.** Assume that (A1)–(A4) hold. There exists a constant  $K$ , which is independent of  $N$ , such that

$$\mathcal{J}_{soc}^{(N)}(u_0^*; u^*) \leq NK.$$

*Proof.* See Appendix B. □

**Proposition 6.5.** *Assume that (A1)–(A4) hold. For all  $(\check{u}_0; \check{u}) \in \mathcal{U}_c$ , there exists a constant  $K$ , which is independent of  $N$ , such that*

$$\alpha N \|\check{u}_0\|_{L^2}^2 + \|\check{u}\|_{L^2}^2 \leq NK.$$

*Proof.* By Lemma 6.4, we have

$$\mathbb{E} \int_0^T \alpha N \|\check{u}_0\|^2 + \|\check{u}\|^2 dt \leq \inf_{(\check{u}_0; \check{u})} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) \leq \mathcal{J}_{soc}^{(N)}(u_0^*; u^*) \leq NK, \quad (\alpha > 0).$$

Therefore,  $\alpha N \|\check{u}_0\|_{L^2}^2 + \|\check{u}\|_{L^2}^2 \leq NK$ , where  $K$  is independent of  $N$ . The proposition follows.  $\square$

The following two propositions will give the rigorous proofs for the approximations in Section 4.

**Proposition 6.6.** *Assume that (A1)–(A4) hold. Then, for (3.1),  $\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x_0\|^2 = O(\frac{1}{N^2})$ ,  $\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x^{(N)}\|^2 = O(\frac{1}{N^2})$  and  $\langle \Theta^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x^{(N)} \rangle + \langle \Theta_1^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x_0 \rangle + \langle \hat{\Theta}^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T) \rangle + \langle \hat{\Theta}_1^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_0(T) \rangle = o(1)$ .*

*Proof.* See Appendix C.  $\square$

**Proposition 6.7.** *Assume that (A1)–(A4) hold. Then,  $N\delta x_j$ ,  $N\delta x_0$ ,  $N\delta x_j$  converge to  $\sum_{j \neq i} \delta x_j$ ,  $\delta x_0^\dagger$ ,  $\delta x^\dagger$  such that*

$$\mathbb{E} \int_0^T \|N\delta x_j - \sum_{j \neq i} \delta x_j\|^2 = O\left(\frac{1}{N^2}\right), \quad \mathbb{E} \int_0^T \|N\delta x_0 - \delta x_0^\dagger\|^2 = O\left(\frac{1}{N^2}\right), \quad \mathbb{E} \int_0^T \|N\delta x_j - \delta x^\dagger\|^2 = O\left(\frac{1}{N^2}\right).$$

*Proof.* See Appendix C.  $\square$

By the lemmas and propositions, we discussed above, we give the main result.

**Theorem 6.8.** *Assume that (A1)–(A4) hold. Then  $(u_0^*, u^*)$  given in (4.6) and (3.13) is a Stackelberg equilibrium with respect to the social cost functional.*

*Proof.* For  $(\check{u}_0; \check{u}) \in \mathcal{U}_c$ , let

$$\begin{aligned} & \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; u^*) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) = \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) \\ & + \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) := \Delta_1 + \Delta_2, \end{aligned}$$

where  $\Delta_1 = \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0))$ ,  $\Delta_2 = \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u})$ . Since  $\check{u}_0$  is fixed, by following the standard method in Huang *et al* [22], we obtain  $\|\Delta_2\|^2 \leq c(\|\check{u}_0\|_{L^2}^2)^{\frac{1}{N}}$ . Specifically, we denote  $\hat{x}_i$  as the state of the  $i$ th follower when its control is  $\mathcal{M}_i(\check{u}_0)$ , thus  $\hat{x}_i$  is equivalent to  $\bar{x}_i$  in Section 4. Let

