A Perturbation Approach to Optimal Investment, Liability Ratio, and Dividend Strategies

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Abstract

TBA

1 Introduction

Objective: apply the perturbation method from Herdegen et al. (2020) to study an optimal dividend, investment, and risk control problem, in which an insurer aims to maximize the expected discounted utility of dividend over an infinite horizon.

The rest of the paper is organized as follows. In Section 2, we introduce the market model and formulate the main stochastic control problem of the paper. We first solve the problem for a restricted class of strategies–constant strategies in Section 3 and next verify the optimal constant strategy is also globally optimal among all the unrestricted admissible strategies in Section 4. In Section 5, we conduct a thorough economic analysis along both the analytic and the numerical directions to investigate the impact of various parameters on the optimal strategy. Finally, we conclude in Section 6.

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2 Problem Formulation

2.1 The Markets

We consider a representative insurer ("she"), who has access to a financial market consisting of a risk-free asset and a risky asset (e.g., a stock index or a mutual fund). The risk-free asset earns at a constant rate r continuously. The price process $S = (S_t)_{t \geq 0}$ of the risky asset is modeled by

$$\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t^{(1)}, \tag{2.1}$$

where $\mu, \sigma > 0$ and $W^{(1)}$ is a standard Brownian motion. We assume the underlying financial market is ideal and frictionless.

The insurer's main business is to underwrite policies against insurable risks. Following Zou and Cadenillas (2014b), we model such risks on a unit basis (e.g., per policy) by

$$\mathrm{d}R_t = \alpha \, \mathrm{d}t + \beta\rho \, \mathrm{d}W_t^{(1)} + \beta\sqrt{1-\rho^2} \, \mathrm{d}W_t^{(2)} + \gamma \, \mathrm{d}N_t, \tag{2.2}$$

where $\alpha, \beta, \gamma > 0$, $-1 < \rho < 1$, $W^{(2)}$ is another standard Brownian motion, and N is a homogeneous Poisson process with constant intensity $\lambda > 0$. Let p represent the *unit* premium rate, corresponding to the unit risk process R in (2.2). We further impose the following assumptions on the model (2.1)-(2.2) throughout the rest of this paper:

$$\mu > r \qquad \text{and} \qquad p > \alpha + \lambda\gamma. \tag{2.3}$$

We discuss the economic interpretations of the above assumptions in (2.3) in Remark 2.1. On the technical level, we assume the processes $W^{(1)}$, $W^{(2)}$, and N are independent under a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is generated by these three processes, augmented with $\mathbb{P}$-null sets.

Remark 2.1. *As argued in Stein (2012)[Chapter 6], a major mistake in the AIG's business operation during the financial crisis of 2007-2008 is ignore or underestimate the* negative *correlation between the financial market and its insurance liabilities. Such a negative correlation can be easily captured by the risk model of R in (2.2) by setting $\rho \in (-1, 0)$; see, e.g., Zou and Cadenillas (2014b), Peng and Wang (2016), Bo and Wang (2017), and Zhou et al. (2017).*

The two assumptions in (2.3) both have reasonable economic meanings. The first condition implies that the Sharpe ratio of the risky asset, $\Lambda := \frac{\mu - r}{\sigma}$, is strictly positive, which holds true for all "good" assets (e.g., stock indexes) over long run in the financial market. Since $\mathbb{E}[\mathrm{d}R_t] = (\alpha + \lambda\gamma)\mathrm{d}t$, the second condition implies that the insurer should charge the unit premium rate p greater than $\alpha + \lambda\gamma$, the so-called "actuarial fair price". Otherwise (i.e., if $p \leq \alpha + \lambda\gamma$), the insurer's ruin probability (without investment) is equal to 1. If we apply the expected value principle to calculate insurance premium (i.e., $p = (1+\theta)(\alpha + \lambda\gamma)$), then assuming $p > \alpha + \lambda\gamma$ is equivalent to setting $\theta > 0$, where θ is often called the loading factor. Note that we do not *assume that p is given by the above expected value principle, which is nevertheless imposed in many works in order to derive the optimal insurance in closed form; see, e.g., Arrow (1974) and Moore and Young (2006).*

2.2 The Insurer's Strategies and Wealth Process

The representative insurer chooses a triplet strategy (control) that consists of an investment strategy, a liability ratio strategy, and a dividend strategy in its business operations, as described in what follows.

- In the financial market, the insurer chooses a dynamic investment strategy $\pi = (\pi_t)_{t\geq 0}$, where π_t denotes the *proportion* of wealth invested in the risky asset at time t.
- In the insurance market, we assume the insurer can directly control the amount of liabilities (number of units or policies) in the underwriting, denoted by $L = (L_t)_{t\geq 0}$. Following Stein (2012), we define $\kappa_t := L_t/X_t$, where X_t is the insurer's wealth at time t (defined later in (2.4)), and call $\kappa = (\kappa_t)_{t\geq 0}$ the insurer's *liability ratio* strategy.
- The insurer chooses a dividend strategy $D = (D_t)_{t\geq 0}$ to distribute profits to the shareholders, where D_t denotes the continuous dividend *rate* payable at time t. Namely, the dividend payment over $[t, t + \mathrm{d}t]$ is given by $D_t \mathrm{d}t$.

We denote the insurer's strategy by $u := (\pi, \kappa, D)$. We offer some explanations to the insurer's liability strategy κ and dividend strategy D in the following remark.

Remark 2.2. *Motivated by the AIG case during the financial crisis of 2007-2008, Stein (2012) finds that the liability ratio provides an early warning signal to the AIG's failure and sets up a model in which the insurer directly controls its liability ratio. Such a setup attracts considerable attention in the actuarial science literature, as evidenced by a series of follow-up works on the optimal control study for an insurer; see Zou and Cadenillas (2014b), Jin et al. (2015), Peng and Wang (2016), and Shen and Zou (2020), among many others. From an economic point of view, the assumption that the insurer can dynamically control the liability ratio is not unreasonable, as major insurers have monopoly power in the insurance market and can "discriminate" policy holders. For instance, in the business of health insurance, insurers often reject potential policy applications based on certain risk factors. Lapham et al. (1996) study a group of participants with genetic disorders in the family and find that 25% of them believed they were refused life insurance and 22% were refused health insurance. A recent work of Bernard et al. (2020) confirms that, under certain circumstances, "it may become optimal for the insurer to refuse to sell insurance to some prospects".*

The dividend strategy D described above is absolutely continuous with respect to the Lebesgue measure, which is used in many related works; see, e.g, Asmussen and Taksar (1997), Avanzi and Wong (2012), and Jin et al. (2015) for a short list. As such, the corresponding control problem is a classical one, instead of a singular or impulse one. We refer to Jeanblanc-Picqué and Shiryaev (1995) and Sotomayor and Cadenillas (2011) for both classical and singular control of optimal dividend problems. As argued in Avanzi and Wong (2012), optimal dividend strategies obtained in the literature are often volatile (e.g., "bang-bang" strategies), which are unlikely to be adopted by managers. In consequence, they consider a setup where dividends are *paid at a constant rate g of the company's surplus and seek to find the optimal rate* g^**. Note that the above dividend strategy D includes the threshold strategies (also called refracting strategies), where* $D_t = constant$ *when the wealth (surplus) at time t is greater than a given threshold and* $D_t = 0$ *otherwise. For studies on threshold-type dividend strategies, please see the review articles Albrecher and Thonhauser (2009) and Avanzi (2009), and the recent works of Albrecher et al. (2018) and Renaud and Simard (2020).*

Let us denote by $X = (X_t)_{t\geq 0}$ the insurer's wealth process under a triplet strategy of investment, liability ratio, and dividend $u := (\pi, \kappa, D)$. The dynamics of X is obtained by

$$\begin{aligned} \mathrm{d}X_t = &\big(X_{t-}(r + (\mu - r)\pi_t + (p - \alpha)\kappa_t) - D_t\big)\,\mathrm{d}t + (\sigma\pi_t - \beta\rho\kappa_t)X_{t-}\,\mathrm{d}W_t^{(1)} \\ &- \beta\sqrt{1-\rho^2}\,\kappa_t X_{t-}\,\mathrm{d}W_t^{(2)} - \gamma\kappa_t X_{t-}\,\mathrm{d}N_t, \qquad X_0 = x > 0. \end{aligned} \tag{2.4}$$

It is clear from (2.4) that the insurer's wealth process depends on her initial wealth x and strategy u, and we write it as X for national simplicity.

We next define the admissible strategies in Definition 2.3 and close this subsection with some remarks.