$$\begin{cases} \tilde{u}_0 = \check{u}_0 - \check{u}_0 = 0, & \tilde{u} = \check{u} - \mathcal{M}(\check{u}_0), & \tilde{u}_i = \check{u}_i - \mathcal{M}_i(\check{u}_0), \\ \tilde{x}_0 = \check{x}_0 - \hat{x}_0, & \tilde{x}_i = \check{x}_i - \hat{x}_i. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) &= \alpha N \mathcal{J}_0(\check{u}_0; \check{u}) + \sum_{i=1}^N \mathcal{J}_i(\check{u}_0; \check{u}) \\ &= \alpha N \mathcal{J}_0(\check{u}_0; \mathcal{M}(\check{u}_0)) + \alpha N H_0 + \alpha N I_0 + \sum_{i=1}^N \mathcal{J}_i(\check{u}_0; \mathcal{M}(\check{u}_0)) + \sum_{i=1}^N H_i + \sum_{i=1}^N I_i, \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_0(\check{u}_0; \mathcal{M}(\check{u}_0)) &= \mathbb{E} \left\{ \int_0^T \|\dot{x}_0 - \Theta_0 \dot{x}^{(N)} - \eta_0\|_{Q_0}^2 + \|\check{u}_0\|_{R_0}^2 dt + \|\dot{x}_0(T) - \hat{\Theta}_0 \dot{x}^{(N)}(T) - \hat{\eta}_0\|_{G_0}^2 \right\}, \\ H_0 &= \mathbb{E} \left\{ \int_0^T \|\tilde{x}_0 - \Theta_0 \tilde{x}^{(N)}\|_{Q_0}^2 dt + \|\tilde{x}_0(T) - \hat{\Theta}_0 \tilde{x}^{(N)}(T)\|_{G_0}^2 \right\}, \\ \mathcal{J}_i(\check{u}_0; \mathcal{M}(\check{u}_0)) &= \mathbb{E} \left\{ \int_0^T \|\dot{x}_i - \Theta \dot{x}^{(N)} - \Theta_1 \dot{x}_0 - \eta\|_Q^2 + \|\mathcal{M}_i(\check{u}_0)\|_R^2 dt \right. \\ &\quad \left. + \|\dot{x}_i(T) - \hat{\Theta} \dot{x}^{(N)}(T) - \hat{\Theta}_1 \dot{x}_0(T) - \hat{\eta}\|_G^2 \right\}, \\ H_i &= \mathbb{E} \left\{ \int_0^T \|\tilde{x}_i - \Theta \tilde{x}^{(N)} - \Theta_1 \tilde{x}_0\|_Q^2 + \|\tilde{u}_i\|_R^2 dt + \|\tilde{x}_i(T) - \hat{\Theta} \tilde{x}^{(N)}(T) - \hat{\Theta}_1 \tilde{x}_0(T)\|_G^2 \right\}, \\ I_0 &= \mathbb{E} \left\{ \int_0^T (\dot{x}_0 - \Theta_0 \dot{x}^{(N)} - \eta_0)^T Q_0 (\tilde{x}_0 - \Theta_0 \tilde{x}^{(N)}) dt \right. \\ &\quad \left. + (\dot{x}_0(T) - \hat{\Theta}_0 \dot{x}^{(N)}(T) - \hat{\eta}_0)^T G_0 (\tilde{x}_0(T) - \hat{\Theta}_0 \tilde{x}^{(N)}(T)) \right\}, \\ I_i &= \mathbb{E} \left\{ \int_0^T (\dot{x}_i - \Theta \dot{x}^{(N)} - \Theta_1 \dot{x}_0 - \eta)^T Q (\tilde{x}_i - \Theta \tilde{x}^{(N)} - \Theta_1 \tilde{x}_0) + \mathcal{M}_i^T(\check{u}) R \check{u}_i dt \right. \\ &\quad \left. + (\dot{x}_i(T) - \hat{\Theta} \dot{x}^{(N)}(T) - \hat{\Theta}_1 \dot{x}_0(T) - \hat{\eta})^T G (\tilde{x}_i(T) - \hat{\Theta} \tilde{x}^{(N)}(T) - \hat{\Theta}_1 \tilde{x}_0(T)) \right\}. \end{aligned}$$

By straightforward computation

$$\begin{aligned} \alpha N I_0 &= \mathbb{E} \left\{ \int_0^T \alpha N [\Psi_1^T Q_0 - (\Theta_0 v_1)^T Q_0] \tilde{x}_0 - \alpha [\Psi_1^T Q_0 \Theta_0 - (\Theta_0 v_1)^T Q_0 \Theta_0] \sum_{i=1}^N \tilde{x}_i dt \right. \\ &\quad \left. + \alpha N [\Psi_4(T)^T G_0 - (\hat{\Theta}_0 v_1(T))^T G_0] \tilde{x}_0(T) - \alpha [\Psi_4(T)^T G_0 \hat{\Theta}_0 - (\hat{\Theta}_0 v_1(T))^T G_0 \hat{\Theta}_0] \sum_{i=1}^N \tilde{x}_i(T) \right\}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \sum_{i=1}^N I_i &= \mathbb{E} \left\{ \int_0^T \sum_{i=1}^N (\Psi_2^i)^T Q \tilde{x}_i - [(\Theta v_1)^T Q + \Psi_3^T Q \Theta - ((I - \Theta) v_1)^T Q \Theta] \sum_{i=1}^N \tilde{x}_i - N [\Psi_3^T Q \Theta_1 \right. \\ &\quad \left. - [(I - \Theta) v_1]^T Q \Theta_1] \tilde{x}_0 + \mathcal{M}_i^T(\check{u}) R \check{u}_i dt + \sum_{i=1}^N (\Psi_5^i(T))^T G \tilde{x}_i(T) - [(\hat{\Theta} v_1(T))^T G + \Psi_6(T)^T G \hat{\Theta} \right. \\ &\quad \left. - ((I - \hat{\Theta}) v_1(T))^T G \hat{\Theta}] \sum_{i=1}^N \tilde{x}_i(T) - N [\Psi_6(T)^T G \hat{\Theta}_1 - [(I - \hat{\Theta}) v_1(T)]^T G \hat{\Theta}_1] \tilde{x}_0(T) \right\}. \end{aligned} \tag{6.3}$$

where  $v_1 = \dot{x}^{(N)} - \hat{x}$ . By (4.5), (5.6) and Itô formula, we obtain following relations:

$$\begin{aligned} N\langle k_1(T), \tilde{x}_0(T) \rangle &= \langle \alpha N G_0 \Psi_4, \tilde{x}_0(T) \rangle - \langle N \hat{\Theta}_1^T G \Psi_6, \tilde{x}_0(T) \rangle \\ &= \mathbb{E} \int_0^T -\langle \alpha N Q_0 \Psi_1, \tilde{x}_0 \rangle + \langle N \Theta_1^T Q \Psi_3, \tilde{x}_0 \rangle - \langle k_2, N F \tilde{x}_0 \rangle + \left\langle C_0^T k_1, \sum_{i=1}^N \tilde{x}_i \right\rangle dt, \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} \sum_{i=1}^N \langle p_i(T), \tilde{x}_i(T) \rangle &= \mathbb{E} \int_0^T \left\langle \Theta^T Q \Psi_3, \sum_{i=1}^N \tilde{x}_i \right\rangle - \left\langle Q \Psi_2^i, \sum_{i=1}^N \tilde{x}_i \right\rangle + \left\langle \alpha \Theta_0^T Q_0 \Psi_1, \sum_{i=1}^N \tilde{x}_i \right\rangle \\ &\quad - \left\langle C_0^T k_1, \sum_{i=1}^N \tilde{x}_i \right\rangle - \left\langle p^{(N)} - k_2, C \sum_{i=1}^N \tilde{x}_i \right\rangle + \sum_{i=1}^N \langle p_i, B \tilde{u}_i \rangle + \langle p^{(N)}, N F \tilde{x}_0 \rangle dt. \end{aligned} \quad (6.5)$$

Meanwhile, by (3.13), we have

$$\sum_{i=1}^N \langle \mathcal{M}_i(\tilde{u}), R \tilde{u}_i \rangle + \sum_{i=1}^N \langle p_i, B \tilde{u}_i \rangle = \sum_{i=1}^N \langle R \mathcal{M}_i(\tilde{u}) + B^T p_i, \tilde{u}_i \rangle = \sum_{i=1}^N \langle R(-R^{-1} B^T p_i) + B^T p_i, \tilde{u}_i \rangle = 0. \quad (6.6)$$

Combining (6.2)–(6.6), Lemmas 6.3 and 6.4, it follows that

$$\frac{1}{N} \left( \alpha N I_0 + \sum_{i=1}^N I_i \right) = O\left(\frac{1}{\sqrt{N}}\right).$$

Moreover,  $\frac{1}{N} \left( \alpha N H_0 + \sum_{i=1}^N H_i \right) \geq 0$ . Thus, we have

$$\Delta_2 = \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) \leq c(\|\check{u}_0\|_{L^2}) \frac{1}{\sqrt{N}}. \quad (6.7)$$

For  $\Delta_1$ , we decompose it as follows:

$$\begin{aligned} \Delta_1 &= \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) = \frac{1}{N} \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) \\ &\quad - \frac{1}{N} \mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) + \frac{1}{N} \mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) - \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)). \end{aligned}$$

Note that  $u_0^{(N)}$  is the centralized social optimal control in (4.3), thus one can easily obtain that

$$\frac{1}{N} \mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) \leq \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)). \quad (6.8)$$

We know that  $\mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0))$  continuously depends on  $u_0$ . Since  $\mathcal{M}(u_0)$  is the solution of FBSDE (5.6) which continuously depends on parameters, we have  $\mathcal{M}(u_0)$  is continuous in  $u_0$ . Note that  $\mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0))$  is a quadratic functional and  $u_0^*$  is fixed. Let  $\check{x}_0^{(N)}$  and  $\check{x}_i^{(N)}$  be the state of the leader and the  $i$ th follower when

the control of the leader is  $u_0^{(N)}$ . Denote

$$\begin{cases} \dot{u}_0 = u_0^{(N)} - u_0^*, & \delta\mathcal{M}(u_0) = \mathcal{M}(u_0^{(N)}) - \mathcal{M}(u_0^*), \\ \delta\mathcal{M}_i(u_0) = \mathcal{M}_i(u_0^{(N)}) - \mathcal{M}_i(u_0^*), & \dot{x}_0 = \check{x}_0^{(N)} - x_0^*, \quad \dot{x}_i = \check{x}_i^{(N)} - x_i^*. \end{cases}$$

Then we have

$$\begin{aligned} & \left| \mathcal{J}_{soc}^{(N)}(u_0^{(N)}; \mathcal{M}(u_0^{(N)})) - \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) \right| \\ &= \left| \mathcal{J}_{soc}^{(N)}(u_0^{(N)} - u_0^* + u_0^*; \mathcal{M}(u_0^{(N)}) - \mathcal{M}(u_0^*) + \mathcal{M}(u_0^*)) - \mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) \right|, \end{aligned}$$

and

$$\mathcal{J}_{soc}^{(N)}(u_0^{(N)}; \mathcal{M}(u_0^{(N)})) = \alpha N [\mathcal{J}_0(u_0^*; \mathcal{M}(u_0^*)) + H'_0 + I'_0] + \sum_{i=1}^N [\mathcal{J}_i(u_0^*; \mathcal{M}(u_0^*)) + H'_i + I'_i],$$

where

$$\mathcal{J}_0(u_0^*; \mathcal{M}(u_0^*)) = \mathbb{E} \left\{ \int_0^T \|x_0^* - \Theta_0(x^*)^{(N)} - \eta_0\|_{Q_0}^2 dt + \|u_0^*\|_{R_0}^2 dt + \|x_0^*(T) - \hat{\Theta}_0(x^*)^{(N)}(T) - \hat{\eta}_0\|_{G_0}^2 \right\},$$