Definition 2.3 (Admissible Strategies)**.** *A strategy* $u = (\pi, \kappa, D)$ *is called admissible, denoted by* $u \in \mathcal{A}$*, if (1) u is predictable with respect to the filtration* $\mathbb{F}$*; (2) for all* $t \geq 0$*, we have* $\int_0^t \pi_s^2\,\mathrm{d}s < \infty$*,* $0 \leq \kappa_t < \frac{1}{\gamma}$*, and* $\mathbb{E}[\int_0^t D_s\,\mathrm{d}s] < \infty$ *with* $D_t \geq 0$*; (3) there exists a unique strong solution to (2.4) such that* $X_t \geq 0$ *for all* $t \geq 0$*.*

Remark 2.4. *The constraint* $0 \leq \kappa_t < \frac{1}{\gamma}$ *stipulates that the wealth process X will not become negative or zero at the jump times of the Poisson process N. According to Definition 2.3, given an admissible control u, the corresponding wealth* X^u *may hit 0 (ruin or bankruptcy). Let us define the ruin time by* $\tau^u := \inf\{t \geq 0 : X^u = 0,\, u \in \mathcal{A}\}$*. We follow the standard treatment to set* u_t *to be any constant within the feasible region for all* $t \geq \tau^u$*. For instance, Zou and Cadenillas (2014a) and Jin et al. (2015) set* $u_t \equiv 0$ *after ruin. Last, we mention that the condition on the investment strategy* π *is weaker than the standard square integrability condition* $\mathbb{E}[\int_0^t \pi_s^2\,\mathrm{d}s] < \infty$ *for all* $t \geq 0$*.*

2.3 The Problem

Given a triplet strategy u, we define the insurer's objective functional J by

$$J(x; u) := \mathbb{E}\left[\int_0^\infty e^{-\delta t}\, U(D_t)\,\mathrm{d}t\right], \tag{2.5}$$

where $\mathbb{E}$ denotes taking expectation under the physical measure $\mathbb{P}$, $\delta > 0$ is the subjective discount factor, and U is a standard utility function (strictly increasing and strictly concave). The concavity of U implies risk aversion in the insurer's decision making. Applying a concave utility function to the dividend payments is in fact not unusual in the optimal dividend literature. Cadenillas et al. (2007) provide economic justifications for incorporating risk aversion in the objective (see pp.85 therein); see also Thonhauser and Albrecher (2011), Jin et al. (2015), and Xu et al. (2020).

We now formulate the main optimal investment, liability ratio, and dividend problem of this paper.

Problem 2.5. *The insurer seeks an optimal strategy $u^* = (\pi^*, \kappa^*, D^*)$ to maximize the expected discounted utility of dividend over an infinite horizon. Equivalently, the insurer solves the following stochastic control problem:*

$$V(x) := \sup_{u \in \mathcal{A}} J(x; u) = J(x; u^*), \tag{2.6}$$

where the admissible set $\mathcal{A}$ is defined in Definition 2.3 and the objective functional J is defined in (2.5).

Using the ruin time τ^u defined in Remark 2.4, we can rewrite J in (2.5) as

$$J = \mathbb{E}\left[\int_0^{\tau^u} e^{-\delta t}\, U(D_t)\,\mathrm{d}t + \int_{\tau^u}^{\infty} e^{-\delta t}\, U(D_t)\,\mathrm{d}t\right].$$

If we further set the second integral to 0 and take $U(x) \equiv x$ (risk neutrality), the above J reduces to the standard setup in optimal dividend literature (see Albrecher and Thonhauser (2009) and Avanzi (2009)). However, such an extra assumption is *not* necessary under our setup, as it turns out that ruin will not happen under the optimal strategy u^* (i.e., $\tau^{u^*} = \infty$).

In this work, we consider the utility function U is of power form, which belongs to the hyperbolic absolute risk aversion (HARA) family. In particular, U is given by

$$U(x) = \frac{1}{1-\eta}\, x^{1-\eta}, \qquad \eta > 0, \tag{2.7}$$

where the limit case of $\eta = 1$ is treated as log utility $U(x) = \ln x$. Note that U in (2.7) is well defined for all $x > 0$. If U is given by log utility ($\eta = 1$) or negative power utility ($\eta > 1$), we set $U(0) = -\infty$. The choice of power utility is dominant in the optimal investment literature (see the classical papers of Merton (1969, 1971)). We comment that the relative risk aversion η in (2.7) can be any positive number, and is in particular allowed to be greater than 1 (arguably the case in real life, see Meyer and Meyer (2005)). In comparison, $0 < \eta \le 1$ is assumed in Cadenillas et al. (2007), Thonhauser and Albrecher (2011), Jin et al. (2015), and Xu et al. (2020).

3 Analysis of Constant Strategies

In this section, we study Problem (2.6) over a restricted set of *constant* strategies $\mathcal{A}_c$, which is defined by

$$\mathcal{A}_c := \{u = (\pi, \kappa, D) \,|\, \pi_t \equiv \pi_c,\ \kappa_t \equiv \kappa_c,\ D_t \equiv \xi_c X_t, \quad \text{where } \pi_c, \kappa_c, \xi_c \in \mathbb{R},\ \kappa_c \in [0, 1/\gamma),\ \xi_c \ge 0\}. \tag{3.1}$$

We denote constant strategies in $\mathcal{A}_c$ by $u_c := (\pi_c, \kappa_c, \xi_c)$, which is slightly different from $u = (\pi, \kappa, D) \in \mathcal{A}$ introduced in Section 1. For any $u_c \in \mathcal{A}_c$, there exists a unique strong (positive) solution X to (2.4) and $\mathbb{E}[X_t^2] < \infty$ for all $t \ge 0$ (see, e.g., Theorem 1.19 in Øksendal and Sulem (2005)). This result, along with the definition in (3.1), implies that all the conditions in Definition 2.3 are satisfied. Therefore, we conclude that the set of constant strategies $\mathcal{A}_c$ is a (proper) subset of the set of admissible strategies $\mathcal{A}$.

To begin, we solve (2.4) explicitly and obtain the insurer's wealth X_t at time t (for all $t \geq 0$) by

$$X_t = x \exp\left(\left(f(\pi_c, \kappa_c) - \xi_c\right)t + \left(\sigma\pi_c - \beta\rho\kappa_c\right)W_t^{(1)} - \beta\sqrt{1-\rho^2}\kappa_c W_t^{(2)} + \ln(1-\gamma\kappa_c)\,\widetilde{N}_t\right), \tag{3.2}$$

where $\widetilde{N} = (\widetilde{N}_t)_{t\in[0,T]}$, with $\widetilde{N}_t := N_t - \lambda t$, is the compensated Poisson process. In addition, the function f is defined over $\mathbb{R} \times [0, 1/\gamma)$ by

$$f(y_1, y_2) := r + (\mu - r)y_1 + (p - \alpha)y_2 - \frac{1}{2}\left(\sigma^2 y_1^2 - 2\beta\rho\sigma y_1 y_2 + \beta^2 y_2^2\right) + \lambda \ln(1 - \gamma y_2). \tag{3.3}$$

For future convenience, we define three constants A, B, and C by

$$A := \gamma\beta^2(1-\rho^2), \qquad B := \beta^2(1-\rho^2) + \gamma(p - \alpha + \beta\rho\Lambda), \qquad C := p - \alpha + \beta\rho\Lambda - \lambda\gamma, \tag{3.4}$$

where Λ is the Sharpe ratio of the risky asset, i.e.,

$$\Lambda := \frac{\mu - r}{\sigma}. \tag{3.5}$$

Due to $\rho \in (-1, 1)$ and (2.3), we have $A > 0$ and $B > 0$. If $\rho \geq 0$, then $C > 0$; but if $\rho < 0$, C may be negative.

We first analyze the case of log utility $U(x) = \ln x$, corresponding to $\eta = 1$ in (2.7). Using (3.1)-(3.2) along with the definition of J in (2.5), we obtain

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t} U(D_t)\,\mathrm{d}t\right] = \frac{1}{\delta}\ln x + \frac{1}{\delta^2} f(\pi_c, \kappa_c) + \frac{1}{\delta}\ln \xi_c - \frac{1}{\delta^2}\xi_c. \tag{3.6}$$

Solving Problem (2.6) over $\mathcal{A}_c$ when $U(x) = \ln x$ is now equivalent to maximizing the right hand side of (3.6), which is solved in the proposition below.

Proposition 3.1. *Suppose $U(x) = \ln x$ and the constant C defined in (3.4) is non-negative. The optimal constant strategy $u_c^* = (\pi_c^*, \kappa_c^*, \xi_c^*)$ to Problem (2.6) over $\mathcal{A}_c$ is given by*

$$\pi_c^* = \frac{\mu - r}{\sigma^2} + \frac{\rho\beta}{\sigma}\kappa_c^*, \qquad \kappa_c^* = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \qquad \xi_c^* = \delta, \tag{3.7}$$

where the constants A, B, and C are defined in (3.4). Furthermore, the value function V_c is obtained by

$$V_c(x) := \sup_{u_c \in \mathcal{A}_c} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\ln(D_t)\,\mathrm{d}t\right] = J(x; u_c^*) = \frac{1}{\delta}\ln(\delta x) + \frac{1}{\delta^2}(\mathfrak{f}^* - \delta), \tag{3.8}$$

where $\mathfrak{f}^* := f(\pi_c^*, \kappa_c^*) = \max\limits_{(y_1,y_2)\in\mathbb{R}\times[0,1/\gamma)} f(y_1, y_2)$, with π_c^* and κ_c^* derived in (3.7) and f given in (3.3).