$$H'_0 = \mathbb{E} \left\{ \int_0^T \|\dot{x}_0 - \Theta_0 \dot{x}^{(N)}\|_{Q_0}^2 dt + \|\dot{u}_0\|_{R_0}^2 dt + \|\dot{x}_0(T) - \hat{\Theta}_0 \dot{x}^{(N)}(T)\|_{G_0}^2 \right\},$$

$$\begin{aligned} \mathcal{J}_i(u_0^*; \mathcal{M}(u_0^*)) &= \mathbb{E} \left\{ \int_0^T \|x_i^* - \Theta(x^*)^{(N)} - \Theta_1 x_0^* - \eta\|_Q^2 dt + \|\mathcal{M}_i(u_0^*)\|_R^2 dt \right. \\ &\quad \left. + \|x_i^*(T) - \hat{\Theta}(x^*)^{(N)}(T) - \hat{\Theta}_1 x_0^*(T) - \hat{\eta}\|_G^2 \right\}, \end{aligned}$$

$$H'_i = \mathbb{E} \left\{ \int_0^T \|\dot{x}_i - \Theta \dot{x}^{(N)} - \Theta_1 \dot{x}_0\|_Q^2 dt + \|\delta\mathcal{M}_i(u_0)\|_R^2 dt + \|\dot{x}_i(T) - \hat{\Theta} \dot{x}^{(N)}(T) - \hat{\Theta}_1 \dot{x}_0(T)\|_G^2 \right\},$$

$$\begin{aligned} I'_0 &= \mathbb{E} \left\{ \int_0^T (x_0^* - \Theta_0(x^*)^{(N)} - \eta_0)^T Q_0 (\dot{x}_0 - \Theta_0 \dot{x}^{(N)}) dt \right. \\ &\quad \left. + (x_0^*(T) - \hat{\Theta}_0(x^*)^{(N)}(T) - \hat{\eta}_0)^T G_0 (\dot{x}_0(T) - \hat{\Theta}_0 \dot{x}^{(N)}(T)) \right\}, \end{aligned}$$

$$\begin{aligned} I'_i &= \mathbb{E} \left\{ \int_0^T (x_i^* - \Theta(x^*)^{(N)} - \Theta_1 x_0^* - \eta)^T Q (\dot{x}_i - \Theta \dot{x}^{(N)} - \Theta_1 \dot{x}_0) + \mathcal{M}_i^T(u_0^*) R \delta\mathcal{M}_i(u_0) dt \right. \\ &\quad \left. + (x_i^*(T) - \hat{\Theta}(x^*)^{(N)}(T) - \hat{\Theta}_1 x_0^*(T) - \hat{\eta})^T G (\dot{x}_i(T) - \hat{\Theta} \dot{x}^{(N)}(T) - \hat{\Theta}_1 \dot{x}_0(T)) \right\}. \end{aligned}$$

By using similar arguments in Lemma A.1 to Lemma A.2 and  $\|\Delta_2\|^2 \leq c(\|\check{u}_0\|_{L^2}^2) \frac{1}{N}$ , we obtain

$$\frac{1}{N} H'_0 + \frac{1}{N} H'_i + \alpha I'_0 + \frac{1}{N} \sum_{i=1}^N I'_i = O\left(\frac{1}{\sqrt{N}}\right).$$

Hence, we have

$$-\frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) + \frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^*; \mathcal{M}\left(u_0^*\right)\right) \leq K\left(\frac{1}{\sqrt{N}}\right) = O\left(\frac{1}{\sqrt{N}}\right), \quad (6.9)$$

where  $K$  is independent of  $N$ . By (6.9) and (6.8), it follows that

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^*; \mathcal{M}\left(u_0^*\right)\right) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) = O\left(\frac{1}{\sqrt{N}}\right),$$

and

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^{(N)}; \mathcal{M}\left(u_0^{(N)}\right)\right) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(\check{u}_0; \mathcal{M}\left(\check{u}_0\right)\right) \leq 0,$$

respectively. Thus, we have

$$\Delta_1 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(u_0^*; \mathcal{M}\left(u_0^*\right)\right) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}\left(\check{u}_0; \mathcal{M}\left(\check{u}_0\right)\right) \leq O\left(\frac{1}{\sqrt{N}}\right). \quad (6.10)$$

By Proposition 6.5, there exists  $K$  independent of  $N$  such that  $\|\check{u}_0\|_{L^2} \leq K$ . Then, combining (6.7), (6.10), we can obtain:

$$\Delta_1 + \Delta_2 \leq O\left(\frac{1}{\sqrt{N}}\right) + c(\|\check{u}_0\|_{L^2})\left(\frac{1}{\sqrt{N}}\right) \leq K \cdot O\left(\frac{1}{\sqrt{N}}\right) = O\left(\frac{1}{\sqrt{N}}\right),$$

where  $K$  is independent of  $N$ . The theorem follows.  $\square$

## 7. NUMERICAL EXAMPLES

We now give a numerical example for Lemma 6.3. By (5.5) and (5.4),  $\mathbb{K}$  and  $\kappa$  can be easily computed. Consider  $\mathbb{Y} = \mathbb{K}\mathbb{X} + \kappa$ , we can obtain that

$$d\mathbb{X} = [(A + \mathbb{B}\mathbb{K})\mathbb{X} + \mathbb{B}\kappa + b]dt + \mathbb{D}dW_0, \quad \mathbb{Y} = \mathbb{K}\mathbb{X} + \kappa,$$

where  $\mathbb{X} = ((\hat{x})^T (\bar{x}_0)^T (q^*)^T (l_1^*)^T (l_2^*)^T)^T$ ,  $\mathbb{Y} = ((y^*)^T (\hat{y}^*)^T (y_0^*)^T (k_1)^T (k_2)^T)^T$ . Since  $p_i = \bar{P}\bar{x}_i + \bar{\varphi}$ , by the following equations below (5.6), we have

$$d\bar{x}_i = [(A - BR^{-1}B^T\bar{P})\bar{x}_i - BR^{-1}B^T\bar{\varphi} + C\hat{x} + F\bar{x}_0]dt + DdW_i.$$

The realized decentralized state  $x_0^*$  and  $(x^*)^{(N)}$ , can be derived by (6.1). Combining them with (4.5), one can obtain

$$\begin{cases} d\begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix} = \left[ \begin{pmatrix} A_0 & C_0 \\ F & A + C \end{pmatrix} \begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix} - \begin{pmatrix} 0 \\ BR^{-1}B^T \end{pmatrix} (p^{(N)} - \hat{p}) \right] dt + \frac{1}{N} \begin{pmatrix} 0 \\ \sum_1^N D \end{pmatrix} dW_i, \\ \begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix} (0) = \begin{pmatrix} 0 \\ \frac{1}{N} \sum_1^N \xi_i - \hat{\xi} \end{pmatrix}, \end{cases}$$

where  $\hat{p} = k_2$ .

We continuously use the parameters in Example 5.1. The population  $N = 100$  and the time interval is  $[0, 12]$ . By Matlab computation, the trajectories of the realized state  $x_i^*$  are shown in Figure 1(a).



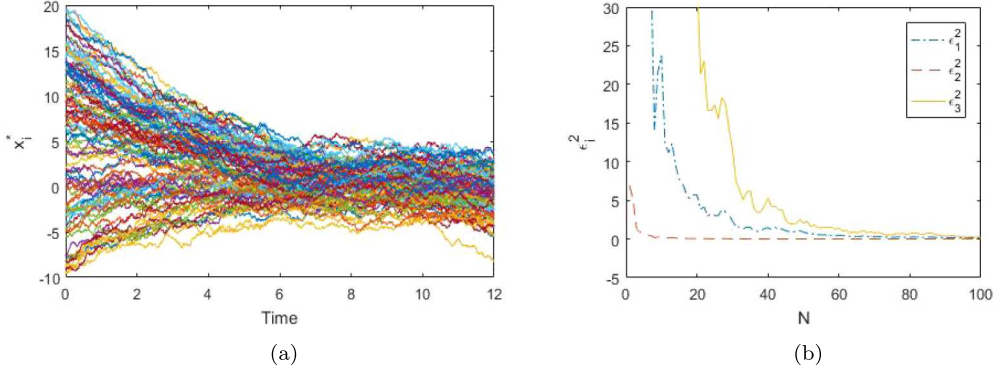


FIGURE 1. (a) is the trajectories of  $x_i^*$ ,  $i = 1, \dots, 100$  and (b) is the curves of  $\varepsilon_i^2$ ,  $i = 1, 2, 3$  when time interval is  $[0, 12]$ .

We defined  $\varepsilon_1^2 = \mathbb{E} \int_0^{12} \|(x^*)^{(N)} - \hat{x}\|^2 dt$ ,  $\varepsilon_2^2 = \mathbb{E} \int_0^{12} \|x_0^* - \bar{x}_0\|^2 dt$ ,  $\varepsilon_3^2 = \mathbb{E} \int_0^{12} \|p^{(N)} - \hat{p}\|^2 dt$ . When  $N$  increases from 1 to 100, the curves of  $\varepsilon_1^2$ ,  $\varepsilon_2^2$  and  $\varepsilon_3^2$  are shown in Figure 1b. The  $X$  axis indicates  $N$  and the  $Y$  axis indicates  $\varepsilon_i^2$ ,  $i = 1, 2, 3$ . It can be seen that they are approaching to zero when  $N$  is growing larger and larger.

## 8. CONCLUSION

This paper has analyzed the social optima in a class of LQ mean field control problem. We obtain the decentralized form of the optimal controls for the leader and  $N$  followers. By Ricatti equation method, we discuss the solvability of the FBSDE. Finally, a Stackelberg equilibrium theorem is established. For future work, one can extend the results of this paper to the hierarchical control with many leaders case.

### APPENDIX A. PROOF OF LEMMA 6.3

By (5.6) and (6.1), we have

$$\begin{cases} d(x^*)^{(N)} = [A(x^*)^{(N)} - BR^{-1}B^T p^{(N)} + C(x^*)^{(N)} + Fx_0^*]dt + \frac{1}{N} \sum_{i=1}^N DdW_i, & (x^*)^{(N)}(0) = \frac{1}{N} \sum_{i=1}^N \xi_i, \\ d\bar{x}^{(N)} = [A\bar{x}^{(N)} - BR^{-1}B^T p^{(N)} + C\hat{x} + F\bar{x}_0]dt + \frac{1}{N} \sum_{i=1}^N DdW_i, & \bar{x}^{(N)}(0) = \frac{1}{N} \sum_{i=1}^N \xi_i, \\ dp^{(N)} = -[A^T p^{(N)} + \chi_1]dt + \zeta_0 dW_0 + \frac{1}{N} \sum_{i=1}^N \zeta_i dW_i, & p^{(N)}(T) = Gx^{(N)}(T) + \chi_2. \end{cases} \quad (\text{A.1})$$

To prove Lemma 6.3, we need the following two lemmas.