Proof. By applying the first-order condition to maximizing the right hand side of (3.6), we obtain that π_c^* and ξ_c^* are given in (3.7) and κ_c^* solves a quadratic equation $A\,y^2 - B\,y + C = 0$, where the constant A, B, and C are defined in (3.4). We compute $B^2 - 4AC = \left(\beta^2(1-\rho^2) - \gamma(\lambda\gamma + C)\right)^2 + 4\lambda B^2\gamma^2(1-\rho^2) > 0$, implying that there are two candidate solutions to the quadratic equation. We next verify that the bigger solution is greater than $1/\gamma$ and the smaller solution is less than $1/\gamma$. Consequently, κ_c^* is given by the smaller solution as shown in (3.7), which is non-negative if and only if $C \geq 0$. By verifying the second-order condition, we show that u_c^* in (3.7) is indeed an optimal constant strategy over $\mathcal{A}_c$ to Problem (2.6). Plugging u_c^* back into J in (2.5), after tedious computations, leads to the value function in (3.8). □

We next study the case of power utility $U(x) = x^{1-\eta}/(1-\eta)$, where $\eta > 0$ and $\eta \neq 1$. For any $u_c \in \mathcal{A}_c$, we obtain

$$(3.9) \qquad J(x; u_c) = \mathbb{E}\left[\int_0^\infty e^{-\delta t} \frac{(\xi_c X_t)^{1-\eta}}{1-\eta} \, \mathrm{d}t\right] = \frac{x^{1-\eta}}{1-\eta} \frac{\xi_c^{1-\eta}}{\delta - (1-\eta) \cdot g(\pi_c, \kappa_c) + (1-\eta)\xi_c},$$

provided $\delta - (1-\eta)g(\pi_c, \kappa_c) + (1-\eta)\xi_c > 0$. Here, the function g is defined over $\mathbb{R} \times [0, 1/\gamma)$ by

$$(3.10) \quad g(y_1, y_2) := r + (\mu - r)y_1 + (p - \alpha)y_2 - \frac{\eta}{2}\left(\sigma^2 y_1^2 - 2\beta\rho\sigma y_1 y_2 + \beta^2 y_2^2\right) + \frac{\lambda}{1-\eta}\left((1-\gamma y_2)^{1-\eta} - 1\right).$$

We now present the main result for power utility as follows.

Proposition 3.2. *Suppose $U(x) = x^{1-\eta}/(1-\eta)$, where $\eta > 0$ and $\eta \neq 1$, and the following two conditions hold*

$$(3.11) \qquad C \geq 0 \qquad \text{and} \qquad \psi := \frac{\delta - (1-\eta)\mathfrak{g}^*}{\eta} > 0,$$

where C is defined in (3.4) and $\mathfrak{g}^* := \max_{(y_1, y_2) \in \mathbb{R} \times [0, 1/\gamma)} g(y_1, y_2)$, with g defined in (3.10). The optimal constant strategy $u_c^* = (\pi_c^*, \kappa_c^*, \xi_c^*)$ to Problem (2.6) over $\mathcal{A}_c$ is given by

$$(3.12) \qquad \pi_c^* = \frac{\mu - r}{\eta\sigma^2} + \frac{\rho\beta}{\sigma}\kappa_c^*, \qquad \kappa_c^* = \text{unique solution of (3.13)}, \qquad \xi_c^* = \psi,$$

where κ_c^* solves the following non-linear equation

$$(3.13) \qquad (1-\gamma y)^{-\eta} + \frac{\eta A}{\lambda\gamma^2} y - \frac{C}{\lambda\gamma} - 1 = 0,$$

with the constants A and C defined in (3.4). Furthermore, the value function V_c is obtained by

$$(3.14) \qquad V_c(x) := \sup_{u_c \in \mathcal{A}_c} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(D_t) \, \mathrm{d}t\right] = J(x; u_c^*) = \frac{\psi^{-\eta}}{1-\eta} x^{1-\eta}.$$

Proof. Applying the first-order condition to the maximizing problem of the right hand side of (3.9) leads to the results of u_c^* in (3.12) and the non-linear equation of κ_c^* in (3.13), which has a unique solution in $[0, 1/\gamma)$ due to the condition $C \geq 0$. By imposing $\psi > 0$, we obtain $\xi_c^* = \psi > 0$ and $\delta - (1-\eta)g(\pi_c, \kappa_c) > 0$ for all $(\pi_c, \kappa_c) \in \mathbb{R} \times [0, 1/\gamma)$. A straightforward verification process then completes the proof. □

By (3.11), we have $\lim_{\eta \to 1} \psi = \delta$. We also notice that, when $\eta \to 1$, the non-linear equation (3.13) reduces to the quadratic equation satisfied by κ_c^* in the log utility case. As such, Proposition 3.1 for log utility can be seen as the limit result of Proposition 3.2 for power utility. In the literature, a diffusion process *without* jumps is commonly used to approximate the classical Craḿer-Lundberg model; see, e.g., Browne (1995) and Højgaard and Taksar (1998). Such a setup corresponds to setting $\lambda = 0$ in the risk process (2.2). The results in Proposition 3.2 can be simplified when $\lambda = 0$, as shown in the corollary below.

Corollary 3.3. *Suppose the utility function U is given by (2.7) and there are no jumps ($\lambda = 0$) in the risk process (2.2). Further suppose two technical conditions hold:*

$$p - \alpha + \beta\rho\Lambda > 0 \qquad \text{and} \qquad \delta > (1-\eta)\hat{\mathfrak{g}}^*, \tag{3.15}$$

where the constant $\hat{\mathfrak{g}}^*$ is defined by

$$\hat{\mathfrak{g}}^* := r + \frac{(p-\alpha+\beta\rho\Lambda)^2}{2\eta\beta^2(1-\rho^2)} + \frac{\Lambda^2}{2\eta}. \tag{3.16}$$

The optimal constant strategy $u_c^* = (\pi_c^*, \kappa_c^*, \xi_c^*)$ to Problem (2.6) over $\mathcal{A}_c$ is given by

$$\pi_c^* = \frac{\mu - r}{\eta\sigma^2} + \frac{\rho\beta}{\sigma}\kappa_c^*, \qquad \kappa_c^* = \frac{p-\alpha+\beta\rho\Lambda}{\eta\beta^2(1-\rho^2)}, \qquad \xi_c^* = \frac{\delta - (1-\eta)\hat{\mathfrak{g}}^*}{\eta}. \tag{3.17}$$

Proof. Introduce $\hat{g} := g|_{\lambda=0}$, where g is defined by (3.10). Maximizing $\hat{g}$ and (3.9) leads to the above optimal strategy (3.17). Note that $\hat{\mathfrak{g}}^* = \max \hat{g}(y_1, y_2) = \hat{g}(\pi_c^*, \kappa_c^*)$, where π_c^* and κ_c^* are obtained in (3.17). □

We end this section by offering some explanations on the technical assumptions $C \geq 0$ and $\psi > 0$ imposed in Proposition 3.2 (or $C \geq 0$ in Proposition 3.1). First, since $\lim_{\eta \to 1} \psi = \delta > 0$ automatically holds, $\psi > 0$ is not needed for the log utility case in Proposition 3.1. By the model assumption in (2.3) and the definition of C in (3.4), if $\rho \geq 0$ (recall ρ measures the correlation between the risky asset and the insurable risk process), we always have $C \geq 0$ and such an assumption becomes redundant for $\rho \geq 0$. If $\rho < 0$ is the true scenario (as argued in Stein (2012) and Zou and Cadenillas (2014b) among many others), the assumption of $C \geq 0$ means that the risky investment opportunity cannot be too "good", comparing to the insurance business. To see this, we rewrite $C < 0$ as $\Lambda = \frac{\mu - r}{\sigma} > \frac{p-\alpha-\lambda\gamma}{-\rho\beta}$, where the left hand side is the Sharpe ratio of the risky asset, and the numerator of the right hand side measures the insurer's expected profit (including operation costs) from the insurance business. In practice, on the one hand, insurers are only allowed to invest in "safe" risky assets (with relatively low Sharpe ratio); while on the other hand, many insurance businesses are lucrative and insurers charge sufficient safe loading in premiums for solvency reasons. Therefore, imposing $C \geq 0$ not only guarantees the optimal liability ratio κ_c^* is non-negative but also makes economic sense. The other technical assumption $\psi > 0$ makes Problem (2.6) well-posed. Indeed, if this assumption fails (i.e. $\psi \leq 0$), we have $V(x) = +\infty$ when $0 < \eta < 1$ and $V(x) = -\infty$ when $\eta > 1$ (see Corollary 6.5 in Herdegen et al. (2020)). By (3.11), $\psi > 0 \Leftrightarrow \delta > \eta + (1-\eta)\mathfrak{g}^*$, i.e., the subjective discount rate δ should be greater than a threshold. Similar conditions are commonly imposed for infinite-horizon control problems; see, e.g., Jin et al. (2015)[Eq.(4.11)].

4 Verification for Admissible Strategies

We consider the HARA type utility function U, given by (2.7), in the formulation of our main stochastic problem (see Problem (2.6)). The special *scale property* of the value function inherited from the HARA utility and the classical results from optimal investment problems (see Merton (1969, 1971)) motivate us to make the following conjecture:

> The optimal strategy u^* over the admissible set $\mathcal{A}$ to Problem (2.6) is a constant strategy, and hence coincides with the optimal constant strategy u_c^* obtained in Section 3.

The goal of this section is to verify that the above conjecture is indeed correct when U is given by (2.7).

4.1 Notations

Previously in Sections 2 and 3, we have used simplified notations to make presentation more concise, since there is no risk of confusion there. Now we need to introduce notations in a more rigorous way for the general analysis. Given an initial wealth $x > 0$ and an admissible control u, we denote the insurer's wealth at time t by $X_t^{x,u}$, for all $t \geq 0$, which satisfies the stochastic differential equation (SDE) in (2.4). By Definition 2.3, the set of admissible strategies $\mathcal{A}$ depends on the insurer's initial wealth x ($x > 0$), and we will write it as $\mathcal{A}(x)$. However, it is still safe to use $\mathcal{A}_c$ to denote the set of constant strategies, as its definition in (3.1) is independent of the initial wealth. Introduce $X_t^{(1)} := X_t^{1,u_c^*}$ and $D_t^{(1)} := \xi_c^* X_t^{(1)}$ for all $t \geq 0$, where u_c^* is the optimal constant strategy obtained in (3.12). Namely, $X^{(1)}$ (resp. $D^{(1)}$) is the corresponding wealth process (resp. dividend process) under the unit initial wealth ($x = 1$) and the optimal constant strategy u_c^*.