**Lemma A.1.** *Assume that (A1)–(A4) hold. Let  $\bar{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N \bar{x}_i$  and  $p^{(N)} = \frac{1}{N} \sum_{i=1}^N p_i$ . Then*

$$\sup_{0 \leq t \leq T} \mathbb{E} \|\bar{x}^{(N)} - \hat{x}\|^2 = O\left(\frac{1}{N}\right), \quad \sup_{0 \leq t \leq T} \mathbb{E} \|p^{(N)} - \hat{p}\|^2 = O\left(\frac{1}{N}\right).$$

*Proof.* Combining (A.1) and (3.15), we can obtain

$$\begin{cases} d\mu_1 = [A\mu_1 - BR^{-1}B^T\mu_2]dt + \frac{1}{N} \sum_{i=1}^N DdW_i, & \mu_1(0) = \frac{1}{N} \sum_{i=1}^N \xi_i - \hat{\xi}, \\ d\mu_2 = -[A^T\mu_2 + Q\mu_1 + \frac{1}{N} \sum_{i=1}^N \zeta_i dW_i], & \mu_2(T) = G\mu_1, \end{cases}$$

where  $\mu_1 = \bar{x}^{(N)} - \hat{x}$  and  $\mu_2 = p^{(N)} - \hat{p}$ . Denote  $\mu_2 = P\mu_1 + \varphi$ ,  $t \in [0, T]$ , where  $P \in C^1([0, T]; \mathbb{S}^n)$  is the solution of the following Riccati equation and  $\varphi \in C^1([0, T]; \mathbb{R}^n)$  satisfies

$$\begin{cases} \dot{P} + PA - PBR^{-1}B^TP + A^TP + Q = 0, & t \in [0, T], \quad P(T) = G, \\ d\varphi = -(A - BR^{-1}B^TP)^T\varphi dt + \frac{1}{N} \sum_{i=1}^N (PD - \zeta_i)dW_i, & t \in [0, T], \quad \varphi(T) = 0. \end{cases}$$

This is a standard Riccati equation and the latter BSDE has a unique solution  $\varphi = 0$ ,  $t \in [0, T]$ . Thus  $\mu_2 = P\mu_1$  and

$$d\mu_1 = [A - BR^{-1}B^TP]\mu_1 dt + \frac{1}{N} \sum_{i=1}^N DdW_i.$$

By Cauchy-Schwarz inequality and Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E} \|\mu_1\|^2 &= \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t (A - BR^{-1}B^TP)\mu_1 ds + \int_0^t \frac{1}{N} \sum_{i=1}^N DdW_i \right\|^2 \\ &\leq 2 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t (A - BR^{-1}B^TP)\mu_1 ds \right\|^2 + 2 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t \frac{1}{N} \sum_{i=1}^N DdW_i \right\|^2 \\ &\leq 2K \left\{ \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t \|\mu_1\|^2 ds + \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \int_0^T \|D\|^2 ds \right\} = 2K \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t \|\mu_1\|^2 ds + O\left(\frac{1}{N}\right), \end{aligned}$$

where constant  $K$  is independent of  $N$ . Then, by Gronwall's inequality and  $\mu_2 = P\mu_1$ , we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E} \|\mu_1\|^2 = O\left(\frac{1}{N}\right), \quad \sup_{0 \leq t \leq T} \mathbb{E} \|\mu_2\|^2 = O\left(\frac{1}{N}\right).$$

The lemma follows. □

**Lemma A.2.** Assume that (A1)–(A4) hold. Let  $(x^*)^{(N)} = \frac{1}{N} \sum_{i=1}^N x_i^*$ . Then

$$\sup_{0 \leq t \leq T} \mathbb{E} \|x_0^* - \bar{x}_0\|^2 = O\left(\frac{1}{N}\right), \quad \sup_{0 \leq t \leq T} \mathbb{E} \|(x^*)^{(N)} - \bar{x}^{(N)}\|^2 = O\left(\frac{1}{N}\right).$$

*Proof.* Denote  $\mu_3 = x_0^* - \bar{x}_0$  and  $\mu_4 = (x^*)^{(N)} - \bar{x}^{(N)}$ . By (A.1), we can obtain

$$d \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} = \left[ \begin{pmatrix} A_0 & C_0 \\ F & A + C \end{pmatrix} \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} + \begin{pmatrix} C_0 \\ C \end{pmatrix} \mu_1 \right] dt, \quad \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} (0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For some constant  $K$  which is independent of  $N$  such that

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E} \left\| \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} \right\|^2 &= \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t \left[ \begin{pmatrix} A_0 & C_0 \\ F & A+C \end{pmatrix} \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} + \begin{pmatrix} C_0 \\ C \end{pmatrix} \mu_1 \right] ds \right\|^2 \\ &\leq 2K \left\{ \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t \left\| \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} \right\|^2 ds + \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t \|\mu_1\|^2 ds \right\} = 2K \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t \left\| \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} \right\|^2 ds + O\left(\frac{1}{N}\right). \end{aligned}$$

By Gronwall's inequality, one can obtain

$$\sup_{0 \leq t \leq T} \mathbb{E} \left\| \begin{pmatrix} \mu_3 \\ \mu_4 \end{pmatrix} \right\|^2 = O\left(\frac{1}{N}\right).$$

Thus, the lemma follows.  $\square$

*Proof of Lemma 6.3.* Since

$$\|(x^*)^{(N)} - \hat{x}\|^2 = \|(x^*)^{(N)} - \bar{x}^{(N)} + \bar{x}^{(N)} - \hat{x}\|^2 \leq 2\|(x^*)^{(N)} - \bar{x}^{(N)}\|^2 + 2\|\bar{x}^{(N)} - \hat{x}\|^2.$$

Combining Lemma A.1 and Lemma A.2, it leads to

$$\mathbb{E} \int_0^T \|(x^*)^{(N)} - \hat{x}\|^2 dt + \mathbb{E} \int_0^T \|p^{(N)} - \hat{p}\|^2 dt + \mathbb{E} \int_0^T \|x_0^* - \bar{x}_0\|^2 dt \leq T \cdot O\left(\frac{1}{N}\right) = O\left(\frac{1}{N}\right).$$

The lemma follows.  $\square$

## APPENDIX B. PROOF OF LEMMA 6.4

*Proof.* By (4.5), (A.1), (6.1) and using a similar argument in Lemma A.2, one obtain that for some constant  $K$  which is not dependent on  $N$  such that

$$\sup_{0 \leq t \leq T} \mathbb{E} \|x_0^*\|^2 \leq K, \quad \sup_{0 \leq t \leq T} \mathbb{E} \|(x^*)^{(N)}\|^2 \leq K.$$

By Cauchy-Schwarz inequality and Burkholder-Davis-Gundy's inequality, we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E} \|x_i^*\|^2 &= \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t (Ax_i^* - BR^{-1}B^T p_i + C(x^*)^{(N)} + Fx_0^*) ds + DdW_i \right\|^2 \\ &\leq \sup_{0 \leq t \leq T} \mathbb{E} \int_0^t 2\|Ax_i^*\|^2 + 2\|BR^{-1}B^T p_i\|^2 + 2\|C(x^*)^{(N)}\|^2 + 2\|Fx_0^*\|^2 ds + \sup_{0 \leq t \leq T} \int_0^t 2\|DdW_i\|^2 \\ &\leq 2\mathbb{E} \int_0^T \|Ax_i^*\|^2 + \|BR^{-1}B^T p_i\|^2 + \|C(x^*)^{(N)}\|^2 + \|Fx_0^*\|^2 ds + \int_0^T \|D\|^2 ds \\ &\leq 2A^2 \sup_{0 \leq t \leq T} \mathbb{E} \int_0^T \|x_i^*\|^2 ds + K, \end{aligned}$$

where constant  $K$  is independent of  $N$ . By Gronwall's inequality, we have

$$\sup_{1 \leq i \leq N} \left[ \sup_{0 \leq t \leq T} \mathbb{E} \|x_i^*\|^2 \right] \leq K,$$

where  $K$  is not dependent on  $N$ . Then, according to Cauchy-Schwarz inequality, Burkholder-Davis-Gundy's inequality and the above discussion, we have

$$\begin{aligned}
\mathcal{J}_{soc}^{(N)}(u_0^*; u^*) &= \alpha N \mathcal{J}_0(u_0^*; u^*) + \sum_{i=1}^N \mathcal{J}_i(u_0^*; u^*) \\
&= \alpha N \mathbb{E} \left\{ \int_0^T [\|x_0^* - \Theta_0(x^*)^{(N)} - \eta_0\|_{Q_0}^2 + \| -(\alpha R_0)^{-1} B_0^T y_0^*\|_{R_0}^2] dt + \|x_0^*(T) \right. \\
&\quad \left. - \hat{\Theta}_0(x^*)^{(N)}(T) - \hat{\eta}_0\|_{G_0}^2 \right\} + \sum_{i=1}^N \mathbb{E} \left\{ \int_0^T [\|x_i^* - \Theta(x^*)^{(N)}(t) - \Theta_1 x_0^* - \eta\|_Q^2 \right. \\
&\quad \left. + \| -R^{-1} B^T p_i\|_R^2] dt + \|x_i^*(T) - \hat{\Theta}(x^*)^{(N)}(T) - \hat{\Theta}_1 x_0^*(T) - \hat{\eta}\|_G^2 \right\} \leq NK,
\end{aligned}$$

where  $K$  is independent of  $N$ . The lemma follows.  $\square$

### APPENDIX C. PROOF OF PROPOSITIONS 6.6 AND 6.7

*Proof of Proposition 6.6.* Since

$$\begin{cases} d\delta x_0 = (A_0 \delta x_0 + C_0 \delta x^{(N)}) dt, & \delta x_0(0) = 0, \\ d\delta x^{(N)} = [(A + C) \delta x^{(N)} + \frac{B}{N} \delta u_i] dt, & \delta x^{(N)}(0) = 0, \end{cases}$$

and by Proposition 6.5, we have  $\|\delta u_i\|_{L^2}^2 \leq K$ ,  $K$  is independent of  $N$ . Using Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} \|\delta x^{(N)}\|^2 &= \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_0^s [(A + C) \delta x^{(N)} + \frac{B}{N} \delta u_i] dr \right\|^2 \\
&\leq K \mathbb{E} \int_0^t \|\delta x^{(N)}\|^2 dr + \frac{1}{N^2} \|K\|^2 dr \leq K \mathbb{E} \int_0^t \|\delta x^{(N)}\|^2 dr + O\left(\frac{1}{N^2}\right),
\end{aligned}$$

where  $K$  is independent of  $N$ . By Gronwall's inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x^{(N)}\|^2 = O\left(\frac{1}{N^2}\right).$$