Let us take two arbitrary admissible controls $u_1 = (\pi_1, \kappa_1, D_1) \in \mathcal{A}(x_1)$ and $u_2 = (\pi_2, \kappa_2, D_2) \in \mathcal{A}(x_2)$, where $x_1, x_2 > 0$. We define a new control $u = (\pi, \kappa, D)$, denoted by $u := u_1 \oplus u_2$, along with the corresponding wealth $X = (X_t^{x,u})_{t\geq 0}$ such that the following holds true: $x = x_1 + x_2$, $\pi_t X_t^{x,u} = \pi_{1,t} X_t^{x_1,u_1} + \pi_{2,t} X_t^{x_2,u_2}$, $\kappa_t X_t^{x,u} = \kappa_{1,t} X_t^{x_1,u_1} + \kappa_{2,t} X_t^{x_2,u_2}$, and $D_t = D_{1,t} + D_{1,t}$, which imply $X_t^{x,u} = X_t^{x_1,u_1} + X_t^{x_2,u_2}$. By this definite and the linearity of (2.4), we have $u_1 \oplus u_2 \in \mathcal{A}(x_1 + x_2)$ and $V(x_1) + V(x_2) \leq V(x_1 + x_2)$.

For any $u \in \mathcal{A}(x)$ and $\epsilon > 0$, define a *perturbed* objective functional J_ϵ by

$$J_\epsilon(x; u) := J(x + \epsilon; u \oplus u_c^*) = \mathbb{E}\left[\int_0^\infty e^{-\delta t} \, \frac{\left(D_t + \epsilon D_t^{(1)}\right)^{1-\eta}}{1-\eta} \, \mathrm{d}t\right], \tag{4.1}$$

where the second equality comes from the definitions of the operator $\oplus$ above and J in (2.5). Let us define the corresponding value function V_ϵ by

$$V_\epsilon(x) := \sup_{u \in \mathcal{A}(x)} J_\epsilon(x; u), \tag{4.2}$$

where J_ϵ is defined in (4.1).

4.2 Main Results

We now present the main result of this paper in Theorem 4.1.

Theorem 4.1. *Suppose $U(x) = x^{1-\eta}/(1-\eta)$, where $\eta > 0$ and $\eta \neq 1$, C defined in (3.4) is non-negative, and ψ defined in (3.11) is positive. For any $\epsilon > 0$ and $x > 0$, we have*

$$V_\epsilon(x) = V_c(x+\epsilon),$$

where V_ϵ is defined in (4.2) and V_c is obtained in (3.14).

Proof. Using the definition of J_ϵ in (4.1), we easily see that $J_\epsilon(x; u_c^*) = J(x+\epsilon; u_c^*) = V_c(x+\epsilon)$, where u_c^* is the optimal constant strategy derived in (3.12). Since $u_c^* \in \mathcal{A}(x)$, we obtain $V_\epsilon(x) \geq V_c(x+\epsilon)$. In the remaining of the proof, we aim to show the converse inequality, $V_\epsilon(x) \leq V_c(x+\epsilon)$, also holds.

Let $\epsilon > 0$ and $x > 0$ be given, take any admissible control $u \in \mathcal{A}(x)$. We define a new control $u^\epsilon := u \oplus u_c^*$, where the initial wealth associated with the strategy u_c^* is ϵ, and denote it by $u^\epsilon = (\pi^\epsilon, \kappa^\epsilon, D^\epsilon)$. Introduce the corresponding wealth process by $X^\epsilon = (X_t^\epsilon)_{t\geq 0}$, i.e., $X_t^\epsilon := X_t^{x+\epsilon, u^\epsilon}$. Recall from (3.2) that the wealth process under a constant strategy is always positive, which leads to

$$X_t^\epsilon = X_t^{x,u} + \epsilon X_t^{(1)} \geq \epsilon X_t^{(1)} > 0, \qquad \forall\, t \geq 0. \tag{4.3}$$

Define another new process M^ϵ by

$$M_t^\epsilon := \int_0^t e^{-\delta s} U(D_s^\epsilon)\, \mathrm{d}s + e^{-\delta t}\, V_c(X_t^\epsilon), \qquad \forall\, t \geq 0. \tag{4.4}$$

Recall from (3.14) that $V_c(x) = \psi^{-\eta} x^{1-\eta}/(1-\eta)$. Since V_c is a smooth function over $(0,\infty)$ and $X^\epsilon > 0$, we can apply Itô's lemma to M^ϵ. Using the SDE (2.4) of X^ϵ, we obtain

$$\begin{aligned} \mathrm{d}M_t^\epsilon = {} & e^{-\delta t} V_c(X_{t-}^\epsilon) \Big[(\sigma \pi_t^\epsilon - \beta\rho\kappa_t^\epsilon)\, \mathrm{d}W_t^{(1)} - \beta\sqrt{1-\rho^2}\kappa_t^\epsilon\, \mathrm{d}W_t^{(2)} + \big((1-\gamma\kappa_t^\epsilon)^{1-\eta} - 1\big)\, \mathrm{d}\widetilde{N}_t \Big] + \\ & e^{-\delta t} \bigg\{ -\delta V_c(X_t^\epsilon) + U(D_t^\epsilon) + V_c'(X_t^\epsilon)\big[(r + (\mu - r)\pi_t^\epsilon + (p-\alpha)\kappa_t^\epsilon) X_t^\epsilon - D_t^\epsilon \big] \\ & \frac{1}{2} V_c''(X_t^\epsilon)\big(\sigma^2 (\pi_t^\epsilon)^2 - 2\beta\rho\sigma\pi_t^\epsilon\kappa_t^\epsilon + \beta^2 (\kappa_t^\epsilon)^2\big)(X_t^\epsilon)^2 + \lambda\big(V_c((1-\gamma\kappa_t^\epsilon)X_{t-}^\epsilon) - V_c(X_{t-}^\epsilon)\big) \bigg\}\, \mathrm{d}t. \end{aligned} \tag{4.5}$$

We decompose the $\mathrm{d}t$ term in (4.5) into two parts as $e^{-\delta t}(Y_t^{(1)} + Y_t^{(2)})\mathrm{d}t$, where

$$Y_t^{(1)} := U(D_t^\epsilon) - D_t^\epsilon\, V_c'(X_t^\epsilon) - \frac{\eta}{1-\eta}\, \big(V_c'(X_t^\epsilon)\big)^{1-\frac{1}{\eta}},$$

and $Y_t^{(2)}$ is the remaining part of the $\mathrm{d}t$ term in (4.5). Using the first-order condition, we show that the inequality $\frac{1}{1-\eta}\, y^{1-\eta} - y - \frac{\eta}{1-\eta} \leq 0$ holds true for all $y \geq 0$ (the maximum value is 0 taken at $y = 1$). By substituting $D_t^\epsilon\, (V_c'(X_t^\epsilon))^{1/\eta}$ for y in the above inequality, we obtain that $Y_t^{(1)} \leq 0$ for all $t \geq 0$. Recall the result of V_c in (3.14), from which we get $V_c'(x) = (\psi x)^{-\eta}$ and $V_c''(x) = -\eta\psi^{-\eta} x^{-1-\eta}$. We then analyze $Y_t^{(2)}$ as follows

$$Y_t^{(2)} := \frac{\eta}{1-\eta}\, \big(V_c'(X_t^\epsilon)\big)^{1-\frac{1}{\eta}} - \delta V_c(X_t^\epsilon) + X_t^\epsilon\, V_c'(X_t^\epsilon)\big(r + (\mu - r)\pi_t^\epsilon + (p-\alpha)\kappa_t^\epsilon\big)$$

$$\frac{1}{2}V_c''(X_t^\epsilon)\big(\sigma^2(\pi_t^\epsilon)^2 - 2\beta\rho\sigma\pi_t^\epsilon\kappa_t^\epsilon + \beta^2(\kappa_t^\epsilon)^2\big)(X_t^\epsilon)^2 + \lambda\big(V_c((1-\gamma\kappa_t^\epsilon)X_{t-}^\epsilon) - V_c(X_{t-}^\epsilon)\big)$$
$$= (g(\pi_t^\epsilon,\kappa_t^\epsilon) - \mathfrak{g}^*)\cdot(1-\eta)V_c(X_t^\epsilon) = (g(\pi_t^\epsilon,\kappa_t^\epsilon) - \mathfrak{g}^*)\cdot\psi^{-\eta}(X_t^\epsilon)^{1-\eta},$$

where we have used the definitions of g in (3.10) and ψ in (3.11) to derive the last equality. Since $\mathfrak{g}^*$ is the maximum value of the function g over $\mathbb{R}\times[0,1/\gamma)$, we obtain $Y_t^{(2)} \le 0$ for all $t \ge 0$. In addition, we notice that $Y_t^{(1)} = Y_t^{(2)} = 0$ if and only if $u^\epsilon = u_c^*$.

Let us denote L^ϵ the local martingale part of $\mathrm{d}M_t^\epsilon$ in (4.5). We claim that L^ϵ is a supermartingale. To see this result, we define M_0^ϵ similar to that of M^ϵ in (4.4), but under $x = 0$ and $u \equiv 0$. By repeating the above analysis, we easily show that M_0^ϵ is a uniformly integrable martingale (recall with $u \equiv 0$, $u^\epsilon = u_c^*$). Now using the monotonicity of both U and V_c, we deduce that $M^\epsilon \ge M_0^\epsilon$, which immediately proves that L^ϵ is indeed a supermartingale as claimed. By the previous finding on $Y_t^{(1)}$ and $Y_t^{(2)}$, we get

$$\mathbb{E}\left[M_t^\epsilon\right] \le V_c(x+\epsilon) + \mathbb{E}\left[L_t^\epsilon\right] \le V_c(x+\epsilon). \tag{4.6}$$

In the final step, using (4.1), (4.3), (4.4), and the above results, we obtain

$$\begin{aligned} J_\epsilon(x;u) &= \lim_{t\to\infty}\mathbb{E}\left[\int_0^t e^{-\delta s}\,U(D_s^\epsilon)\,\mathrm{d}s\right] = \lim_{t\to\infty}\mathbb{E}\left[M_t^\epsilon - e^{-\delta t}\,V_c(X_t^\epsilon)\right] \\ &\le \limsup_{t\to\infty}\mathbb{E}\left[M_t^\epsilon\right] - \liminf_{t\to\infty}\mathbb{E}\left[e^{-\delta t}\,V_c(X_t^\epsilon)\right] \\ &\le \limsup_{t\to\infty}\mathbb{E}\left[M_t^\epsilon\right] \le V_c(x+\epsilon), \end{aligned}$$

where, to prove the second inequality above, we have used the following result

$$\liminf_{t\to\infty}\mathbb{E}\left[e^{-\delta t}\,V_c(X_t^\epsilon)\right] \ge \psi^{-\eta}\epsilon^{1-\eta}\,\liminf_{t\to\infty}\mathbb{E}\left[e^{-\delta t}\,\frac{(X_t^{(1)})^{1-\eta}}{1-\eta}\right] = 0. \tag{4.7}$$

By taking supremum over $\mathcal{A}(x)$, we obtain $V_\epsilon(x) \le V_c(x+\epsilon)$. The proof is now complete. □

Using Theorem 4.1 and the monotonicity result $J(x;u) \le J_\epsilon(x;u)$, we have the following corollary that eventually verifies the conjecture in the opening of this section.

Corollary 4.2. *Under the same assumptions as in Theorem 4.1, we have*

$$V(x) = V_c(x) = J(x, u_c^*), \qquad \forall\, x > 0,$$

where u_c^* and V_c are given respectively by (3.12) and (3.14), and V is the value function to Problem (2.6).

In both Theorem 4.1 and Corollary 4.2, the utility function is of power form, with $\eta \ne 1$. When $\eta = 1$ (i.e., the log utility case), the same result holds, namely, we still have $V(x) = V_c(x)$, where now V_c is given by (3.8) from Proposition 3.1. The analysis leading to this conclusion is similar to the one given above for the power utility case, and is thus omitted.

Corollary 4.3. *Suppose $U(x) = \ln x$ and the constant C defined in (3.4) is non-negative. We have*

$$V(x) = V_c(x) = J(x, u_c^*), \qquad \forall\, x > 0,$$

where u_c^* and V_c are given respectively by (3.7) and (3.8), and V is the value function to Problem (2.6).

4.3 Comparisons with the HJB Approach

In this subsection, we compare the perturbation approach adopted in this paper (see Herdegen et al. (2020)) with the standard HJB approach of solving stochastic control problems (see, e.g., Merton (1969, 1971)).

The perturbation approach works brilliantly for the insurer's problem under HARA (power and log) utility considered in Section 4.2. In addition, the proof to Theorem 4.1 applies to all the cases of the relative risk aversion η: $0 < \eta < 1$, $\eta = 1$, and $\eta > 1$. In comparison, the HJB approach encounters technical issues when $\eta > 1$. For this reason, many related existing works that rely on the HJB approach assume $0 < \eta \leq 1$; see Cadenillas et al. (2007), Thonhauser and Albrecher (2011), Jin et al. (2015), and Xu et al. (2020). Below we discuss some of these issues in details.

- A key step in the verification theorem under the HJB approach is to guarantee the validness of interchanging the order of a limit and a conditional expectation (integral) for a family of processes when passing the limit to infinity. A sufficient condition for such an interchange is the uniform integrability of the processes, which can be achieved by imposing the growth condition on the value function or the utility function. For instance, Korn and Korn (2001)[Theorem 5.17, p.229] assume that $|V(t,x)| \leq K(1+|x|^k)$ holds for the candidate value function $V(t,x)$, where $K > 0$ and $k \in \mathbb{N}$; Sotomayor and Cadenillas (2009)[Eq.(2.2)] assume that $U(x) \leq K(1+x)$ holds for the utility function U, where $K, x > 0$; see also a similar condition in Capponi and Figueroa-López (2014)[Eq.(4.10)]. Alternatively, one may "simply" impose a much stronger condition of uniform boundedness on both the drift and the diffusion terms; see Fleming and Soner (2006)[Section IV.5, Eq.(5.2), p.164]. It is trivial to see that both "solutions" are restrictive and do not apply to Problem (2.6) when $\eta > 1$.

- The second technical issue we discuss is the possibility of bankruptcy (i.e., $X_t^{x,u} = 0$ at some $t < \infty$). A full and rigorous treatment of bankruptcy under the standard approaches requires lengthy technical arguments. Karatzas et al. (1986) introduce a free parameter P as the bequest value at the bankruptcy time and consider a modified version of the original problem (which has a boundary condition $\lim_{x \downarrow 0} V^P(x) = P$), and take pages to show that the value function $V^P(x)$ of the modified problem converges to the value function $V(x)$ of the original problem as $P \to -\infty$.

- Last, a commonly required condition for an infinite horizon problem is the so-called transversality condition, which is given by

$$\lim_{t \to \infty} \mathbb{E}\left[e^{-\delta t} \frac{(X_t^u)^{1-\eta}}{1-\eta}\right] = 0 \quad \text{or in a weaker version} \quad \liminf_{t \to \infty} \mathbb{E}\left[e^{-\delta t} \frac{(X_t^u)^{1-\eta}}{1-\eta}\right] \geq 0. \tag{4.8}$$

  The above transversality condition (4.8) is satisfied automatically if $0 < \eta < 1$ but may fail in general when $\eta > 1$. Proving (4.8) is certainly not trivial when $\eta > 1$, although many applications "simply" assume (4.8) holds to avoid a proof. In fact, (4.8) may not be expected; see Herdegen et al. (2020)[Remark 4.7].

Further technical discussions can be found in Herdegen et al. (2020). We remark that another standard approach, the martingale (duality) method, faces the "dual" side of the technical issues/assumptions in the HJB approach. For that, we refer readers to Karatzas and Shreve (1998)[Section 3.9] on optimal investment problems over an infinite horizon.

Having seen some of the technical difficulties under the HJB approach, we now explain why the perturbation approach proposed in Herdegen et al. (2020) works so well in the proof of Theorem 4.1. First, by (4.3), the perturbed wealth process X^ϵ is *strictly* positive, immediately yielding a significant advantage. That is, we no longer need to deal with bankruptcy in the proof of Theorem 4.1. Second, by utilizing the explicit results from Section 3 and the monotonicity of the utility/value functions, we easily show that M_0^ϵ is uniformly integrable and $M_t^\epsilon \geq M_{0,t}^\epsilon$ for all $t \geq 0$. (Note M^ϵ is defined by (4.4) and M_0^ϵ is a special version of M^ϵ under $x = 0$ and $u = 0$.) There results together verify that the process L^ϵ, the local martingale part of M^ϵ, is a supermartingale, which leads to an essential inequality in (4.6). In other words, a carefully-chosen special (optimal) strategy enables us to obtain a lower bound on the process M^ϵ, bypassing the difficulty of proving uniform integrability of $(e^{-\delta t} V(X_t^u))_{t \geq 0}$. Note that in the definition of M^ϵ in (4.4), the second term is $e^{-\delta t} V_c(X_t^\epsilon)$, not $e^{-\delta t} V(X_t^u)$ as in the standard HJB approach. Here the advantages are at least twofold: (1) V_c is already obtained explicitly in (3.14) and is smooth, while the value function V is assumed to be smooth and is yet to be solved from the associated HJB equation, and (2) $X_t^\epsilon > 0$ but $X_t^u \geq 0$ for all t. Last, we point out that we derive an inequality in (4.7), which is in the same spirit to the transversality condition in (4.8). To be precise, replacing X^u by X^ϵ in (4.8) leads to (4.7). In our case, proving (4.7) is very easy, by the monotone increasing property of V_c and (4.3). However, as already pointed out previously, the transversality condition (4.8) does not hold in general, and even for specific problems when it does hold, proving it is not trivial in the case of $\eta > 1$ due to the possibility of bankruptcy.

In the last part, we discuss the shortcomings of the perturbation approach. Such an approach is restrictive and only applies to the HARA-type utility function under a market model with deterministic parameters. In some sense, this approach can be seen as a tailored method to tackle stochastic control problems under power utility with risk aversion $\eta > 1$. On the other hand, the standard HJB and martingale approaches can be applied to solve a wide range of control problems beyond utility maximization.

5 Economic Analysis

In this section, we conduct an economic analysis to study how the model parameters and risk aversion affect the insurer's optimal strategy u^*. Due to the popularity of applying a diffusion process to approximate the risk model (see Browne (1995) and Højgaard and Taksar (1998)), we assume there are no jumps in the risk process R (setting $\lambda = 0$ in (2.2)), unless stated otherwise. Recall that, in the case of no jumps, we obtain the optimal strategy $u^* = u_c^*$, given by (3.17), in Corollary 3.3. We divide the economic analysis along two directions, with analytic results in Section 5.1 and numerical results in Section 5.2.