For  $\delta x_0$ , we have

$$\mathbb{E} \sup_{0 \leq s \leq t} \|\delta x_0\|^2 = \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_0^s [A_0 \delta x_0 + C_0 \delta x^{(N)}] dr \right\|^2 \leq K \mathbb{E} \int_0^t \|\delta x_0\|^2 dr + O\left(\frac{1}{N^2}\right),$$

where  $K$  is independent of  $N$ . By Gronwall's inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x_0\|^2 = O\left(\frac{1}{N^2}\right).$$

Moreover,

$$\Theta^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta) \leq \frac{1}{N} \inf_{(u_0; u)} J_{soc}^{(N)}(u_0; u) \leq K.$$

Similarly,  $\Theta_1^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta)$ ,  $\hat{\Theta}^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta})$ ,  $\hat{\Theta}_1^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta})$  are bounded. Thus,

$$\begin{aligned} & \langle \Theta^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x^{(N)} \rangle + \langle \Theta_1^T Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x_0 \rangle + \langle \hat{\Theta}^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) \\ & - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T) \rangle + \langle \hat{\Theta}_1^T G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_0(T) \rangle = o(1). \end{aligned}$$

The proposition follows.  $\square$

*Proof of Proposition 6.7.* Since

$$\begin{cases} d \left( \sum_{j \neq i} \delta x_j \right) = \left[ A \left( \sum_{j \neq i} \delta x_j \right) + C \frac{N-1}{N} \delta x^{(N)} + F(N-1) \delta x_0 \right] dt, & \left( \sum_{j \neq i} \delta x_j \right) (0) = 0, \\ d(N \delta x_0) = [A_0(N \delta x_0) + C_0(N \delta x^{(N)})] dt, & (N \delta x_0)(0) = 0, \\ d(N \delta x_j) = [A(N \delta x_j) + C(N \delta x^{(N)}) + F(N \delta x_0)] dt, & (N \delta x_j)(0) = 0. \end{cases}$$

According to equations in (3.3), one can obtain

$$\begin{cases} d \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) = \left[ A \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) + \frac{C}{N} \delta x^{(N)} + F \delta x_0 \right] dt, & \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) (0) = 0, \\ d(N \delta x_0 - \delta x_0^\dagger) = \left[ A_0 \left( N \delta x_0 - \delta x_0^\dagger \right) + C_0 \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) + C_0 \left( N \delta x_j - \delta x_0^\dagger \right) \right] dt, & \left( N \delta x_0 - \delta x_0^\dagger \right) (0) = 0, \\ d(N \delta x_j - \delta x_j^\dagger) = [(A + C)(N \delta x_j - \delta x_j^\dagger) + C \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) + F(N \delta x_0 - \delta x_0^\dagger)] dt, & (N \delta x_j - \delta x_j^\dagger)(0) = 0. \end{cases}$$

Combing with the results in Proposition 6.6, we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} \left\| N \delta x_j - \sum_{j \neq i} \delta x_j \right\|^2 &= \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_0^s \left[ A \left( N \delta x_j - \sum_{j \neq i} \delta x_j \right) + \frac{C}{N} \delta x^{(N)} + F \delta x_0 \right] dr \right\|^2 \\ &\leq K \mathbb{E} \int_0^t \left\| N \delta x_j - \sum_{j \neq i} \delta x_j \right\|^2 dr + O\left(\frac{1}{N^2}\right), \end{aligned}$$

where constant  $K$  is independent of  $N$ . By Gronwall's inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \left\| N \delta x_j - \sum_{j \neq i} \delta x_j \right\|^2 = O\left(\frac{1}{N^2}\right).$$

Since  $N\delta x_0 - \delta x_0^\dagger$  and  $N\delta x_j - \delta x_j^\dagger$  are coupled, we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq t} \left\| \begin{pmatrix} N\delta x_0 - \delta x_0^\dagger \\ N\delta x_j - \delta x_j^\dagger \end{pmatrix} \right\|^2 \\ &= \mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_0^s \left[ \begin{pmatrix} A_0 & C_0 \\ F & A+C \end{pmatrix} \begin{pmatrix} N\delta x_0 - \delta x_0^\dagger \\ N\delta x_j - \delta x_j^\dagger \end{pmatrix} + \begin{pmatrix} C_0 \\ C \end{pmatrix} (N\delta x_j - \sum_{j \neq i} \delta x_j) \right] dr \right\|^2 \\ &\leq K \mathbb{E} \int_0^t \left\| \begin{pmatrix} N\delta x_0 - \delta x_0^\dagger \\ N\delta x_j - \delta x_j^\dagger \end{pmatrix} \right\|^2 dr + O\left(\frac{1}{N^2}\right), \end{aligned}$$

where real-valued matrix  $K$  is independent of  $N$ . By Gronwall's inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \left\| \begin{pmatrix} N\delta x_0 - \delta x_0^\dagger \\ N\delta x_j - \delta x_j^\dagger \end{pmatrix} \right\|^2 = O\left(\frac{1}{N^2}\right).$$

The proposition follows. □

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