5.1 Analytic Results

In the insurance literature, one often assumes that the insurance market is *independent* of the financial market, i.e., $\rho = 0$ in (2.2). But as argued in Stein (2012) and many others, ignoring the possible dependence between the two markets could lead to catastrophic consequences (e.g., the infamous AIG case in the financial crisis of 2007-2008). Hence in the first study, we investigate the impact of ρ on the insurer's optimal strategy. By (3.17), we obtain

$$\frac{\partial \pi_c^*}{\partial \rho} = \frac{\beta}{\sigma}\,\kappa_c^* > 0, \qquad \frac{\partial \kappa_c^*}{\partial \rho} = \frac{\Lambda}{\eta\beta} + 2\rho\kappa_c^*, \qquad \frac{\partial \xi_c^*}{\partial \rho} = -\frac{1-\eta}{\eta^2}\,\Big[\beta\Lambda\kappa_c^* + \rho\big(\kappa_c^*\big)^2\Big], \tag{5.1}$$

where the Sharpe ratio $\Lambda = (\mu - r)/\sigma$ is defined in (3.5). An interesting result is that $\partial\pi_c^*/\partial\rho > 0$, which is also found in Zou and Cadenillas (2014b)[Figures 1-3] and Shen and Zou (2020)[Figure 1]. By (3.17) and (5.1), when $\rho > 0$, we observe $\partial\kappa_c^*/\partial\rho > 0$ and $\text{Sign}\,(\partial\xi_c^*/\partial\rho) = \text{Sign}\,(\eta - 1)$, and further $\pi_c^* > 0$. Note that when $\rho > 0$ the risky asset provides a natural hedge to the insurance business: an increase in the risk R (losses to the insurer) is associated with an increase in the risky asset price S. This observation implies that the insurer should long the risky asset, explaining why $\pi_c^*|_{\rho>0} > 0$. For the same reason, when the *positive* correlation becomes stronger, the hedging effect amplifies, or equivalently, the "risky" insurance business becomes "less risky", indicating $\frac{\partial\kappa_c^*}{\partial\rho}|_{\rho>0} > 0$. Regarding the result of $\partial\xi_c^*/\partial\rho$ in (5.4), observe that the inequality $\beta\Lambda\kappa_c^* + \rho(\kappa_c^*)^2 > 0$ holds if and only if $\rho > -\beta\Lambda/\kappa_c^*$, which can be seen as a good indication of the overall market condition.[1] Next, recall that the power utility function in (2.7) is applied to quantitatively measure the welfare of dividend payments (monetary wealth). In similar settings, empirical evidence often shows that the relative risk aversion η is greater than 1; see for instance Meyer and Meyer (2005)[Table 1]. As a result, it is reasonable for us to assume $\eta > 1$, although we do not rule out the possibility of $0 < \eta \leq 1$. Now when ρ increases in the region $(-\beta\Lambda/\kappa_c^*, 1)$, the overall market condition is likely to become more favorable to the insurer. As such, insurers with high risk aversion ($\eta > 1$) pay dividend at a higher rate, since they prefer to "cash out" now while the market is still in "good" regime; on the other hand, insurers with low risk aversion ($0 < \eta < 1$) reduce the dividend rate, as the desire to invest more capital in the risky asset is dominating. We end this study by commenting that our theoretical result in (5.1) offers insight on $\eta > 1$ as shown in the empirical findings. To see this, let us assume for a moment that $\rho > 0$. Then as ρ increases, the overall market improves for the insurer and naturally we expect all the insurer's strategies π_c^*, κ_c^*, and ξ_c^* to increase at the same time; while (5.1) shows that ξ_c^* increases only if $\eta > 1$.

As seen in the above analysis on the correlation coefficient ρ, the insurer's (relative) risk aversion parameter η is decisive in the comparative statics of the optimal dividend strategy. We next analyze how

[1]Recall that $\pi_c^* = \frac{\mu - r}{\eta\sigma^2} + \frac{\rho\beta}{\sigma}\kappa_c^*$, where all the components are positive except the possibility of ρ being negative. When ρ is negative, the second term of hedging demand becomes negative and acts on the opposite direction of the first term (Merton's strategy) in π_c^*. When $\rho \leq -\beta\Lambda/\kappa_c^*$, the negative second term dominates the positive first term in π_c^*, i.e., the insurer needs to "short sell" the risky asset mainly for the hedge purpose, which may lead to excessive risk taking.

the changes in η affect the insurer's optimal strategy. To that end, we obtain from (3.17) that

$$\frac{\partial \pi_c^*}{\partial \eta} = -\frac{1}{\eta}\,\pi_c^*, \qquad \frac{\partial \kappa_c^*}{\partial \eta} = -\frac{1}{\eta}\,\kappa_c^* < 0, \qquad \frac{\partial \xi_c^*}{\partial \eta} = -\frac{1}{\eta}\,\xi_c^* + \frac{\hat{\mathfrak{g}}^* + (\eta-1)r}{\eta^2}. \tag{5.2}$$

Due to (3.15), $\kappa_c^* > 0$ and that implies a higher risk aversion always leads to a reduction of underwriting in the insurance business. Since $\mathrm{Sign}(\partial \pi_c^*/\partial \eta) = -\,\mathrm{Sign}(\pi_c^*)$, when the optimal investment strategy is a "buy" strategy (resp. a "short-sell" strategy), the investment proportion in the risky asset reduces (resp. increases) as the risk aversion η increases. Notice that when $\pi_c^* < 0$, an increase in π_c^* means the absolute value of π_c^* decreases. Therefore, given an increase in the risk aversion η, the insurer always invests less proportion in absolute values in the risky asset (i.e., less risk taking). Such a result is clearly consistent with the economic definition of risk aversion. Despite the analytical result of $\partial \xi_c^*/\partial \eta$ in (5.2), how risk aversion affects the dividend strategy is still unclear, since the first term is always negative but the second term is always positive (note $\hat{\mathfrak{g}}^* - r > 0$ by (3.16)).

In the third analysis, we focus on the impact of the financial market on the optimal strategy. Introduce the excess return $\bar{\mu}$ and the excess premium $\bar{p}$ by

$$\bar{\mu} := \mu - r \qquad \text{and} \qquad \bar{p} := p - \alpha, \tag{5.3}$$

where $\bar{\mu} > 0$ and $\bar{p} > 0$ by (2.3). To see how the excess return $\bar{\mu}$ affects the insurer's decision, we compute

$$\frac{\partial \pi_c^*}{\partial \bar{\mu}} = \frac{1}{\eta \sigma^2 (1-\rho^2)} > 0, \qquad \frac{\partial \kappa_c^*}{\partial \bar{\mu}} = \frac{\rho}{\eta \beta \sigma (1-\rho^2)}, \qquad \frac{\partial \xi_c^*}{\partial \bar{\mu}} = -\frac{1-\eta}{\eta^2}\,\frac{\rho \bar{p} + \beta \Lambda}{\beta \sigma (1-\rho^2)}. \tag{5.4}$$

By (5.4), following an increase in $\bar{\mu}$, the insurer invests more in the risky asset and underwrites more (resp. less) insurance policies if $\rho > 0$ (resp. $\rho < 0$). These findings can be explained using the discussions of ρ in the above study and the "fact" that the higher the excess return $\bar{\mu}$, the more attractive the risky asset. Note that in our framework the insurer is exposed to both the investment risk from the risky asset S and the insurable risk R. But according to (5.4), $\partial \pi_c^*/\partial \bar{\mu} > 0$ always holds, regardless of the sign of ρ. That is, even though we allow a non-zero correlation ρ in our model, the result $\partial \pi_c^*/\partial \bar{\mu} > 0$ is still in line with the standard optimal investment literature; see Merton (1969, 1971) and Herdegen et al. (2020). We mention that ρ impacts the sensitivity magnitude of π_c^* with respect to $\bar{\mu}$: the stronger the correlation, the more sensitive π_c^* is to the changes of $\bar{\mu}$. Assuming $\rho \bar{p} + \beta \Lambda > 0$,[2] as $\bar{\mu}$ increases, insurers with low risk aversion ($0 < \eta < 1$) reduce the dividend rate but insurers with high risk aversion ($\eta > 1$) reacts exactly the opposite way. But, if $\rho \bar{p} + \beta \Lambda < 0$,[3] the converse of the above statement holds true. Another key parameter in the financial market is the volatility σ of the risky asset, and the related sensitivity results are obtained by

$$\frac{\partial \pi_c^*}{\partial \sigma} = -\frac{(2-\rho^2)\Lambda}{\eta \sigma^2 (1-\rho^2)} - \frac{\rho \beta}{\sigma^2}\,\kappa_c^*, \qquad \frac{\partial \kappa_c^*}{\partial \sigma} = -\frac{\rho \bar{\mu}}{\eta \beta \sigma^2 (1-\rho^2)}, \qquad \frac{\partial \xi_c^*}{\partial \sigma} = \frac{1-\eta}{\eta^2}\,\frac{\bar{\mu}}{\sigma^2}\,\frac{\rho \bar{p} + \beta \Lambda}{\beta (1-\rho^2)}. \tag{5.5}$$

[2] A sufficient condition for this inequality is $\rho \geq 0$, as $\bar{p}, \beta, \Lambda > 0$.

[3] Equivalently, $\rho < 0$ and $\bar{p} > -\beta\Lambda/\rho > 0$. The economic meaning of this condition is that, in an "adverse" market (since correlation is negative), the insurer sets the net insurance premium above a threshold.

If we set $\rho = 0$ in (5.5), we have $\partial\pi_c^*/\partial\sigma = -2\Lambda/(\eta\sigma^2) < 0$, but such a result is *not* expected in general, which is different from the standard literature; see Merton (1969, 1971). From (5.5), we observe that $\partial\pi_c^*/\partial\sigma < 0$ whenever $\rho \geq 0$ and $\operatorname{Sign}(\partial\kappa_c^*/\partial\sigma) = -\operatorname{Sign}(\rho)$. Consequently, assuming the two markets are positively correlated, when the volatility σ increases, the insurer invests less in the risky asset and reduces insurance liabilities. Under a negatively correlated condition, the insurer's reaction to an increase of the volatility σ is to underwrite more policies. However, given $\rho < 0$, how the insurer should adjust her investment strategy to an increase of σ is not easily seen, and will be investigated numerically in the next subsection. Comparing (5.3) with (5.5) shows that $\partial\xi_c^*/\partial\sigma = -\Lambda\,\partial\xi_c^*/\partial\bar{\mu}$, and thus the opposite side of the previous discussions on $\partial\xi_c^*/\partial\bar{\mu}$ applies here.

Our last agenda is to study the impact of the insurance market, the drift parameter α and the diffusion parameter β in the risk process (2.2), on the insurer's optimal strategy. We first focus on the optimal liability strategy κ_c^* and obtain

$$\frac{\partial\kappa_c^*}{\partial\alpha} = -\frac{1}{\eta\beta^2(1-\rho^2)} < 0 \qquad \text{and} \qquad \frac{\partial\kappa_c^*}{\partial\beta} = -\frac{2\bar{p}}{\eta\beta^3(1-\rho^2)} < 0. \tag{5.6}$$

The sensitivity results in (5.6) fit exactly our intuition: when α or β increases, the insurable risk R becomes more risky and the insurer should reduce underwriting in response. Regarding the optimal investment strategy π_c^*, we have

$$\frac{\partial\pi_c^*}{\partial\alpha} = \frac{\rho\beta}{\sigma}\frac{\partial\kappa_c^*}{\partial\alpha} \qquad \text{and} \qquad \frac{\partial\pi_c^*}{\partial\beta} = \frac{\rho\beta}{\sigma}\left(\frac{\kappa_c^*}{\beta} + \frac{\partial\kappa_c^*}{\partial\beta}\right). \tag{5.7}$$

An immediate result is that $\operatorname{Sign}(\partial\pi_c^*/\partial\alpha) = \operatorname{Sign}(\rho)$, which is consistent with the preceding analysis on ρ. But the story of $\partial\pi_c^*/\partial\beta$ in (5.7) is more complex; it not only depends on the correlation ρ but also κ_c^* (positive by (3.15)) and its derivative with respect to β (negative by (5.6)). Last, we analyze the optimal dividend strategy ξ_c^* and obtain

$$\frac{\partial\xi_c^*}{\partial\alpha} = \frac{1-\eta}{\eta}\,\kappa_c^* \qquad \text{and} \qquad \frac{\partial\xi_c^*}{\partial\beta} = \frac{1-\eta}{\eta^2}\,\frac{\bar{p}^2 + \beta\rho\bar{p}\Lambda}{\beta^3(1-\rho^2)}. \tag{5.8}$$

From (5.8), we easily see that $\operatorname{Sign}(\partial\xi_c^*/\partial\alpha) = \operatorname{Sign}(1-\eta)$ and $\operatorname{Sign}(\partial\xi_c^*/\partial\beta) = \operatorname{Sign}(1-\eta)$ if $\bar{p} > \max\{0, -\beta\rho\Lambda\}$. Now if we assume $\eta > 1$ and $p > \max\{0, -\beta\rho\Lambda\}$, both derivatives in (5.8) are negative, implying that the insurer should reduce the optimal dividend payment rate when the insurance business becomes more risky. This finding may further support that $\eta > 1$ is a more reasonable scenario.

5.2 Numerical Results

In Section 5.1, we analyze the sensitivity of the optimal strategy with respect to various model parameters *analytically*. Here, in this subsection, we continue the same analysis, but from a numerical point of view. In particular, we focus on two important parameters: the correlation coefficient ρ and the relative risk aversion η.

Table 1: Default Model Parameters

Parameters	r	μ	σ	α	β	p	δ
Values	0.01	0.05	0.25	0.1	0.1	0.15	0.15

Note. δ is the subjective discount factor, p is the premium rate, and all other parameters are from the model (2.1) and (2.2).

Since our sensitivity analysis is *qualitative*, we set the base values for the model parameters of (2.1)-(2.2) in Table 1 and allow *one* parameter to vary over a reasonable range in each study. We compute the insurer's optimal investment, liability ratio, and dividend rate strategies $u^* = (\pi^*, \kappa^*, \xi^*)$, where $u^* = u_c^*$ in (3.17), when the correlation coefficient ρ varies from -0.8 to 0.8. We plot the results under four different risk aversion levels ($\eta = 0.8, 1, 2, 5$) in Figure 1. Our main findings are summarized as follows:

- (Optimal investment π^* in the upper panel of Figure 1)

  We observe that π^* is an increasing function of ρ, which verifies the result in (5.1). In consequence, the insurer invests more in the risky asset as ρ increases. There exists a *negative* threshold $\hat{\rho}$[4] ($\hat{\rho}$ is about -0.32 in the numerical example), below which the optimal investment π^* is negative, i.e., short selling is optimal. We also notice that the absolute investment weight $|\pi^*|$ in the risky asset is a decreasing function of the relative risk aversion η. Economically, that means insurers with higher relative risk aversion invest more conservatively than those with lower relative risk aversion, which is obviously consistent with the definition of risk aversion.

- (Optimal liability ratio κ^* in the middle panel of Figure 1)

  κ^* is *not* a monotone function of ρ globally. Instead, there exists a negative threshold $\hat{\rho}$ ($\hat{\rho}$ is around -0.15 in the numerical example) such that κ^* is increasing over $(\hat{\rho}, 1)$ and decreasing over $(-1, \hat{\rho})$. In addition, the increasing part is "steeper" than the decreasing part, i.e., the same increment of ρ over $(\hat{\rho}, 1)$ has a bigger impact on κ^* than the one over $(-1, \hat{\rho})$. On the other hand, we observe that κ^* is a decreasing function of the risk aversion parameter η. Therefore, an increase in risk aversion always leads to a decrease in the optimal liability ratio κ^*.

- (Optimal dividend rate ξ^* in the bottom panel of Figure 1)

  An immediate and also important observation is that the shape of ξ^* differs dramatically over different regions of η. When $\eta = 1$ (corresponding to log utility), the optimal dividend rate ξ^* is a straight line, equal to δ ($\delta = 0.15$ from Table 1), confirming the result in (3.7). When $0 < \eta < 1$, ξ^* is a concave function of ρ, increasing first from -1 to a negative threshold (around -0.25 in Figure 1) and decreasing afterwards. When $\eta > 1$, ξ^* is a convex function of ρ, decreasing first and then increasing. Furthermore, when η increases over $(1, \infty)$, the insurer reduces the optimal dividend rate ξ^*.

[4]We use $\hat{\rho}$ to denote a genetic threshold constant of ρ, which may differ content by content.

Our sensitivity analysis so far is based on the assumption that there are no jumps in the risk process R, given by (2.2). In what follows, we relax this assumption and numerically study how the jump intensity λ affects the insurer's optimal strategy u^*, which is now given by (3.12) in Proposition 3.2. Due to the inclusion of jumps, we now set the premium rate by the expected value principle: $p = (1+\theta) \times (\alpha + \lambda\gamma)$, where $\theta = 50\%$.[5] We further assume the insurer's risk aversion is $\eta = 2 > 1$ and fix the jump size $\gamma = 0.3$ (3 times the diffusion parameter $\beta = 0.1$). We plot the insurer's optimal investment π^*, liability ratio κ^*, and dividend rate ξ^* as a function of λ over $(0.01, 0.2)$ in Figure 2. The key results are summarized as follows:

- (Optimal investment π^* in the upper panel of Figure 2)

  As anticipated, the results are divided into two regions of the correlation coefficient ρ by a negative threshold $\hat{\rho}$. When ρ is greater than $\hat{\rho}$, the optimal investment strategy π^* is a decreasing function of the jump intensity λ, and thus the insurer invests less in the risky asset as λ increases. When ρ is less than $\hat{\rho}$, π^* becomes an increasing function of λ. In particular, for a very negative ρ (e.g., $\rho = -0.6$ in Figure 2), π^* may even change from a negative value to a positive value when λ increases. These results can be explained by the expression of π^* in (3.12), which has a positive myopic term, independent of λ, and a hedging term with the same sign as ρ, which depends on λ through κ^*. Last, Figure 2 also confirms the finding from Figure 1 that π^* is an increasing function of ρ.

- (Optimal liability ratio κ^* in the middle panel of Figure 2)

  We observe that κ^* is a decreasing function of λ. The economic explanation for such a result is that, as λ increase, the insurable risk becomes more risky and less profitable, and the insurer's rational response is to reduce underwriting. Another observation from the numerical example is that κ^* increases as the correlation ρ increases.

- (Optimal dividend rate ξ^* in the bottom panel of Figure 2)

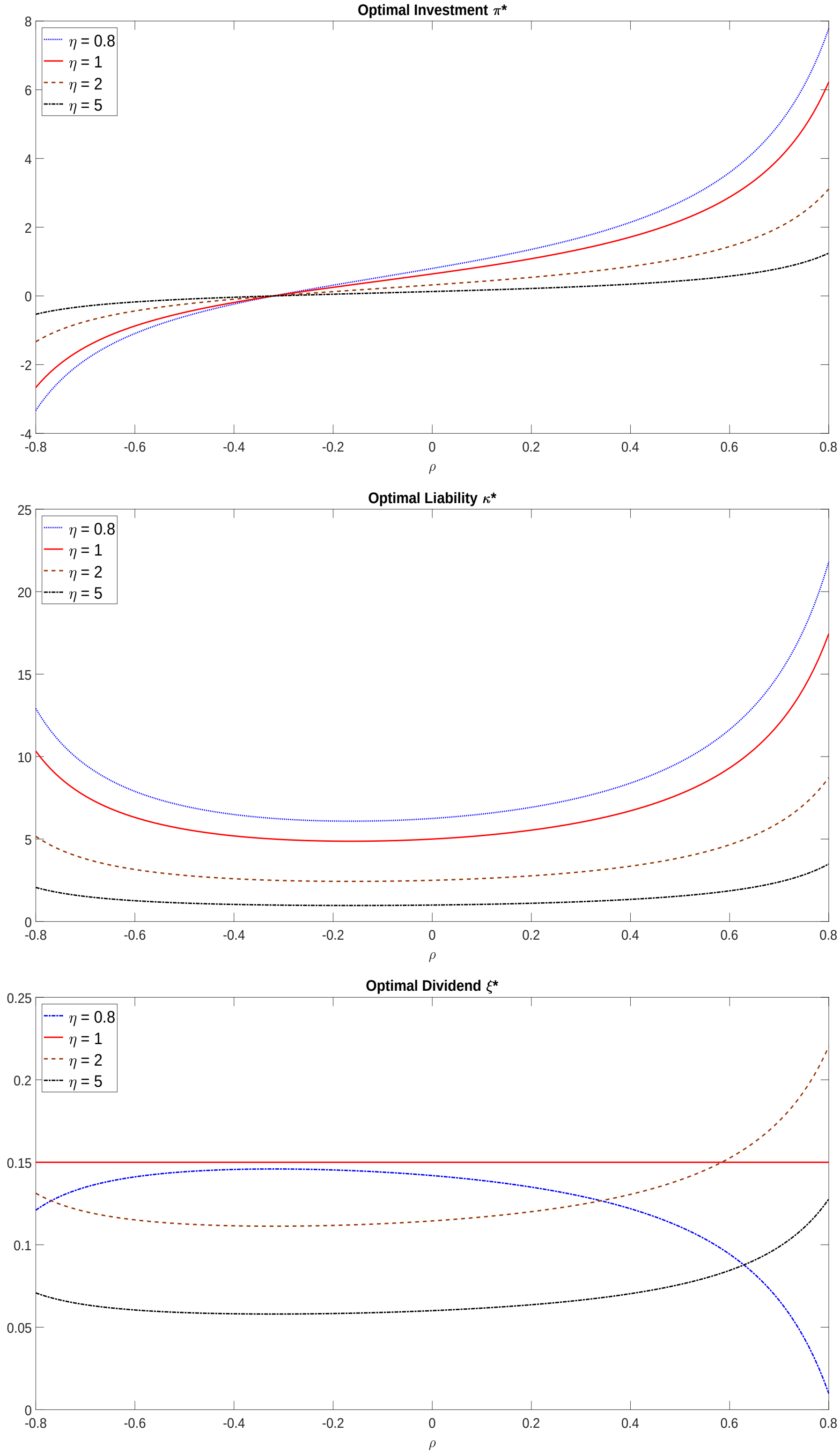
  Similar to the result of π^*, how the jump intensity λ affects the optimal dividend strategy ξ^* also depends on the value of ρ. When ρ is greater than a negative threshold $\hat{\rho}$, a negative relation between ξ^* and λ is shown by Figure 2. However, for a very negative ρ (e.g., $\rho = -0.6$ in Figure 2), it is possible that ξ^* goes up as λ increases. Following the discussions on ρ in Section 5.1, let us interpret $\rho > \hat{\rho}$ as a "preferable" market condition and $\rho < \hat{\rho}$ an "adverse" market condition. In a preferable market, although an increase in λ makes the insurance business less profitable, the insurer still wants to maintain the surplus at a relatively high level (since the market is preferable), and to achieve so, the insurer reduces the optimal dividend rate. To the contrary, in an adverse market, the insurer pays dividend at a higher rate when λ increases, that is because keeping a large surplus is no longer attractive when the available business opportunities are unfavorable. In other words, when

[5] Such a rule on premium guarantees that the assumptions in (3.11) hold true. Note that the choice of $p = 0.15$ in Table 1 is equivalent to $p = (1 + 50\%) \times (\alpha + \lambda\gamma)$, with $\lambda = 0$ there.

the market is in adverse condition, the "utility" of cashing out dividends now outweighs the "utility" of saving for business opportunities in future.

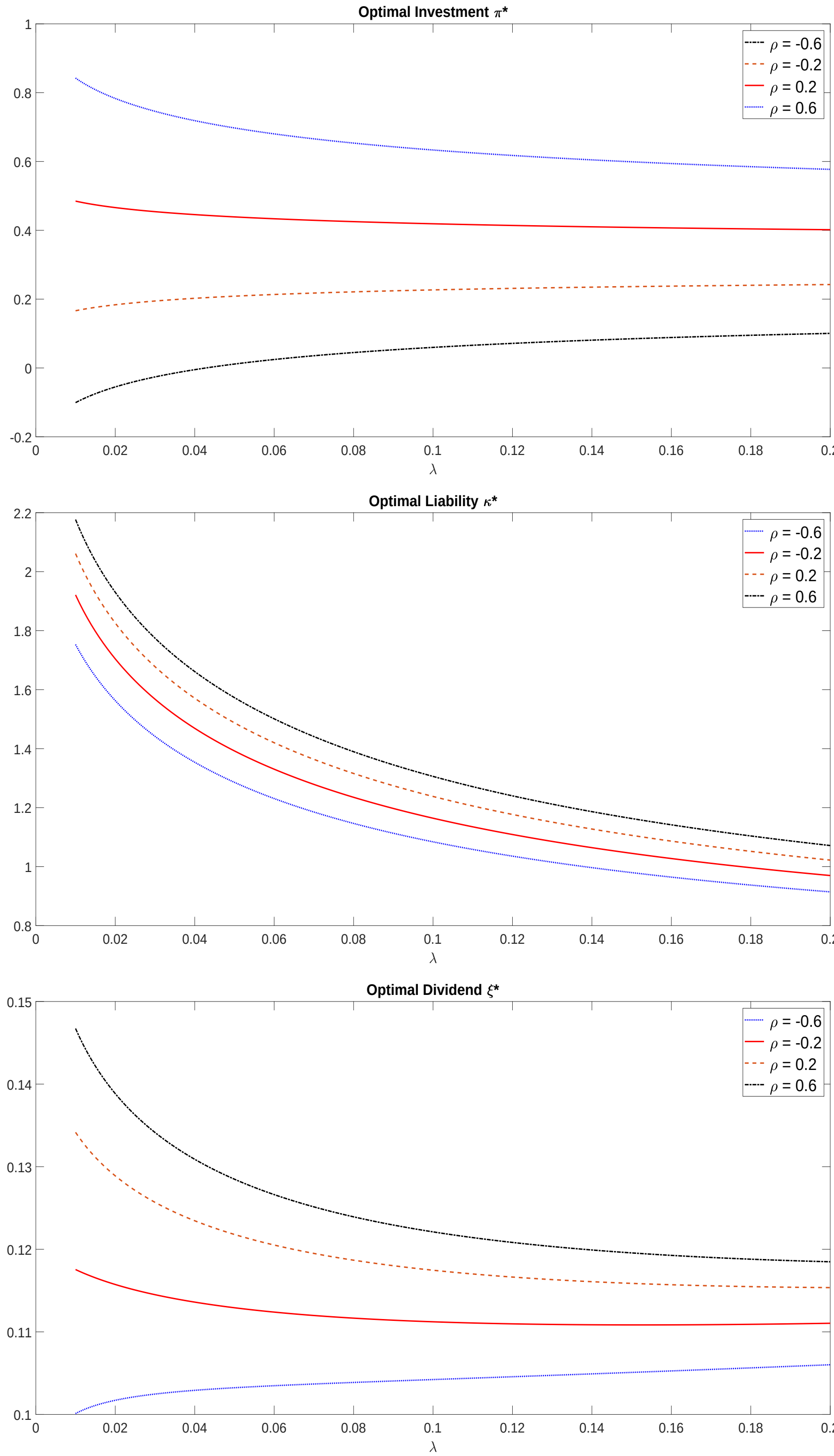
6 Conclusions

Figure 1: Impact of the Correlation Coefficient ρ and Risk Aversion η on the Optimal Strategies



Note. We plot the insurer's optimal investment π^* (upper panel), liability κ^* (middle panel), and dividend ξ^* (lower), as a function of the correlation coefficient ρ over $(-0.8, 0.8)$, under four different risk aversion levels $\eta = 0.8$ (dotted blue), $\eta = 1$ (solid red), $\eta = 2$ (dashed brown), and $\eta = 5$ (dash-dot black). The parameters are chosen from Table 1.

Figure 2: Impact of the Jump Intensity λ on the Optimal Strategies



Note. We plot the insurer's optimal investment π^* (upper panel), liability κ^* (middle panel), and dividend ξ^* (lower), as a function of the jump intensity λ over $(0.01, 0.2)$, under four different correlation levels $\rho = -0.6$ (dotted blue), $\rho = -0.2$ (solid red), $\rho = 0.2$ (dashed brown), and $\rho = 0.6$ (dash-dot black). We set $p = 1.5(\alpha + \lambda\gamma)$, $\eta = 2$, $\gamma = 0.3$, and the rest by Table 1.

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